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A note on F_B-convergence

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1. Introduction

Let m and c be the Banach spaces of bounded and convergent sequences $\mathbf{x}=\{x_k\}$ with the usual norm $\|\mathbf{x}\|_\infty=\sup_k |x_k|$, and let v be the space of sequences of bounded variation, i.e.

$$v=\Big\{\mathbf{x}:\ \|\mathbf{x}\| \equiv \sum_{k=0}^{\infty} |x_k-x_{k-1}| <\infty \ (x_{-1}=0)\Big\}.$$

Suppose that $B=(B_i)$ is a sequence of infinite complex matrices with $B_i=\big(b_{np}(i)\big)$. Then $\mathbf{x}\in m$ is said to be *F_B-convergent* [5] to the value $\operatorname{Lim} B\mathbf{x}$, if

$$\lim_{n\to\infty} (B_i\mathbf{x})_n = \lim_{n\to\infty} \sum_{p=0}^{\infty} b_{np}(i)x_p = \operatorname{Lim} B\mathbf{x},$$

uniformly for $i=0, 1, 2, \dots$.

The space F_B of F_B-convergent sequences depends on the fixed chosen sequence $B=(B_i)$. In case $B=B_0=(I)$ (unit matrix) the space F_B is same as c and for $B=B_1=(B_i^{(1)})$ it is the same as the space f of almost convergent sequences [2], where $B_i^{(1)}=(b_{np}^{(1)}(i))$ with

$$b_{np}^{(1)}(i)=\begin{cases} \dfrac{1}{n+1}, & i\leqq p\leqq i+n,\\ 0 & \text{otherwise.}\end{cases}$$

Let s be the space of all complex sequences and define

$$\mathbf{e}_k=\{0,0,\dots,0,\overset{k}{\breve{1}},0,\dots\},\quad \mathbf{e}=\{1,1,1,\dots\},$$
$$d_B=\{\mathbf{x}\in s:\ B\mathbf{x}=\big((B_i\mathbf{x})_n\big)\ \ exists\},$$

and

$$F_B=\{\mathbf{x}\in(d_B\cap m):\ \lim_{n\to\infty} t_n(i,\mathbf{x})\ \textit{exists uniformly in } i\geqq 0 \textit{ and is independent of } i\},$$

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where

$$
(1.1) \qquad t_n(i, \mathbf{x}) = \begin{cases} \sum_{p=0}^{\infty} b_{np}(i)x_p & (n \geqq 1), \\ \sum_{p=0}^{\infty} b_{op}(i)x_p & (n = 0), \end{cases}
$$

and

$$
b_{op}^{(i)} = \begin{cases} 1 & if \;\; p = i \\ 0 & otherwise. \end{cases}
$$

Let X and Y be two nonempty subsets of s. Let $A=(a_{nk})$ $(n, k=0, 1, 2, \ldots)$ be an infinite matrix of complex numbers. We write $A\mathbf{x}=(A_n(\mathbf{x}))$ if $(A\mathbf{x})_n= = \sum_{k=0}^{\infty} a_{nk}x_k$ converges for each n. If $\mathbf{x}\in X \Rightarrow A\mathbf{x}\in Y$, we say that A is (X, Y)-matrix.

In [5], STIEGLITZ has characterized (c, F_B)-, (m, F_B)-, and (f, F_B)-matrices. These classes of matrices give directly the known characterizations in special cases of the matrix sequence B. In this paper, we characterize (v, F_B)-matrices which will give directly the characterizations for (v, c)-, and (v, f)-matrices as choosing $B=B_0$ and $B=B_1$ respectively.

2. Main result

By (1.1), we get

$$
t_n(i, A\mathbf{x}) = \begin{cases} \sum_{p=0}^{\infty} b_{np}(i)A_p\mathbf{x}, & (n \geqq 1), \\ \sum_{p=0}^{\infty} b_{op}(i)A_p\mathbf{x}, & (n = 0), \end{cases}
$$

$$
(2.1) \qquad = \sum_{p=0}^{\infty} g_{nk}(i)x_k, \quad \text{say,}
$$

where

$$
g_{nk}(i) = \begin{cases} \sum_{p=0}^{\infty} b_{np}(i)a_{pk} & (n \geqq 1), \\ \sum_{p=0}^{\infty} b_{op}(i)a_{pk} & (n = 0). \end{cases}
$$

Theorem. *Let $A=(a_{nk})$ be an infinite matrix and $B=(B_i)$ be a sequence of infinite matrices with*

$$
\sup_{0 \leqq n < \infty} \sum_{p=0}^{\infty} |b_{np}(i)| < \infty, \quad \textit{for each } i.
$$

Then A is (v, F_B)-matrix, if and only if

$$
\text{(i)} \qquad \sup_{p,k} \Big| \sum_{l=k}^{\infty} a_{pl} \Big| < \infty,
$$

(ii) *there is a constant M such that for r, $i=0, 1, 2, \ldots$*

$$\sup_n \left|\sum_{k=r}^{\infty} g_{nk}(i)\right| \leqq M,$$

(iii) $\lim_{n\to\infty} g_{nk}(i)=\alpha_k$ *uniformly for* $n=0, 1, 2, \ldots,$ *and*

(iv) $\lim_{n\to\infty} \sum_{k=0}^{\infty} g_{nk}(i)=\alpha$ *uniformly for* $n=0, 1, 2, \ldots.$

Proof. Necessity. Condition (i) follows from the fact that A is (v, m)-matrix. Since $\mathbf{e}_k, \mathbf{e}\in v$, necessity of (iii) and (iv) is obvious.

