INDEX BOUNDEDNESS CONDITION FOR MAPPINGS WITH BOUNDED DISTORTION

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1. Let $U \subset R^n$ be an open domain, i.e., a connected open set, and let $f:U \to R^n$ be a certain mapping. We say that the mapping f belongs to the class $W_n^1(U)$ if each of the real functions $f_1, f_2, \ldots, f_n$ constituting the vector function f is locally additive in U and has therein generalized first derivatives (see [7]) locally additive in U in the n-th power.

Let $f \in W_n^1(U)$. We set

$$\lambda(x, f) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{\partial f_i}{\partial x_j}(x)\right]^2, \quad J(x, f) = \det\left(\frac{\partial f_i}{\partial x_j}\right) \quad (i, j = 1, 2, \ldots, n).$$

A mapping $f:U \to R^n$ of the class $W_b^1(U)$ is called a mapping with bounded distortion if there exists a constant $K (1 \le K < \infty)$ such that for almost all $x \in U$

$$[\lambda(x, f)]^{n/2} \leqslant n^{n/2} K^{n/2} J(x, f).$$

We denote the least possible value of the constant K by the symbol q(f) and call it the distortion coefficient of the mapping f (in the terminology introduced in [3] q(f) is the distortion coefficient of f in the conformal norm M_2).

The notion of a mapping with bounded distortion was introduced in [1]. Its characteristics are investigated in [2] and [3].

The notion of a mapping with bounded distortion is similar to that of a quasi-conformal mapping in space. Mappings with bounded distortion are essentially multisheeted quasi-conformal mappings in space.

Let $G \subset R^n$ be a compact domain, i.e., a compact set, the open kernel of which is connected and is such that the closure of the open kernel of G coincides with G. We consider the arbitrary continuous mapping $f:G \to R^n$. Every point $y \in R^n$ such that $y \notin f(\mathrm{Fr}\, G)$ (Fr A is the boundary of a set $A \subset R^n$) can be compared with a certain number $\mu(y, f)$ representing the index of the point y with respect to the mapping f (see [5, 6], for example, for a definition of the index function). Additional information needed with regard to the index may also be found in [2].

Let $U \subset R^n$ be an open domain in R^n. We say that a mapping $f:U \to R^n$ satisfies the index boundedness condition if for every compact domain $G \subset U$ the function $y \to \mu(y, f\, G)$ is bounded.

Every mapping with bounded distortion is continuous, as proved in [1]. The purpose of the present article is to demonstrate that every mapping with bounded distortion satisfies the index boundedness condition.

The index boundedness condition was introduced in [2] as an additional requirement on the mapping in the investigation of mappings with bounded distortion. It follows from the results of the present article, therefore, that the requirement is in fact not a restriction.

We can lean heavily on the results of [4] for the proof of the fundamental theorem.

Let $f:U \to R^n$ be a mapping with bounded distortion. We use the following notation:

$$f_{i,j} = \partial f_i / \partial x_j, \quad f_{,j} = \partial f / \partial x_j.$$

The iterated subscripts indicate summation from 1 to n. We let $L_f(x)$ denote the matrix $\|f_{i,j}\|$. The matrix $L_f(x)$ is defined almost everywhere in U. At every point x where $L_f(x) \neq 0$ we have $L_f(x) = J(x, f) \neq 0$. We put

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$$\theta(x) = \theta(x, f) = (L_f)^{-1}(L_f^*)^{-1}(\det L_f)^{2/n} \tag{2.1}$$

at a point where $\det L_f \neq 0$ (L_f^* is the transpose of the matrix L_f), and

$$\theta(x) = I \tag{2.2}$$

(where I is the unit matrix) at a point where $\det L_f = 0$. The matrix $\theta(x)$ is a symmetric positive definite matrix, where $\det \theta(x) = 1$, and the smallest and largest eigenvalues of the matrix $\theta(x)$ lie in the interval $[\alpha, 1/\alpha]$, where $0 < \alpha \leq 1$, and α depends only on the distortion coefficient $q = q(f)$ of the mapping f.

We also note the following property of the matrix θ:

$$L_f \theta L_f^* = DI, \tag{2.3}$$

where $D = (\det L_f)^{2/n}$.

Let $f: U \to R^n$ represent a continuous mapping of the class $W_n{}^1$. We assume that f is not identically equal to zero in the domain U, and we let V be the set of all $x \in U$ for which $f(x) \neq 0$. The set V is open, and the following exterior differential form ω_f of order n-1 is defined in it:

$$\omega_f = \frac{1}{|f|^n} \sum_{k=1}^{n} (-1)^{k-1} f_k df_1 \wedge \ldots \wedge \widehat{df_k} \wedge \ldots \wedge df_n$$

(the sign ^ over the subscript means that the corresponding term is to be dropped).

