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Homogenization of Free Discontinuity Problems

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Introduction

Following Griffith's theory, hyperelastic brittle media subject to fracture can be modeled by the introduction, in addition to the elastic volume energy, of a surface term which accounts for crack initiation. In its simplest formulation, the energy of a deformation u is of the form

$$E(u, K) = \int_{\Omega \setminus K} f(\nabla u)\, dx + \lambda \mathscr{H}^{n-1}(K), \tag{1}$$

where ∇u is the deformation gradient, Ω is the reference configuration, and K is the crack surface. The bulk energy density f accounts for elastic deformations outside the crack, while λ is a constant given by Griffith's criterion for fracture initiation (see [49, 50, 54, 53, 14]). The existence of equilibria, under appropriate boundary conditions, can be deduced from the study of minimum pairs (u, K) for the energy (1), and a description of crack growth can be obtained by a limit of successive minimizations at fixed time steps, as outlined in [36] (see also [27] and [40]).

The presence of two unknowns, the surface K and the deformation u, can be overcome by a weak formulation of the problem in spaces of discontinuous functions. The space $SBV(\Omega; \mathbf{R}^m)$ of "special functions of bounded variation" was introduced by DE GIORGI & AMBROSIO [37] as the subset of $\mathbf{R}^m$-valued functions of bounded variation on the open set $\Omega \subset \mathbf{R}^n$, whose measure first derivative can be written in the form

$$Du = \nabla u \mathscr{L}^n \llcorner \Omega + (u^+ - u^-) \otimes \nu_u \mathscr{H}^{n-1} \llcorner S_u, \tag{2}$$

where ∇u is now the approximate gradient of u, S_u is the complement of the set of Lebesgue points of u, which admits a unit normal ν_u, and u^+, u^- are the approximate values of u on both sides of S_u. The measures $\mathscr{L}^n$ and $\mathscr{H}^{n-1}$ are the n-dimensional Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure, respectively. The energy in (1) can be rewritten as

$$\mathscr{E}(u) = \int_{\Omega} f(\nabla u)\, dx + \lambda \mathscr{H}^{n-1}(S_u), \tag{3}$$

which makes sense on $SBV(\Omega; \mathbf{R}^m)$. If f is quasiconvex and satisfies some standard growth conditions, then we can apply the direct methods of the calculus of variations to obtain minimum points for problems involving $\mathscr{E}$, using Ambrosio's lower semicontinuity and compactness theorems (see [4–7]). A complete regularity theory for minimum points u for $\mathscr{E}$ has not yet been developed, but in some cases it is possible to prove that the jump set S_u is $\mathscr{H}^{n-1}$-equivalent to its closure (see [38, 31]) or is even more regular (see [12, 11]), and that u is smooth on $\Omega \setminus \bar{S}_u$, and thus to obtain minimizing pairs $(u, K) = (u, \bar{S}_u)$ for the functional E.

The functionals $\mathscr{F}$ on $SBV(\Omega; \mathbf{R}^m)$ which have bulk and surface parts, and which satisfy the translation invariance condition $\mathscr{F}(u) = \mathscr{F}(u + c)$ for all constant vectors c, can be written in the form

$$\mathscr{F}(u) = \int_\Omega f(x, \nabla u)\, dx + \int_{S_u} g(x, (u^+ - u^-) \otimes \nu_u)\, d\mathscr{H}^{n-1} \tag{4}$$

(we adopt the equivalent notation $g(x, u^+ - u^-, \nu_u)$ in the course of the paper). Necessary and sufficient conditions for the lower semicontinuity of such functionals $\mathscr{F}$ are described in [6, 9, 7]. In the formulation (4) are included non-isotropic, non-homogeneous Griffith materials, when

$$g(x, a \otimes \nu) = \tilde{g}(x, \nu), \tag{5}$$

where the condition $\tilde{g}(x, \nu) = \tilde{g}(x, -\nu)$ must be imposed to have a good definition of the surface integral. We can also include in this setting surface problems in the framework of BARENBLATT's models, taking

$$g(x, a \otimes \nu) = \tilde{g}(|a|). \tag{6}$$

We shall not treat Barenblatt materials directly, but we remark that their study can be carried on by a singular perturbation approach from the study of models of the type (5) (see [24]). Many other problems in mathematical physics and computer vision involve minimum pairs with a "free discontinuity set" K and an unknown function u as above (see, e.g., [52, 13, 2, 25, 8, 9, 32, 33]). We shall be content to interpret our results in the framework of nonlinear fracture mechanics.

In this paper we study the asymptotic behaviour of functionals of the type (4) modelling cellular elastic materials with fine microstructure. The study of this kind of nonlinear media, but without considering the possiblity of fracture (i.e., in the framework of Sobolev functions), has been carried on by S. MÜLLER [51] and A. BRAIDES [16] (see also [17, 18, 19, 21, 26, 47]; a wide literature exists for the linear case, or when u is scalar-valued; we refer the interested reader to the rich bibliography of [35]). Here we consider functionals

$$\mathscr{F}_\varepsilon(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u\right) dx + \int_{S_u} g\left(\frac{x}{\varepsilon}, (u^+ - u^-) \otimes \nu_u\right) d\mathscr{H}^{n-1}, \tag{7}$$

where f and g are Borel functions, periodic in the first variable, which respectively model the response of the material to elastic deformation and fracture at a microscopic scale (which is given by the small parameter ε). The behaviour of sequences of minima for problems involving $\mathscr{F}_\varepsilon$, and of the corresponding minimizers, can be deduced from the Γ-convergence of this sequence (see [39, 35]). This analysis is usually referred to as homogenization. The main result of this paper is showing that, under the growth conditions

$$\alpha|\xi|^p \leqq f(x,\xi) \leqq \beta(1+|\xi|^p), \quad \alpha \leqq g(x,\xi) \leqq \beta \tag{8}$$

for all $x \in \mathbf{R}^n$, $\xi \in M^{m\times n}$, with $p > 1$, α, $\beta > 0$, we obtain, in the limit when $\varepsilon \to 0$, a minimum problem for the functional

$$\mathscr{F}_{\hom}(u) = \int_\Omega f_{\hom}(\nabla u)\,dx + \int_{S_u} g_{\hom}((u^+ - u^-)\otimes \nu_u)\,d\mathscr{H}^{n-1}. \tag{9}$$

The integrands $f_{\hom}$ and $g_{\hom}$ can be characterized by asymptotic formulas. The homogenized bulk energy density is the same integrand as obtained in [16] in the case without fracture:

$$f_{\hom}(\xi) = \lim_{T\to+\infty} \inf\left\{\frac{1}{T^n}\int_{]0,T[^n} f(x,\nabla u+\xi)\,dx : u \in W_0^{1,p}(]0,T[^n;\mathbf{R}^m)\right\}, \tag{10}$$

while the function $g_{\hom}$ is given on rank-one matrices by

$$\begin{aligned} g_{\hom}(z\otimes\nu) = \lim_{T\to+\infty}\frac{1}{T^{n-1}}\inf\Bigg\{&\int_{TQ_\nu\cap S_u} g(x,(u^+-u^-)\otimes\nu_u)\,d\mathscr{H}^{n-1}: \\ &u\in SBV(TQ_\nu;\mathbf{R}^m),\ \nabla u = 0 \text{ a.e.},\ u = u_{z,\nu} \text{ on } \partial(TQ_\nu)\Bigg\}, \end{aligned} \tag{11}$$

where Q_ν is any unit cube in $\mathbf{R}^n$ with centre at the origin and one face orthogonal to ν, and

$$u_{z,\nu}(x) = \begin{cases} z & \text{if } \langle x,\nu\rangle \geqq 0, \\ 0 & \text{if } \langle x,\nu\rangle < 0. \end{cases} \tag{12}$$

Note that by (9) it is sufficient to define $g_{\hom}$ on rank-one matrices. From (9)–(11) we obtain that the overall behaviour of the medium described by (7) at the scale ε is that of a homogeneous material whose bulk elastic response is given by the study of $\mathscr{F}_\varepsilon$ only on elastic deformations without cracks, and whose response to fracture can be derived by the examination of 'stiff deformations' (for which $\nabla u = 0$). In particular, note that the homogenized surface energy density is not influenced by f; this phenomenon is particular to the process of homogenization, since in general we do have an interaction (see [3, Theorem 4.1]). We also mention that the homogenization under SBV-growth conditions (8) gives rise to phenomena different from those that occur when a growth of order one is allowed; i.e.,

$$f(x,\xi) \leqq \gamma|\xi| \quad \text{or} \quad g(x,\xi) \leqq \gamma|\xi| \tag{13}$$

(e.g., if $g(x,\cdot)$ is positively homogeneous of degree one), in which case the homogenized functional is defined and finite on the whole $BV(\Omega;\mathbf{R}^m)$ (see [19]).

The paper is organized as follows. In Section 1 we recall the main definitions and preliminaries on SBV functions, and we introduce the space $SBV^p(\Omega;\mathbf{R}^m)$ of SBV-functions whose approximate gradient is p-summable and whose jump set is $\mathscr{H}^{n-1}$-finite. Section 2 is devoted to the statement of the homogenization result. In

Sections 2–7 we deal with functionals like (7), with g satisfying the technical assumption that

$$\alpha(1+|\xi|) \leqq g(x,\xi) \leqq \beta(1+|\xi|), \tag{14}$$

which allows us to limit our analysis to $SBV^p(\Omega;\mathbf{R}^m)$. The treatment of the case with g satisfying the growth condition (8) is carried on in Section 8 by a singular perturbation approach. The proof of the homogenization theorem relies on several technical results. In Section 3 we give a compactness theorem with respect to Γ-convergence for functionals defined in $SBV^p(\Omega;\mathbf{R}^m)$. Its proof is based on a "fundamental estimate" (Proposition 3.1), which allows the application of the localization techniques of Γ-convergence (see [35]). We also prove a truncation lemma (Lemma 3.5), which, in several cases, permits us to deal with equibounded sequences. In Section 4 we apply the techniques of BUTTAZZO & DAL MASO [29] and of AMBROSIO & BRAIDES [8] to give an integral representation on $W^{1,p}(\Omega;\mathbf{R}^m)$ and on spaces of "partitions" $BV(\Omega;T)$ ($T\subset\mathbf{R}^m$ is any fixed finite set) of the functionals given by the compactness argument of Section 3. The characterization by formula (10) of the volume energy density which describes the integral representation on $W^{1,p}(\Omega;\mathbf{R}^m)$ is obtained in Section 5. In order to use the homogenization results of [16] and [51], we apply a technique introduced by AMBROSIO (see [7]), which allows us to pass from sequences of SBV-functions with vanishing surface energy to sequences of Lipschitz functions in the description of the Γ-limit process. The construction of minimizing sequences with surface energy tending to 0 is obtained by a scaling argument, which is based on the periodic structure of the problem. A similar procedure leads in Section 6 to the characterization by formula (11) of the homogenized surface energy density: after a scaling argument, which again is possible by the periodicity assumptions, we can pass from sequences in $SBV^p(\Omega;\mathbf{R}^m)$ with vanishing bulk energy to sequences with $\nabla u=0$. This passage is carried on by a careful use of the coarea formula. In Section 7 we prove the integral representation (9) on $SBV^p(\Omega;\mathbf{R}^m)\cap L^\infty(\Omega;\mathbf{R}^m)$, from which the general result follows by approximation. The two key points are the application of the strong convergence results in $SBV^p(\Omega;\mathbf{R}^m)$ of piecewise smooth functions proved by BRAIDES & CHIADÒ PIAT [23], which gives an inequality in (9) (Proposition 7.1), and, for the opposite inequality, a blow-up argument which locally reduces the problem to the case of linear or piecewise constant functions. The characterization of the Γ-limits through formulas (10) and (11), together with the compactness argument of Section 2 conclude the proof. Finally in Section 8 we describe the applications of the homogenization theorem to problems in fracture mechanics.

1. Notation and preliminaries

Let $m\geqq 1$ and $n\geqq 1$ be fixed integers. If Ω is an open subset of $\mathbf{R}^n$, we denote by $\mathscr{A}(\Omega)$ and $\mathscr{B}(\Omega)$ families of the open and Borel subsets of Ω, respectively; moreover, we set $\mathscr{A}=\mathscr{A}(\mathbf{R}^n)$ and $\mathscr{B}=\mathscr{B}(\mathbf{R}^n)$, while $\mathscr{A}_0$ stands for the family of the bounded open subsets of $\mathbf{R}^n$. If $x, y\in\mathbf{R}^n$, then $\langle x, y\rangle$ denotes their scalar product; $B_\rho(x)$ is the open ball with centre x and radius ρ, and S^{n-1} the surface of the unit ball $B_1(0)$; $M^{m\times n}$ is the space of the $m\times n$ real matrices. The usual product of a matrix $\xi\in M^{m\times n}$ and a vector $x\in\mathbf{R}^n$ is denoted by $\xi\cdot x$.

The Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$ are denoted by $\mathscr{L}^n$ and $\mathscr{H}^{n-1}$, respectively, but we also write $|E|$ in place of $\mathscr{L}^n(E)$. Moreover, $\omega_n = |B_1(0)|$.

If $\Omega\in\mathscr{A}$, we use standard notation for the Lebesgue and Sobolev spaces $L^p(\Omega;\mathbf{R}^m)$ and $W^{1,p}(\Omega;\mathbf{R}^m)$.

Functions of bounded variation

For the general theory of the functions of bounded variation we refer to [43, 48, 42, 56]; here we just recall some definitions and results we shall use in the sequel.

Let Ω be an open subset of $\mathbf{R}^n$ and $u:\Omega\to\mathbf{R}^m$ be a Borel function. We say that $z\in\mathbf{R}^m$ is the approximate limit of u in x and we write $z = \text{ap-lim}_{y\to x}u(y)$ if for every $\varepsilon>0$,

$$\lim_{\rho\to 0}\rho^{-n}|\{y\in B_\rho(x)\cap\Omega : |u(y)-z|>\varepsilon\}| = 0.$$

We define S_u as the subset of Ω where the approximate limit of u does not exist. It turns out that S_u is a Borel set, $|S_u|=0$ and u is approximately continuous a.e. in Ω; more precisely, $u(x) = \text{ap-lim}_{y\to x}u(y)$ for a.e. $x\in\Omega\backslash S_u$.

We say that $u=(u^1,\ldots,u^m)\in L^1(\Omega;\mathbf{R}^m)$ is a *function of bounded variation* if its distributional first derivatives D_iu^j are (Radon) measures with finite total variation in Ω. This space is denoted by $BV(\Omega;\mathbf{R}^m)$. We use Du to indicate the matrix-valued measure whose entries are D_iu^j.

If $u\in BV(\Omega;\mathbf{R}^m)$, then S_u is countably $(n-1)$-rectifiable, i.e.,

(1.1) $$S_u = N\cup\Big(\bigcup_{i\in N}K_i\Big),$$

where $\mathscr{H}^{n-1}(N)=0$ and (K_i) is a sequence of compact sets, each contained in a C^1 hypersurface Γ_i. Moreover, there exist Borel functions ν_u: $S_u\to S^{n-1}$ and u^+, u^-: $S_u\to\mathbf{R}^m$ such that for $\mathscr{H}^{n-1}$-a.e. $x\in S_u$

$$\lim_{\rho\to 0}\rho^{-n}\int_{B_\rho^+(x)\cap\Omega}|u(y)-u^+(x)|\,dy = 0, \qquad \lim_{\rho\to 0}\rho^{-n}\int_{B_\rho^-(x)\cap\Omega}|u(y)-u^-(x)|\,dy = 0,$$

where $B_\rho^+(x) = \{y\in B_\rho(x): \langle y-x,\nu_u(x)\rangle>0\}$ and $B_\rho^-(x) = \{y\in B_\rho(x): \langle y-x,\nu_u(x)\rangle<0\}$. Hence, for $\mathscr{H}^{n-1}$-a.e. $x\in S_u$,

$$\lim_{\rho\to 0}\rho^{-n}|\{y\in B_\rho(x)\cap\Omega:\ \langle y-x,\ \pm\nu_u(x)\rangle>0, |u(y)-u^\pm(x)|>\varepsilon\}| = 0$$

for every $\varepsilon>0$. The triple $(u^+(x), u^-(x), \nu_u(x))$ is uniquely determined up to a change of sign of $\nu_u(x)$ and an interchange between $u^+(x)$ and $u^-(x)$. The vector ν_u is normal to S_u, in the sense that, if S_u is represented as in (1.1), then $\nu_u(x)$ is normal to Γ_i for $\mathscr{H}^{n-1}$-a.e. $x\in K_i$. In particular, it follows that $\nu_u(x) = \pm\nu_v(x)$ for $\mathscr{H}^{n-1}$-a.e. $x\in S_u\cap S_v$ and u, $v\in BV(\Omega;\mathbf{R}^m)$. If $x\in S_u$, we define $u^+(x)=u^-(x)=$ ap-lim$_{y\to x}u(y)$.

We denote by ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure. $\nabla u(x)$ turns out to be the approximate differential of u at x for a.e. $x \in \Omega$, in the sense that

$$\lim_{\rho \to 0} \rho^{-n} \int_{B_\rho(x) \cap \Omega} \frac{|u(y) - u(x) - \nabla u(x) \cdot (y - x)|}{|y - x|} \, dy = 0.$$

We point out that if $u, v \in BV(\Omega; \mathbf{R}^m)$, then $\nabla u(x) = \nabla v(x)$ for a.e. $x \in \Omega$ such that $u(x) = v(x)$.

It is easy to verify that if $u, v \in BV(\Omega; \mathbf{R}^m)$ and if φ is a smooth real function on Ω, then $(u + v)^\pm = u^\pm + v^\pm$, $(\varphi u)^\pm = \varphi u^\pm$, $|u^\pm - v^\pm| \leqq \| u - v \|_{L^\infty(\Omega; \mathbf{R}^m)}$ and $\nabla(\varphi u) = u \otimes \nabla \varphi + \varphi \nabla u$.

We say that a function $u \in BV(\Omega; \mathbf{R}^m)$ is a *special function of bounded variation* if the singular part of Du is given by $(u^+ - u^-) \otimes \nu_u \mathscr{H}^{n-1} \llcorner S_u$, i.e., if

$$Du = \nabla u \mathscr{L}^n + (u^+ - u^-) \otimes \nu_u \mathscr{H}^{n-1} \llcorner S_u.$$

We denote the space of the special functions of bounded variation by $SBV(\Omega; \mathbf{R}^m)$. This space was introduced by DE GIORGI & AMBROSIO [37]. For the properties of the functions $u \in SBV(\Omega; \mathbf{R}^m)$ we refer to [5] and [6]. Here we mention the following result (see [10]): If $u \in SBV(\Omega; \mathbf{R}^m)$ and if $\varphi: \mathbf{R}^m \to \mathbf{R}^m$ is a Lipschitz function with Lipschitz constant L, then $\varphi(u) \in SBV(\Omega; \mathbf{R}^m)$, $S_{\varphi(u)} \subseteq S_u$, $(\varphi(u))^\pm = \varphi(u^\pm)$, and $|\nabla \varphi(u)| \leqq L |\nabla u|$ a.e. in Ω.

Let $p > 1$; the space $SBV^p(\Omega; \mathbf{R}^m)$ is defined as the space of the functions $u \in SBV(\Omega; \mathbf{R}^m)$ such that

$$\mathscr{H}^{n-1}(S_u \cap \Omega) < +\infty, \quad \nabla u \in L^p(\Omega; M^{m \times n}).$$

Sets of finite perimeter

Let $\Omega \in \mathscr{A}$ and $E \in \mathscr{B}$. We say that E has *finite perimeter* in Ω if the characteristic function χ_E of E belongs to $BV(\Omega; \mathbf{R})$. Define the *essential boundary* of E as

$$\partial^* E = \left\{ x \in \mathbf{R}^n : \limsup_{\rho \to 0} \rho^{-n} |B_\rho(x) \cap E| > 0 \text{ and } \limsup_{\rho \to 0} \rho^{-n} |B_\rho(x) \backslash E| > 0 \right\}.$$

If E is a set of finite perimeter in Ω, then

$$\int_\Omega |D\chi_E| = \mathscr{H}^{n-1}(\Omega \cap \partial^* E);$$

this value is the perimeter of E in Ω. If $u \in BV(\Omega; \mathbf{R})$, then $\{x \in \Omega: u(x) > t\}$ has finite perimeter in Ω for a.e. $t \in \mathbf{R}$, and the following Fleming-Rishel *coarea formula* holds:

$$\int_B |Du| = \int_{-\infty}^{+\infty} \mathscr{H}^{n-1}(B \cap \partial^* \{x \in \Omega: u(x) > t\}) \, dt \tag{1.2}$$

for every $B \in \mathscr{B}(\Omega)$. For an exposition of the theory of sets of finite perimeter see the books quoted above for the functions of bounded variation.

Approximation of BV functions by Lipschitz functions

Let μ be a non-negative finite Radon measure on $Y =]0, 1[^n$. For every $x \in Y$ let us define

$$M(\mu)(x) = \sup\left\{\frac{\mu(B_\rho(x))}{|B_\rho(x)|} : \rho > 0 \text{ such that } B_\rho(x) \subseteq Y\right\}.$$

$M(\mu)$ is called the (local) maximal function of μ. If μ is absolutely continuous with respect to the Lebesgue measure and h is its density, we also set $M(h) = M(\mu)$. In [7] $M(\mu)$ is defined with respect to the unit ball $B_1(0)$ instead of Y. However, it is easy to see that the analogues of Proposition 2.2 and Theorem 2.3 in [7] still hold, as in the following two statements.

Proposition 1.1. *Let μ be as above. Then there exists a constant $c(n) > 0$ such that*

$$|\{x \in Y : M(\mu)(x) > \lambda\}| \leqq \frac{c(n)\mu(Y)}{\lambda}$$

for every $\lambda > 0$. Moreover, if μ is absolutely continuous with respect to the Lebesgue measure and its density h belongs to $L^p(Y)$ for some $p > 1$, then

$$\int_Y (M(\mu)(x))^p\, dx \leqq C(n, p) \int_Y (h(x))^p\, dx,$$

with $C(n, p) = p2^p c(n)/(p-1)$.

Theorem 1.2. *Let $\lambda > 0$, $u \in BV(Y; \mathbf{R}^m) \cap L^\infty(Y; \mathbf{R}^m)$, and let*

$$E = \{x \in Y : M(|Du|)(x) > \lambda\}.$$

Then for every $0 < \varepsilon < 1$ there is a Lipschitz function $v : Y_\varepsilon \to \mathbf{R}^m$, where $Y_\varepsilon =]\varepsilon, 1-\varepsilon[^n$, such that $u = v$ a.e. on $Y_\varepsilon \backslash E$, and the Lipschitz constant $\mathrm{Lip}(v, Y_\varepsilon)$ of v on Y_ε satisfies the inequality

$$\mathrm{Lip}(v, Y_\varepsilon) \leqq m\left(c'(n)\lambda + \frac{2}{\varepsilon}\|u\|_{L^\infty(Y;\mathbf{R}^m)}\right),$$

for a suitable positive constant $c'(n)$.

Γ-*convergence*

We recall briefly the notion of Γ-convergence [39]. Let (X, d) be a metric space, let $F_h : X \to \bar{\mathbf{R}}$ be a sequence of functionals on X, and let $F : X \to \bar{\mathbf{R}}$.

We say that (F_h) Γ-converges to F at the point $x \in X$ with respect to the topology induced by d if the following conditions are satisfied:

(i) for every sequence (x_h) in X such that $d(x_h, x) \to 0$, we have $F(x) \leqq \liminf_{h\to+\infty} F_h(x_h)$;

(ii) there exists a sequence (x_h) in X such that $d(x_h, x) \to 0$ and $F(x) = \lim_{h\to+\infty} F_h(x_h)$.

We say that (F_h) Γ-converges to F on the space X with respect to the topology induced by d if (i) and (ii) hold for every $x \in X$. In this case F is called the Γ-limit of (F_h), and we write $F = \Gamma\text{-}\lim_{h \to +\infty} F_h$.

For a complete treatment of the subject we refer to [35]. Here we only recall the following facts. If (F_h) Γ-converges to F, then F is lower semicontinuous. If (F_h) is a constant sequence, i.e., if F_h is equal to the same functional G for every $h \in \mathbf{N}$, then the Γ-limit exists and coincides with the *lower semicontinuous envelope* (or *relaxed functional*) $\bar{G}$ of G on the space X with respect to the topology induced by d (see [28]). Under suitable coercivity conditions, Γ-convergence guarantees the convergence of the minimum values of the functional F_h to the minimum value of their Γ-limit.

Quasiconvexity

We finally recall that a continuous function $f: M^{m \times n} \to \mathbf{R}$ is *quasiconvex* if for every open set Ω and $\xi \in M^{m \times n}$ we have $|\Omega| f(\xi) \leqq \int_\Omega f(\xi + Du)\, dx$ for all $u \in \mathscr{C}_0^1(\Omega; \mathbf{R}^m)$. Quasiconvexity is a well-known necessary and sufficient condition for the weak lower semicontinuity of integral functionals defined on Sobolev spaces (see [15, 1, 34, 28]).

2. Statement of the main result

Let $f: \mathbf{R}^n \times M^{m \times n} \to [0, +\infty[$ and $g: \mathbf{R}^n \times \mathbf{R}^m \times S^{n-1} \to [0, +\infty[$ be two Borel functions. We suppose that f satisfies

(i) for every $\xi \in M^{m \times n}$ the function $f(\cdot, \xi)$ is 1-periodic, i.e., $f(x + e_i, \xi) = f(x, \xi)$ for every $i = 1, \ldots, n$ and $x \in \mathbf{R}^n$;

(ii) there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 |\xi|^p \leqq f(x, \xi) \leqq c_2(1 + |\xi|^p)$$

for a.e. $x \in \mathbf{R}^n$ and for every $\xi \in M^{m \times n}$,

and that g satisfies

(i) $g(x, s, v) = g(x, -s, -v)$ for every $(x, s, v) \in \mathbf{R}^n \times \mathbf{R}^m \times S^{n-1}$;

(ii) $g(\cdot, s, v)$ is 1-periodic for every $(s, v) \in \mathbf{R}^m \times S^{n-1}$;

(iii) there exist a function $\omega: [0, +\infty[\to [0, +\infty[$, continuous and non-decreasing with $\omega(0) = 0$, and a constant $L > 0$, such that $\omega(t) \leqq Lt$ for $t \geqq 1$ and

$$|g(x, s, v) - g(x, t, v)| \leqq \omega(|s - t|) \text{ for every } x \in \mathbf{R}^n, s, t \in \mathbf{R}^m, v \in S^{n-1};$$

(iv) there exist two constants $c_3 > 0$ and $c_4 > 0$ such that

$$c_3(1 + |s|) \leqq g(x, s, v) \leqq c_4(1 + |s|)$$

for every $(x, s, v) \in \mathbf{R}^n \times \mathbf{R}^m \times S^{n-1}$.