Now, we can easily see that for fixed p and j

$$\sum_{k=0}^{j} a_{pk}x_k$$

is a continuous linear functional on v. We are given that, for all $\mathbf{x}\in v$ it tends to a limit as $j\to\infty$ (for fixed p) and hence by Banach—Steinhaus theorem [3], this limit, $(A\mathbf{x})_p$ is also a continuous linear functional on v.

Put for $i\geqq 0$

$$q_i(\mathbf{x}) = \sup_n |t_n(i, A\mathbf{x})|,$$

then q_i is a continuous seminorm on v, and $\{q_i\}$ is pointwise bounded on v. Therefore, by another application of the Banach—Steinhaus theorem, there exists a constant M, such that

$$q_i(\mathbf{x}) \leqq M\|\mathbf{x}\|. \tag{2.2}$$

Apply (2.2) with $\mathbf{x}=\{x_k\}$ defined by

$$x_k = \begin{cases} 1 & (k \geqq r), \\ 0 & (k < r). \end{cases}$$

Hence, (ii) must hold.

Sufficiency. Suppose that conditions (i)—(iv) hold and that $\mathbf{x}\in v$. By virtue of condition (i), it is clear that $A\mathbf{x}$ is bounded. Now, by conditions (iv) and (i), it follows that

$$\sum_{k=0}^{\infty} g_{nk}(i)$$

converges for all i, n. Hence if we write

$$h_{nk}(i) = \sum_{l=k}^{\infty} g_{nl}(i),$$

then $h_{nk}(i)$ exists, also for fixed i, n, we have

$$h_{nk}(i) \to 0 \tag{2.1}$$

as $k\to\infty$. Since

(2.2) $$h_{nk}(i) = h_{no}(i) - \sum_{l=0}^{k-1} g_{nl}(i).$$

Now,

(2.3) $$\sum_{k=0}^{\infty} g_{nk}(i)x_k = \sum_{k=0}^{\infty} [h_{nk}(i) - h_{n,k+1}(i)]x_k = \sum_{k=0}^{\infty} h_{nk}(i)(x_k - x_{k-1})$$

by (2.1) and boundedness of x_k. Therefore,

$$|t_n(i, A\mathbf{x})| = \left|\sum_{k=0}^{\infty} g_{nk}(i)x_k\right| \leqq \sum_{k=0}^{\infty} |x_k - x_{k-1}|\,|h_{nk}(i)| \leqq M\|\mathbf{x}\|$$

(by condition (ii)) for $\mathbf{x}\in v$. Also,

(2.4) $$\lim_{n\to\infty} t_n(i, A\mathbf{x}) = \lim_{n\to\infty} \sum_{k=0}^{\infty} g_{nk}(i)x_k = \sum_{k=0}^{\infty} (x_k - x_{k-1}) \lim_{n\to\infty} h_{nk}(i).$$

By (2.2) and conditions (iii) and (iv), we have

(2.5) $$\lim_{n\to\infty} h_{nk}(i) = \lim_{n\to\infty} h_{no}(i) - \sum_{l=0}^{k-1} \lim_{n\to\infty} g_{nl}(i) = \alpha - \sum_{l=0}^{k-1} \alpha l.$$

Therefore, (2.4) and (2.5) give

$$\lim_{n\to\infty} t_n(i, A\mathbf{x}) = \sum_{k=0}^{\infty} (x_k - x_{k-1})\Big(\alpha - \sum_{l=0}^{k-1} \alpha_l\Big) = \alpha \lim_{k\to\infty} x_k + \sum_{k=0}^{\infty} x_k \alpha_k.$$

Hence $A\mathbf{x}\in F_B$ for $\mathbf{x}\in v$.

This completes the proof.

Remark. For $B=B_0$ in the above theorem, we get the known characterization for (v, c)-matrices due to HAHN [1], and in case $B=B_1$, we have (v, f)-matrices due to NANDA [4].

References

[1] H. HAHN, Über Folgen linearer Operationen, *Monatsh. Math.*, **32** (1922), 3—88.
[2] G. G. LORENTZ, A contribution to the theory of divergent series, *Acta Math.*, **80** (1948), 167—190.
[3] I. J. MADDOX, *Elements of functional analysis*, Cambridge Univ. Press. 1970.
[4] S. NANDA, Matrix transformations and almost convergence (Russian), *Math. Vestnik*, **13** (28) (1976), 305—312.
[5] M. STIEGLITZ, Eine Verallgemeinerung des Begriffs der Fastkonvergenz, *Math. Japon.*, **18** (1973), 53—70.

Заметка об F_B-сходимости

МУРСАЛЕН

М. Штиглиц ввел понятие F_B-сходимости, дающее широкое обобщение классического понятия почти сходимости, данного Г. Г. Лоренцом. Штиглиц получил при этом необходимые и достаточные условия для характеризации $(c, F_B)-(m, F_B)$- и (f, F_B)-матриц. В настоящей работе получена характеризация (v, F_B)-матриц.

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