<u>LEMMA 1.</u> For every function $\eta \in W_n{}^1(U)$ whose carrier is compact and is contained in the set V

$$\int_V d\eta \wedge \omega_f = 0. \tag{2.4}$$

<u>Proof.</u> We assume first that the functions η and f are infinitely differentiable. The function η is clearly representable in the form $\eta = \eta_1 + \eta_2 + \ldots + \eta_m$, where each of the functions $\eta_s \in C^\infty$ and is such that its carrier is contained in some closed sphere $Q_s \subset V$. Let Γ_s be suitably oriented boundary of the sphere Q_s. Then

$$\int_{\Gamma_s} \eta_s \omega = \int_{Q_s} d\eta_s \wedge \omega_f + \int_{Q_s} \eta_s d\omega_f.$$

Inasmuch as $\eta_s = 0$ on Γ_s, we have

$$\int_{Q_s} d\eta_s \wedge \omega_f = -\int_{Q_s} \eta_s d\omega_f.$$

An elementary calculation shows that $d\omega_f = 0$, whence it follows that

$$\int_V d\eta_s \wedge \omega_f = \int_{Q_s} d\eta_s \wedge \omega_f = 0.$$

Summing over s, we obtain

$$\int_V d\eta \wedge \omega_f = 0.$$

We now consider the general case. Let $S \subset V$ be the carrier of a function η, and let V' be an open set such that $S \subset V'$, while the closure of V' is compact and contained in V. Let η_h and f_h be average functions in the sense of [7] for the functions η and f, respectively. Then for sufficiently small h ($h < h_0$) the functions η_h and f_h are defined on the set V', where the carrier of η_h is contained in V'. As $h \to 0$ we find $f_h \to f$ uniformly on V'. It follows that there exists an $h_1 \leq h_0$ ($h_1 > 0$) such that $\|f_h(x)\| > 0$ for $0 < h < h_1$ and for all $x \in V'$, so that the form ω_{f_h} is defined on V'. We have $\|\eta_h - \eta\|_{W_n(V)} \to 0$ and $\|f_h - f\|_{W^n(V)} \to 0$ as $h \to 0$. Hence (see [8], Lemma 6)

$$\int_{V} d\eta_h \wedge \omega_{f_h} = \int_{V_t} d\eta_h \wedge \omega_{f_h} \to \int_{V_t} d\eta \wedge \omega_f = \int_{V} d\eta \wedge \omega_f$$

as $h \to 0$ (it is assumed in the left equation that $h < h_t$). For $h < h_t$, by what has been proved,

$$\int_{V_t} d\eta_h \wedge \omega_{f_h} = 0$$

so that

$$\int_{V} d\eta \wedge \omega_f = 0,$$

which it was required to prove.

Now let $f: U \to R^n$ be a mapping with bounded distortion. We assume that f is not identically equal to zero in U, and we say, as before, that V is the set of all $x \in U$ for which $f(x) \neq 0$. Let us examine the function

$$u(x) = u_f(x) = \ln \frac{1}{|f(x)|}. \tag{2.5}$$

The function u is defined and continuous on the set V. Moreover, if $A = U \setminus V$ is the set of all $x \in U$ for which $f(x) = 0$, then for every point $x' \in A$ that is an accumulation point for V we have $u(x) \to +\infty$ when $x \to x'$. It is readily seen also that $u \in W_n^1(V)$.

We now examine the functional

$$K(v, f, A) = \int_{A} \{\theta_{ij}(x)\, v_{,i}(x)\, v_{,j}(x)\}^{n/2} dx, \tag{2.6}$$

where $A \subset V$ is an arbitrary compact set, v is a function of the class $W_n^1(U)$, and $\theta(x) = (\theta_{ij}(x)) = \theta_f(x)$ is a matrix defined by the conditions (2.1) and (2.2). It is a simple matter to show that the function

$$F(x, p) = \{\theta_{ij}(x) p_i p_j\}^{n/2}$$

meets all the conditions of [4]. The exponent α specified in these conditions is equal to n in this case.

LEMMA 2. The function u defined by Eq. (2.5) according to the mapping $f: U \to R^n$ is an extremal function for the functional (2.6) on the set V.