For every $\varepsilon > 0$, $A \in \mathscr{A}$, $u \in SBV_{\mathrm{loc}}(A; \mathbf{R}^m)$ and $B \in \mathscr{B}(A)$ we define

$$F_\varepsilon(u, B) = \int_B f\left(\frac{x}{\varepsilon}, \nabla u\right) dx + \int_{S_u \cap B} g\left(\frac{x}{\varepsilon}, u^+ - u^-, \nu_u\right) d\mathscr{H}^{n-1}. \tag{2.1}$$

We remark that there exists a one-to-one correspondence between $(\mathbf{R}^m \setminus \{0\}) \times S^{n-1}$ modulo the equivalence relation $(s, \nu) \sim (-s, -\nu)$ and the space of matrices of rank equal to 1. Hence we could as well write the functional in (2.1) in the form

$$F_\varepsilon(u, B) = \int_B f\left(\frac{x}{\varepsilon}, \nabla u\right) dx + \int_{S_u \cap B} g\left(\frac{x}{\varepsilon}, (u^+ - u^-) \otimes \nu_u\right) d\mathscr{H}^{n-1},$$

with the identification $g(x, s \otimes \nu) = g(x, s, \nu)$, to have a symmetric notation in the two integrals. However, in the sequel we shall always use the notation (2.1) to highlight the different behaviour of the surface integral with respect to $u^+ - u^-$ and ν_u.

The following propositions introduce the functions f_{hom} and g_{hom} which will appear in the integral representation of the limit functional of the family $(F_\varepsilon)_{\varepsilon > 0}$.

Proposition 2.1. *For every $\xi \in M^{m \times n}$ there exists*

$$f_{\mathrm{hom}}(\xi) = \lim_{\varepsilon \to 0} \inf \left\{ \int_{]0,1[^n} f\left(\frac{x}{\varepsilon}, \nabla u + \xi\right) dx : u \in W_0^{1,p}(]0, 1[^n; \mathbf{R}^m) \right\}.$$

The function f_{hom} is quasiconvex, and for every $\xi \in \mathbf{R}^m$

$$c_1 |\xi|^p \leqq f_{\mathrm{hom}}(\xi) \leqq c_2(1 + |\xi|^p).$$

Moreover, for every sequence (ε_h) of positive numbers converging to 0 and for every $\Omega \in \mathscr{A}_0$ the sequence $u \mapsto \int_\Omega f(x/\varepsilon_h, \nabla u)\, dx$ Γ-converges to the functional $u \mapsto \int_\Omega f_{\mathrm{hom}}(\nabla u)\, dx$ on $W^{1,p}(\Omega; \mathbf{R}^m)$ with respect to the L^p-topology.

Proof. For the proof see [16] and [17, Theorem 2.3 and the subsequent remark, Proposition 1.8 and Remark 1.7]. □

Proposition 2.2. *For every $(z, \nu) \in \mathbf{R}^m \times S^{n-1}$ there exists*

$$g_{\mathrm{hom}}(z, \nu) = \lim_{\varepsilon \to 0} \inf \left\{ \int_{S_u \cap Q_\nu} g\left(\frac{x}{\varepsilon}, u^+ - u^-, \nu_u\right) d\mathscr{H}^{n-1} : \right.$$
$$\left. u \in SBV(Q_\nu; \mathbf{R}^m),\ \nabla u = 0 \text{ a.e.},\ u = u_{z,\nu} \text{ on } \partial Q_\nu \right\},$$

where Q_ν is any unit cube in $\mathbf{R}^n$ with centre at the origin and one face orthogonal to ν (the limit being independent of such a choice), and

$$u_{z,\nu}(x) = \begin{cases} z & \text{if } \langle x, \nu \rangle \geqq 0, \\ 0 & \text{if } \langle x, \nu \rangle < 0. \end{cases}$$

The function $g_{\hom}$ *is continuous on* $(\mathbf{R}^m\backslash\{0\})\times S^{n-1}$, *and*

$$c_3(1+|z|)\leqq g_{\hom}(z,\nu)\leqq c_4(1+|z|)$$

for every $(z,\nu)\in(\mathbf{R}^m\backslash\{0\}\times S^{n-1}$.

The proof of Proposition 2.2 is postponed to Section 6.

For every $A\in\mathscr{A}$, $u\in SBV_{\mathrm{loc}}(A;\mathbf{R}^m)$ and $B\in\mathscr{B}(A)$ we define

$$F_{\hom}(u,B)=\int_B f_{\hom}(\nabla u)\,dx+\int_{S_u\cap B} g_{\hom}(u^+-u^-,\nu_u)\,d\mathscr{H}^{n-1}.$$

The main result of the paper is the following homogenization theorem.

Theorem 2.3. *Let* $(F_\varepsilon)_{\varepsilon>0}$ *and* $F_{\hom}$ *be as above. Let* (ε_h) *be a sequence of positive real numbers converging to* 0. *Then for every* $A\in\mathscr{A}_0$,

$$F_{\hom}(\cdot,A)=\Gamma\text{-}\lim_{h\to+\infty}F_{\varepsilon_h}(\cdot,A)$$

on the space $SBV^p(A;\mathbf{R}^m)$ *with respect to the* $L^1(A;\mathbf{R}^m)$*-topology, and on the space* $SBV^p(A;\mathbf{R}^m)\cap L^p(A;\mathbf{R}^m)$ *with respect to the* $L^p(A;\mathbf{R}^m)$*-topology.*

In the case when f and g are constant with respect to the first variable, we immediately obtain the following relaxation result (see also [44]).

Corollary 2.4. *Let* $\Omega\in\mathscr{A}_0$ *and*

$$F(u)=\int_\Omega f(\nabla u)\,dx+\int_{S_u\cap\Omega} g(u^+-u^-,\nu_u)\,\mathscr{H}^{n-1}$$

for $u\in SBV^p(\Omega;\mathbf{R}^m)$. *Then the lower semicontinuous envelope of* F *on* $SBV^p(\Omega;\mathbf{R}^m)$ *with respect to the* $L^1(\Omega;\mathbf{R}^m)$*-topology (or on* $SBV^p(\Omega;\mathbf{R}^m)\cap L^p(\Omega;\mathbf{R}^m)$ *with respect to the* $L^p(\Omega;\mathbf{R}^m)$*-topology) is given by*

$$\bar F(u)=\int_\Omega \bar f(\nabla u)\,dx+\int_{S_u\cap\Omega}\bar g(u^+-u^-,\nu_u)\,\mathscr{H}^{n-1},$$

where

$$\bar f(\xi)=\inf\left\{\int_{]0,1[^n} f(\nabla u+\xi)\,dx : u\in W_0^{1,p}(]0,1[^n;\mathbf{R}^m)\right\},$$

i.e., $\bar f$ *is the quasiconvex envelope of* f *(see* [34]), *and*

$$\bar g(z,\nu)=\inf\left\{\int_{S_u\cap\Omega_\nu} g(u^+-u^-,\nu_u)\,\mathscr{H}^{n-1}:\right.$$

$$\left. u\in SBV(Q_\nu;\mathbf{R}^m),\ \nabla u=0\ \text{a.e.},\ u=u_{z,\nu}\ \text{on}\ \partial Q_\nu\right\},$$

for every $\xi\in M^{m\times n}$ *and* $(z,\nu)\in\mathbf{R}^m\times S^{n-1}$, *with* Q_ν *and* $u_{z,\nu}$ *as in Proposition* 2.2.

3. A compactness result on $SBV^p(\Omega; \mathbf{R}^m)$

In this section we prove some general properties for functionals of the form

$$F_\varepsilon(u, A) = \int_A f_\varepsilon(x, \nabla u)\, dx + \int_{S_u \cap A} g_\varepsilon(x, u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1},$$

where $f_\varepsilon : \mathbf{R}^n \times M^{m \times n} \to [0, +\infty[$ and $g_\varepsilon : \mathbf{R}^n \times \mathbf{R}^m \times S^{n-1} \to [0, +\infty[$ are Borel functions satisfying

$$c_1 |\xi|^p \leqq f_\varepsilon(x, \xi) \leqq c_2(1 + |\xi|^p) \quad \text{for a.e. } x \in \mathbf{R}^n \text{ and for every } \xi \in M^{m \times n},$$

$$c_3(1 + |s|) \leqq g_\varepsilon(x, s, \nu) \leqq c_4(1 + |s|) \quad \text{for every } (x, s, \nu) \in \mathbf{R}^n \times \mathbf{R}^m \times S^{n-1}$$

for suitable positive constants c_i. Moreover, we suppose that $g_\varepsilon(x, s, \nu) = g_\varepsilon(x, -s, -\nu)$ for every $(x, s, \nu) \in \mathbf{R}^n \times \mathbf{R}^m \times S^{n-1}$. In particular, we can have $f_\varepsilon(x, \xi) = f(x/\varepsilon, \xi)$ and $g_\varepsilon(x, s, \nu) = g(x/\varepsilon, s, \nu)$, where f and g are the functions introduced in Section 2.

For every $A \in \mathscr{A}$, $u \in SBV_{\text{loc}}(A; \mathbf{R}^m)$ and $B \in \mathscr{B}(A)$ let

$$H(u, B) = \int_B |\nabla u|^p\, dx + \int_{S_u \cap B} (1 + |u^+ - u^-|)\, d\mathscr{H}^{n-1}. \tag{3.1}$$

The functional $H(\cdot, A)$ is lower semicontinuous on $SBV_{\text{loc}}(A; \mathbf{R}^m)$ with respect to the $L^1_{\text{loc}}(A; \mathbf{R}^m)$-topology; see [6, Theorems 2.2 and 3.7] or [7, Theorem 4.5 and Remark 4.6].

In view of the growth conditions satisfied by f_ε and g_ε, there exist $\gamma_1, \gamma_2 > 0$ such that for every $\varepsilon > 0$,

$$\gamma_1 H(u, A) \leqq F_\varepsilon(u, A) \leqq \gamma_2 (H(u, A) + |A|). \tag{3.2}$$

Hence, for every $A \in \mathscr{A}_0$ the Γ-limit of any sequence $(F_{\varepsilon_h}(\cdot\, A))$ $(\varepsilon_h \to 0)$ on a subspace of $SBV_{\text{loc}}(A; \mathbf{R}^m)$ with respect to a topology stronger than $L^1_{\text{loc}}(A; \mathbf{R}^m)$ is finite exactly on $SBV^p(A; \mathbf{R}^m)$. Thus in the sequel we shall restrict our attention to the space $SBV^p(A; \mathbf{R}^m)$. The crucial properties of the Γ-limit are based on the so-called fundamental estimate ([35]), of which we give now an SBV-version.

Proposition 3.1 (Fundamental Estimate). *Let (F_ε) be the family of functionals defined in* (2.1). *For every $\eta > 0$ and for every $A', A'', B \in \mathscr{A}$, with $A' \subset\subset A''$, there exists a constant $M > 0$ with the following property: For every $\varepsilon > 0$ and for every $u \in SBV^p(A''; \mathbf{R}^m)$, $v \in SBV^p(B; \mathbf{R}^m)$ there exists a function $\varphi \in C_0^\infty(A'')$ with $\varphi = 1$ in a neighbourhood of $\bar{A}'$ and $0 \leqq \varphi \leqq 1$ such that*

$$F_\varepsilon(\varphi u + (1 - \varphi)v, A' \cup B) \leqq (1 + \eta)[F_\varepsilon(u, A'') + F_\varepsilon(v, B)] + M \| u - v \|^p_{L^p(S; \mathbf{R}^m)} + \eta,$$

where $S = (A'' \backslash A') \cap B$.

Remark 3.2. From the proof it follows that the cut-off function φ can be chosen in a finite family depending only on η, A', and A''.

Proof of Proposition 3.1. Let $\eta > 0$, A', A'' and B be fixed as in the statement. Let $k \in \mathbf{N}$ satisfy

$$\frac{1}{k} \max\left(2^{p-1} \frac{c_2}{c_1}, \frac{c_4}{c_3}, c_2 |(A'' \setminus A') \cap B|\right) \leqq \eta. \tag{3.3}$$

Let $A_1, \ldots, A_{k+1}$ be open subsets of $\mathbf{R}^n$ such that $A' \subset\subset A_1 \subset\subset A_2 \subset\subset \cdots \subset\subset A_{k+1} \subset\subset A''$. For every $i = 1, \ldots, k$ let φ_i be a function in $C_0^\infty(A_{i+1})$ with $\varphi_i = 1$ on a neighbourhood V_i of $\bar{A}_i$. Define

$$M = 2^{p-1} \frac{c_2}{k} \max_{1 \leqq i \leqq k} \| \nabla \varphi_i \|_{L^\infty}^p.$$

For fixed $\varepsilon > 0$, $u \in SBV^p(A'')$, and $v \in SBV^p(B)$, define on $A' \cup B$ the function $w_i = \varphi_i u + (1 - \varphi_i) v$ (where u and v are extended arbitrarily outside A'' and B, respectively). Then for $i = 1, \ldots, k$,

$$\begin{aligned} F_\varepsilon(w_i, A' \cup B) &\leqq F_\varepsilon(u, (A' \cup B) \cap V_i) \\ &\quad + F_\varepsilon(v, B \setminus \operatorname{spt} \varphi_i) + F_\varepsilon(w_i, B \cap (A_{i+1} \setminus \bar{A}_i)) \\ &\leqq F_\varepsilon(u, A'') + F_\varepsilon(u, B) + F_\varepsilon(w_i, B \cap (A_{i+1} \setminus \bar{A}_i)). \end{aligned} \tag{3.4}$$

Set $T_i = B \cap (A_{i+1} \setminus \bar{A}_i)$. We estimate the last term:

$$\begin{aligned} F_\varepsilon(w_i, T_i) &\leqq c_2 \int_{T_i} (1 + |\nabla w_i|^p)\, dx + c_4 \int_{S_{w_i} \cap T_i} (1 + |w_i^+ - w_i^-|)\, d\mathcal{H}^{n-1} \\ &\leqq c_2 \left(|T_i| + \int_{T_i} |\varphi_i \nabla u + (1 - \varphi_i) \nabla v + (\nabla \varphi_i)(u - v)|^p\, dx \right) \\ &\quad + c_4 \left(\int_{(S_u \setminus S_v) \cap T_i} (1 + |u^+ - u^-|)\, d\mathcal{H}^{n-1} \right. \\ &\quad + \int_{(S_v \setminus S_u) \cap T_i} (1 + |v^+ - v^-|)\, d\mathcal{H}^{n-1} \\ &\quad \left. + \int_{S_u \cap S_v \cap T_i} (1 + |\varphi_i(u^+ - u^-) + (1 - \varphi_i)(v^+ - v^-)|)\, d\mathcal{H}^{n-1} \right) \\ &\leqq c_2 \left(|T_i| + 2^{p-1} \int_{T_i} (|\nabla u|^p + |\nabla v|^p + |\nabla \varphi_i|^p |u - v|^p)\, dx \right) \\ &\quad + c_4 \left(\int_{S_u \cap T_i} (1 + |u^+ - u^-|)\, d\mathcal{H}^{n-1} + \int_{S_v \cap T_i} (1 + |v^+ - v^-|)\, d\mathcal{H}^{n-1} \right) \end{aligned}$$

$$\leqq 2^{p-1}\frac{c_2}{c_1}\left(\int_{T_i} f\left(\frac{x}{\varepsilon}, \nabla u\right)dx + \int_{T_i} f\left(\frac{x}{\varepsilon}, \nabla v\right)dx\right)$$

$$+\frac{c_4}{c_3}\left(\int_{S_u\cap T_i} g\left(\frac{x}{\varepsilon}, u^+ - u^-, \nu_u\right)d\mathscr{H}^{n-1}\right.$$

$$\left.+\int_{S_v\cap T_i} g\left(\frac{x}{\varepsilon}, v^+ - v^-, \nu_v\right)d\mathscr{H}^{n-1}\right)$$

$$+ c_2(|T_i| + 2^{p-1}(\|\nabla\varphi_i\|_{L^\infty}\|u-v\|_{L^p(T_i)})^p) \leqq \gamma(F_\varepsilon(u, T_i) + F_\varepsilon(v, T_i))$$

$$+ c_2\left(|T_i| + 2^{p-1}\left(\max_{1\leqq i\leqq k}\|\nabla\varphi_i\|^p_{L^\infty}\right)\|u-v\|^p_{L^p(T_i)}\right),$$

where $\gamma = \max(2^{p-1}c_2/c_1, c_4/c_3)$. Hence there exists $i_0\in\{1,\dots,k\}$ such that

$$F_\varepsilon(w_{i_0}, T_{i_0}) \leqq \frac{1}{k}\sum_{i=0}^{k} F_\varepsilon(w_i, T_i)$$

$$\leqq \frac{\gamma}{k}(F_\varepsilon(u, A'') + F_\varepsilon(v, B)) + \frac{c_2}{k}|S| + M\|u-v\|^p_{L^p(S;\mathbf{R}^m)}$$

where $S = (A''\backslash A')\cap B$. From (3.4) it follows that

$$F_\varepsilon(w_{i_0}, A'\cap B) \leqq \left(1+\frac{\gamma}{k}\right)(F_\varepsilon(u, A'') + F_\varepsilon(v, B)) + \frac{c_2}{k}|S| + M\|u-v\|^p_{L^p(S;\mathbf{R}^m)}.$$

By (3.3) the proof of Proposition 3.1 is accomplished. □

Proposition 3.3. *Let* (ε_h) *be a sequence of positive numbers converging to* 0. *Then there exist a subsequence* $(\varepsilon_{\sigma(h)})$ *of* (ε_h) *and a functional* F_0 *defined on the set* $\{(u, A)$: $A\in\mathscr{A}, u\in SBV^p(A;\mathbf{R}^m)\cap L^p(A;\mathbf{R}^m)\}$ *with values in* $[0, +\infty]$ *such that for every* $A\in\mathscr{A}_0$,

$$F_0(\cdot, A) = \Gamma\text{-}\lim_{h\to+\infty} F_{\varepsilon_{\sigma(h)}}(\cdot, A)$$

on the space $SBV^p(A;\mathbf{R}^m)\cap L^p(A;\mathbf{R}^m)$ *endowed with the* $L^p(A;\mathbf{R}^m)$-*topology. Moreover, for every* $\Omega\in\mathscr{A}_0$ *and* $u\in SBV^p(\Omega;\mathbf{R}^m)\cap L^p(\Omega;\mathbf{R}^m)$ *the set function* $F_0(u,\cdot)$ *is the restriction to* $\mathscr{A}(\Omega)$ *of a Borel measure on* Ω.

Proof. For every $\varepsilon > 0$ let $G_\varepsilon: L^p(\mathbf{R}^n;\mathbf{R}^m)\times\mathscr{A}\to[0, +\infty]$ be defined by

$$G_\varepsilon(u, A) = \begin{cases} F_\varepsilon(u, A) & \text{if } u|_A\in SBV^p(A;\mathbf{R}^m),\\ +\infty & \text{otherwise.}\end{cases}$$

By Theorem 16.9 of [35] there exist a subsequence $(\varepsilon_{\sigma(h)})$ of (ε_h) and a functional $G_0: L^p(\mathbf{R}^n;\mathbf{R}^m)\times\mathscr{A}\to[0, +\infty]$ such that $G_0 = \bar{\Gamma}(L^p(\mathbf{R}^n;\mathbf{R}^m))\text{-}\lim_{h\to+\infty} G_{\varepsilon_{\sigma(h)}}$. In the notation of [35], which we refer to for details and precise definitions, this means that G_0 is the inner regular envelope of both the Γ-lower and the Γ-upper limit of the sequence $(G_{\varepsilon_{\sigma(h)}})$. By (3.2)

$$\gamma_1 H(u, A) \leqq G_\varepsilon(u, A) \leqq \gamma_2(H(u, A) + |A|)$$

for every $\varepsilon > 0$, $A \in \mathscr{A}$ and $u \in L^p(\mathbf{R}^n; \mathbf{R}^m)$ with $u|_A \in SBV^p(A; \mathbf{R}^m)$. Taking into account Proposition 3.1 we can apply the same method of proof as in Theorem 18.7 in [35]. Thus we obtain that for every $A \in \mathscr{A}_0$ the sequence $(G_{\varepsilon_{\sigma(h)}}(\cdot, A))$ of functionals on $L^p(\mathbf{R}^n; \mathbf{R}^m)$ Γ-converges to $G_0(\cdot, A)$ with respect to the $L^p(\mathbf{R}^n; \mathbf{R}^m)$-topology at all points $u \in L^p(\mathbf{R}^n; \mathbf{R}^m)$ with $u|_A \in SBV^p(A; \mathbf{R}^m)$.

For every $A \in \mathscr{A}$ and $u \in SBV^p(A; \mathbf{R}^m) \cap L^p(A; \mathbf{R}^m)$ define $F_0(u, A) = G_0(\tilde{u}, A)$, where $\tilde{u}$ is any L^p-extension of u to $\mathbf{R}^n$. This definition is well-posed since from the $\bar{\Gamma}$-convergence of $(G_{\varepsilon_{\sigma(h)}})$ to G_0 it follows that for every $u, v \in L^p(\mathbf{R}^n; \mathbf{R}^m)$, if $u|_A = v|_A$, then $G_0(u, A) = G_0(v, A)$. The stated convergence of $F_{\varepsilon_{\sigma(h)}}(\cdot, A)$ is easily proved. Observe now that $G_\varepsilon(u, \cdot)$ is the restriction to $\mathscr{A}$ of a Borel measure on $\mathbf{R}^n$ for every u. Then, by Proposition 3.1 and by Theorem 18.5 in [35] (which holds with the same proof for vector-valued L^p-functions), for every $u \in L^p(\mathbf{R}^n; \mathbf{R}^m)$ the set function $G_0(u, \cdot)$ is the restriction to $\mathscr{A}$ of a Borel measure on $\mathbf{R}^n$. From this we obtain the stated measure property of F_0. □

We now prove some further properties of the Γ-limit F_0.

Proposition 3.4. *Let (ε_h) be a sequence of positive numbers converging to* 0, *and let $A \in \mathscr{A}_0$ be such that the limit $F_0(\cdot, A) = \Gamma\text{-}\lim_{h \to +\infty} F_{\varepsilon_h}(\cdot, A)$ exists on the space $SBV^p(A; \mathbf{R}^m) \cap L^p(A; \mathbf{R}^m)$ endowed with the $L^p(A; \mathbf{R}^m)$-topology. Then for every sequence (u_h) in $SBV^p(A; \mathbf{R}^m)$ converging in $L^1(A; \mathbf{R}^m)$ to a function $u \in SBV^p(A; \mathbf{R}^m) \cap L^\infty(A; \mathbf{R}^m)$ we have $F_0(u, A) \leqq \liminf_{h \to +\infty} F_{\varepsilon_h}(u_h, A)$.*

For the proof we need a technical lemma (see also [30]).

Lemma 3.5. *Let $A \in \mathscr{A}_0$ and let (u_h) be a sequence in $SBV^p(A; \mathbf{R}^m)$ which is bounded in $L^1(A; \mathbf{R}^m)$ and such that $(H(u_h, A))$ is bounded. Then for every $\eta > 0$, $M_0 > 0$ and $h \in \mathbf{N}$ there exists a Lipschitz function $\varphi_h : \mathbf{R}^m \to \mathbf{R}^m$ with Lipschitz constant less than or equal to 1 satisfying*

$$\varphi_h(y) = \begin{cases} y & \text{if } |y| \leqq a_h, \\ 0 & \text{if } |y| > b_h \end{cases}$$

for suitable constants $a_h, b_h \in \mathbf{R}$ with $M_0 \leqq a_h < b_h$, such that

$$F_{\varepsilon_h}(\varphi_h(u_h), A) \leqq F_{\varepsilon_h}(u_h, A) + \eta$$

for every $h \in \mathbf{N}$. The function φ_h can be chosen in a finite family independent of h.

