Analysis Mathematica, **8** (1982), 257—261

Holomorphic retracts and intrinsic metrics in convex domains

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§ 1.

The three objects we shall be concerned here with are holomorphic retracts, the Carathéodory and the Kobayashi distances.

1) *Holomorphic retracts.* A subset S of a domain $D\subset \mathbf{C}^n$ is called a holomorphic retract if there is a holomorphic mapping $r\colon D\to D$ such that $r(D)\subset S$ and $r(z)=z$ for $z\in S$. Such a set is necessarily an analytic subset of D, since

$$S = \{z\in D\colon r(z)-z = 0\}.$$

2) *The Carathéodory distance.* Let again $D\subset \mathbf{C}^n$ be a domain and U the unit disc in $\mathbf{C}$. The Carathéodory distance $c_D(z, w)$ of two points $z, w\in D$ is defined as

$$c_D(z, w)=\sup_F \{\text{hyp dist}\,(F(z), F(w))\colon F\colon D\to U \text{ is holomorphic}\}.$$

Here hyp dist stands for the hyperbolic distance in U.

3) *The Kobayashi distance* is defined in a dual way: for $z, w\in D$ put

(1) $\delta_D(z, w) = \inf\{\text{hyp dist}\,(\zeta, \omega)\colon \exists f\colon U\to D \text{ holomorphic, with } f(\zeta)=z,\, f(\omega) = w\}.$

The only difference is that δ_D does not always satisfy the triangle inequality, so that one is obliged to define the quantity

$$k_D(z, w) = \inf\left\{\sum_{j=1}^{m} \delta_D(z_{j-1}, z_j)\colon z_0=z,\, z_m = w\right\}$$

and call it the Kobayashi distance.

Theorem 1. *If D is convex, then $k_D=c_D$.*

The proof of this theorem depends on the description of the one-dimensional holomorphic retracts in a strongly convex smoothly bounded domain D. (Strong convexity means that the normal curvatures of the boundary are everywhere positive.)

Received June 19, 1981.

Given $z, w \in D$, $z \neq w$, consider the extremum problem (1). A holomorphic mapping $f\colon U \to D$ for which the infimum is attained will be called extremal. If f is extremal, the analytic disc $f(U)$ will be called extremal disc. In [3] we proved that through any pair of distinct points there goes exactly one extremal disc (and, moreover, these discs have been characterized in a simple way).

Theorem 2. *The one-dimensional holomorphic retracts in a strongly convex smoothly* ($=C^\infty$) *bounded domain are precisely the extremal discs.*

In particular, any couple of points can be joined by a unique holomorphic retract.

Both theorems hold for strongly linearly convex domains as well. D is called linearly convex if for every $z \in \partial D$ there is a complex hyperplane through z which is disjoint from D (see Aizenberg [1]). D is strongly linearly convex if it is bounded by a C^2 boundary and its small C^2 perturbations are linearly convex. The point is that the results of [3] hold also for such domains. We shall not, however, go into details.

In the last paragraph we collect some negative results. We shall indicate why it is incidental for a domain to contain holomorphic retracts of dimension greater than one. (An exceptional case is the ball, see Rudin [4].)

We shall also consider the case of a strictly pseudoconvex domain. It will be shown that

(i) an extremal disc is not necessarily a holomorphic retract;

(ii) there may exist points which cannot be joined by holomorphic retracts.

2. §

Lemma. *Let $D \subset \mathbf{C}^n$ be a strongly convex smoothly bounded domain. Then for any $z_1, z_2 \in D$, $z_1 \neq z_2$, there exist two mappings $f\colon U \to D$ and $F\colon D \to U$, both holomorphic, such that*

1) $z_1, z_2 \in f(U)$,

2) $F \circ f = id_U$.