Proof. As shown in [4], it is sufficient to prove that u is a stationary function of the functional (2.6), i.e., that for every function $\eta \in W_n^1(V)$ finite in V

$$\int_{V} F_{p_i}(x, \nabla u)\, \eta_{,i}\, dx = 0,$$

where $F(x, p) = (\theta_{ij}(x) p_i p_j)^{n/2}$. We have

$$F_{p_j}(x, \nabla u)\, \eta_{,j} = n\, (\theta_{ij} u_{,i} u_{,j})^{n/2-1} \theta_{ij} u_{,i} \eta_{,j}.$$

Moreover,

$$u_{,i} = f_k f_{k,i} / |f|^2 \quad (i = 1, 2, \ldots, n).$$

Hence

$$\theta_{ij} u_{,i} u_{,j} = \frac{1}{|f|^4} \theta_{ij} f_{k,i} f_{l,j} f_k f_l.$$

By virtue of (2.3), $f_{k,i}\theta_{l,j} = D\delta_{kl}$, where $D = [J(x,f)]^{2/n}$, $\delta_{kl} = 1$ for $k = l$, and $\delta_{kl} = 0$ for $k \neq l$. Thus

$$\theta_{ij} u_{,i} u_{,j} = \frac{1}{|f|^4} D \delta_{kl} f_k f_l. \tag{2.7}$$

Furthermore, it follows from (2.3) that

$$L_f 0 = D(L_f^*)^{-1} \tag{2.8}$$

at every point where $L_f \neq 0$. We denote by $Y_{k,i}$ the signed minor of the element $f_{k,i}$ of the matrix $L_f = (f_{k,i})$. Then

$$(L_f^*)^{-1} = (Y_{ki} / D^{n/2}). \tag{2.9}$$

From the relations (2.8) and (2.9) we obtain

$$\theta_{ij} u_{,i} \eta_{,j} = \frac{1}{|f|^2} \theta_{ij} f_k f_{k,i} \eta_{,j} = \frac{1}{|f|^2} (L_f^* 0)_{kj} f_k \eta_{,j} = \frac{Y_{kj} f_k \eta_{,j}}{|f|^2 D^{n/2-1}}$$

and, finally,

$$(\theta_{ij} u_{,i} u_{,j})^{n/2-1} \theta_{ij} u_{,i} \eta_{,j} = \frac{D^{n/2-1}}{|f|^{n-2}} \frac{Y_{kj} f_k \eta_{,j}}{|f|^2 D^{n/2-1}} = \frac{Y_{kj} f_k \eta_{,j}}{|f|^n}. \tag{2.10}$$

Equation (2.10) has thus been proved at points x where $I_f \neq 0$. Wherever $L_f = 0$ we have $f_{k,i} = 0$ for all k and i, whence it is clear that (2.10) is also true wherever $L_f = 0$. We have

$$Y_{kj}\eta_j = \begin{vmatrix} f_{1,1}, & f_{1,2}, & \ldots, f_{1,n} \\ \cdots & \cdots & \cdots \\ f_{k-1,1}, & f_{k-1,2}, & \ldots, f_{k-1,n} \\ \eta_{,1} & \eta_{,2} & \ldots, \eta_{,n} \\ f_{k+1,1}, & f_{k+1,2}, & \ldots, f_{k+1,n} \\ \cdots & \cdots & \cdots \\ f_{n,1}, & f_{n,2} & \ldots, f_{n,n} \end{vmatrix}.$$

Hence

$$\begin{aligned} Y_{kj}\eta_{,j} dx_1 \wedge \ldots \wedge dx_n &= df_1 \wedge \ldots \wedge df_{k-1} \wedge d\eta \wedge df_{k+1} \wedge \ldots \wedge df_n \\ &= (-1)^{k-1} d\eta \wedge df_1 \wedge \ldots \wedge \widehat{df_k} \wedge \ldots \wedge df_n. \end{aligned}$$

It follows from this equation that

$$\begin{aligned} &(\theta_{ij} u_{,i} u_{,j})^{n/2-1} \theta_{ij} u_{,i} \eta_{,j} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n \\ &= d\eta \wedge \left(\frac{1}{|f|^n} \sum_{k=1}^{n} (-1)^{k-1} f_k \, df_1 \wedge \ldots \wedge \widehat{df_k} \wedge \ldots \wedge df_n \right) = d\eta \wedge \omega_f. \end{aligned}$$

By Lemma 1 we obtain

$$\int_V F_{p_j}(x, u_{,i}) \eta_{,j} \, dx = \int_V d\eta \wedge \omega_f = 0.$$

<u>LEMMA 3.</u> Let $f: U \to R^n$, where U is a domain in R^n, be a mapping with bounded distortion. If f is not identically constant in U, then the complete image of a point in the mapping f is a set of zero n-capacity.

<u>Proof.</u> Let $u \in R^n$ be an arbitrary point. The set $A = f^{-1}(y)$ is closed with respect to U, and $V = U \setminus A$ is an open set. Replacing f(x) by g(x) = f(x)-y in the preceding arguments, we deduce that the function

$$v(x) = \ln \frac{1}{|f(x) - y|}$$

is continuous on the set V and is an extremal function for the functional

$$\int_G (\theta_{ij} u_{,i} u_{,j})^{n/2} dx$$

on the set V. As x tends to any boundary point of A, clearly, $v(x) \to \infty$. We see, therefore, that all the conditions of the theorem of [4] are met, so that the n-capacity of the set A is equal to zero. The lemma is proved.