Proof. Fix $\eta > 0$, $M_0 > 0$. Let (a_j) be a strictly increasing sequence of positive real numbers such that for every $j \in \mathbf{N}$ there exists a Lipschitz function $\varphi_j : \mathbf{R}^m \to \mathbf{R}^m$ with Lipschitz constant less than or equal to 1 satisfying

$$\varphi_j(y) = \begin{cases} y & \text{if } |y| \leqq a_j, \\ 0 & \text{if } |y| > a_{j+1}. \end{cases}$$

The sequence (a_j) will be determined subsequently in a suitable way (see (3.6)) and will depend only on η and M_0. For every $h \in \mathbf{N}$ and $j \in \mathbf{N}$ let $w_h^j = \varphi_j(u_h)$. Consider the volume part of $F_{\varepsilon_h}(w_h^j, A)$; we have

$$\begin{aligned}\int_A f_{\varepsilon_h}(x, \nabla w_h^j)\,dx &= \int_{A \cap \{|u_h| \leqq a_j\}} f_{\varepsilon_h}(x, \nabla u_h)\,dx + \int_{A \cap \{|u_h| > a_{j+1}\}} f_{\varepsilon_h}(x, 0)\,dx \\ &\quad + \int_{A \cap \{a_j < |u_h| \leqq a_{j+1}\}} f_{\varepsilon_h}(x, \nabla w_h^j)\,dx \\ &\leqq \int_A f_{\varepsilon_h}(x, \nabla u_h)\,dx + c_2 |A \cap \{|u_h| > a_{j+1}\}| \\ &\quad + c_2 \int_{A \cap \{a_j < |u_h| \leqq a_{j+1}\}} (1 + |\nabla u_h|^p)\,dx.\end{aligned}$$

As for the surface part, it is not restrictive to assume that $|u_h^-| \leqq |u_h^+|$ $\mathscr{H}^{n-1}$-a.e. on S_{u_h}. Since $(w_h^j)^\pm = \varphi_j(u_h^\pm)$ we have

$$\begin{aligned}&\int_{S_{w_h^j} \cap A} g_{\varepsilon_h}(x, (w_h^j)^+ - (w_h^j)^-, \nu_{w_u^j})\,d\mathscr{H}^{n-1} \\ &\qquad \leqq \int_{(S_{u_h} \setminus \{a_{j+1} \leqq |u_h^-|\}) \cap A} g_{\varepsilon_h}(x, \varphi_j(u_h^+) - \varphi_j(u_h^-), \nu_{u_h})\,d\mathscr{H}^{n-1}.\end{aligned}$$

The set $S_{u_h} \setminus \{a_{j+1} \leqq |u_h^-|\}$ can be decomposed as $\bigcup_{i=1}^5 S_i^j$, where

$$\begin{aligned}&S_1^j = \{|u_h^+| < a_j\}, \quad S_2^j = \{|u_h^-| < a_j,\ a_{j+1} \leqq |u_h^+|\}, \\ &S_3^j = \{|u_h^-| < a_j \leqq |u_h^+| < a_{j+1}\}, \quad S_4^j = \{a_j \leqq |u_h^-|,\ |u_h^+| < a_{j+1}\}, \\ &S_5^j = \{a_j \leqq |u_h^-| < a_{j+1} \leqq |u_h^+|\}.\end{aligned}$$

Hence, taking into account the Lipschitz continuity of φ_i, we have

$$\begin{aligned}&\int_{S_{w_h^j} \cap A} g_{\varepsilon_h}(x, (w_h^j)^+ - (w_h^j)^-, \nu_{w_h^j})\,d\mathscr{H}^{n-1} \\ &\qquad \leqq \int_{S_1^j \cap A} g_{\varepsilon_h}(x, u_h^+ - u_h^-, \nu_{u_h})\,d\mathscr{H}^{n-1} + c_4 \int_{S_2^j \cap A} (1 + |u_h^-|)\,d\mathscr{H}^{n-1} \\ &\qquad\quad + \sum_{i=3}^5 c_4 \int_{S_i^j \cap A} (1 + |u_h^+ - u_h^-|)\,d\mathscr{H}^{n-1}.\end{aligned}$$

We now use these inequalities to estimate $1/N \sum_{j=1}^N F_{\varepsilon_h}(w_h^j, A)$, for every fixed $h \in \mathbf{N}$ and $N \in \mathbf{N}$. Note that each of the families $(\{a_j < |u_h| \leqq a_{j+1}\})_{j \in \mathbf{N}}$, $(S_i^j)_{j \in \mathbf{N}}$ $(i = 3, 4, 5)$ consists of pairwise disjoint sets. Then

$$\begin{aligned}\text{(3.5)} \qquad \frac{1}{N} \sum_{j=1}^N F_{\varepsilon_h}(w_h^j, A) &\leqq F_{\varepsilon_h}(u_h, A) \\ &\quad + \frac{1}{N} \sum_{j=1}^N \Big(c_2 |A \cap \{|u_h| > a_{j+1}\}| + c_4 \int_{S_2^j \cap A} (1 + |u_h^-|)\,d\mathscr{H}^{n-1}\Big) \\ &\quad + \frac{1}{N}\Big(c_2 \int_A (1 + |\nabla u_h|^p)\,dx + 3c_4 \int_{S_{u_h} \cap A} (1 + |u_h^+ - u_h^-|)\,d\mathscr{H}^{n-1}\Big).\end{aligned}$$

By assumption there exists a constant $c > 0$ such that

$$c_2 \int_A (1 + |\nabla u_h|^p)\, dx + 3c_4 \int_{S_{u_h} \cap A} (1 + |u_h^+ - u_h^-|)\, d\mathscr{H}^{n-1} \leqq c$$

for every $h \in \mathbf{N}$. Choose $N \in \mathbf{N}$ such that $c/N \leqq \eta/3$. Moreover (we may suppose that $c_4 \geqq 1$)

$$c \geqq \int_{S_2^j \cap A} |u_h^+ - u_h^-|\, d\mathscr{H}^{n-1} \geqq \int_{S_2^j \cap A} (|u_h^+| - |u_h^-|)\, d\mathscr{H}^{n-1}$$
$$\geqq (a_{j+1} - a_j)\, \mathscr{H}^{n-1}(S_2^j \cap A),$$

whence

$$\int_{S_2^j \cap A} (1 + |u_h^-|)\, d\mathscr{H}^{n-1} \leqq c\, \frac{1 + a_j}{a_{j+1} - a_j}.$$

The sequence (a_j) is now defined recursively by the following conditions

$$(3.6) \qquad \begin{aligned} & c_2 |A \cap \{|u_h| > a_1\}| \leqq \frac{\eta}{3} \quad \text{for every } h \in \mathbf{N}, \quad a_1 \geqq M_0, \\ & c_4 c\, \frac{1 + a_j}{a_{j+1} - a_j} \leqq \frac{\eta}{3} \quad \text{for every } j \in \mathbf{N}, \end{aligned}$$

which is possible by the assumed boundedness of (u_h) in $L^1(A; \mathbf{R}^m)$. From (3.5) we now obtain

$$\frac{1}{N} \sum_{j=1}^{N} F_{\varepsilon_h}(w_h^j, A) \leqq F_{\varepsilon_h}(u_h, A) + \eta.$$

Therefore for every $h \in \mathbf{N}$ there exists $j(h) \in \{1, \ldots, N\}$ such that $F_{\varepsilon_h}(w_h^{j(h)}, A) \leqq F_{\varepsilon_h}(u_h, A) + \eta$. The function $\varphi_h = \varphi_{j(h)}$ is the Lipschitz function we were looking for. Note that N is independent of h. □

Remark 3.6. From its proof it follows immediately that the previous lemma still holds for the functionals of the type $F_{\varepsilon_h}(u_h, A) = \int_A f_{\varepsilon_h}(x, \nabla u)\, dx$ or $F_{\varepsilon_h}(u, A) = \int_{S_u \cap A} g_{\varepsilon_h}(x, u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1}$, with f_{ε_h} and g_{ε_h} as above.

Proof of Proposition 3.4. We can assume that $(F_{\varepsilon_h}(u_h, A))$ converges to a finite value. Fix $\eta > 0$. By applying Lemma 3.5 to the sequence (u_h) with $M_0 = \|u\|_\infty$, we obtain a sequence (v_h) in $SBV^p(A; \mathbf{R}^m) \cap L^\infty(A; \mathbf{R}^m)$, bounded in $L^\infty(A; \mathbf{R}^m)$, such that $v_h \to u$ in $L^p(A; \mathbf{R}^m)$ and $\liminf_{h \to +\infty} F_{\varepsilon_h}(v_h, A) \leqq \liminf_{h \to +\infty} F_{\varepsilon_h}(u_h, A) + \eta$. By the Γ-convergence of $(F_{\varepsilon_{r(h)}})$ we have $F_0(u, A) \leqq \liminf_{h \to +\infty} F_{\varepsilon_h}(u_h, A)$. The arbitrariness of η yields the proof. □

In the following lemma we assume that $f_{\varepsilon_h}(x, \xi) = f(x/\varepsilon_h, \xi)$ and $g_{\varepsilon_h}(x, s, \nu) = g(x/\varepsilon_h, s, \nu)$ where f and g are the functions introduced in Section 2.

Lemma 3.7. *Let* (ε_h) *be a sequence of positive numbers converging to* 0 *such that for every* $A \in \mathscr{A}_0$ *the limit* $F_0(\cdot, A) = \Gamma\text{-}\lim_{h \to +\infty} F_{\varepsilon_h}(\cdot, A)$ *exists on* $SBV^p(A; \mathbf{R}^m) \cap L^p(A; \mathbf{R}^m)$ *with respect to the* $L^p(A; \mathbf{R}^m)$*-topology. Then for every* $u \in SBV^p(A; \mathbf{R}^m) \cap L^p(A; \mathbf{R}^m)$, $a \in \mathbf{R}^m$ *and* $y \in \mathbf{R}^n$,

$$\text{(i)}\ F_0(u + a, A) = F_0(u, A), \qquad \text{(ii)}\ F_0(\tau_y u, \tau_y A) = F_0(u, A),$$

where $(\tau_y u)(x) = u(x - y)$ *and* $\tau_y A = A + y$.

Proof. First we prove (i). Let (u_h) in $SBV^p(A; \mathbf{R}^m) \cap L^p(A; \mathbf{R}^m)$ be a sequence converging to u in $L^p(A; \mathbf{R}^m)$ and such that $(F_{\varepsilon_h}(u_h, A))$ converges to $F_0(u, A)$. Then $(u_h + a)$ converges to $u + a$ in $L^p(A; \mathbf{R}^m)$ and

$$F_0(u + a, A) \leqq \liminf_{h \to +\infty} F_{\varepsilon_h}(u_h + a, A) = \liminf_{h \to +\infty} F_{\varepsilon_h}(u_h, A) = F_0(u, A).$$

On the other hand, $F_0(u, A) = F_0((u + a) + (-a), A) \leqq F_0(u + a, A)$.

We now prove (ii). There exists a sequence (z_h) in $\mathbf{Z}^n$ such that $y_h = \varepsilon_h z_h$ converges to y. Let (u_h) be a sequence as in the proof of (i). Set $v_h = \tau_{y_h} u_h : A + y_h \to \mathbf{R}^m$. By taking the periodicity assumptions on f and g into account we get

$$\begin{aligned} F_{\varepsilon_h}(u_h, A) &= \int_A f\left(\frac{x + y_h}{\varepsilon_h}, \nabla u_h\right) dx + \int_{S_{u_h} \cap A} g\left(\frac{x + y_h}{\varepsilon_h}, u_h^+ - u_h^-, v_{u_h}\right) d\mathscr{H}^{n-1} \\ &= \int_{A + y_h} f\left(\frac{x}{\varepsilon_h}, \nabla u_h\right) dx + \int_{S_{v_h} \cap (A + y_h)} g\left(\frac{x}{\varepsilon_h}, v_h^+ - v_h^-, v_{v_h}\right) d\mathscr{H}^{n-1}. \end{aligned}$$

Let $B \subset\subset A$; for h sufficiently large we may assume $A + y_h \supseteq B + y$; hence

$$F_{\varepsilon_h}(u_h, A) \geqq \int_{B + y} f\left(\frac{x}{\varepsilon_h}, \nabla v_h\right) dx + \int_{S_{v_h} \cap (B + y)} g\left(\frac{x}{\varepsilon_h}, v_h^+ - v_h^-, v_{v_h}\right) d\mathscr{H}^{n-1},$$

which yields $F_0(u, A) \geqq F_0(\tau_y u, B + y)$, since (v_h) converges to $\tau_y u$. By the arbitrariness of $B \subset\subset A$ we also have $F_0(u, A) \geqq F_0(\tau_y u, \tau_y A)$. We conclude the proof of (ii) by noticing that $F_0(\tau_y u, \tau_y A) \geqq F_0(\tau_{-y}(\tau_y u), \tau_{-y}(\tau_y A)) = F_0(u, A)$. □

4. Integral representation on $W^{1,p}(\Omega; \mathbf{R}^m)$ and on partitions

On account of Proposition 3.3 we shall try to identify the Γ-limit of convergent sequences of functionals F_ε. Therefore, up to Section 7 we assume that a sequence (ε_h) of positive numbers converging to 0 is given, such that for every $A \in \mathscr{A}_0$ the limit

$$F_0(\cdot, A) = \Gamma\text{-}\lim_{h \to +\infty} F_{\varepsilon_h}(\cdot, A) \tag{4.1}$$

exists on the space $SBV^p(A; \mathbf{R}^m) \cap L^p(A; \mathbf{R}^m)$ endowed with the $L^p(A; \mathbf{R}^m)$-topology. In particular, $F_0(\cdot, A)$ is lower semicontinuous with respect to the $L^p(A; \mathbf{R}^m)$-topology. As we have seen, for every $\Omega \in \mathscr{A}_0$ and $u \in SBV^p(\Omega; \mathbf{R}^m) \cap L^p(\Omega; \mathbf{R}^m)$ the set function $F_0(u, \cdot)$ can be extended to a Borel measure on Ω. Such a measure is given by (see [35, Theorem 14.23])

$$F_0(u, B) = \inf\{F_0(u, A) : A \in \mathscr{A}(\Omega), B \subseteq A\} \tag{4.2}$$

for every $B \in \mathscr{B}(\Omega)$. Moreover, from the considerations at the beginning of Section 3 it follows that there exist two constants $\gamma_1, \gamma_2 > 0$ such that for every $u \in SBV^p(A; \mathbf{R}^m) \cap L^p(A; \mathbf{R}^m)$ we have $\gamma_1 H(u, A) \leqq F_0(u, A) \leqq \gamma_2(H(u, A) + |A|)$, where H is defined in (3.1). By (4.2) we immediately obtain

$$\gamma_1 H(u, B) \leqq F_0(u, B) \leqq \gamma_2(H(u, B) + |B|) \tag{4.3}$$

for every $B \in \mathscr{B}(\Omega)$.

Proposition 4.1. *There exists a unique quasiconvex function* $\tilde{f}: M^{m \times n} \to [0, +\infty[$ *with the following properties*:

(i) *There exist* $\gamma_1, \gamma_2 > 0$ *such that* $\gamma_1 |\xi|^p \leqq \tilde{f}(\xi) \leqq \gamma_2(1 + |\xi|^p)$ *for every* $\xi \in M^{m \times n}$.

(ii) $F_0(u, A) = \int_A \tilde{f}(\nabla u)\, dx$ *for every* $A \in \mathscr{A}_0$ *and* $u \in W^{1,p}(A; \mathbf{R}^m)$.

Proof. Let $\Omega \in \mathscr{A}_0$ and consider $F_0: W^{1,p}(\Omega; \mathbf{R}^m) \times \mathscr{A}(\Omega) \to [0, +\infty[$. This functional satisfies the assumptions of Theorem 1.1 in [29], i.e., for every $u, v \in W^{1,p}(\Omega; \mathbf{R}^m)$ and $A \in \mathscr{A}(\Omega)$:

(a) $F_0(u, A) = F_0(v, A)$ whenever $u|_A = v|_A$.

(b) The set function $F_0(u, \cdot)$ is the restriction to $\mathscr{A}(\Omega)$ of a Borel measure on Ω.

(c) $F_0(u, A) \leqq c \int_A (1 + |Du|^p)\, dx$, with c a positive constant.

(d) $F_0(u + a, A) = F_0(u, A)$ for every $a \in \mathbf{R}^m$.

(e) $F_0(\cdot, A)$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbf{R}^m)$.

In fact, properties (b), (c), and (d) follow from Proposition 3.3, estimate (4.3) and Lemma 3.7, respectively, while (a) and (e) can be obtained immediately from the fact that $F_0(\cdot, A)$ is the Γ-limit (4.1).

By [29, Theorem 1.1] the Carathéodory function $\tilde{f}: \mathbf{R}^n \times M^{m \times n} \to [0, +\infty[$ defined by

$$\tilde{f}(x, \xi) = \limsup_{\rho \to 0} \frac{F_0(u_\xi, B_\rho(x))}{|B_\rho(x)|} \tag{4.4}$$

(u_ξ is the linear function defined by $u_\xi(x) = \xi \cdot x$) gives the integral representation

$$F_0(u, \Omega) = \int_\Omega \tilde{f}(x, Du)\, dx$$

for every $u \in W^{1,p}(\Omega; \mathbf{R}^m)$. The function $\tilde{f}(x, \cdot)$ is quasiconvex for a.e. $x \in \mathbf{R}^n$, and, from (4.4) and Lemma 3.7, we deduce that $\tilde{f}(x, \xi)$ is constant with respect to $x \in \mathbf{R}^n$. Consequently we can drop the dependence on x. Finally, the uniqueness of $\tilde{f}$ follows from (4.4), while (i) follows from (ii) and (4.3). □

The next step is to obtain an integral representation formula for F_0 on finite partitions, i.e., on those BV functions which take only a finite set of values. It will be achieved by applying a theorem due to AMBROSIO & BRAIDES. Given $\Omega \in \mathscr{A}_0$ and a finite subset T of $\mathbf{R}^m$ we denote by $BV(\Omega; T)$ the set of functions $u:\Omega \to T$ which belong to $BV(\Omega; \mathbf{R}^m)$. It turns out that $BV(\Omega; T) \subseteq SBV^p(\Omega; \mathbf{R}^m) \cap L^\infty(\Omega; \mathbf{R}^m)$.

Proposition 4.2. *There exists a unique function* $\tilde{g}:\mathbf{R}^m \times S^{n-1} \to [0, +\infty[$ *continuous in the second variable and such that*
(i) $\tilde{g}(-s, -v) = \tilde{g}(s, v)$ *for every* $(s, v) \in \mathbf{R}^m \times S^{n-1}$;
(ii) *for every finite subset* T *of* $\mathbf{R}^m$,

$$F_0(u, S) = \int_S \tilde{g}(u^+ - u^-, v_u)\, d\mathscr{H}^{n-1}$$

for every $A \in \mathscr{A}_0$, $u \in BV(A; T)$ *and* S *a Borel subset of* $S_u \cap A$.

Proof. Let T be a finite subset of $\mathbf{R}^m$ and let $\Omega \in \mathscr{A}_0$. For every $A \in \mathscr{A}(\Omega)$ and $u \in BV(\Omega; T)$ we define $G_T(u, A) = F_0(u, S_u \cap A)$, where $F_0(u, S_u \cap A)$ is defined in (4.2). Let us show that the assumptions of Theorem 3.1 in [8] are satisfied by $G_T: BV(\Omega; T) \times \mathscr{A}(\Omega) \to [0, +\infty[$ in the following form:
(i) $0 \leqq G_T(u, A) \leqq \Lambda \mathscr{H}^{n-1}(A \cap S_u)$ with $\Lambda \in \mathbf{R}$ fixed.
(ii) $G_T(u, \cdot)$ is the restriction to $\mathscr{A}(\Omega)$ of a Borel measure on Ω.
(iii) $G_T(u, A) = G_T(v, A)$ whenever $u = v$ a.e. in A.
(iv) If $u_h \to u$ a.e. in A, then $G_T(u, A) \leqq \liminf_{h\to+\infty} G_T(u_h, A)$.
(v) $G_T(\tau_y u, \tau_y A) = G_T(u, A)$ (where $(\tau_y u)(x) = u(x - y)$ and $\tau_y A = A + y$) for every $y \in \mathbf{R}^n$ whenever $\tau_y A \subseteq \Omega$.

Property (i) follows immediately from the definition of G_T and from estimate (4.3). As for (ii), the Borel measure $F_0(u, \cdot)$ on Ω (see (4.2)), restricted to S_u, is an extension of $G_T(u, \cdot)$. The proof of (iii) follows immediately from (4.1) and the definition (4.2) of $F_0(u, \cdot)$ on Borel sets.

Let us come to (iv). If $u_h \to u$ a.e. on A, by the equiboundedness of (u_h), it turns out that $u_h \to u$ in $L^p(A; \mathbf{R}^m)$. For every open subset E of A with $S_u \cap A \subseteq E$ we have

$$F_0(u, E) \leqq \liminf_{h\to+\infty} F_0(u_h, E).$$

Furthermore, by (4.3),

$$F_0(u_h, E) = F_0(u_h, S_{u_h} \cap E) + F_0(u_h, E \backslash S_{u_h}) \leqq F_0(u_h, S_{u_h} \cap A) + \gamma_2 |E|.$$

Thus,

$$F_0(u, E) \leqq \liminf_{h\to+\infty} F_0(u_h, S_{u_h} \cap A) + \gamma_2 |E|.$$

By taking the infimum over all open sets $E \supseteq S_u \cap A$, we get

$$F_0(u, S_u \cap A) \leqq \liminf_{h\to+\infty} F_0(u_h, S_{u_h} \cap A).$$

Finally, for the proof of (v) it is enough to refer to the property of translation invariance shown for F_0 in Lemma 3.7.

At this point we can apply Theorem 3.1 of [8], which yields the existence of a unique continuous function $g_T^\Omega : \Omega \times T \times T \times S^{n-1} \to [0, +\infty[$ such that $g_T^\Omega(x, a, b, \nu) = g_T^\Omega(x, b, a, -\nu)$ and

$$(4.5) \qquad F_0(u, S_u \cap A) = G_T(u, A) = \int_{S_u \cap A} g_T^\Omega(x, u^+, u^-, \nu_u)\, d\mathscr{H}^{n-1}$$

for every $u \in BV(\Omega; T)$ and $A \in \mathscr{A}(\Omega)$. Define

$$u_{x_0,\nu}^{a,b}(x) = \begin{cases} a & \text{if } \langle x - x_0, \nu \rangle \geqq 0, \\ b & \text{if } \langle x - x_0, \nu \rangle < 0, \end{cases} \qquad \Pi_{x_0,\nu} = \{ y \in \mathbf{R}^n : \langle y - x_0, \nu \rangle = 0 \}$$

whenever $x, x_0 \in \mathbf{R}^n$, $a, b \in \mathbf{R}^m$ and $\nu \in S^{n-1}$. The continuity of $g_T^\Omega(\cdot, a, b, \nu)$ yields that

$$(4.6) \qquad g_T^\Omega(x_0, a, b, \nu) = \lim_{\rho \to 0} \frac{F_0(u_{x_0,\nu}^{a,b}, \Pi_{x_0,\nu} \cap B_\rho(x_0))}{\mathscr{H}^{n-1}(\Pi_{x_0,\nu} \cap B_\rho(x_0))}$$

for every $(x_0, a, b, \nu) \in \Omega \times T \times T \times S^{n-1}$. This allows us to replace the integrand in (4.5) by a function $\tilde{g}(x, a, b, \nu)$, independent of Ω and T, defined on $\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^m \times S^{n-1}$. From (4.6) and Lemma 3.7 we also obtain that $\tilde{g}$ is independent of x and depends on (a, b) through the difference $a - b$. Therefore we can write

$$F_0(u, S_u \cap A) = \int_{S_u \cap A} \tilde{g}(u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1}$$

for every finite subset T of $\mathbf{R}^m$, $A \in \mathscr{A}_0$ and $u \in BV(A; T)$. Since $F_0(u, \cdot) \llcorner S_u$ is a regular Borel measure, this immediately yields the integral representation on the Borel subsets of S_u stated in the proposition. □

5. Characterization of the homogenized bulk energy density

The goal of this section is to prove that the function $\tilde{f}$ given in Proposition 4.1 is precisely the function f_{hom} introduced in Proposition 2.1. This is achieved in the next two propositions. We use the notation $Y =]0, 1[^n$.

Proposition 5.1. *$\tilde{f}(\xi) \leqq f_{\text{hom}}(\xi)$ for every $\xi \in M^{m \times n}$.*

Proof. Fix $\xi \in M^{m \times n}$. From the definition of f_{hom}, for every $\sigma > 0$ there exists $\varepsilon = \varepsilon(\sigma) > 0$ and a function $v \in W_0^{1,p}(Y; \mathbf{R}^m)$ such that

$$\int_Y f(x/\varepsilon, \xi + \nabla v(x))\, dx \leqq f_{\text{hom}}(\xi) + \sigma.$$

We still denote by v the 1-periodic extension of v to $\mathbf{R}^n$. For every $h \in \mathbf{N}$ define $u_h(x) = \xi \cdot x + \frac{\varepsilon_h}{\varepsilon} v\left(\frac{\varepsilon}{\varepsilon_h} x\right)$ for $x \in \mathbf{R}^n$. Since $\left(v\left(\frac{\varepsilon}{\varepsilon_h} x\right)\right)_h$ is bounded in $L^p(Y; \mathbf{R}^m)$, we have that (u_h) converges to $\xi \cdot x$ in $L^p(Y; \mathbf{R}^m)$. We may assume that $\varepsilon = \frac{1}{k}$ for a suitable $k \in \mathbf{N}$, so that the function $x \mapsto f\left(\frac{x}{\varepsilon}, \eta\right)$ is Y-periodic for every η. Hence, by the definition of F_0,

$$F_0(\xi \cdot x, Y) \leqq \liminf_{h \to +\infty} \int_Y f\left(\frac{x}{\varepsilon_h}, \xi + (\nabla v)\left(\frac{\varepsilon}{\varepsilon_h} x\right)\right) dx$$

$$\leqq \liminf_{h \to +\infty} \left(\frac{\varepsilon_h}{\varepsilon}\right)^n \int_{([\varepsilon/\varepsilon_h] + 1) Y} f\left(\frac{x}{\varepsilon}, \xi + \nabla v(x)\right) dx$$

$$= \liminf_{h \to +\infty} \left(\frac{\varepsilon_h}{\varepsilon}\right)^n \left(\left[\frac{\varepsilon}{\varepsilon_h}\right] + 1\right)^n \int_Y f\left(\frac{x}{\varepsilon}, \xi + \nabla v(x)\right) dx.$$

Therefore $F_0(\xi \cdot x, Y) \leqq f_{\mathrm{hom}}(\xi) + \sigma$. The conclusion follows from Proposition 4.1 and the arbitrariness of $\sigma > 0$. $\square$

Proposition 5.2. *For every $\xi \in M^{m \times n}$, we have $\tilde{f}(\xi) \geqq f_{\mathrm{hom}}(\xi)$.*

Proof. For the sake of clarity, the proof is divided into three steps. Fix $\xi \in M^{m \times n}$.