Proof. Given $z_1, z_2 \in D$, consider the following extremum problem:

$$\inf \{\xi > 0\colon \exists f\colon U \to D \text{ holomorphic, with } f(0) = z_1,\ f(\xi) = z_2\}. \tag{2}$$

In [3] we showed that there exists exactly one f that yields the minimal value in (2). Moreover, this extremal f was proved to be "stationary" by which we meant that

1) f extends continuously to $\bar{U}$;

2) Using the same letter f for this extension, we have $f(\partial U) \subset \partial D$;

3) Denote by $\nu(z)$ the outward unit normal vector to ∂D in $z \in \partial D$. Then there is a continuous positive function p: $\partial U \to \mathbf{R}^+$ such that the mapping $\partial U \to \mathbf{C}^n$ defined by

$$\zeta \mapsto \zeta p(\zeta) \overline{\nu(f(\zeta))}$$

extends to a continuous mapping $\tilde{f}$: $\overline{U} \to \mathbf{C}^n$, holomorphic in U.

Obviously this extremal f satisfies the first condition of the Lemma. To see that it has a holomorphic left-inverse, consider the equation

(3) $$\langle z - f(\zeta), \tilde{f}(\zeta) \rangle = 0$$

for the unknown $\zeta \in U$, z being a fixed point in D. Here

$$\langle a, b \rangle = \sum_{j=1}^{n} a_j b_j \quad \text{for} \quad a = (a_j), \quad b = (b_j) \in \mathbf{C}^n.$$

For each $z \in D$ (3) has exactly one solution $\zeta \in U$. Indeed, the number of the solutions equals wind φ, the winding number of the function $\varphi(\zeta) = \langle z - f(\zeta), \tilde{f}(\zeta) \rangle$ on ∂U. Now

(4) $$\text{wind}\, \varphi = \text{wind}\, \zeta + \text{wind}\, \langle z - f(\zeta),\ p(\zeta) \overline{\nu(f(\zeta))} \rangle = 1,$$

since

$$\text{Re}\, \langle z - f(\zeta),\ p(\zeta) \overline{\nu(f(\zeta))} \rangle < 0 \quad \text{for} \quad \zeta \in U$$

by convexity, thus the second summand in (4) is zero.

Therefore one has a function F: $D \to U$ which assigns to each $z \in D$ the corresponding solution $\zeta \in U$ of (3). F is obviously holomorphic and satisfies $F \circ f = id_U$. Thus the Lemma is proved.

Proof of Theorem 1. First suppose that D is as in the Lemma and let $z_1, z_2 \in D$, $z_1 \neq z_2$. Furthermore, let f and F be as in the Lemma, $F(z_1) = \zeta_1$, $F(z_2) = \zeta_2$, i.e. $f(\zeta_1) = z_1$, $f(\zeta_2) = z_2$. Then by definition

$$k_D(z_1, z_2) \leqq \delta_D(z_1, z_2) \leqq \text{hyp dist}\, (\zeta_1, \zeta_2) \leqq c_D(z_1, z_2).$$

On the other hand, as a trivial consequence of the Schwarz lemma one has

$$c_D(z_1, z_2) \leqq k_D(z_1, z_2)$$

(for any domain; see e.g. KOBAYASHI [2]). Therefore $k_D = c_D$.

The general case (of an arbitrary convex domain) can be obtained from this special one by a simple approximation process.

Proof of Theorem 2. Any extremal disc is of the form $f(U)$, where f is a mapping which gives the minimum in (2) (for some fixed z_1, z_2). Furthermore, let F be a left-inverse of f, whose existence has been proved in the Lemma. Then $f \circ F$: $D \to f(U)$ is a retraction on $f(U)$.

Conversely, suppose that $r\colon D\to S$ is a holomorphic retraction on S. Choose two distinct points z_1, z_2 in S and denote again by f the unique mapping $U\to D$ for which the minimum in (2) is attained. Then $r\circ f$ is another mapping with the same property, so that by uniqueness $r\circ f=f$, whence $f(U)\subset S$. S must be irreducible, since D is; therefore, if it is one-dimensional, $f(U)=S$. This completes the proof of Theorem 2.

§ 3.

a) *Higher dimensional retracts.* It would be a nice situation if (in convex domains) every triplet of points were contained in some two-dimensional holomorphic retract. That this is not the case can be seen from the following considerations.