THEOREM. Let $f: U \to R^n$ be an arbitrary mapping with bounded distortion. Then f satisfies the index boundedness condition.

Proof. The theorem is obvious if f is identically constant on U'. Let us suppose that this is not so.

Let $x_0 \in U$ be an arbitrary point of the domain U, and let $y_0 = f(x_0)$. By Lemma 2 the set $A = f^{-1}(y_0)$ has zero n-capacity. Hence A is a set whose linear Hausdorff measure is equal to zero ([2], Corollary 1 to Theorem 1, Section 3). We denote by $Q_r(x_0)$ the ball $\{x \in R^n: |x-x_0| < r\}$, and by $S_r(x_0)$ the sphere $\{x \in R^n: |x-x_0| = r\}$. Then for almost all $r \in (0, \infty)$ the sphere $S_r(x_0)$ does not contain points of the set A. We assign an $r_0 > 0$ such that $\overline{Q}_{r_0}(x_0) \subset U$ and $S_{r_0}(x_0) \cap A = \emptyset$. The point y_0 does not belong to the set $f(\mathrm{Fr}\, Q_0)$, hence the quantity $\mu_0 = \mu(y_0, f|Q_0)$ is defined. Let V be the connected component of the set $R^n \setminus f(\mathrm{Fr}\, Q_0)$ containing the point y_0. Then for all $y \in V$ we have $\mu(y, f|Q_0) = \mu_0$.

Now let r_1 be such that $0 < r_1 < r_0$ and $f[\overline{Q}_{r_1}(x_0)] \subset V$. We set $Q_1 = Q_{r_1}(x_0)$. For $y \notin V$ we have, clearly, $\mu(y, f|Q_1) = 1$. For $y \in V$ we have $\mu(y, f|Q_1) \leq \mu(y, f|Q_0)$ by virtue of the properties of the index of mappings with bounded distortion. We infer, therefore, that the function $\mu(y, f|Q_1)$ is bounded, and $\mu(y, f|Q_1) \leq \mu_0$ for all $y \notin f(\mathrm{Fr}\, Q_1)$. For every compact domain G lying in the open ball Q_1 we have $\mu(y, f|G) \leq \mu(y, f|Q_1) \leq \mu_0$. Hence, by Theorem 3 of [2], Section 4, it follows that $f^{-1}(y_0) \cap Q_1$ comprises at most μ_0 elements. Consequently, there exists a ball $Q_{r_2}(x_0)$ such that $0 < r_2 \leq r_1$ and $f(x) \neq f(x_0)$ for $\in Q_{r_2}(x_0)$ if $x \neq x_0$.

Consequently, every point $x_0 \in U$ has a spherical neighborhood Q such that $\overline{Q} \subset U$, the function $\mu(y, f|\overline{Q})$ is bounded, and $f(x) \neq f(x_0)$ for $x \in \overline{Q}$ if $x \neq x_0$. This means that for every point $x_0 \in U$ it is possible to specify an integer $j(x_0, f)$ as the index of the mapping f at the point x_0. Here

$$j(x_0, f) = \mu(y_0, f|\overline{Q}),$$

where $\overline{Q}$ is the neighborhood indicated above. It follows that $j(x_0, f) \geq 1$ for every $x_0 \in U$. We call a neighborhood Q having the stated properties a normal neighborhood of the point x_0.

Now let $G \subset U$ be an arbitrary compact domain. For every point $x \in G$ we construct its normal neighborhood. By the compactness of G there exists a finite system of normal neighborhoods $Q_1, Q_2, \ldots, Q_m$ covering G. Let $y \in f(G)$, and let $y \notin f(\mathrm{Fr}\, G)$. Each of the sets $f^{-1}(y) \cap Q_i$ is finite. Hence it follows that the set $A = f^{-1}(y) \cap G$ is finite. We have

$$\mu(y, f|G) = \sum_{x \in A} j(x, f_0).$$

For every i

$$\sum_{x \in A \cap Q_i} j(x, f_0) = \mu(y, f|Q_i) \leqslant \mu_i,$$

where $\mu_1, \mu_2, \ldots, \mu_m$ are constants. Hence

$$\mu(y, f|G) \leqslant \mu_1 + \mu_2 + \ldots + \mu_m = M < \infty.$$

Since y is any point of the set $f(G) \setminus f(\mathrm{Fr}\, G)$, the theorem is proved.

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