Step 1. *There exist a sequence of positive real numbers (α_h) converging to 0 and a sequence of functions (w_h) in $SBV^p(Y; \mathbf{R}^m) \cap L^p(Y; \mathbf{R}^m)$ such that*

(5.1)
$$\text{(i)}\quad w_h \to \xi \cdot x \quad \text{in } L^p(Y; \mathbf{R}^m),$$
$$\text{(ii)}\quad \lim_{h \to +\infty} \mathscr{H}^{n-1}(S_{w_h} \cap Y) = 0,$$
$$\text{(iii)}\quad \limsup_{h \to +\infty} \int_Y f\left(\frac{x}{\alpha_h}, \nabla w_h(x)\right) dx \leqq \tilde{f}(\xi).$$

Proof. By assumption there exists a sequence (v_h) in $SBV^p(Y; \mathbf{R}^m) \cap L^p(Y; \mathbf{R}^m)$ such that $v_h \to 0$ in $L^p(Y; \mathbf{R}^m)$ and

$$\lim_{h \to +\infty} F_{\varepsilon_h}(\xi \cdot x + v_h, Y) = F_0(\xi \cdot x, Y) = \tilde{f}(\xi).$$

Set $u_h(x) = \xi \cdot x + v_h(x)$, $\| v_h \|_{L^p(Y; \mathbf{R}^m)} = \sigma_h^{(n/p)+1}$, $\beta_h = k_h \varepsilon_h$. It is easy to see that there exists a divergent sequence (k_h) of natural numbers such that

$$\beta_h \to 0, \quad \frac{\sigma_h}{\beta_h} \to 0.$$

Now, in order to be able to introduce the sequence (w_h), let us consider for every $h \in \mathbf{N}$ and $\lambda \in \mathbf{N}^n$ the set $Q_{h,\lambda} = \beta_h(\lambda + Y)$. Let $\lambda(h) \in \mathbf{N}^n$ be the index of a "minimal cube", i.e.,

$$F_{\varepsilon_h}(u_h, Q_{h,\lambda(h)}) \leqq F_{\varepsilon_h}(u_h, Q_{h,\lambda})$$

for every $\lambda \in \mathbf{N}^n$ with $Q_{h,\lambda} \subseteq Y$, and set

$$Q_h = Q_{h,\lambda(h)}, \quad x_h = \beta_h \lambda(h), \quad w_h(x) = \xi \cdot x + \frac{1}{\beta_h} v_h(x_h + \beta_h x), \quad x \in Y.$$

Let us prove (5.1) (i). Easy computations show that

$$\begin{aligned} \| w_h - \xi \cdot x \|_{L^p(Y;\mathbf{R}^m)} &= \frac{1}{(\beta_h)^{(n/p)+1}} \left(\int_{Q_h} |v_h(y)|^p \, dy \right)^{1/p} \\ &\leqq \frac{1}{(\beta_h)^{(n/p)+1}} \| v_h \|_{L^p(Y;\mathbf{R}^m)} = \left(\frac{\sigma_h}{\beta_h} \right)^{(n/p)+1}. \end{aligned}$$

The conclusion is now immediate by our assumptions on σ_h/β_h.

Let us prove (5.1) (ii). For every $h \in \mathbf{N}$,

$$F_{\varepsilon_h}(u_h, Y) \geqq \left[\frac{1}{\beta_h} \right]^n F_{\varepsilon_h}(u_h, Q_h) \geqq \left[\frac{1}{\beta_h} \right]^n c_3 \mathscr{H}^{n-1}(S_{u_h} \cap Q_h).$$

Therefore, since the sequence $(F_{\varepsilon_h}(u_h, Y))$ is bounded, there exists a constant $C > 0$ such that

$$\mathscr{H}^{n-1}(S_{u_h} \cap Q_h) \leqq \frac{C}{[1/b_h]^n} \quad \text{for every } h \in \mathbf{N}.$$

On the other hand, $S_{w_h} \cap Y = \dfrac{1}{\beta_h}((S_{v_h} \cap Q_h) - x_h) = \dfrac{1}{\beta_h}((S_{u_h} \cap Q_h) - x_h)$, which implies that

$$\mathscr{H}^{n-1}(S_{w_h} \cap Y) = \frac{1}{(\beta_h)^{n-1}} \mathscr{H}^{n-1}(S_{u_h} \cap Q_h) \leqq C \left(\frac{1/\beta_h}{[1/\beta_h]} \right)^{n-1} \cdot \frac{1}{[1/\beta_h]}.$$

Since (β_h) converges to 0 as h tends to $+\infty$, the proof of (5.1) (ii) is accomplished.

Let us now define $\alpha_h = \varepsilon_h/\beta_h$. Then $\alpha_h \to 0$ as $h \to +\infty$. We now prove (5.1) (iii). By taking into account the periodicity assumption on f and the fact that $\beta_h/\varepsilon_h \in \mathbf{N}$, we have

$$\begin{aligned} \int_Y f\left(\frac{x}{\alpha_h}, \nabla w_h(x) \right) dx &= \int_Y f\left(\frac{x}{\alpha_h}, \xi + (\nabla v_h)(x_h + \beta_h x) \right) dx \\ &= \frac{1}{(\beta_h)^n} \int_{Q_h} f\left(\frac{y - x_h}{\varepsilon_h}, \xi + \nabla v_h(y) \right) dy \\ &= \frac{1}{(\beta_h)^n} \int_{Q_h} f\left(\frac{y}{\varepsilon_h}, \xi + \nabla v_h(y) \right) dy \\ &\leqq \left(\frac{1/\beta_h}{[1/\beta_h]} \right)^n \left[\frac{1}{\beta_h} \right]^n F_{\varepsilon_h}(u_h, Q_h) \leqq \left(\frac{1/\beta_h}{[1/\beta_h]} \right)^n F_{\varepsilon_h}(u_h, Y). \end{aligned}$$

Hence $\limsup_{h \to +\infty} \int_Y f\left(\dfrac{x}{\alpha_h}, \nabla w_h(x) \right) dx \leqq \lim_{h \to +\infty} F_{\varepsilon_h}(u_h, Y) = \tilde{f}(\xi)$, which proves (5.1) (iii).

Step 2. *For every fixed $\eta > 0$ the sequence (w_h) can be chosen so that*

$$
(5.2)\qquad
\begin{aligned}
&\text{(i) } \exists \Lambda > 0 \text{ such that } \| w_h \|_{L^\infty(Y;\mathbf{R}^m)} \leqq \Lambda \quad \forall h \in \mathbf{N},\\
&\text{(ii) } w_h \to \xi \cdot x \quad \text{in } L^p(Y;\mathbf{R}^m),\\
&\text{(iii) } \lim_{h\to+\infty} \mathscr{H}^{n-1}(S_{w_h} \cap Y) = 0,\\
&\text{(iv) } \limsup_{h\to+\infty} \int_Y f\left(\frac{x}{\alpha_h}, \nabla w_h(x)\right) dx \leqq \tilde{f}(\xi) + \eta.
\end{aligned}
$$

Proof. Apply Remark 3.6 to the sequence of functionals $u \mapsto \int_Y f(\frac{x}{\alpha_h}, \nabla u(x))\, dx$. Thus, Lemma 3.5 applied to (w_h), with $M_0 = \| \xi \cdot x \|_{L^\infty(Y;\mathbf{R}^m)}$, furnishes a sequence $\varphi_h(w_h)$ which satisfies properties (5.2), as one can easily check on account of properties (5.1) and the fact that $S_{\varphi_h(w_h)} \subseteq S_{w_h}$. Now the functions $\varphi_h(w_h)$ are renamed w_h.

Step 3. We replace the sequence (w_h) with a suitable sequence in $W^{1,p}$ (actually Lipschitz functions), still satisfying properties similar to (5.2) (ii) and (5.2) (iv), and apply the homogenization results in $W^{1,p}$.

Proof. We first need a preliminary remark. By Proposition 1.1, for every $h \in \mathbf{N}$ we have

$$\int_Y M^p(|\nabla w_h|)\, dx \leqq C(p,n) \int_Y |\nabla w_h|^p\, dx \leqq \frac{C(p,n)}{c_1} \int_Y f\left(\frac{x}{\alpha_h}, \nabla w_h\right) dx.$$

Then the sequence $(M^p(|\nabla w_h|))$ is bounded on $L^1(Y)$. This ensures, as proved in [1, Lemma 1.7], a weak equi-integrability property for $(M^p(|\nabla w_h|))$. More precisely, for any $\varepsilon > 0$ there exist a Borel set $C_\varepsilon \subseteq Y$, $\delta > 0$ and an infinite set $S \subseteq \mathbf{N}$ such that $|C_\varepsilon| < \varepsilon$ and for all $C \in \mathscr{B}(Y)$,

$$\text{if } C \cap C_\varepsilon = \emptyset \quad \text{and} \quad |C| < \delta, \quad \text{then} \quad \int_C M^p(|\nabla w_h|) dx < \varepsilon \quad \text{for all } h \in S.$$

Fix $\varepsilon > 0$ and let C_ε and δ enjoy this property. It is not restrictive to assume that $S = \mathbf{N}$. Since $(M^p(|\nabla w_h|))$ is bounded in $L^1(Y)$, we can choose $\lambda_\varepsilon \geqq 1$ such that for every $\lambda \geqq \lambda_\varepsilon$ and $h \in \mathbf{N}$,

$$|\{x \in Y : M(|\nabla w_h|) > \lambda\}| < \delta, \quad \frac{\Lambda}{\varepsilon} < \lambda,$$

where Λ is given in (5.2) (i).

By Theorem 1.2, for every $h \in \mathbf{N}$ and $\lambda \geqq \lambda_\varepsilon$ fixed, there exists a Lipschitz function $w_{h,\lambda}: Y_\varepsilon \to \mathbf{R}^m$ whose Lipschitz constant does not exceed $2m(c'(n)+1)\lambda$ such that $w_{h,\lambda} = w_h$ a.e. on $Y_\varepsilon \setminus E_{h,\lambda}$, where

$$E_{h,\lambda} = \{x \in Y : M(|Dw_h|) > 2\lambda\}.$$

Since $M(|Dw_h|) \leqq M(|\nabla w_h|) + M(|D^s w_h|)$, by Proposition 1.1 we have

$$\begin{aligned} |E_{h,\lambda} \setminus C_\varepsilon| &\leqq |\{x \in Y \setminus C_\varepsilon : M(|\nabla w_h|)(x) > \lambda\}| + |\{x \in Y : M(|D^s w_h|)(x) > \lambda\}| \\ &\leqq |\{x \in Y \setminus C_\varepsilon : M^p(|\nabla w_h|)(x) > \lambda^p\}| + \lambda^{-1} c(n) |D^s w_h|(Y) \\ &\leqq \lambda^{-p} \int_{\{M(|\nabla w_h|) > \lambda\} \setminus C_\varepsilon} M^p(|\nabla w_h|)\, dx \\ &\quad + 2\lambda^{-1} c(n) \| w_h \|_{L^\infty(Y;\mathbf{R}^m)} \mathscr{H}^{n-1}(S_{w_h} \cap Y) \\ &\leqq \varepsilon \lambda^{-p} + 2\lambda^{-1} c(n) \Lambda \mathscr{H}^{n-1}(S_{w_h} \cap Y). \end{aligned}$$

This implies by (5.2) (iii) that

$$\limsup_{h \to +\infty} \lambda^p |E_{h,\lambda} \setminus C_\varepsilon| \leqq \varepsilon. \tag{5.3}$$

Moreover, for a suitable constant $K = K(n, m) > 0$ we have

$$0 \leqq f\left(\frac{x}{\alpha_h}, \nabla w_{h,\lambda}\right) \leqq K\lambda^p \quad \text{on } Y_\varepsilon. \tag{5.4}$$

Thus, since $\nabla w_{h,\lambda} = \nabla w_h$ on $Y_\varepsilon \setminus E_{h,\lambda}$, we get

$$\begin{aligned} \int_{Y_\varepsilon \setminus C_\varepsilon} f\left(\frac{x}{\alpha_h}, \nabla w_{h,\lambda}\right) dx &\leqq \int_{(Y_\varepsilon \setminus C_\varepsilon) \setminus E_{h,\lambda}} f\left(\frac{x}{\alpha_h}, \nabla w_h\right) dx + K\lambda^p |E_{h,\lambda} \setminus C_\varepsilon| \\ &\leqq \int_Y f\left(\frac{x}{\alpha_h}, \nabla w_h\right) dx + K\lambda^p |E_{h,\lambda} \setminus C_\varepsilon|. \end{aligned}$$

By taking into account (5.2) (iv) and (5.3), the previous inequality yields

$$\begin{aligned} \limsup_{h \to +\infty} \int_{Y_\varepsilon \setminus C_\varepsilon} f\left(\frac{x}{\alpha_h}, \nabla w_{h,\lambda}\right) dx &\leqq \limsup_{h \to +\infty} \int_Y f\left(\frac{x}{\alpha_h}, \nabla w_h\right) dx + K\varepsilon \\ &\leqq \tilde{f}(\xi) + K\varepsilon + \eta. \end{aligned} \tag{5.5}$$

From (5.4) we also deduce that for every $\lambda \geqq \lambda_\varepsilon$ there exist an increasing sequence $(\sigma(h))$ of natural numbers and a function $\varphi \in L^\infty(Y_\varepsilon)$ such that $f(x/\alpha_{\sigma(h)}, \nabla w_{\sigma(h),\lambda}) \rightharpoonup \varphi$ in w^*-$L^\infty(Y_\varepsilon)$. In particular, for all $B \in \mathscr{B}(Y_\varepsilon)$,

$$\int_B \varphi(x)\, dx = \lim_{h \to +\infty} \int_B f\left(\frac{x}{\alpha_{\sigma(h)}}, \nabla w_{\sigma(h),\lambda}\right) dx. \tag{5.6}$$

On the other hand, by passing, if necessary, to a further subsequence (still depending on λ) we can assume with no loss of generality that there exists $w_\lambda \in W^{1,\infty}(Y_\varepsilon)$ such that $w_{\sigma(h),\lambda} \rightharpoonup w_\lambda$ in w^*-$W^{1,\infty}(Y_\varepsilon)$. By Proposition 2.1,

$$\int_A f_{\text{hom}}(\nabla w_\lambda)\, dx \leqq \liminf_{h \to +\infty} \int_A f\left(\frac{x}{\alpha_{\sigma(h)}}, \nabla w_{\sigma(h),\lambda}\right) dx = \int_A \varphi(x)\, dx$$

for all $A \in \mathscr{A}(Y_\varepsilon)$. It follows that $f_{\text{hom}}(\nabla w_\lambda(x)) \leqq \varphi(x)$ for a.e. $x \in Y_\varepsilon$. Then (5.5) and (5.6) with $B = Y_\varepsilon \backslash C_\varepsilon$ imply that

$$\int_{Y_\varepsilon \backslash C_\varepsilon} f_{\text{hom}}(\nabla w_\lambda)\, dx \leqq \limsup_{h \to +\infty} \int_{Y_\varepsilon \backslash C_\varepsilon} f\left(\frac{x}{\alpha_h}, \nabla w_{h,\lambda}\right) dx \leqq \tilde{f}(\xi) + K\varepsilon + \eta.$$

Therefore,

$$f_{\text{hom}}(\xi) | Y_\varepsilon \backslash C_{\varepsilon,\lambda}| = \int_{Y_\varepsilon \backslash C_{\varepsilon,\lambda}} f_{\text{hom}}(\xi)\, dx \leqq \tilde{f}(\xi) + K\varepsilon + \eta, \tag{5.7}$$

where $C_{\varepsilon,\lambda} = C_\varepsilon \cup \{x \in Y_\varepsilon : w_\lambda \neq \xi \cdot x\}$, and $\lambda \geqq \lambda_\varepsilon$.

Since $(|Dw_h|(Y))$ is a bounded sequence, by Proposition 1.1 there exists a constant $C > 0$ such that

$$|\{x \in Y_\varepsilon : w_{\sigma(h),\lambda} \neq w_{\sigma(h)}\}| \leqq |E_{\sigma(h),\lambda}| \leqq \frac{C}{\lambda}.$$

By the lower semicontinuity of the functional $w \mapsto |\{x \in Y_\varepsilon : w(x) \neq 0\}|$ with respect to the convergence in measure we infer that

$$|\{x \in Y_\varepsilon : w_\lambda(x) \neq \xi \cdot x\}| \leqq \frac{C}{\lambda}.$$

Hence, from (5.7), $f_{\text{hom}}(\xi)\,((1-\varepsilon)^n - \varepsilon - C/\lambda) \leqq \tilde{f}(\xi) + K\varepsilon + \eta$. The conclusion now follows if we consider successively the limits as $\lambda \to +\infty$, $\varepsilon \to 0^+$, and $\eta \to 0^+$. □

6. Characterization of the homogenized surface energy density

Here we first prove Proposition 2.2, where the function g_{hom} is introduced. Then, as we did in the previous section for the volume part, we prove that the function $\tilde{g}$, which represents F_0 on finite partitions according to Proposition 4.2, is actually the function g_{hom}.

Proof of Proposition 2.2. For every orthonormal basis $\nu = (\nu_0, \nu_1, \dots, \nu_{n-1})$ of $\mathbf{R}^n$, set

$$Q_\nu = \{\alpha_0 \nu_0 + \cdots + \alpha_{n-1}\nu_{n-1} : \alpha_0, \dots, \alpha_{n-1} \in]-\tfrac{1}{2}, \tfrac{1}{2}[\},$$

and for every $z \in \mathbf{R}^m$ and $\varepsilon > 0$, set

$$I_\varepsilon(\nu) = \inf\left\{ \int_{S_u \cap Q_\nu} g(x/\varepsilon, u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1} : u \in SBV^p(Q_\nu; \mathbf{R}^m), \right. \tag{6.1}$$

$$\left. \nabla u = 0 \text{ a.e.}, u = u_{z,\nu_0} \text{ on } \partial Q_\nu \right\}.$$

For the following four steps we consider $z \in \mathbf{R}^m$ fixed.

Step 1. *Let $\nu = (\nu_0, \nu_1, \dots, \nu_{n-1})$ and $\nu' = (\nu_0, \nu'_1, \dots, \nu'_{n-1})$ be two orthonormal bases of $\mathbf{R}^n$ with equal first vector. Suppose that ν is an orthonormal "rational basis",* *i.e., for all* $i \in \{0, \ldots, n-1\}$ *there exists* $r_i \in \mathbf{R} \setminus \{0\}$ *such that* $r_i v_i \in \mathbf{Z}^n$. *Then* $\limsup_{\varepsilon \to 0} I_\varepsilon(v') \leqq \liminf_{\varepsilon \to 0} I_\varepsilon(v)$.

Proof. For every $i = 1, \ldots, n-1$ let $\gamma_i > 0$ be such that $v_i = \gamma_i v_i \in \mathbf{Z}^n$. If we set

$$P = \{\alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1} : \alpha_1, \ldots, \alpha_{n-1} \in [-\tfrac{1}{2}, \tfrac{1}{2}[\},$$

then the function g is P-periodic in the first variable, in the sense that $g(x + l_1 v_1 + \cdots + l_{n-1} v_{n-1}, w, \mu) = g(x, w, \mu)$ whenever $(x, w, \mu) \in \mathbf{R}^n \times \mathbf{R}^m \times S^{n-1}$ and $l_1, l_2, \ldots, l_{n-1} \in \mathbf{Z}$.

Let $\varepsilon > 0$, $\eta > 0$ and $\sigma > 0$ be fixed. Let $u_\varepsilon \in SBV(Q_v; \mathbf{R}^m)$ with $\nabla u_\varepsilon = 0$ a.e. and $u_\varepsilon = u_{z, v_0}$ on ∂Q_v be such that

$$\int_{S_{u_\varepsilon} \cap Q_v} g(x/\varepsilon, u_\varepsilon^+ - u_\varepsilon^-, v_{u_\varepsilon}) \, d\mathscr{H}^{n-1} \leqq I_\varepsilon(v) + \sigma. \tag{6.2}$$

For every $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbf{Z}^{n-1}$ set

$$x_\lambda = \eta(\lambda_1 v_1 + \cdots + \lambda_{n-1} v_{n-1}), \quad Q_\lambda = x_\lambda + \frac{\eta}{\varepsilon} Q_v.$$

We have to choose the centres x_λ properly. Let

$$\Lambda = \Lambda(\eta, \varepsilon) = \Big\{ \lambda \in \mathbf{Z}^{n-1} : Q_\lambda \subseteq Q_{v'}, \quad \exists l = (l_1, \ldots, l_{n-1}) \in \mathbf{Z}^{n-1},$$
$$\text{such that } x_\lambda \in \sum_{i=1}^{n-1} l_i(\eta/\varepsilon + \eta\gamma_i) v_i + \eta P \Big\}.$$

It is easy to see that

(i) The cubes of the family $(Q_\lambda)_\lambda$ are pairwise disjoint.

(ii) Denoting by $S = S_{u_{x, v_0}}$ the hyperplane $\langle x, v_0 \rangle = 0$, we have

$$\lim_{\eta \to 0} \mathscr{H}^{n-1}\Big(S \cap \Big(Q_{v'} \setminus \bigcup_{\lambda \in \Lambda} Q_\lambda \Big) \Big) = 0$$

(i.e., $\lim_{\eta \to 0} (\eta/\varepsilon)^{n-1} \# \Lambda(\eta, \varepsilon) = 1$).

Define $u_\eta : Q_{v'} \to \mathbf{R}^m$ by

$$u_\eta(x) = \begin{cases} u_\varepsilon(\varepsilon(x - x_\lambda)/\eta) & \text{if } x \in Q_\lambda, \lambda \in \Lambda, \\ u_{z, v_0}(x) & \text{otherwise.} \end{cases}$$

It turns out that $S_{u_\eta} \subseteq S \cup \bigcup_{\lambda \in \Lambda} Q_\lambda$, and clearly

$$I_\eta(v') \leqq \int_{S_{u_\eta} \cap Q_{v'}} g(x/\eta, u_\eta^+ - u_\eta^-, v_{u_\eta}) \, d\mathscr{H}^{n-1}.$$

We can estimate this integral. We have

$$\begin{aligned} I_1 &\equiv \int_{S_{u_\eta} \cap \bigcup_{\lambda \in \Lambda} Q_v} g(x/\eta, u_\eta^+ - u_\eta^-, v_{u_\eta}) \, d\mathscr{H}^{n-1} \\ &= \sum_\lambda (\eta/\varepsilon)^{n-1} \int_{S_{u_\varepsilon} \cap Q_v} g(y/\varepsilon + \lambda_1 v_1 + \cdots + \lambda_{n-1} v_{n-1}, u_\varepsilon^+ - u_\varepsilon^-, v_{u_\varepsilon}) \, d\mathscr{H}^{n-1}, \end{aligned}$$

where the change of variable $x \mapsto \varepsilon(x - x_\lambda)/\eta$ has been applied on Q_λ. Then, by the P-periodicity of g and by (6.2)

$$I_1 \leqq (\eta/\varepsilon)^{n-1} \# \Lambda(\eta, \varepsilon)(I_\varepsilon(v) + \sigma).$$

Let us consider $S_{u_\eta} \cap (S \setminus \bigcup_{\lambda \in \Lambda} Q_\lambda)$:

$$I_2 \equiv \int_{S_{u_\eta} \cap (S \setminus \bigcup_{\lambda \in \Lambda} Q_\nu)} g(x/\eta, u_\eta^+ - u_\eta^-, \nu_{u_\eta})\, d\mathscr{H}^{n-1}$$
$$\leqq c_4(1 + |z|)\mathscr{H}^{n-1}\Bigg(S \cap \Bigg(Q_{\nu'} \setminus \bigcup_{\lambda \in \Lambda} Q_\lambda\Bigg)\Bigg).$$

The estimates now obtained for I_1 and I_2, together with property (ii) satisfied by Λ, yield

$$\limsup_{\eta \to 0} I_\eta(v') \leqq I_\varepsilon(v) + \sigma.$$

We conclude by taking the lower limit for $\varepsilon \to 0$ and by considering the arbitrariness of $\sigma > 0$.

Step 2. *Let $v = (v_0, v_1, \ldots, v_{n-1})$ and $v' = (v_0, v_1', \ldots, v_{n-1}')$ be two orthonormal rational bases of $\mathbf{R}^n$ with equal first vector. Then the limits $\lim_{\varepsilon \to 0} I_\varepsilon(v)$ and $\lim_{\varepsilon \to 0} I_\varepsilon(v')$ exist and are equal.*

Proof. By applying Step 1 with $v = v'$ we obtain the existence of the limits. By exchanging the roles of v and v' we obtain that they are equal.

Step 3. *For every $\sigma > 0$ there exists $\delta > 0$ (independent of $z \in \mathbf{R}^m$) such that if $v = (v_0, v_1, \ldots, v_{n-1})$ and $v' = (v_0', v_1', \ldots, v_{n-1}')$ are two orthonormal bases of $\mathbf{R}^n$ with $|v_i - v_i'| < \delta$ for every $i = 0, \ldots, n-1$, then*

$$\liminf_{\varepsilon \to 0} I_\varepsilon(v) - K\sigma \leqq \liminf_{\varepsilon \to 0} I_\varepsilon(v') \leqq \limsup_{\varepsilon \to 0} I_\varepsilon(v') \leqq \limsup_{\varepsilon \to 0} I_\varepsilon(v) + K\sigma,$$

where $K = 1 + 2c_4(1 + |z|)$.