Suppose that S is a two-dimensional holomorphic retract in a strongly convex smoothly bounded domain $D\subset \mathbf{C}^3$. Choose three points $z_1, z_2, z_3\in S$, not contained in any one-dimensional retract. Denote by S_1, S_2, S_3 the one-dimensional holomorphic retracts determined by z_2 and z_3, z_3 and z_1, z_1 and z_2, respectively. In the second part of the proof of Theorem 2 we have seen that if S contains two points, it must also contain the one-dimensional retract determined by them. Therefore $S_j\subset S$. Furthermore, if S_z denotes the one-dimensional retract that joins z_1 and $z\in S_1$, then we also have $S_z\subset S$.

Fix now $z_0\in S_1\setminus\{z_2, z_3\}$ and a point $w\in \bar{S}_{z_0}\setminus S_{z_0}\subset\partial D$. It is easy to see that $w\notin \bar{S}_1\cup\bar{S}_2\cup\bar{S}_3$. If the boundary ∂D is perturbed in a closed neighbourhood of w disjoint from $\bar{S}_1, \bar{S}_2, \bar{S}_3$, then S_1, S_2 and S_3 still remain holomorphic retracts in the new domain D' (see the characterization of extremal discs by conditions 1)—3) in the Lemma). Also, for $z\in S_1$ near to z_2 or z_3, S_z will persist to be a holomorphic retract. Thus, if S' is a two-dimensional holomorphic retract in D' containing z_1, z_2, z_3, we will have $S_z\subset S'$ for $z\in S_1$ near to z_2 or z_3.

Now we shall suppose that $D'\subset D$. Then $S\cap D'$ and S' are analytic subsets in D', and their intersection contains a real four-dimensional manifold (a portion of the union of certain S_z's). Since S' is irreducible, this implies $S'\subset S\cap D\subset S$. At the same time, the perturbation can be performed in such a way (in fact, generic perturbations are such) that the extremal disc in D' determined by z_1 and z_0 does not lie in S. (As a consequence of the behaviour of extremal discs under perturbations of the domain described in [3].) Consequently, this extremal disc does not lie in S' either, which is a contradiction, since S' contains two of its points. This contradiction shows that in general there is no two-dimensional retract containing three given points.

b) *Nonconvex domains.* Here we shall give an example of a strictly pseudoconvex domain D, where no couple of points can be joined by some one-dimensional holomorphic retract.

Let D be any strictly pseudoconvex but not convex Reinhard domain. To be specific; let

$$D = \{(x, y)\in \mathbf{C}^2\colon\ (1+|x|^2)(1+|y|^2) < 25\}. \tag{5}$$

Let $Z_0\in\partial D$ be a nonconvex point, i.e. where D does not admit a supporting hyperplane (e.g. $Z_0=(2, 2)$), and let $Z=Z_0/2$. We claim that there is no one-dimensional holomorphic retract containing 0 and Z.

Indeed, suppose there were such a retract S and denote by r the retraction $D\to S$.

It is a simple consequence of the Schwarz lemma that the unique extremal disc through $0, Z$ is $U_{Z_0}=\{\lambda Z_0\colon\ \lambda\in U\}$. It follows then as in the proof of Theorem 2 that $S=U_{Z_0}$. So, all we have to show is that there is no holomorphic retraction $r\colon D\to U_{Z_0}$.

To this end, let the mapping R be defined by

$$R(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\vartheta} r(ze^{i\vartheta})\, d\vartheta.$$

Then R is also a holomorphic retraction $D\to U_{Z_0}$. Moreover, R is linear, as can be seen by expanding r into a series of homogeneous polynomials.

Consider R as a linear mapping $\mathbf{C}^n\to\mathbf{C}^n$. It follows that $D\subset R^{-1}(U_{Z_0})$ and $R^{-1}(U_{Z_0})$ is convex. Since Z_0 is a boundary point of $R^{-1}(U_{Z_0})$, this latter domain admits a supporting hyperplane in Z_0, which would be a supporting hyperplane to D, as well. However, this contradicts the choice of Z_0. Therefore, there is no one-dimensional holomorphic retract in D containing 0 and Z.

Note also that U_{Z_0} is an extremal disc in D, which fails to be a holomorphic retract.

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Голоморфные ретракты и внутренние метрики в выпуклой области

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Доказывается, что в выпуклых областях в $\mathbf{C}^n$ метрики Каратеодори и Кобаяши совпадают. Устанавливается связь между геодезическими этих метрик и одномерными голоморфными ретрактами.

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