Proof. We use the notation

$$Q_{v,\eta} = (1 - \eta)Q_v \quad \text{(with } v \text{ an orthonormal basis for } \mathbf{R}^n,\ 0 < \eta < 1).$$

Let $\sigma > 0$ be fixed and let $0 < \eta < 1$ be such that

$$2(1 - (1 - 2\eta)^{n-1}) < \sigma. \tag{6.3}$$

It is easy to see that there exists $\delta > 0$ with the property that for every pair $v = (v_0, v_1, \ldots, v_{n-1})$ and $v' = (v_0', v_1', \ldots, v_{n-1}')$ of orthonormal bases of $\mathbf{R}^n$, if $|v_i - v_i'| < \delta$ $(i = 0, \ldots, n-1)$, then

(i) $\partial Q_{v',\eta} \subseteq Q_v \setminus \bar{Q}_{v,2\eta}$,

(ii) $\mathscr{H}^{n-1}((\partial Q_{v',\eta}) \cap (H \Delta H')) < \sigma$,

where H and H' denote the half spaces $\langle x, v_0 \rangle > 0$ and $\langle x, v_0' \rangle > 0$, respectively. Fix v and v' with this property. Given $\varepsilon > 0$ there exists $u \in SBV(Q_v; \mathbf{R}^m)$ such that

$\nabla u = 0$ a.e., $u = u_{z,\nu_0}$ on ∂Q_ν and

$$\int_{S_u \cap Q_\nu} g(x/\varepsilon, u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1} \leqq I_\varepsilon(\nu) + \sigma.$$

We consider u extended with value u_{z,ν_0} on the whole $\mathbf{R}^n$. Then we can define

$$v : \mathbf{R}^n \to \mathbf{R}^m, \quad v(x) = u(x/(1-2\eta)).$$

We have $S_v \cap Q_\nu = ((1-2\eta)S_u) \cap Q_\nu$, and

$$\int_{S_u \cap Q_\nu} g\left(\frac{x}{\varepsilon(1-2\eta)}, v^+ - v^-, \nu_v\right) d\mathscr{H}^{n-1}$$

$$= (1-2\eta)^{n-1} \int_{S_u \cap (\frac{1}{1-2\eta} Q_\nu)} g(y/\varepsilon, u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1}$$

$$\leqq \int_{S_u \cap Q_\nu} g(y/\varepsilon, u^+ - u^-, \nu_v)\, d\mathscr{H}^{n-1} + c_4(1 + |z|)[1 - (1-2\eta)^{n-1}];$$

hence

$$\int_{S_u \cap Q_\nu} g\left(\frac{x}{\varepsilon(1-2\eta)}, v^+ - v^-, \nu_v\right) d\mathscr{H}^{n-1} \tag{6.4}$$

$$\leqq I_\varepsilon(\nu) + \sigma + c_4(1 + |z|)[1 - (1-2n)_n^{-1}].$$

Define $w : Q_{\nu'} \to \mathbf{R}^m$ by

$$w(x) = \begin{cases} v & \text{on } Q_{\nu',\eta}, \\ u_{z,\nu_0'} & \text{on } Q_{\nu'} \setminus Q_{\nu',\eta}. \end{cases}$$

Clearly

$$I_{\varepsilon(1-2\eta)}(\nu') \leqq \int_{S_w \cap Q_{\nu'}} g\left(\frac{x}{\varepsilon(1-2\eta)}, w^+ - w^-, \nu_w\right) d\mathscr{H}^{n-1}. \tag{6.5}$$

We estimate the integral. We have

$$S_w \cap Q_{\nu'} \subseteq (S_v \cap Q_{\nu',\eta}) \cup (S_w \cap \partial Q_{\nu',\eta}) \cup (S_{u_{z,\nu_0'}} \cap (Q_{\nu'} \setminus Q_{\nu',\eta})).$$

Note that $(\partial Q_{\nu',\eta}) \cap (H \cap H')$ is contained in the open set $(Q_\nu \setminus \bar{Q}_{\nu,2\eta}) \cap H \cap H'$ (property (i)); since on this open set the function v takes the value z, we conclude that $S_w \cap (\partial Q_{\nu',\eta}) \cap H \cap H' = \emptyset$. In the same way we obtain that $S_w \cap ((\partial Q_{\nu',\eta}) \setminus (\bar{H} \cup \bar{H}')) = \emptyset$. Therefore, up to a $\mathscr{H}^{n-1}$-negligible set,

$$S_w \cap (\partial Q_{\nu',\eta}) \subseteq (\partial Q_{\nu',\eta}) \cap (H \Delta H').$$

Moreover, $|w^+ - w^-| \leqq |z|$ on $(\partial Q_{v',\eta}) \cap (H \cap H')$ (v takes the values 0 or z on the open set $Q_v \setminus \bar{Q}_{v,2\eta} \supseteq \partial Q_{v',\eta}$). Thus, in view of property (ii) above,

$$\int_{S_w \cap Q_{v'}} g\left(\frac{x}{\varepsilon(1-2\eta)}, w^+ - w^-, \nu_w\right) d\mathscr{H}^{n-1}$$

$$\leqq \int_{S_v \cap Q_v} g\left(\frac{x}{\varepsilon(1-2\eta)}, v^+ - v^-, \nu_v\right) d\mathscr{H}^{n-1} + c_4(1+|z|)(\sigma + 1 - (1-\eta)^{n-1}),$$

and, by (6.4) and (6.5),

$$I_{\varepsilon(1-2\eta)}(v') \leqq I_\varepsilon(v) + \sigma + c_4(1+|z|)(\sigma + 2(1-(1-2\eta)^{1-n})).$$

In view of our choice of η,

$$I_{\varepsilon(1-2\eta)}(v') \leqq I_\varepsilon(v) + K\sigma$$

(with $K = 1 + 2c_4(1+|z|)$). Finally, by letting ε tend to 0 we have

$$\liminf_{\varepsilon\to 0} I_\varepsilon(v') \leqq \liminf_{\varepsilon\to 0} I_\varepsilon(v) + K\sigma,$$

$$\limsup_{\varepsilon\to 0} I_\varepsilon(v') \leqq \limsup_{\varepsilon\to 0} I_\varepsilon(v) + K\sigma.$$

The symmetry of the roles of v and v' allows us to exchange them, thus concluding the proof of Step 3.

Step 4. *Let $v = (v_0, v_1, \dots, v_{n-1})$ and $v' = (v_0, v'_1, \dots, v'_{n-1})$ be two orthonormal bases of $\mathbf{R}^n$ with equal first vector. Then the limits $\lim_{\varepsilon\to 0} I_\varepsilon(v)$ and $\lim_{\varepsilon\to 0} I_\varepsilon(v')$ exist and are equal.* (For this reason and for our purposes we take the freedom to denote by v both a vector of S^{n-1} and any orthonormal basis of $\mathbf{R}^n$ with v as first element.)

Proof. Let $\sigma > 0$ be fixed, and let $\delta > 0$ as in Step 3. Let $\mu = (\mu_0, \mu_1, \dots, \mu_{n-1})$ and $\mu' = (\mu_0, \mu'_1, \dots, \mu'_{n-1})$ be two rational orthonormal bases of $\mathbf{R}^n$ with

$$|\mu_0 - v_0| < \delta, \quad |\mu_i - v_i| < \delta, \quad |\mu'_i - v'_i| < \delta \quad (i = 1, \dots, n-1).$$

$I_\varepsilon(\mu)$ and $I_\varepsilon(\mu')$ converge to the same limit l when $\varepsilon \to 0$. Then it is enough to observe that Step 3 yields

$$l - K\sigma \leqq \liminf_{\varepsilon\to 0} I_\varepsilon(v) \leqq \limsup_{\varepsilon\to 0} I_\varepsilon(v) \leqq l + K\sigma$$

and the analogous inequalities for v'.

Step 5. *For every $z \in \mathbf{R}^m$ the function $g_{\text{hom}}(z, \cdot)$ is continuous on S^{n-1}, uniformly with respect to z when the latter varies on bounded sets.*

Proof. The proof follows immediately from Steps 3 and 4.

Step 6. *For every $v \in S^{n-1}$ the function $g_{\text{hom}}(\cdot, v)$ is continuous on $\mathbf{R}^m \setminus \{0\}$.*

Proof. Let $v \in S^{n-1}$ and let Q_v be any unit cube in $\mathbf{R}^n$ with centre at the origin and one face orthogonal to v. Fix $a, z \in \mathbf{R}^m \setminus \{0\}$.

For every $\sigma > 0$, from the definition of g_{hom} it follows that there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ we can find $u \in SBV(Q_\nu; \mathbf{R}^m)$ with $\nabla u = 0$ a.e., $u = u_{a,\nu}$ on ∂Q_ν and

$$\int_{S_u \cap Q_\nu} g(x/\varepsilon, u^+ - u^-, \nu_u)\, d\mathcal{H}^{n-1} \leqq g_{\text{hom}}(a, \nu) + \sigma. \tag{6.6}$$

We introduce the notation $\| s \| = \max_{1 \leqq i \leqq m} |s_i|$, when $s = (s_1, \ldots, s_m) \in \mathbf{R}^m$.

Assuming, for simplicity, that $\| a \| = |a_1|$, we define a matrix $C = (c_{ij}) \in M^{m \times m}$ by

$$c_{i1} = (z_i - a_i)/a_1 \quad \text{for } i = 1, \ldots, m, \quad c_{ij} = 0 \text{ otherwise.}$$

Note that $(C + I)a = z$; hence, for every $\varepsilon < \varepsilon_0$, if we denote by I_ε the infimum (6.1) corresponding to Q_ν, we have

$$\begin{aligned} I_\varepsilon &\leqq \int_{S_u \cap Q_\nu} g(x/\varepsilon, (C + I)(u^+ - u^-), \nu_u)\, d\mathcal{H}^{n-1} \\ &\leqq \int_{S_u \cap Q_\nu} g(x/\varepsilon, u^+ - u^-, \nu_u)\, d\mathcal{H}^{n-1} + \int_{S_u \cap Q_\nu} \omega(|C(u^+ - u^-|)\, d\mathcal{H}^{n-1}; \end{aligned}$$

hence, from (6.6),

$$I_\varepsilon \leqq g_{\text{hom}}(a, \nu) + \sigma + \int_{S_u \cap Q_\nu} \omega\left(\frac{|z - a|}{\| a \|} |u^+ - u^-| \right) d\mathcal{H}^{n-1}.$$

We estimate the last term on the right-hand side. Note that there exists a constant $K > 0$ such that $\omega(t) \leqq K(1 + t)$ for every $t \geqq 0$. Then, for every $\alpha > 0$,

$$\begin{aligned} &\int_{S_u \cap Q_\nu} \omega\left(\frac{|z - a|}{\| a \|} |u^+ - u^-| \right) d\mathcal{H}^{n-1} \\ &\leqq K \int_{S_u \cap Q_\nu \cap \{|u^+ - u^-| > \alpha\}} \left(1 + \frac{|z - a|}{\| a \|} |u^+ - u^-| \right) d\mathcal{H}^{n-1} \\ &\quad + \int_{S_u \cap Q_\nu} \omega\left(\frac{|z - a|}{\| a \|} \alpha \right) d\mathcal{H}^{n-1} \\ &\leqq K \left(\frac{1}{\alpha} + \frac{|z - a|}{\| a \|} \right) \int_{S_u \cap Q_\nu} |u^+ - u^-|\, d\mathcal{H}^{n-1} + \omega\left(\frac{|z - a|}{\| a \|} \alpha \right) \mathcal{H}^{n-1}(S_u \cap Q_\nu) \\ &\leqq \left[K \left(\frac{1}{\alpha} + \frac{|z - a|}{\| a \|} \right) + \omega\left(\frac{|z - a|}{\| a \|} \alpha \right) \right] \frac{1}{c_3} \int_{S_u \cap Q_\nu} g(x/\varepsilon, u^+ - u^-, \nu_u)\, d\mathcal{H}^{n-1}. \end{aligned}$$

Inserting this in the estimate of I_ε and recalling (6.6) we get

$$I_\varepsilon \leqq g_{\text{hom}}(a, \nu) + \sigma + \frac{1}{c_3} \left(K \left(\frac{1}{\alpha} + \frac{|z - a|}{\| a \|} \right) + \omega\left(\frac{|z - a|}{\| a \|} \alpha \right) \right) (g_{\text{hom}}(a, \nu) + \sigma).$$

Now we let ε and σ tend to 0 successively, obtaining

$$(6.7)\quad g_{\text{hom}}(z,v) \leqq g_{\text{hom}}(a,v) + \frac{1}{c_3}\left(K\left(\frac{1}{\alpha} + \frac{|z-a|}{\|a\|}\right) + \omega\left(\frac{|z-a|}{\|a\|}\alpha\right)\right) g_{\text{hom}}(a,v).$$

This inequality holds for every $a, z \in \mathbf{R}^m \backslash \{0\}$ and $\alpha > 0$. Now fix $a \in \mathbf{R}^m \backslash \{0\}$. From (6.7), with $\alpha = 1$, we deduce that there exists a constant $K_1(a) > 0$ such that

$$g_{\text{hom}}(z,v) \leqq K_1(a) \quad \text{for every } z \in B_\delta(a), \quad \delta = |a|.$$

Observe now that (6.7) clearly holds with the roles of a and z reversed. Hence for every $z \in B_\delta(a)$,

$$(6.8)\quad g_{\text{hom}}(a,v) \leqq g_{\text{hom}}(z,v) + \frac{1}{c_3}\left[K\left(\frac{1}{\alpha} + \frac{|z-a|}{\|z\|}\right) + \omega\left(\frac{|z-a|}{\|z\|}\alpha\right)\right] K_1(a).$$

Let z tend to a in (6.7) and (6.8). Then for every $\alpha > 0$,

$$\limsup_{z\to a} g_{\text{hom}}(z,v) \leqq g_{\text{hom}}(a,v) + \frac{K}{\alpha c_3} g_{\text{hom}}(a,v),$$

$$\liminf_{z\to a} g_{\text{hom}}(z,v) \geqq g_{\text{hom}}(a,v) - \frac{K}{\alpha c_3} K_1(a).$$

We complete the proof by letting α tend to $+\infty$.

Step 7. We prove the stated estimates of g_{hom}.

Proof. From the analogous estimates for g it follows that it is enough to prove that

$$\inf\left\{\int_{S_u \cap Q_v} (1 + |u^+ - u^-|)\, d\mathscr{H}^{n-1} : u \in SBV(Q_v;\mathbf{R}^m),\ \nabla u = 0 \text{ a.e.},\ u = u_{z,v} \text{ on } \partial Q_v\right\} \geqq 1 + |z|.$$

Let u be an admissible function for this infimum; we regard u as extended to the whole of $\mathbf{R}^n$ with value $u_{z,v}$ on $\{x \in \mathbf{R}^n : |\langle x, v\rangle| > \frac{1}{2}\}$ and by periodicity on $\{x \in \mathbf{R}^n : |\langle x, v\rangle| < \frac{1}{2}\}$. For every $k \in \mathbf{N}$ let $u_k : Q_v \to \mathbf{R}^m$ be defined by $u_k(x) = u(kx)$. It is easy to see that (u_k) converges to $u_{z,v}$ in $L^1(Q_v;\mathbf{R}^m)$. Since the functional $u \mapsto \int_{S_u \cap Q_v} (1 + |u^+ - u^-|)\, d\mathscr{H}^{n-1}$ is $L^1(Q_v;\mathbf{R}^m)$-lower semicontinuous (see [6, Theorem 3.7]), we have

$$1 + |z| \leqq \liminf_{k\to+\infty} \int_{S_{u_k} \cap Q_v} (1 + |u_k^+ - u_k^-|)\, d\mathscr{H}^{n-1} = \int_{S_u \cap Q_v} (1 + |u^+ - u^-|)\, d\mathscr{H}^{n-1}.$$

This concludes the proof. □

Proposition 6.1. *$\tilde{g}(z,v) \leqq g_{\text{hom}}(z,v)$ for every $(z,v) \in \mathbf{R}^m \times S^{n-1}$.*

Proof. Since $\tilde{g}$ and g_{hom} are continuous in the second variable, we can restrict our attention to the case in which $tv \in \mathbf{Q}^n$ for some $t \in \mathbf{R}\backslash\{0\}$. Without loss of generality we can suppose that $v = e_1$. Indeed, let $(e'_1, \ldots, e'_n)$ be a "rational basis" of $\mathbf{R}^n$ (see Step 1 of the previous proposition) with $e'_1 = v$. Then there exist $r_1, \ldots, r_n \in \mathbf{R}\backslash\{0\}$

such that $v_1 = r_1 e'_1, \dots, v_n = r_n e'_n$ belong to $\mathbf{Z}^n$. It follows that a 1-periodic function φ on $\mathbf{R}^n$ satisfies $\varphi(x) = \varphi(x + v_i)$ for every $i = 1, \dots, n$ and $x \in \mathbf{R}^n$. As a consequence, if A is the $n \times n$ matrix such that $Av_i = e_i$ ($i = 1, \dots, n$), then the change of variable $x \mapsto y = Ax$ transforms 1-periodic functions into 1-periodic functions and the direction of v into the direction of e_1.

Define $Q =]-\frac{1}{2}, \frac{1}{2}[^n$ and

$$u_z(x) = \begin{cases} z & \text{if } \langle x, e_1 \rangle \geqq 0, \\ 0 & \text{if } \langle x, e_1 \rangle < 0. \end{cases} \tag{6.9}$$

Fix $\sigma > 0$; there exist $\varepsilon = \varepsilon(\sigma) > 0$ and $v \in SBV(Q; \mathbf{R}^m)$ with $\nabla v = 0$ a.e. and $v = u_z$ on ∂Q such that

$$\int_{S_v \cap Q} g(x/\varepsilon, v^+ - v^-, \nu_v)\, d\mathscr{H}^{n-1} \leqq g_{\text{hom}}(z, e_1) + \sigma. \tag{6.10}$$

We regard the function v as extended to $\mathbf{R}^n$ with value u_z on $\mathbf{R}^n \setminus \{x \in \mathbf{R}^n : |\langle x, e_1 \rangle| > \frac{1}{2}\}$ and by periodicity on $\{x \in \mathbf{R}^n : |\langle x, e_1 \rangle| < \frac{1}{2}\}$. For every $h \in \mathbf{N}$ we define

$$w_h(x) = v\left(\frac{\varepsilon}{\varepsilon_h} x\right), \quad x \in Q.$$

The sequence (w_h) converges to u_z in $L^p(Q; \mathbf{R}^m)$:

$$\begin{aligned} \int_Q |w_h - u_z|^p\, dx &= \left(\frac{\varepsilon_h}{\varepsilon}\right)^n \int_{\varepsilon Q/\varepsilon_h} |(v - u_z)(y)|^p\, dy \\ &= \left(\frac{\varepsilon_h}{\varepsilon}\right)^n \int_{(\frac{\varepsilon}{\varepsilon_h} Q) \cap \{|\langle x, e_1 \rangle| < \frac{1}{2}\}} |(v - u_z)(y)|^p\, dy \\ &\leqq \left(\frac{\varepsilon_h}{\varepsilon}\right)^n \left(\left[\frac{\varepsilon}{\varepsilon_h}\right] + 2\right)^{n-1} \int_Q |v(y) - u_z(y)|^p\, dy, \end{aligned}$$

which tends to 0 as h tends to $+\infty$. Fix $\eta > 0$ and set $Q_\eta = \{x \in Q : |\langle x, e_1 \rangle| < \eta\}$. By the definition of F_0 we have

$$\begin{aligned} F_0(u_z, Q_\eta) &\leqq \liminf_{h \to +\infty} \left(\int_{Q_\eta} f(x/\varepsilon_h, 0)\, dx + \int_{S_{w_h} \cap Q_\eta} g(x/\varepsilon_h, w_h^+ - w_h^-, \nu_{w_h})\, d\mathscr{H}^{n-1} \right) \\ &\leqq c_2 |Q_\eta| + \liminf_{h \to +\infty} \left(\frac{\varepsilon_h}{\varepsilon}\right)^{n-1} \int_{(\frac{\varepsilon}{\varepsilon_h} Q_\eta) \cap S_v} g(y/\varepsilon, v^+ - v^-, \nu_v)\, d\mathscr{H}^{n-1}. \end{aligned}$$

We may suppose that $\varepsilon = 1/k$ for a suitable $k \in \mathbf{N}$. Then the function $y \mapsto g(y/\varepsilon, s, \nu)$ is 1-periodic. Hence

$$F_0(u_z, Q_\eta) \leqq c_2 |Q_\eta| + \liminf_{h \to +\infty} \left(\frac{\varepsilon_h}{\varepsilon}\right)^{n-1} \left(\left[\frac{\varepsilon}{\varepsilon_h}\right] + 2\right)^{n-1} \int_{Q \cap S_v} g(y/\varepsilon, v^+ - v^-, \nu_v)\, d\mathscr{H}^{n-1}.$$

By (6.10),

$$F_0(u_z, Q_\eta) \leqq c_2|Q_\eta| + g_{\mathrm{hom}}(z, e_1) + \sigma,$$

and, by means of Propositions 4.1 and 4.2,

$$\tilde{f}(0)|Q_\eta| + \tilde{g}(z, e_1) \leqq c_2|Q_\eta| + g_{\mathrm{hom}}(z, e_1) + \sigma.$$

As $\eta \to 0^+$ and $\sigma \to 0^+$ we obtain $\tilde{g}(z, e_1) \leqq g_{\mathrm{hom}}(z, e_1)$. □

Proposition 6.2. $\tilde{g}(z, \nu) \geqq g_{\mathrm{hom}}(z, \nu)$ *for every* $(z, \nu) \in \mathbf{R}^m \times S^{n-1}$.

Proof. Fix $(z, \nu) \in (\mathbf{R}^m \backslash \{0\}) \times S^{n-1}$. As in the proof of the previous proposition, without loss of generality we may suppose that $\nu = e_1$. Let u_z be defined by (6.9). Fix $\sigma \in]0, 1[$ and define $Q_\sigma =]-\frac{\sigma}{2}, \frac{\sigma}{2}[\times]-\frac{1}{2}, \frac{1}{2}[^{n-1}$. There exists a sequence (v_h) in $SBV^p(Q_\sigma; \mathbf{R}^m) \cap L^p(Q_\sigma; \mathbf{R}^m)$ such that

$$v_h \to u_z \quad \text{in } L^p(Q_\sigma; \mathbf{R}^m), \quad F_{\varepsilon_h}(v_h, Q_\sigma) \to F_0(u_z, Q_\sigma).$$

By applying the truncation Lemma 3.5 for every $\eta \in]0, 1[$ we can find a sequence (u_h) in $SBV^p(Q_\sigma; \mathbf{R}^m) \cap L^\infty(Q_\sigma; \mathbf{R}^m)$ such that (u_h) is equibounded, and $u_h \to u_z$ in $L^p(Q_\sigma; \mathbf{R}^m)$ and $F_{\varepsilon_h}(u_h, Q_\sigma) \leqq F_{\varepsilon_h}(v_h, Q_\sigma) + \eta$ for every $h \in \mathbf{N}$, so that

$$\limsup_{h \to +\infty} F_{\varepsilon_h}(u_h, Q_\sigma) \leqq F_0(u_z, Q_\sigma) + \eta = \tilde{f}(0)\sigma + \tilde{g}(z, e_1) + \eta.$$

In the last equality we have taken into account Propositions 4.1 and 4.2.

It is easy to see that there exists a nondecreasing sequence (n_h) of natural numbers such that (n_h) tends to $+\infty$, $(n_h\varepsilon_h)$ tends to 0 and

$$\frac{1}{(n_h\varepsilon_h)^n} \int_{Q_\sigma} |u_h - u_z|^p \, dx \to 0.$$

For every $h \in \mathbf{N}$ we set $\beta_h = n_h\varepsilon_h$, and for $\lambda \in \{0\} \times \mathbf{Z}^{n-1}$,

$$x_{h,\lambda} = \beta_h\lambda, \quad Q_{h,\lambda} = x_{h,\lambda} + \beta_h Q_\sigma.$$

Let $\lambda(h) \in \{0\} \times \mathbf{Z}^{n-1}$ be the index of a "minimal cube", i.e.,

$$F_{\varepsilon_h}(u_h, Q_{h,\lambda(h)}) \leqq F_{\varepsilon_h}(u_h, Q_{h,\lambda})$$

for every $\lambda \in \{0\} \times \mathbf{Z}^{n-1}$ with $Q_{h,\lambda} \subseteq Q_\sigma$. We define

$$x_h = x_{h,\lambda(h)}, \quad Q_h = Q_{h,\lambda(h)}, \quad w_h(x) = u_h(x_h + \beta_h x), \quad x \in Q_\sigma.$$

Clearly $w_h \in SBV^p(Q_\sigma; \mathbf{R}^m) \cap L^\infty(Q_\sigma; \mathbf{R}^m)$. Let us prove that

(6.11)

(i) (w_h) is equibounded,

(ii) $w_h \to u_z$ in $L^p(Q_\sigma; \mathbf{R}^m)$,

(iii) $\int_{Q_\sigma} |\nabla w_h|^p \, dx \to 0$,

(iv) $\limsup_{h \to +\infty} \int_{S_{w_h} \cap Q_\sigma} g\left(\frac{x}{\alpha_h}, w_h^+ - w_h^-, \nu_{w_h}\right) d\mathscr{H}^{n-1} \leqq \tilde{f}(0)\sigma + \tilde{g}(z, e_1) + \eta$,

where (α_h) is a suitable sequence tending to 0.

Property (i) is obvious, while (ii) is checked as follows:

$$\int_{Q_\sigma} |w_h - u_z|^p \, dx = \frac{1}{\beta_h^n} \int_{Q_\sigma} |u_h - u_z|^p \, dy \leqq \frac{1}{\beta_h^n} \int_{Q_\sigma} |u_h - u_z|^p \, dy,$$

which tends to 0 as h tends to $+\infty$ thanks to our assumptions on β_h. Let C be an upper bound for the sequence $(F_{\varepsilon_h}(u_h, Q_\sigma))$. Then

$$C \geqq F_{\varepsilon_h}(u_h, Q_\sigma) \geqq \sum_{\substack{\lambda \in \{0\} \times \mathbf{Z}^{n-1} \\ Q_{h,\lambda} \subseteq Q_\sigma}} F_{\varepsilon_h}(u_h, Q_{h,\lambda}) \geqq \left[\frac{1}{\beta_h}\right]^{n-1} F_{\varepsilon_h}(u_h, Q_h) \tag{6.12}$$

$$\geqq \left[\frac{1}{\beta_h}\right]^{n-1} \int_{Q_h} f\left(\frac{x}{\varepsilon_h}, \nabla u_h\right) dx$$

$$\geqq c_1 \left[\frac{1}{\beta_h}\right]^{n-1} \int_{Q_h} |\nabla u_h|^p \, dx.$$

We now estimate $\int_{Q_\sigma} |\nabla w_h|^p \, dx$. We have

$$\int_{Q_\sigma} |\nabla w_h|^p \, dx = \int_{Q_\sigma} |\beta_h (\nabla u_h)(x_h + \beta_h x)|^p \, dx = \beta_h^{p-n} \int_{Q_\sigma} |\nabla u_h(y)|^p \, dy;$$

hence, by (6.12) $\int_{Q_\sigma} |\nabla w_h|^p \, dx \leqq \beta_h^{p-n} \dfrac{C/c_1}{[1/\beta_h]^{n-1}}$, which tends to 0 as h tends to $+\infty$. Thus (6.11) (iii) holds.

Finally, we prove (6.11) (iv), with $\alpha_h = \varepsilon_h / \beta_h$. Taking into account that $x_h / \varepsilon_h \in \mathbf{Z}^n$ we have

$$\int_{S_{w_h} \cap Q_\sigma} g\left(\frac{x}{\alpha_h}, w_h^+ - w_h^-, \nu_{w_h}\right) d\mathscr{H}^{n-1}$$

$$= \int_{S_{w_h} \cap Q_\sigma} g\left(\frac{x}{\varepsilon_h / \beta_h}, u_h^+(x_h + \beta_h x) - u_h^-(x_h + \beta_h x), \nu_{w_h}\right) d\mathscr{H}^{n-1}$$

$$= \frac{1}{\beta_h^{n-1}} \int_{x_h + \beta_h (S_{w_h} \cap Q_\sigma)} g\left(\frac{y - x_h}{\varepsilon_h}, u_h^+(y) - u_h^-(y), \nu_{u_h}\right) d\mathscr{H}^{n-1}(y)$$

$$= \frac{1}{\beta_h^{n-1}} \int_{S_{u_h} \cap Q_h} g\left(\frac{y}{\varepsilon_h}, u_h^+ - u_h^-, \nu_{u_h}\right) d\mathscr{H}^{n-1}(y)$$

$$\leqq \frac{1}{\beta_h^{n-1}} F_{\varepsilon_h}(u_h, Q_h) \leqq \frac{1}{\beta_h^{n-1}} \frac{1}{[1/\beta_h]^{n-1}} F_{\varepsilon_h}(u_h, Q_\sigma).$$

Therefore,

$$\limsup_{h\to+\infty} \int_{S_{w_h}\cap Q_\sigma} g\left(\frac{x}{\alpha_h}, w_h^+ - w_h^-, v_{w_h}\right) d\mathscr{H}^{n-1} \leqq \limsup_{h\to+\infty} F_{\varepsilon_h}(u_h, Q_\sigma) \leqq \tilde{f}(0)\sigma + \tilde{g}(z, e_1) + \eta,$$

which proves (6.11) (iv).

We now modify the sequence (w_h) to satisfy the boundary condition u_z on ∂Q_σ. Let K_η be a compact subset of Q_σ such that

$$c_2 |Q_\sigma \setminus K_\eta| + c_4(1 + |z|)\mathscr{H}^{n-1}(S_{u_z} \cap (Q_\sigma \setminus K_\eta)) \leqq \eta. \tag{6.13}$$

We apply the fundamental estimate (Proposition 3.1) with A' a neighbourhood of K_η strictly contained in Q_σ, $A'' = Q_\sigma$ and $B = Q_\sigma \setminus K_\eta$. Then there exist a constant $M > 0$ and, for every $h \in \mathbf{N}$, a function $\varphi_h \in C_0^\infty(Q_\sigma)$ with $0 \leqq \varphi_h \leqq 1$ such that

$$F_{\alpha_h}(\varphi_h w_h + (1 - \varphi_h)u_z, Q_\sigma) \leqq (1 + \eta)[F_{\alpha_h}(w_h, Q_\sigma) + F_{\alpha_h}(u_z, Q_\sigma \setminus K_\eta)] + M \| w_h - u_z \|^p_{L^p(Q_\sigma; \mathbf{R}^m)} + \eta.$$

Set $\hat{w}_h = \varphi_h w_h + (1 - \varphi_h)u_z$; by (6.13) we have

$$F_{\alpha_h}(\hat{w}_h, Q_\sigma) \leqq (1 + \eta)[F_{\alpha_h}(w_h, Q_\sigma) + \eta] + M \| w_h - u_z \|^p_{L^p(Q_\sigma; \mathbf{R}^m)} + \eta,$$

and $\limsup_{h\to+\infty} F_{\alpha_h}(\hat{w}_h, Q_\sigma) \leqq (1 + \eta) \limsup_{h\to+\infty} F_{\alpha_h}(u_h, Q_\sigma) + 3\eta$. Moreover, by Remark 3.2, the functions φ_h can be chosen in a finite family independent of h. This implies that the sequence $(\| \nabla\varphi_h \|_{L^\infty(Q_\sigma; \mathbf{R}^n)})$ is bounded, so that $(\int_{Q_\sigma} |\nabla \hat{w}_h|^p \, dx)$ converges to 0.

In summary, for every $\eta > 0$ we have found a sequence $(\hat{w}_h)$ in $SBV^p(Q_\sigma; \mathbf{R}^m) \cap L^\infty(Q_\sigma; \mathbf{R}^m)$ such that

$$\begin{cases} \text{(i)} & (\hat{w}_h) \text{ is equibounded,} \\ \text{(ii)} & \displaystyle\int_{Q_\sigma} |\nabla \hat{w}_h|^p \, dx \to 0, \\ \text{(iii)} & \displaystyle\limsup_{h\to+\infty} \int_{S_{\hat{w}_h}\cap Q_\sigma} g\left(\frac{x}{\alpha_h}, \hat{w}_h^+ - \hat{w}_h^-, v_{\hat{w}_h}\right) d\mathscr{H}^{n-1} \\ & \qquad \leqq (1 + \eta)[(c_2 + \tilde{f}(0))\sigma + \tilde{g}(z, e_1) + \eta] + 3\eta, \\ \text{(iv)} & \hat{w}_h = u_z \text{ on a neighbourhood of } \partial Q_\sigma. \end{cases} \tag{6.14}$$

Our next aim is to obtain from $(\hat{w}_h)$ a suitable sequence (v_h) with $\nabla v_h = 0$ a.e. Let (k_h) be a divergent sequence in $\mathbf{N}$ such that

$$\lim_{h\to+\infty} k_h \left(\int_{Q_\sigma} |\nabla \hat{w}|^p \, dx \right)^{1/p} = 0.$$

It is not restrictive to assume that $2/k_h < |z^i|$ for every $i = 1, \ldots, m$ with $z^i \neq 0$. Let $M \in \mathbf{N}$ be such that $\| \hat{w}_h \|_{L^\infty(Q_\sigma; \mathbf{R}^m)} < M$ for every $h \in \mathbf{N}$. By the coarea formula (1.2), for every $i = 1, \dots, m$,

$$\int_{Q_\sigma} |\nabla \hat{w}_h^i|\,dx = \int_{Q_\sigma \setminus S_{\hat{w}_h}} |\nabla \hat{w}_h^i|\,dx = \int_{Q_\sigma \setminus S_{\hat{w}_h}} |D\hat{w}_h^i|$$

$$= \int_{-M}^{M} \mathscr{H}^{n-1}((Q_\sigma \setminus S_{\hat{w}_h}) \cap \partial^*(\{x \in Q_\sigma : \hat{w}_h^i(x) > t\}))\,dt.$$

Then for every $l \in \mathbf{Z}$, with $-k_h M \leqq l < k_h M$, there exists $t_l^i \in \left]\frac{l}{k_h}, \frac{l+1}{k_h}\right[$ such that

$$\int_{Q_\sigma} |\nabla \hat{w}_h^i|\,dx \geqq \sum_{l=-k_h M}^{k_h M - 1} \frac{1}{k_h} \mathscr{H}^{n-1}((Q_\sigma \setminus S_{\hat{w}_h}) \cap \partial^*(\{x \in Q_\sigma : \hat{w}_h^i(x) > t_l^i\})). \tag{6.15}$$

In addition, we define $t_l^i = -M$ if $l = -k_h M - 1$ and $t_l^i = M$ if $l = k_h M$. For every $l \in \mathbf{Z}$, with $-k_h M - 1 \leqq l < k_h M$, set $Z_l^i = \{x \in Q_\sigma : t_l^i < \hat{w}_l^i(x) \leqq t_{l+1}^i\}$. We have $Q_\sigma = \bigcup_l Z_l^i$; therefore we can define $v_h : Q_\sigma \to \mathbf{R}^m$ by

$$v_h^i|_{Z_l^i} = \begin{cases} 0 & \text{if } t_l^i < 0 \leqq t_{l+1}^i, \\ z^i & \text{if } t_l^i < z^i \leqq t_{l+1}^i, \\ t_l^i & \text{otherwise.} \end{cases}$$

This definition is well-posed since $2/k_h < |z^i|$ if $z^i \neq 0$. Each set Z_l^i has finite perimeter in Q_σ since it is the difference $\{\hat{w}_h^i > t_l^i\} \setminus \{\hat{w}_l^i > t_{l+1}^i\}$ of two sets of finite perimeter. It follows that $v_h \in SBV(Q_\sigma; \mathbf{R}^m)$. Furthermore, by (6.14) (iv), it is clear that $v_h = u_z$ on a neighbourhood of ∂Q_σ. From the definition of v_h we also have that $\max_{1 \leqq i \leqq m} \| v_h^i - \hat{w}_h^i \|_{L^\infty(Q_\sigma; \mathbf{R}^m)} \leqq 2/k_h$. If we now consider v_h extended to $Q =]-\frac{1}{2}, \frac{1}{2}[^n$ with value u_z on Q/Q_σ, then v_h is an admissible function for the formula defining $g_{\text{hom}}(z, e_1)$. Hence let us estimate $\int_{S_{v_h} \cap Q} g(x/\alpha_h, v_h^+ - v_h^-, v_{v_h})\,d\mathscr{H}^{n-1}$. First of all, notice that $S_{v_h} \cap Q_\sigma \subseteq \bigcup_i \bigcup_l (\partial^* Z_l^i) \cap Q_\sigma$. Since $Z_l^i = \{\hat{w}_h^i > t_l^i\} \setminus \{\hat{w}_h^i > t_{l+1}^i\}$, it is easy to see that

$$\partial^* Z_l^i \subseteq \partial^* \{x \in Q_\sigma : \hat{w}_h^i(x) > t_l^i\} \cup \partial^* \{x \in Q_\sigma : \hat{w}_h^i(x) > t_{l+1}^i\}$$

and therefore

$$S_{v_h} \cap Q_\sigma \subseteq \bigcup_{i=1}^{m} \bigcup_{l=-k_h M}^{k_h M - 1} (\partial^* \{x \in Q_\sigma : \hat{w}_h^i(x) > t_l^i\}) \cap Q_\sigma$$

(the terms corresponding to $l = -k_h M - 1$ and $l = k_h M$ do not contribute). Then we have

$$\int_{(S_{v_h} \cap Q) \setminus S_{\hat{w}_h}} g(x/\alpha_h, v_h^+ - v_h^-, v_{v_h})\,d\mathscr{H}^{n-1}$$

$$\leqq c_4(1 + 2M) \mathscr{H}^{n-1}(S_{v_h} \cap (Q_\sigma \setminus S_{\hat{w}_h}))$$

$$\leqq c_4(1 + 2M) \sum_{i=1}^{m} \sum_{l=-k_h M}^{k_h M - 1} \mathscr{H}^{n-1}((Q_\sigma \setminus S_{\hat{w}_h}) \cap \partial^* \{x \in Q_\sigma : \hat{w}_h^i(x) > t_l^i\}).$$

Thus, by (6.15),

$$\int_{(S_{v_h}\cap Q)\setminus S_{\hat{w}_h}} g(x/\alpha_h, v_h^+ - v_h^-, \nu_{v_h})\, d\mathscr{H}^{n-1} \leqq c_4(1+2M)k_h \sum_{i=1}^m \int_{Q_\sigma} |\nabla \hat{w}_h^i|\, dx$$

$$\leqq ck_h \left(\int_{Q_\sigma} |\nabla \hat{w}_h|^p\, dx \right)^{1/p}$$

for a suitable positive constant c. Hence, by the condition on (k_h) we get

$$\lim_{h\to+\infty} \int_{(S_{v_h}\cap Q)\setminus S_{\hat{w}_h}} g(x/\alpha_h, v_h^+ - v_h^-, \nu_{v_h})\, d\mathscr{H}^{n-1} = 0. \tag{6.16}$$

On the other hand, by taking into account the continuity properties of g and that $\nu_{v_h} = \nu_{\hat{w}_h}$ $\mathscr{H}^{n-1}$-a.e. on $S_{v_h}\cap S_{\hat{w}_h}$, we get

$$\int_{(S_{v_h}\cap Q)\cap S_{\hat{w}_h}} g(x/\alpha_h, v_h^+ - v_h^-, \nu_{v_h})\, d\mathscr{H}^{n-1}$$

$$\leqq \int_{S_{v_h}\cap S_{\hat{w}_h}\cap Q_\sigma} g(x/\alpha_h, \hat{w}_h^+ - \hat{w}_h^-, \nu_{\hat{w}_h})\, d\mathscr{H}^{n-1}$$

$$+ \int_{S_{v_h}\cap S_{\hat{w}_h}\cap Q_\sigma} \omega(|v_h^+ - \hat{w}_h^+| + |v_h^- - \hat{w}_h^-|)\, d\mathscr{H}^{n-1}$$

$$\leqq \int_{S_{\hat{w}_h}\cap Q_\sigma} g(x/\alpha_h, \hat{w}_h^+ - \hat{w}_h^-, \nu_{\hat{w}_h})\, d\mathscr{H}^{n-1}$$

$$+ \mathscr{H}^{n-1}(S_{v_h}\cap S_{\hat{w}_h}\cap Q_\sigma)\,\omega(4/k_h).$$

From (6.14) (iii) and the coercivity condition of g we obtain that $\mathscr{H}^{n-1}(S_{\hat{w}_h}\cap(Q_\sigma))$ is bounded and that

$$\limsup_{h\to+\infty} \int_{(S_{v_h}\cap Q)\cap S_{\hat{w}_h}} g(x/\alpha_h, v_h^+ - v_h^-, \nu_{v_h})\, d\mathscr{H}^{n-1}$$

$$\leqq \limsup_{h\to+\infty} \int_{S_{\hat{w}_h}\cap Q_\sigma} g(x/\alpha_h, \hat{w}_h^+ - \hat{w}_h^-, \nu_{\hat{w}_h})\, d\mathscr{H}^{n-1}$$

$$\leqq (1+\eta)\,[(c_2 + \tilde{f}(0))\sigma + \tilde{g}(z, e_1) + \eta] + 3\eta.$$

From this and (6.16) we conclude that

$$g_{\text{hom}}(z, e_1) \leqq \limsup_{h\to+\infty} \int_{S_{v_h}\cap Q} g(x/\alpha_h, v_h^+ - v_h^-, \nu_{v_h})\, d\mathscr{H}^{n-1}$$

$$\leqq (1+\eta)((c_2 + \tilde{f}(0))\sigma + \tilde{g}(z, e_1) + \eta) + 3\eta.$$

By letting η and σ tend to 0^+, we finally obtain $g_{\text{hom}}(z, e_1) \leqq \tilde{g}(z, e_1)$. □

7. Integral representation of the homogenized functional

We first study the homogenized functional F_0 on the space $SBV^p(\Omega; \mathbf{R}^m)\cap L^\infty(\Omega; \mathbf{R}^m)$, with $\Omega \in \mathscr{A}_0$.

Proposition 7.1. *Let* $\Omega \in \mathscr{A}_0$ *and* $u \in SBV^p(\Omega; \mathbf{R}^m) \cap L^\infty(\Omega; \mathbf{R}^m)$. *Then*

$$F_0(u, \Omega) \leqq \int_\Omega f_{\text{hom}}(\nabla u)\,dx + \int_{S_u \cap \Omega} g_{\text{hom}}(u^+ - u^-, \nu_u)\,d\mathscr{H}^{n-1}.$$

Proof. We proceed as in [23, Lemma 6.2]. Let K be a compact subset of S_u with finite $(n-1)$-dimensional upper Minkowski content, i.e., with the property that there exists a constant $C > 0$ such that for every $h \in \mathbf{N}$,

$$h|B_h| \leqq C, \quad \text{where } B_h = \{x \in \mathbf{R}^n : \text{dist}(x, K) < 1/h\}. \tag{7.1}$$

We can assume that $B_h \subseteq \Omega$ for every $h \in \mathbf{N}$.

For any given sequence (σ_h) of positive numbers, tending to 0, with the same technique used in the last part of the proof of Proposition 6.2, we can find a sequence $v_h \in BV(B_h; T_h)$, for suitable finite subsets T_h of $\mathbf{R}^m$, such that

$$\begin{aligned} &\| u - v_h \|_{L^\infty(B_h; \mathbf{R}^m)} \leqq \sigma_h, \\ &\mathscr{H}^{n-1}((S_{v_h} \cap B_h) \setminus S_u) \leqq \frac{\sqrt{m}}{\sigma_h} \int_{B_h} |\nabla u|\,dx. \end{aligned} \tag{7.2}$$

For every $h \in \mathbf{N}$ consider a function $\varphi_h \in C_0^\infty(B_h)$ with $\varphi_h = 1$ on B_{2h}, $0 \leqq \varphi_h \leqq 1$ and $\|\nabla \varphi_h\|_{L^\infty(\Omega; \mathbf{R}^n)} \leqq c_0 h$, where $c_0 > 0$ is independent of h. Define $u_h = \varphi_h v_h + (1 - \varphi_h) u$. Then (v_h) converges to u in $L^\infty(\Omega; \mathbf{R}^m)$, so that

$$F_0(u, \Omega) \leqq \liminf_{h \to +\infty} F_0(u_h, \Omega). \tag{7.3}$$

We have

$$\begin{aligned} F_0(u_h, \Omega) &\leqq F_0(v_h, B_{2h}) + F_0(u_h, B_h \setminus K) + F_0(u, \Omega \setminus \text{spt}\varphi_h) \\ &\leqq F_0(v_h, B_{2h} \cap S_u \cap S_{v_h}) + F_0(v_h, B_{2h} \setminus (S_u \cap S_{v_h})) \\ &\quad + F_0(u_h, B_h \setminus K) + F_0(u, \Omega \setminus \bar{S}_u) + F_0(u, \bar{S}_u \setminus K). \end{aligned} \tag{7.4}$$

We now estimate each term.

By the integral representation of F_0 on finite partitions (Propositions 4.2, 6.1 and 6.2) and on $W^{1,p}(\Omega; \mathbf{R}^m)$ (Propositions 4.1, 5.1 and 5.2) we have

$$\begin{aligned} F_0(v_h, B_{2h} \cap S_u \cap S_{v_h}) &= \int_{S_{v_h} \cap S_u \cap B_{2h}} g_{\text{hom}}(v_h^+ - v_h^-, \nu_{v_h})\,d\mathscr{H}^{n-1} \\ &\leqq \int_{S_u \cap \Omega} g_{\text{hom}}(v_h^+ - v_h^-, \nu_u)\,d\mathscr{H}^{n-1}, \end{aligned}$$

$$F_0(u, \Omega \setminus \bar{S}_u) \leqq \int_\Omega f_{\text{hom}}(\nabla u)\,dx.$$

Moreover, by (4.3), (7.1) and (7.2) we have

$$\begin{aligned}F_0(v_h, B_{2h}\backslash(S_u\cap S_{v_h})) &\leqq \gamma_2\left(|B_{2h}| + \int_{(S_{v_h}\cap B_{2h})\backslash S_u}(1+|v_h^+ - v_h^-|)\,d\mathscr{H}^{n-1}\right)\\ &\leqq c(|B_{2h}| + \mathscr{H}^{n-1}((S_{v_h}\cap B_{2h})\backslash S_u))\\ &\leqq c\left(|B_{2h}| + \frac{1}{\sigma_h}\int_{B_h}|\nabla u|\,dx\right)\end{aligned}$$

(c denotes a positive constant independent of h and K and which may vary from line to line),

$$\begin{aligned}F_0(u_h, B_h\backslash K) \leqq c\Bigg(&|B_h| + \int_{B_h}|v_h - u|^p|\nabla\varphi_h|^p\,dx + \int_{B_h}|\nabla u|^p\,dx\\ &+ \int_{(S_{v_h}\cap B_{2h})\backslash S_u}(1+|v_h^+ - v_h^-|)\,d\mathscr{H}^{n-1} + \mathscr{H}^{n-1}((S_u\backslash K)\cap B_h)\Bigg)\\ \leqq c\Bigg(&Ch^{-1} + Ch^{p-1}\sigma_h^p + \int_{B_h}|\nabla u|^p\,dx + \frac{1}{\sigma_h}\int_{B_h}|\nabla u|\,dx + \mathscr{H}^{n-1}(S_u\backslash K)\Bigg).\end{aligned}$$

Since

$$\int_{B_h}|\nabla u|\,dx \leqq |B_h|^{1-1/p}\left(\int_{B_h}|\nabla u|^p\,dx\right)^{1/p} \leqq \left(\frac{C}{h}\right)^{1-1/p}\left(\int_{B_h}|\nabla u|^p\,dx\right)^{1/p},$$

choosing (σ_h) so that $h^{1-1/p}\sigma_h = (\int_{B_h}|\nabla u|^p\,dx)^{1/2p}$, we have

$$h^{p-1}\sigma_h^p \to 0, \quad \frac{1}{\sigma_h}\int_{B_h}|\nabla u|\,dx \to 0 \quad \text{as } h\to +\infty.$$

In conclusion, from (7.3) and (7.4) we obtain

$$\begin{aligned}F_0(u,\Omega) \leqq \int_\Omega f_{\text{hom}}(\nabla u)\,dx + \liminf_{h\to+\infty}\int_{S_u\cap\Omega} g_{\text{hom}}(v_h^+ - v_h^-, \nu_u)\,d\mathscr{H}^{n-1}\\ + c\mathscr{H}^{n-1}(S_u\backslash K) + F_0(u, \bar{S}_u\backslash K),\end{aligned}$$

and, in view of the continuity of $g_{\text{hom}}(\cdot, \nu)$,

$$\begin{aligned}F_0(u,\Omega) \leqq \int_\Omega f_{\text{hom}}(\nabla u)\,dx + \int_{S_u\cap\Omega} g_{\text{hom}}(u^+ - u^-, \nu_u)\,d\mathscr{H}^{n-1}\\ + c\mathscr{H}^{n-1}(S_u\backslash K) + F_0(u, \bar{S}_u\backslash K).\end{aligned}$$

By (1.1) and by Lemma 3.2.38 in [43] we can find a sequence (K_h) of compact subsets of S_u with finite $(n-1)$-dimensional upper Minkowski content and such that $\mathscr{H}^{n-1}(S_u \backslash K_h) \to 0$ as $h \to +\infty$. Therefore

$$F_0(u, \Omega) \leqq \int_\Omega f_{\text{hom}}(\nabla u)\, dx + \int_{S_u \cap \Omega} g_{\text{hom}}(u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1} + F_0(u, \bar{S}_u \backslash S_u). \tag{7.5}$$

We now use the "strong approximation" result in $SBV^p(\Omega; \mathbf{R}^m)$ given by Lemma 5.2 and Remark 5.3 in [23], which guarantees the existence of a sequence (u_h) in $SBV^p(\Omega; \mathbf{R}^m)$ such that $\| u_h \|_{L^\infty(\Omega; \mathbf{R}^m)} \leqq \| u \|_{L^\infty(\Omega; \mathbf{R}^m)}$ and

(i) $$u_h \to u \text{ in } L^p(\Omega; \mathbf{R}^m),$$

(ii) $$\nabla u_h \to \nabla u \text{ in } L^p(\Omega; M^{m \times n}),$$

(iii) $$\mathscr{H}^{n-1}(S_{u_h} \Delta S_u) \to 0,$$

(iv) $$\int_{S_{u_h} \cup S_u} (|u_h^+ - u^+| + |u_h^- - u^-|)\, d\mathscr{H}^{n-1} \to 0,$$

(v) $$\mathscr{H}^{n-1}(\bar{S}_{u_h} \backslash S_{u_h}) \to 0.$$

Then $F_0(u_h, \bar{S}_{u_h} \backslash S_{u_h}) \to 0$ by (4.3), so that the application of (7.5) to the functions u_h yields

$$\begin{aligned} F_0(u, \Omega) &\leqq \liminf_{h \to +\infty} F_0(u_h, \Omega) \\ &\leqq \liminf_{h \to +\infty} \left(\int_\Omega f_{\text{hom}}(\nabla u_h)\, dx + \int_{S_{u_h} \cap \Omega} g_{\text{hom}}(u_h^+ - u_h^-, \nu_{u_h})\, d\mathscr{H}^{n-1} \right). \end{aligned}$$

We conclude the proof by passing to the limit, taking into account the strong convergence of (u_h). $\square$

By (4.3), $F_0(u, \cdot) = F_0(u, \cdot) \llcorner (\Omega \backslash S_u) + F_0(u, \cdot) \llcorner S_u$ is the decomposition of $F_0(u, \cdot)$ into the sum of two Borel measures on Ω which are absolutely continuous with respect to $\mathscr{L}^n$ and $\mathscr{H}^{n-1} \llcorner S_u$, respectively. We denote by $\dfrac{dF_0(u, \cdot)}{d\mathscr{L}^n}$ and $\dfrac{dF_0(u, \cdot)}{d(\mathscr{H}^{n-1} \llcorner S_u)}$, respectively, their densities.

Proposition 7.2. *Let $\Omega \in \mathscr{A}_0$ and $u \in SBV^p(\Omega; \mathbf{R}^m) \cap L^\infty(\Omega; \mathbf{R}^m)$. Then*

$$F_0(u, \Omega \backslash S_u) \geqq \int_\Omega f_{\text{hom}}(\nabla u)\, dx.$$

Proof. It is enough to prove that

$$\frac{dF_0(u, \cdot)}{d\mathscr{L}^n} \geqq f_{\text{hom}}(\nabla u) \quad \mathscr{L}^n\text{-a.e. on } \Omega.$$

For $\mathscr{L}^n$-a.e. $x_0 \in \Omega$ the function u is approximately differentiable in x_0 and

$$\left. \frac{dF_0(u, \cdot)}{d\mathscr{L}^n} \right|_{x_0} = \lim_{\rho \to 0^+} \frac{F_0(u, B_\rho(x_0))}{\omega_n \rho^n}. \tag{7.6}$$

Let $x_0 \in \Omega$ be a point with both these properties. Fix $0 < \delta < 1$ and $0 < \rho < \text{dist}(x_0, \partial\Omega) \wedge 1$. We can find a sequence (u_h) (depending on ρ) in $SBV^p(B_\rho(x_0); \mathbf{R}^m) \cap L^p(B_p(x_0); \mathbf{R}^m)$ such that

$$u_h \underset{h}{\rightarrow} u \text{ in } L^p(B_\rho(x_0); \mathbf{R}^m), \quad F_{\varepsilon_h}(u_h, B_\rho(x_0)) \underset{h}{\rightarrow} F_0(u, B_\rho(x_0)).$$

Consider a sequence (ξ_h) in $\mathbf{R}^n$ converging to x_0 and such that $\xi_h/\varepsilon_h \in \mathbf{Z}^n$. There exists $h_0 \in \mathbf{N}$ (depending on δ and ρ) such that $\xi_h + \rho B_\delta(0) \subseteq B_\rho(x_0)$ for every $h \geqq h_0$. We can then define

$$u_{h,\rho} : B_\delta(0) \rightarrow \mathbf{R}^m, \quad u_{h,\rho}(y) = \frac{1}{\rho}(u_h(\xi_h + \rho y) - u(x_0))$$

for every $h \geqq h_0$ and $u_\rho : B_1(0) \rightarrow \mathbf{R}^m$, $u_\rho(y) = \frac{1}{\rho}(u(x_0 + \rho y) - u(x_0))$. Clearly, $u_{h,\rho} \in SBV^p(B_\delta(0); \mathbf{R}^m)$ and $u_\rho \in SBV^p(B_1(0); \mathbf{R}^m)$. From the convergence of (u_h) to u in $L^p(B_\rho(x_0); \mathbf{R}^m)$ and from the continuity of translations in L^1 it follows that

$$u_{h,\rho} \underset{h}{\rightarrow} u_\rho \quad \text{in } L^1(B_\rho(0); \mathbf{R}^m). \tag{7.7}$$

Moreover, if for every $A \in \mathscr{A}_0$ and $v \in SBV^p(A; \mathbf{R}^m)$ we define

$$F_{h,\rho}(v, A) = \int_A f\left(\frac{y}{\varepsilon_h/\rho}, \nabla u\right) dy + c_3 \int_{S_v \cap A} (1 + |v^+ - v^-|)\, d\mathscr{H}^{n-1}$$

(c_3 is the constant appearing in property (iv) satisfied by g (Section 2)), then

$$\limsup_{h \rightarrow +\infty} F_{h,\rho}(u_{h,\rho}, B_\delta(0)) \leqq \frac{F_0(u, B_\rho(x_0))}{\rho^n}. \tag{7.8}$$

Indeed, for every $v \in SBV^p(B_\delta(0); \mathbf{R}^m)$ we have

$$\begin{aligned} F_{h,\rho}(v, B_\delta(0)) \leqq & \int_{B_\delta(0)} f\left(\frac{y}{\varepsilon_h/\rho}, \nabla v\right) dy \\ & + \frac{c_3}{\rho} \int_{S_v \cap B_\delta(0)} (1 + \rho|v^+ - v^-|)\, d\mathscr{H}^{n-1} \\ \leqq & \int_{B_\delta(0)} f\left(\frac{y}{\varepsilon_h/\rho}, \nabla v\right) dy \\ & + \frac{1}{\rho} \int_{S_v \cap B_\delta(0)} g\left(\frac{y}{\varepsilon_h/\rho}, \rho(v^+ - v^-), v_v\right) d\mathscr{H}^{n-1}. \end{aligned}$$

Then for every $h \geqq h_0$,

$$F_{h,\rho}(u_{h,\rho}, B_\delta(0)) \leqq \int_{B_\delta(0)} f\Big(\frac{\rho y}{\varepsilon_h}, (\nabla u_h)(\xi_h + py)\Big)\, dy$$

$$+ \frac{1}{\rho} \int_{S_{u_{h,\rho}} \cap B_\delta(0)} g\Big(\frac{\rho y}{\varepsilon_h}, (u_h^+ - u_h^-)(\xi_h + py), \nu_{u_{h,\rho}}\Big)\, d\mathscr{H}^{n-1}$$

$$= \frac{1}{\rho^n} \int_{\xi_h + \rho B_\delta(0)} f\Big(\frac{x - \xi_h}{\varepsilon_h}, (\nabla u_h)(x)\Big)\, dx$$

$$+ \frac{1}{\rho^n} \int_{S_{u_h} \cap (\xi_h + \rho B_\delta(0))} g\Big(\frac{x - \xi_h}{\varepsilon_h}, u_h^+ - u_h^- \nu_{u_h}\Big)\, d\mathscr{H}^{n-1}$$

$$\leqq \frac{1}{\rho^n}\Big(\int_{B_\rho(x_0)} f\Big(\frac{x}{\varepsilon_h}, \nabla u_h\Big)\, dx$$

$$+ \int_{S_{u_h} \cap B_\rho(x_0)} g\Big(\frac{x}{\varepsilon_h}, u_h^+ - u_h^- \nu_{u_h}\Big)\, d\mathscr{H}^{n-1}\Big).$$

By taking the upper limit we obtain (7.8).

Now let (ρ_k) be a sequence of positive numbers tending to 0 with $0 < \rho_k < \operatorname{dist}(x_0, \partial\Omega) \wedge 1$. In view of (7.7) and (7.8), for every $k \in \mathbf{N}$ there exists $h_k \in \mathbf{N}$ such that the function $v_k = u_{h_k, \rho_k}$ has the following properties:

$$\| v_k - u_{\rho_k} \|_{L^1(B_\delta(0); \mathbf{R}^m)} \leqq \frac{1}{k}, \tag{7.9}$$

$$F_{h_k, \rho_k}(v_k, B_\delta(0)) \leqq \frac{F_0(u, B_{\rho_k}(x_0))}{\rho_k^n} + \frac{1}{k}. \tag{7.10}$$

Moreover, h_k can be chosen so that the sequence $(\varepsilon_{h_k}/\rho_k)$ tends to 0 as $k \to +\infty$. We can apply Proposition 3.3 to the sequence $(F_{h_k, \rho_k})_k$ to obtain the existence of a strictly increasing sequence $(\sigma(k))$ of natural numbers such that for every $A \in \mathscr{A}_0$ the limit

$$\tilde{F}_0(\cdot, A) = \Gamma\text{-}\lim_{k \to +\infty} F_{h_{\sigma(k)}, \rho_{\sigma(k)}}(\cdot, A)$$

exists on the space $SBV^p(A; \mathbf{R}^m) \cap L^p(A; \mathbf{R}^m)$ endowed with the $L^p(A; \mathbf{R}^m)$-topology.

Since we assumed that u is approximately differentiable in x_0, if we define $w(y) = \nabla u(x_0) \cdot y, (y \in \mathbf{R}^n)$, then for every $\rho > 0$,

$$\int\limits_{B_1(0)} |u_\rho(y) - w(y)|\,dy = \frac{1}{\rho^n} \int\limits_{B_\rho(x_0)} \frac{|u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0)|}{\rho}\,dx$$

$$\leqq \frac{1}{\rho^n} \int\limits_{B_\rho(x_0)} \frac{|u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0)|}{|x - x_0|}\,dx,$$

which tends to 0 as ρ tends to 0. Hence (u_ρ) converges to w in $L^1(B_1(0); \mathbf{R}^m)$ as ρ tends to 0. It follows, by (7.9), that (v_k) converges to w in $L^1(B_\delta(0); \mathbf{R}^m)$. We now apply Proposition 3.4 to the sequence of functionals $(F_{h_{\sigma(k)}, \rho_{\sigma(k)}})$. By (7.10),

$$\tilde{F}_0(w, B_\delta(0)) \leqq \liminf_{k \to +\infty} F_{h_{\sigma(k)}, \rho_{\sigma(k)}}(v_{\sigma(k)}, B_\delta(0))$$

$$\leqq \liminf_{k \to +\infty} \frac{F_0(u, B_{\rho_{\sigma(k)}}(x_0))}{\rho^n_{\sigma(k)}} = \lim_{\rho \to 0^+} \frac{F_0(u, B_\rho(x_0))}{\rho^n}.$$

Propositions 4.1, 5.1 and 5.2 applied to $\tilde{F}_0$ yield

$$\omega_n \delta^n f_{\mathrm{hom}}(\nabla u(x_0)) \leqq \lim_{\rho \to 0^+} \frac{F_0(u, B_\rho(x_0))}{\rho^n}.$$

Let now $\delta \to 1$. The conclusion follows from (7.6). □

Proposition 7.3. *Let $\Omega \in \mathscr{A}_0$ and $u \in SBV^p(\Omega; \mathbf{R}^m) \cap L^\infty(\Omega; \mathbf{R}^m)$. Then*

$$F_0(u, S_u \cap \Omega) \geqq \int\limits_{S_u \cap \Omega} g_{\mathrm{hom}}(u^+ - u^-, \nu_u)\,d\mathscr{H}^{n-1}.$$

Proof. We use the same blow-up technique employed for Proposition 7.2. It is clearly enough to prove that

$$\frac{dF_0(u, \cdot)}{d(\mathscr{H}^{n-1} \llcorner S_u)} \geqq g_{\mathrm{hom}}(u^+ - u^-, \nu_u) \quad \mathscr{H}^{n-1}\text{-a.e. on } S_u.$$

We know that for $\mathscr{H}^{n-1}$-a.e. $x_0 \in S_u$ the triple $(u^+(x_0), u^-(x_0), \nu_u(x_0))$ is defined, and, by the Besicovitch Differentiation Theorem (see, e.g., [55, Theorem 4.7]), that

$$\frac{dF_0(u, \cdot)}{d(\mathscr{H}^{n-1} \llcorner S_u)} = \lim_{\rho \to 0+} \frac{F_0(u, B_\rho(x_0))}{\omega_{n-1} \rho^{n-1}} \tag{7.11}$$

(recall that $\mathscr{H}^{n-1}$-a.e. on S_u the $(n-1)$-dimensional density of $\mathscr{H}^{n-1} \llcorner S_u$ is 1 (see [43, Theorem 3.2.19]). Let $x_0 \in S_u$ be such a point. Fix $0 < \delta < 1$ and $0 < \rho < \operatorname{dist}(x_0, \partial\Omega)$. Let $(u_h), (\xi_h)$ and h_0 be as in the proof of the previous proposition. We can then define

$$u_{h,\rho}: B_\delta(0) \to \mathbf{R}^m, \quad u_{h,\rho}(y) = u_h(\xi_h + \rho y)$$

for every $h \geqq h_0$ and $u_\rho: B_1(0) \to \mathbf{R}^m$, $u_\rho(y) = u(x_0 + \rho y)$. Clearly,

$$u_{h,\rho} \in SBV^p(B_\delta(0); \mathbf{R}^m) \cap L^p(B_\delta(0); \mathbf{R}^m), \quad u_\rho \in SBV^p(B_1(0); \mathbf{R}^m) \cap L^\infty(B_1(0); \mathbf{R}^m),$$

and

$$u_{h,\rho} \underset{h}{\to} u_\rho \quad \text{in } L^p(B_\delta(0); \mathbf{R}^m). \tag{7.12}$$

Let $\eta > 0$ and define $A_{\eta,\delta} = B_\delta(0) \cap \sum_\eta$, with $\sum_\eta = \{x \in \mathbf{R}^n : |\langle x, \nu_u(x_0)\rangle| < \eta\}$. We have

$$\limsup_{h \to +\infty} F_{\varepsilon_h/\rho}(u_{h,\rho}, A_{\eta,\delta}) \leqq \frac{F_0(u, B_\rho(x_0))}{\rho^{n-1}} + c_2 |A_{\eta,\delta}| \tag{7.13}$$

whenever ρ is such that $c_1/\rho^{p-1} \geqq c_2$. Indeed, for every $\rho > 0$ such that $c_2 \leqq c_1/\rho^{p-1}$ and for every $v \in SBV^p(B_\delta(0); \mathbf{R}^m) \cap L^p(B_\delta(0); \mathbf{R}^m)$ we get

$$\begin{aligned} F_{\varepsilon_h/\rho}(v, A_{\eta,\rho}) &\leqq c_2 \int_{A_{\eta,\delta}} (1 + |\nabla v|^p)\, dy + \int_{S_v \cap A_{\eta,\delta}} g\left(\frac{y}{\varepsilon_h/\rho}, v^+ - v^-, \nu_v\right) d\mathscr{H}^{n-1} \\ &\leqq \frac{c_1}{\rho^{p-1}} \int_{B_\delta(0)} |\nabla v|^p\, dy + \int_{S_v \cap B_\delta(0)} g\left(\frac{y}{\varepsilon_h/\rho}, v^+ - v^-, \nu_v\right) d\mathscr{H}^{n-1} + c_2 |A_{\eta,\delta}| \\ &\leqq \rho \int_{B_\delta(0)} f\left(\frac{y}{\varepsilon_h/\rho}, \frac{1}{\rho} \nabla v\right) dy + \int_{S_v \cap B_\delta(0)} g\left(\frac{y}{\varepsilon_h/\rho}, v^+ - v^-, \nu_v\right) d\mathscr{H}^{n-1} \\ &\quad + c_2 |A_{\eta,\delta}|. \end{aligned}$$

Then for $h \geqq h_0$

$$\begin{aligned} F_{\varepsilon_h/\rho}(u_{h,\rho}, A_{\eta,\delta}) &\leqq \rho \int_{B_\delta(0)} f\left(\frac{\rho y}{\varepsilon_h}, (\nabla u_h)(\xi_h + \rho y)\right) dy \\ &\quad + \int_{S_{u_{h,\rho}} \cap B_\delta(0)} g\left(\frac{\rho y}{\varepsilon_h}, u_{h,\rho}^+ - u_{h,\rho}^-, \nu_{u_{h,\rho}}\right) d\mathscr{H}^{n-1} + c_2 |A_{\eta,\delta}| \\ &= \frac{1}{\rho^{n-1}} \int_{\xi_h + \rho B_\delta(0)} f\left(\frac{x - \xi_h}{\varepsilon_h}, \nabla u_h(x)\right) dx \\ &\quad + \frac{1}{\rho^{n-1}} \int_{\xi_h + \rho(S_{u_{h,\rho}} \cap B_\delta(0))} g\left(\frac{x - \xi_h}{\varepsilon_h}, u_h^+ - u_h^-, \nu_{u_h}\right) d\mathscr{H}^{n-1} + c_2 |A_{\eta,\delta}| \\ &\leqq \frac{1}{\rho^{n-1}} \Bigg(\int_{B_\rho(x_0)} f\left(\frac{x}{\varepsilon_h}, \nabla u_h(x)\right) dx \\ &\quad + \int_{S_{u_h} \cap B_\rho(x_0)} g\left(\frac{x}{\varepsilon_h}, u_h^+ - u_h^-, \nu_{u_h}\right) d\mathscr{H}^{n-1} \Bigg) + c_2 |A_{\eta,\delta}|. \end{aligned}$$

By taking the upper limit we get (7.13).

Now let (ρ_k) be a sequence of positive numbers tending to 0. We can assume that the condition $c_1/\rho_k^{p-1} \geqq c_2$ is satisfied for every $k \in \mathbf{N}$. In view of (7.12) and (7.13), for every $k \in \mathbf{N}$ there exists $h_k \in \mathbf{N}$ such that the function $v_k = u_{h_k,\rho_k}$ has the properties:

$$\| v_k - u_{\rho_k} \|_{L^p(B_\delta(0);\, \mathbf{R}^m)} \leqq \frac{1}{k}, \tag{7.14}$$

$$F_{\varepsilon_{h_k}/\rho_k}(v_k, A_{\eta,\delta}) \leqq \frac{F_0(u, B_{\rho_k}(x_0))}{\rho_k^{n-1}} + c_2 |A_{\eta,\delta}| + \frac{1}{k}. \tag{7.15}$$

Moreover, h_k can be chosen so that the sequence $\alpha_k = \varepsilon_{h_k}/\rho_k$ tends to 0 as k tends to $+\infty$. We can apply Proposition 3.3 to the sequence $(F_{\alpha_k})_k$, obtaining the existence of a subsequence $(F_{\alpha_{\sigma(k)}})$ such that for every $A \in \mathscr{A}_0$ the limit

$$\tilde{F}_0(\,\cdot\,, A) = \Gamma\text{-}\lim_{k \to +\infty} F_{\alpha_{\sigma(k)}}(\,\cdot\,, A)$$

exists on the space $SBV^p(A; \mathbf{R}^m) \cap L^p(A; \mathbf{R}^m)$ endowed with the $L^p(A; \mathbf{R}^m)$-topology. It is easy to see that (u_{ρ_k}) converges in $L^p(B_\delta(0); \mathbf{R}^m)$ to the function

$$w(y) = \begin{cases} u^-(x_0) & \text{if } \langle x, \nu_u(x_0) \rangle < 0, \\ u^+(x_0) & \text{if } \langle x, \nu_u(x_0) \rangle \geqq 0. \end{cases}$$

It follows, by (7.14), that (v_k) converges to w in $L^p(A_{\eta,\delta}; \mathbf{R}^m)$. Hence, by (7.15),

$$\begin{aligned} \tilde{F}_0(w, A_{\eta,\delta}) &\leqq \liminf_{k\to\infty} F_{\alpha_{\sigma(k)}}(v_{\sigma(k)}, A_{\eta,\delta}) \leqq \liminf_{k\to\infty} \frac{F_0(u, B_{\rho_{\sigma(k)}}(x_0))}{\rho_{\sigma(k)}^{n-1}} + c_2 |A_{\eta,\delta}| \\ &= \lim_{\rho\to 0^+} \frac{F_0(u, B_\rho(x_0))}{\rho^{n-1}} + c_2 |A_{\eta,\delta}|. \end{aligned}$$

Since $S_w \cap B_\delta(0) \subseteq A_{\eta,\delta}$, we deduce that

$$\tilde{F}_0(w, S_w \cap B_\delta(0)) \leqq \lim_{\rho\to 0^+} \frac{F_0(u, B_\rho(x_0))}{\rho^{n-1}} + c_2 |A_{\eta,\delta}|.$$

We can apply Propositions 4.2, 6.1 and 6.2 to $\tilde{F}_0$, thus obtaining

$$\omega_{n-1}\delta^{n-1} g_{\text{hom}}(u^+(x_0) - u^-(x_0), \nu_{u(x_0)}) \leqq \lim_{\rho\to 0^+} \frac{F_0(u, B_\rho(x_0))}{\rho^{n-1}} + c_2 |A_{\eta,\delta}|.$$

Finally, let $\delta \to 1$, $\eta \to 0$ and recall (7.11). □

Conclusion of the proof of Theorem 2.3. Let (ε_h) be a sequence of positive numbers tending to 0 such that for every $A \in \mathscr{A}_0$ the limit

$$F_0(\,\cdot\,, A) = \Gamma\text{-}\lim_{h\to+\infty} F_{\varepsilon_h}(\,\cdot\,, A) \tag{7.16}$$

exists on the space $SBV^p(A; \mathbf{R}^m) \cap L^p(A; \mathbf{R}^m)$ endowed with the $L^p(A; \mathbf{R}^m)$-topology (see (4.1)). So far, in view of Propositions 7.1, 7.2 and 7.3, we have obtained that

$$F_0(u, A) = F_{\text{hom}}(u, A) \quad \text{for every } u \in SBV^p(A; \mathbf{R}^m) \cap L^\infty(A; \mathbf{R}^m), \tag{7.17}$$

where $F_{\text{hom}}(u, A) = \int_A f_{\text{hom}}(\nabla u)\,dx + \int_{S_u \cap A} g_{\text{hom}}(u^+ - u^-, \nu_u)\,d\mathcal{H}^{n-1}$. We now prove that for every $A \in \mathcal{A}_0$,

$$\Gamma\text{-}\lim_{h \to +\infty} F_{\varepsilon_h}(\cdot, A) = F_{\text{hom}}(\cdot, A) \tag{7.18}$$

on the space $SBV^p(A; \mathbf{R}^m)$ endowed with the $L^1(A; \mathbf{R}^m)$-topology, and on the space $SBV^p(A; \mathbf{R}^m) \cap L^p(A; \mathbf{R}^m)$ with respect to the $L^p(A; \mathbf{R}^m)$-topology. To this end fix $u \in SBV^p(A; \mathbf{R}^m)$.

Step 1. $F_{\text{hom}}(u, A) \leqq \liminf_{h \to +\infty} F_{\varepsilon_h}(u_h, A)$ *for every sequence* (u_h) *in* $SBV^p(A; \mathbf{R}^m)$ *converging to* u *in* $L^1(A; \mathbf{R}^m)$.

Proof. We can assume that the lower limit in the right-hand side is actually a limit and is finite. Fix $\eta > 0$ and $k \in \mathbf{N}$, and apply Lemma 3.5 to the sequence (u_h) with $M_0 = k$. We find a subsequence $(\varepsilon_{\sigma(h)})$ of (ε_h) and a Lipschitz function $\varphi_k : \mathbf{R}^m \to \mathbf{R}^m$ having compact support, having a Lipschitz constant less than or equal to 1 and satisfying $\varphi_k(y) = y$ if $|y| \leqq k$, such that for every $h \in \mathbf{N}$

$$F_{\varepsilon_{\sigma(h)}}(\varphi_k(u_{\sigma(h)}), A) \leqq F_{\varepsilon_{\sigma(h)}}(u_{\sigma(h)}, A) + \eta.$$

Since $(\varphi_k(u_{\sigma(h)}))_h$ converges to $\varphi_k(u)$ in $L^p(A; \mathbf{R}^m)$, from (7.16) and (7.17) it follows that

$$\begin{aligned} F_{\text{hom}}(\varphi_k(u), A) &\leqq \liminf_{h \to +\infty} F_{\varepsilon_{\sigma(h)}}(\varphi_k(u_{\sigma(h)}), A) \\ &\leqq \lim_{h \to +\infty} F_{\varepsilon_{\sigma(h)}}(u_{\sigma(h)}, A) + \eta = \liminf_{h \to +\infty} F_{\varepsilon_h}(u_h, A) + \eta. \end{aligned}$$

By the arbitrariness of $\eta > 0$, it only remains to prove that

$$\lim_{k \to +\infty} F_{\text{hom}}(\varphi_k(u), A) = F_{\text{hom}}(u, A). \tag{7.19}$$

For every $k \in \mathbf{N}$ set

$$E_k = \{x \in \Omega : |u(x)| > k\}, \quad S_k = \{x \in S_u : |u^+(x)| \geqq k \text{ or } |u^-(x)| \geqq k\}.$$

Then

$$\begin{aligned} &|F_{\text{hom}}(\varphi_k(u), A) - F_{\text{hom}}(u, A)| \\ &= \Bigg| \int_{E_k} (f_{\text{hom}}(\nabla(\varphi_k(u)) - f_{\text{hom}}(\nabla u))\,dx \\ &\quad + \int_{S_k} (g_{\text{hom}}(\varphi_k(u^+) - \varphi_k(u^-), \nu_u) - g_{\text{hom}}(u^+ - u^-, \nu_u))\,d\mathcal{H}^{n-1} \Bigg| \\ &\leqq K\left(\int_{E_k} (1 + |\nabla u|^p)\,dx + \int_{S_k} (1 + |u^+ - u^-|)\,d\mathcal{H}^{n-1} \right), \end{aligned}$$

for a suitable constant $K > 0$. Since $|E_k|$ and $\mathcal{H}^{n-1}(\{x \in S_u : |u^\pm(x)| \geqq k)\}$ tend to 0 as k tends to $+\infty$, we obtain (7.19).

Step 2. *There exists a sequence* (u_h) *in* $SBV^p(A;\mathbf{R}^m)\cap L^p(A;\mathbf{R}^m)$ *which converges to* u *in* $L^1(A;\mathbf{R}^m)$ (*or in* $L^p(A;\mathbf{R}^m)$ *if* $u\in L^p(A;\mathbf{R}^m)$) *and which satisfies*

$$\limsup_{h\to+\infty} F_{\varepsilon_h}(u_h, A) \leqq F_{\text{hom}}(u, A).$$

Proof. For every $k\in\mathbf{N}$ let $\varphi_k:\mathbf{R}^m\to\mathbf{R}^m$ be a Lipschitz function with compact support, with a Lipschitz constant less than or equal to 1 and such that $\varphi_k(y)=y$ if $|y|\leqq k$. Since $\varphi_k(u)\in SBV^p(A;\mathbf{R}^m)\cap L^\infty(A;\mathbf{R}^m)$, by (7.16) and (7.17) there exists a sequence $(v_h^k)_h$ in $SBV^p(A;\mathbf{R}^m)\cap L^p(A;\mathbf{R}^m)$ with the properties that

$$v_h^k \underset{h}{\to} \varphi_k(u) \text{ in } L^p(A;\mathbf{R}^m), \quad F_{\varepsilon_h}(v_h^k, A) \underset{h}{\to} F_{\text{hom}}(\varphi_k(u), A)$$

as h tends to $+\infty$. Therefore for every $k\in\mathbf{N}$ we can find $h_k\in\mathbf{N}$ such that $h_k<h_{k+1}$ and

$$\| v_h^k - \varphi_k(u)\|_{L^p(A;\mathbf{R}^m)} \leqq 1/k, \quad |F_{\varepsilon_h}(v_h^k, A) - F_{\text{hom}}(\varphi_k(u), A)| \leqq 1/k$$

for every $h\geqq h_k$. Define $u_h = v_h^k$ if $h_k\leqq h<h_{k+1}$. Taking into account (7.19) and that $(\varphi_k(u))$ converges to u in $L^1(A;\mathbf{R}^m)$ (or in $L^p(A;\mathbf{R}^m)$ if $u\in L^p(A;\mathbf{R}^m)$), we conclude that (u_h) is the required sequence.

In view of Proposition 3.3 and of what we have proved up to now for a convergent sequence of functionals F_ε, Theorem 2.3 follows by applying Proposition 8.3 in [35], which asserts that (F_{ε_h}) converges to F_{hom} if and only if every subsequence of (F_{ε_h}) contains a further subsequence converging to F_{hom}. □

8. Homogenization in fracture mechanics

In this section we consider the homogenization of the functionals F_ε under prescribed boundary conditions. The result is then applied to the homogenization of a boundary-value problem for functionals whose surface energy density satisfies Griffith's growth conditions. Throughout this section $n\geqq 2$.

Let F_ε ($\varepsilon>0$) and F_{hom} be the functionals introduced in Section 2 and let Ω be a bounded open subset of $\mathbf{R}^n$ with piecewise C^1 boundary. In order to prescribe boundary values to functions in $SBV^p(\Omega;\mathbf{R}^m)$, we introduce a bounded open subset Ω' of $\mathbf{R}^n$ containing Ω a function $\phi\in SBV^p(\Omega';\mathbf{R}^m)$ and we define

$$\mathscr{D}_\phi = \{u\in SBV^p(\Omega';\mathbf{R}^m): u=\phi \text{ a.e. on } \Omega'\backslash\Omega\}.$$

Now let $\gamma=\Omega'\cap\partial\Omega$, and for every $u\in\mathscr{D}_\phi$ and $\varepsilon>0$ set

$$F_\varepsilon^\phi = F_\varepsilon(u, \Omega\cup\gamma), \quad F_{\text{hom}}^\phi(u) = F_{\text{hom}}(u, \Omega\cup\gamma).$$

Theorem 8.1. *Assume that* $\phi\in SBV^p(\Omega';\mathbf{R}^m)\cap L^\infty(\Omega';\mathbf{R}^m)$ *and* $\mathscr{H}^{n-1}(\gamma)<+\infty$. *Let* (ε_h) *be a sequence of positive numbers tending to* 0. *Then* $F_{\text{hom}}^\phi = \Gamma\text{-}\lim_{h\to+\infty} F_{\varepsilon_h}^\phi$ *on* $\mathscr{D}_\phi$ *with respect to the* $L^1(\Omega';\mathbf{R}^m)$-*topology*.

For the proof we need two lemmas, the first of which collects simple facts whose proof is omitted.

Lemma 8.2. (a) *Let M be a non-empty subset of $\mathbf{R}^n$ and let $\psi:\mathbf{R}^n \to \mathbf{R}$ be defined by $\psi(x) = \operatorname{dist}(x, M)$. Then ψ is differentiable a.e. in $\mathbf{R}^n$ and $|D\psi(x)| = 1$ at every point of differentiability $x \notin \bar{M}$.*

(b) *Let Ω be an open subset of $\mathbf{R}^n$ with a C^1 boundary, and let M be a non-empty subset of $\partial\Omega$. Assume that M is open in $\partial\Omega$ and let $x_0 \in M$. If $\nu(x_0)$ denotes the inner unit normal of $\partial\Omega$ is x_0, then*

$$\lim_{\substack{x \to x_0 \\ x \in A}} D\psi(x) = \nu(x_0),$$

with A the subset of Ω where ψ is differentiable.

Lemma 8.3. *Let ϕ and γ be as in Theorem 8.1. Then for every $u \in \mathscr{D}_\phi \cap L^\infty(\Omega'; \mathbf{R}^m)$ and $\sigma > 0$ there exists $u_\sigma \in \mathscr{D}_\phi \cap L^\infty(\Omega'; \mathbf{R}^m)$ such that*

(i) $u_\sigma \to u$ *in* $L^p(\Omega'; \mathbf{R}^m)$ *as* $\sigma \to 0$,
(ii) $\mathscr{H}^{n-1}(S_{u_\sigma} \cap \gamma) = 0$ *for every* $\sigma > 0$,
(iii) $\limsup_{\sigma \to 0} F_{\text{hom}}(u_\sigma, \Omega \cup \gamma) \leqq F_{\text{hom}}(u, \Omega \cup \gamma)$.

Proof. By a simple reflection argument, it is not restrictive to assume that $\mathscr{H}^{n-1}(S_\phi \cap \gamma) = 0$. Note that if $u \in SBV(\Omega'; \mathbf{R}^m)$ for $\mathscr{H}^{n-1}$-a.e. $x_0 \in S_u \cap \gamma$, then the vector $\nu_u(x_0)$ is normal to γ; in such a case we agree to choose as $\nu_u(x_0)$ the inner normal with respect to Ω. Then it easily turns out that

$$\text{(8.1)} \qquad \lim_{\rho \to 0} \frac{1}{\omega_n \rho^n} |\{x \in B_\rho(x_0) \cap \Omega : |u(x) - u^+(x_0)| > \varepsilon\}| = 0 \quad \text{for every } \varepsilon > 0.$$

Let $u \in \mathscr{D}_\phi \cap L^\infty(\Omega'; \mathbf{R}^m)$ and let $\sigma > 0$ be fixed. Define $v = u - \phi$. Since $\mathscr{H}^{n-1}(S_\phi \cap \gamma) = 0$, we have

$$u^+(x_0) - u^-(x_0) = v^+(x_0) \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x_0 \in S_u \cap \gamma.$$

For every $x \in \gamma$ we denote by $\nu(x)$ the unit inner normal to γ in x with respect to Ω. Let E be the set of the points $x_0 \in S_u \cap \gamma$ such that $\nu_u(x_0) = \nu(x_0)$ and there exists $\rho_1(x_0) > 0$ with the property for all $\rho \leqq \rho_1(x_0)$,

(8.2)

$$\left| \frac{1}{\omega_{n-1} \rho^{n-1}} \int_{B_\rho(x_0) \cap \gamma} g_{\text{hom}}(u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1} - g_{\text{hom}}(v^+(x_0), \nu_u(x_0))\, d\mathscr{H}^{n-1} \right| \leqq \sigma.$$

By the Besicovitch differentiation theorem for Radon measures and the fact that the $(n-1)$-dimensional density of $\mathscr{H}^{n-1} \llcorner \gamma$ is 1 $\mathscr{H}^{n-1}$-a.e. on γ, we obtain that $\mathscr{H}^{n-1}(S_u \cap \gamma \setminus E) = 0$. Let x_0 be fixed in E. Since g_{hom} is continuous in $(v^+(x_0), \nu(x_0))$ there exists $\eta > 0$ such that for all $(s, \nu) \in \mathbf{R}^m \times S^{n-1}$,

if $|s - v^+(x_0)| \leqq \eta$ and $|\nu - \nu(x_0)| \leqq \eta$, then $|g_{\text{hom}}(s, \nu) - g_{\text{hom}}(v^+(x_0), \nu(x_0))| \leqq \sigma$.

Moreover, by (8.1), there exists $\rho_2(x_0) > 0$ such that

$$\forall \rho \leqq \rho_2(x_0) \quad |\{x \in B_\rho(x_0) \cap \Omega : |v(x) - v^+(x_0)| > \eta\}| \leqq \sigma \rho^n.$$

Let $\psi:\mathbf{R}^n \to [0, +\infty[$ be defined by $\psi(x) = \operatorname{dist}(x, \gamma)$. By Lemma 8.2 we can suppose that

$$|D\psi(x) - \nu(x_0)| \leqq \eta \quad \text{for a.e. } x \in B_{\rho_2(x_0)}(x_0).$$

Then, if we set

$$Z_\rho = \{x \in B_\rho(x_0) \cap \Omega : \psi \text{ is differentiable in } x, \\ |g_{\hom}(v(x), D\psi(x)) - g_{\hom}(v^+(x_0), \nu(x_0))| > \sigma\},$$

we clearly have

$$\forall \rho \leqq \rho_2(x_0), \quad |Z_\rho| \leqq \sigma \rho^n. \tag{8.3}$$

For every $t > 0$ define

$$V_t = \{x \in \Omega : \psi(x) < t\}, \quad \gamma_t = \{x \in \Omega : \psi(x) = t\} = \Omega \cap \partial V_t,$$
$$S_t(x_0) = \{x \in \mathbf{R}^n : |\langle x - x_0, \nu(x_0)\rangle| \leqq t\}.$$

We show that there exists $\rho_3(x_0) > 0$ such that

$$\forall \rho \leqq \rho_3(x_0), \quad \forall t > 0, \quad V_t \cap B_\rho(x_0) \subseteq S_\delta(x_0) \cap B_\rho(x_0) \quad \text{with } \delta = t + \sigma\rho. \tag{8.4}$$

Indeed, from

$$\lim_{\substack{x \to x_0 \\ x \in \gamma}} \frac{|\langle x - x_0, \nu(x_0)\rangle|}{|x - x_0|} = 0$$

(which holds for $\mathscr{H}^{n-1}$-a.e. $x_0 \in \gamma$) we deduce the existence of $\rho_3(x_0) > 0$ such that $B_{2\rho_3(x_0)}(x_0) \cap \partial\Omega \subseteq \gamma$ and

$$\forall \rho \leqq 2\rho_3(x_0), \forall x \in \gamma \cap B_\rho(x_0), \quad |\langle x - x_0, \nu(x_0)\rangle| \leqq \tfrac{1}{2}\sigma\rho.$$

Let $\rho \leqq \rho_3(x_0)$ and $x \in V_t \cap B_\rho(x_0)$. If $\bar{x} \in \bar{\gamma}$ is such that $\psi(x) = |x - \bar{x}|$, then $|\bar{x} - x_0| < 2\rho$, and

$$|\langle x - x_0, \nu(x_0)\rangle| \leqq |\langle x - \bar{x}, \nu(x_0)\rangle| + |\langle \bar{x} - x_0, \nu(x_0)\rangle| \leqq t + \sigma\rho.$$

This proves (8.4).

Finally let us set $\bar{\rho}(x_0) = \min\{\rho_1(x_0), \rho_2(x_0), \rho_3(x_0), \sigma\}$. In addition to (8.2), (8.3) and (8.4), by the regularity of γ we can also suppose (possibly discarding from E a further $\mathscr{H}^{n-1}$-negligible subset) that for every $\rho \leqq \bar{\rho}(x_0)$,

$$\begin{aligned} &\mathscr{H}^{n-1}(\gamma \cap B_\rho(x_0)) \geqq \tfrac{1}{2}\omega_{n-1}\rho^{n-1}, \\ &\mathscr{H}^{n-1}(\gamma_t \cap B_\rho(x_0)) \leqq (1 + \sigma)\omega_{n-1}\rho^{n-1} \text{ for every } t < \bar{\rho}(x_0), \quad \bar{B}_\rho(x_0) \subseteq \Omega'. \end{aligned} \tag{8.5}$$

Let $\mathscr{F} = \{\bar{B}_\rho(x_0) : x_0 \in E, 0 < \rho \leqq \bar{\rho}(x_0)\}$, where $\bar{\rho}(x_0)$ is defined above, and let G be an open set containing E. By the Besicovitch Covering Theorem (see [56]) there exists a countable disjoint subcollection $\mathscr{G} = \{\bar{B}_{\rho_i}(x_i) : i \in \mathbf{N}\}$ of $\mathscr{F}$ which covers E up to a $\mathscr{H}^{n-1}$-negligible set and whose elements are contained in G.

Let $i \in \mathbf{N}$ be fixed and define

$$\bar{t}_i = \sup\{t \in]0, \rho_i] : \mathcal{H}^{n-1}(Z_{\rho_i} \cap \gamma_\tau) > \sqrt{\sigma}\rho_i^{n-1} \text{ for a.e. } \tau \in]0, t]\}.$$

We show that

$$\bar{t}_i < \sqrt{\sigma}\rho_i. \tag{8.6}$$

We can assume that $\bar{t}_i > 0$; otherwise there is nothing to prove. By the coarea formula (1.2) and Lemma 8.3 we have

$$\begin{aligned} |Z_{\rho_i}| &= \int_{Z_{\rho_i}} |D\psi(x)|\, dx = \int_0^{\rho_i} \mathcal{H}^{n-1}(Z_{\rho_i} \cap \gamma_t)\, dt \\ &> \int_0^{\bar{t}_i} \sqrt{\sigma}\rho_i^{n-1}\, dt = \sqrt{\sigma}\rho_i^{n-1}\bar{t}_i, \end{aligned}$$

so that (8.3) implies (8.6).

Since ψ is differentiable a.e. and $|S_u| = |S_\phi| = 0$, from (8.6) and the coarea formula we deduce that there exists $0 < t_i < \sqrt{\sigma}\rho_i$ such that

$$\begin{gathered} \psi \text{ is differentiable } \mathcal{H}^{n-1}\text{-a.e. on } \gamma_{t_i} \cap B_{\rho_i}(x_i), \\ \mathcal{H}^{n-1}(S_u \cap \gamma_{t_i} \cap B_{\rho_i}(x_i)) = 0, \quad \mathcal{H}^{n-1}(S_\phi \cap \gamma_{t_i} \cap B_{\rho_i}(x_i)) = 0, \\ \mathcal{H}^{n-1}(Z_{\rho_i} \cap \gamma_{t_i}) \leqq \sqrt{\sigma}\rho_i^{n-1}. \end{gathered} \tag{8.7}$$

Define

$$u_\sigma = \begin{cases} \phi & \text{on } A, \\ u & \text{otherwise on } \Omega' \end{cases} \qquad \text{where } A = \bigcup_{i \in \mathbf{N}} V_{t_i} \cap B_{\rho_i}(x_i).$$

By the subadditivity of the perimeter we have

$$\mathcal{H}^{n-1}(\Omega \cap \partial^* A) \leqq \sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\Omega \cap \partial(V_{t_i} \cap B_{\rho_i}(x_i))).$$

Therefore, since $\Omega \cap \partial(V_{t_i} \cap B_{\rho_i}(x_i)) \subseteq (\gamma_{t_i} \cap B_{\rho_i}(x_i)) \cup (\bar{V}_{t_i} \cap \partial B_{\rho_i}(x_i))$, we have

$$\mathcal{H}^{n-1}(\Omega \cap \partial^* A) \leqq \sum_{i=1}^{\infty} (\mathcal{H}^{n-1}(\gamma_{t_i} \cap B_{\rho_i}(x_i)) + \mathcal{H}^{n-1}(\bar{V}_{t_i} \cap \partial B_{\rho_i}(x_i))). \tag{8.8}$$

By (8.5) we have $\mathcal{H}^{n-1}(\gamma_{t_i} \cap B_{\rho_i}(x_i)) \leqq 2(1+\sigma)\mathcal{H}^{n-1}(\gamma \cap B_{\rho_i}(x_i))$. Moreover, by (8.4) and the inequality $t_i < \sqrt{\sigma}\rho_i$ we obtain

$$\mathcal{H}^{n-1}(\bar{V}_{t_i} \cap \partial B_{\rho_i}(x_i)) \leqq \mathcal{H}^{n-1}(S_{t_i + \sigma\rho_i}(x_i) \cap \partial B_{\rho_i}(x_i)) \leqq 4(n-1)\omega_{n-1}\rho_i^{n-1}\sqrt{\sigma},$$

so that, by (8.5),

$$\mathcal{H}^{n-1}(\bar{V}_{t_i} \cap \partial B_{\rho_i}(x_i)) \leqq 8(n-1)\mathcal{H}^{n-1}(\gamma \cap B_{\rho_i}(x_i))\sqrt{\sigma}. \tag{8.9}$$

Since the balls $B_{\rho_i}(x_i)$ are pairwise disjoint, and since $\mathcal{H}^{n-1}(\gamma) < +\infty$, we conclude that $\mathcal{H}^{n-1}(\Omega \cap \partial^* A) < +\infty$ and that A is a set of finite perimeter. In particular, $u_\sigma \in SBV^p(\Omega'; \mathbf{R}^m)$.

Taking into account that $D\psi$ is the outer normal to γ_{t_i} with respect to $V_{t_i}\cap B_{\rho_i}(x_i)$, and that u and ϕ are approximately continuous $\mathscr{H}^{n-1}$-a.e. on γ_{t_i}, we have

$$\int_{S_{u_\sigma}\cap(\gamma_{t_i}\cap B_{\rho_i}(x_i))} g_{\text{hom}}(u_\sigma^+ - u_\sigma^-, \nu_{u_\sigma})\,d\mathscr{H}^{n-1}$$
$$= \int_{\gamma_{t_i}\cap B_{\rho_i}(x_i)} g_{\text{hom}}(v, D\psi)\,d\mathscr{H}^{n-1}$$
$$\leqq \int_{\gamma_{t_i}\cap B_{\rho_i}(x_i)\setminus Z_{\rho_i}} (g_{\text{hom}}(v^+(x_i), \nu(x_i)) + \sigma)\,d\mathscr{H}^{n-1}$$
$$+ c_4(1 + \|u - \phi\|_{L^\infty})\mathscr{H}^{n-1}(\gamma_{t_i}\cap Z_{\rho_i}).$$

In view of (8.5) and (8.2) the first of these two terms does not exceed

$$(1+\sigma)(g(v^+(x_i), \nu(x_i)) + \sigma)\omega_{n-1}\rho_i^{n-1}$$
$$\leqq (1+\sigma)\left(\int_{\gamma\cap B_{\rho_i}(x_i)} g_{\text{hom}}(u^+ - u^-, \nu_u)\,d\mathscr{H}^{n-1} + 2\sigma\omega_{n-1}\rho_i^{n-1}\right),$$

and by (8.5) again this is estimated by

$$(1+\sigma)\left(\int_{\gamma\cap B_{\rho_i}(x_i)} g_{\text{hom}}(u^+ - u^-, \nu_u)\,d\mathscr{H}^{n-1} + 4\sigma\mathscr{H}^{n-1}(\gamma\cap B_{\rho_i}(x_i))\right).$$

Moreover, by (8.7) (and (8.5)),

$$\mathscr{H}^{n-1}(\gamma_{t_i}\cap Z_{\rho_i}) \leqq \sqrt{\sigma}\rho_i^{n-1} \leqq \frac{2}{\omega_{n-1}}\mathscr{H}^{n-1}(\gamma\cap B_{\rho_i}(x_i))\sqrt{\sigma}.$$

All these inequalities yield that

$$\sum_{i=1}^{\infty} \int_{S_{u_\sigma}\cap(\gamma_{t_i}\cap B_{\rho_i}(x_i))} g_{\text{hom}}(u_\sigma^+ - u_\sigma^-, \nu_{u_\sigma})\,d\mathscr{H}^{n-1} \tag{8.10}$$
$$\leqq (1+\sigma)\int_{\gamma\cap G} g_{\text{hom}}(u^+ - u^-, \nu_u)\,d\mathscr{H}^{n-1} + C\mathscr{H}^{n-1}(\gamma)\sqrt{\sigma},$$

for a suitable constant $C > 0$ independent of σ.

We finally show that the family (u_σ) satisfies the required properties. Clearly $u_\sigma \in \mathscr{D}_\phi \cap L^\infty(\Omega'; \mathbf{R}^m)$, and property (i) holds. Since $\mathscr{H}^{n-1}$-a.e. $x_0 \in \gamma\setminus S_u$ is a point of approximate continuity for both ϕ and u, we obtain that $\mathscr{H}^{n-1}(S_{u_\sigma}\cap(\gamma\setminus S_u)) = 0$. Moreover, $\mathscr{H}^{n-1}$-a.e. point x_0 in E belongs to $B_{\rho_i}(x_i)$ for some $i \in \mathbf{N}$. Since $u_\sigma = \phi$ in a neighborhood of x_0 and $\mathscr{H}^{n-1}(S_\phi\cap\gamma) = 0$, we conclude that $x_0 \notin S_{u_\sigma}$. Therefore $\mathscr{H}^{n-1}(S_{u_\sigma}\cap\gamma) = 0$.

It remains only to prove (iii). We have

$$F_{\text{hom}}(u_\sigma, \Omega\cup\gamma) \leqq \int_\Omega f_{\text{hom}}(\nabla u)\,dx + \int_{\gamma\cup V_\sigma} f_{\text{hom}}(\nabla\phi)\,dx$$
$$+ \int_{S_u\cap\Omega} g_{\text{hom}}(u^+ - u^-, \nu_u)\,d\mathscr{H}^{n-1}$$
$$+ \int_{S_{u_\sigma}\cap\Omega\cap\partial^* A} g_{\text{hom}}(u_\sigma^+ - u_\sigma^-, \nu_{u_\sigma})\,d\mathscr{H}^{n-1}.$$

Since $\gamma_{t_i} \cap B_{\rho_i}(x_i) \subseteq \partial^* A$ (up to a set of zero $\mathscr{H}^{n-1}$-measure), we have

$$\mathscr{H}^{n-1}\Big(\Omega \cap \partial^* A \setminus \bigcup_{i \in \mathbf{N}} (\gamma_{t_i} \cap B_{\rho_i}(x_i))\Big) = \mathscr{H}^{n-1}(\Omega \cap \partial^* A) - \sum_{i=1}^{\infty} \mathscr{H}^{n-1}(\gamma_{t_i} \cap B_{\rho_i}(x_i));$$

therefore, in view of (8.8) and (8.9),

$$\mathscr{H}^{n-1}\Big(\Omega \cap \partial^* A \setminus \bigcup_{i \in \mathbf{N}} (\gamma_{t_i} \cap B_{\rho_i}(x_i))\Big) \leqq 8(n-1)\mathscr{H}^{n-1}(\gamma)\sqrt{\sigma}.$$

This and (8.10) yield that

$$\int_{S_{u_\sigma} \cap \Omega \cap \partial^* A} g_{\hom}(u_\sigma^+ - u_\sigma^-, \nu_{u_\sigma})\, d\mathscr{H}^{n-1} \leqq (1+\sigma) \int_{\gamma \cap G} g_{\hom}(u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1} + c\sqrt{\sigma}$$

for a suitable $c > 0$ independent of σ. Then

$$\begin{aligned} F_{\hom}(u_\sigma, \Omega \cup \gamma) \leqq & \int_\Omega f_{\hom}(\nabla u)\, dx + \int_{S_u \cap \Omega} g_{\hom}(u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1} \\ & + \int_{V_\sigma} f_{\hom}(\nabla \phi)\, dx + \int_{S_\phi \cap V_\sigma} g_{\hom}(\phi^+ - \phi^-, \nu_u)\, d\mathscr{H}^{n-1} \\ & + (1+\sigma) \int_{\gamma \cap G} g_{\hom}(u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1} + c\sqrt{\sigma}; \end{aligned}$$

hence

$$\begin{aligned} \limsup_{\sigma \to 0} F_{\hom}(u_\sigma, \Omega \cup \gamma) \leqq & \int_\Omega f_{\hom}(\nabla u)\, dx + \int_{S_u \cap \Omega} g_{\hom}(u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1} \\ & + \int_{\gamma \cap G} g_{\hom}(u^+ - u^-, \nu_u)\, d\mathscr{H}^{n-1}. \end{aligned}$$

We conclude the proof by using the arbitrariness of the open set G containing E. □

Proof of Theorem 8.1. *Let* $u \in \mathscr{D}_\phi$, and let (u_h) be a sequence in $\mathscr{D}_\phi$ which converges to u in $L^1(\Omega'; \mathbf{R}^m)$. Then, by Theorem 2.3,

$$\begin{aligned} F^\phi_{\hom}(u) \leqq F_{\hom}(u, \Omega') & \leqq \liminf_{h \to +\infty} F_{\varepsilon_h}(u_h, \Omega') \\ & \leqq \liminf_{h \to +\infty} F^\phi_{\varepsilon_h}(u_h) + c_2 \int_{\Omega' \setminus \bar{\Omega}} (1 + |\nabla \phi|^p)\, dx \\ & \quad + c_4 \int_{S_\phi \cap (\Omega' \setminus \bar{\Omega})} (1 + |\phi^+ - \phi^-|)\, d\mathscr{H}^{n-1}. \end{aligned}$$

Note that the values of $F^\phi_{\varepsilon_h}$ and $F_{\hom}$ do not depend on Ω', provided $\Omega' \cap \partial\Omega$ remains unchanged. Therefore, we can let Ω' shrink to Ω, thus obtaining

$$F^\phi_{\hom}(u) \leqq \liminf_{h \to +\infty} F^\phi_{\varepsilon_h}(u_h).$$

It remains to prove that for every $u \in \mathscr{D}_\phi$ we can find a sequence (u_h) in $\mathscr{D}_\phi$ which converges to u in $L^1(\Omega'; \mathbf{R}^m)$, and

$$\limsup_{h \to +\infty} F_{\varepsilon_h}(u_h, \Omega \cap \gamma) \leqq F_{\hom}(u, \Omega \cup \gamma).$$

For the moment we assume that $u \in L^{\infty}(\Omega'; \mathbf{R}^m)$. Then we can consider the functions $u_\sigma (\sigma > 0)$ given by Lemma 8.3. By (i) and (iii) of this lemma, it is not difficult to realize that it suffices to show that

(8.11)
$$F_{\text{hom}}(u_\sigma, \Omega \cup \gamma) \geqq \inf \left\{ \limsup_{h \to +\infty} F_{\varepsilon_h}(u_h, \Omega \cup \gamma) : (u_h) \text{ in } \mathscr{D}_\phi,\, u_h \to u_\sigma \text{ in } L^1(\Omega'; \mathbf{R}^m) \right\}.$$

Since $(F_{\varepsilon_h}(\cdot, \Omega))$ Γ-converges to $F_{\text{hom}}(\cdot, \Omega)$, for every $\sigma > 0$, there is a sequence (v_h) in $SBV^p(\Omega; \mathbf{R}^m) \cap L^p(\Omega; \mathbf{R}^m)$ converging to u_σ in $L^p(\Omega; \mathbf{R}^m)$ and such that

$$\lim_{h \to +\infty} F_{\varepsilon_h}(v_h, \Omega) = F_{\text{hom}}(u_\sigma, \Omega).$$

For every $\eta > 0$ let K_η be a compact subset of Ω with the property that

(8.12)
$$\int_{\Omega \setminus K_\eta} (1 + |\nabla u_\sigma|^p)\, dx < \eta, \quad \mathscr{H}^{n-1}(S_{u_\sigma} \cap (\Omega \cup \gamma) \setminus K_\eta) < \eta;$$

this is possible since $\mathscr{H}^{n-1}(S_{u_\sigma} \cap \gamma) = 0$. Now we join v_h and u_σ by means of Proposition 3.1 with $A'' = \Omega$, $A' \subset\subset \Omega$ a neighbourhood of K_η and $B = \Omega' \setminus K_\eta$. Then we get a sequence (w_h) in $\mathscr{D}_\phi$ which converges to u_σ in $L^p(\Omega'; \mathbf{R}^m)$ and satisfies the inequality

$$F_{\varepsilon_h}(w_h, \Omega') \leqq (1 + \eta)[F_{\varepsilon_h}(v_h, \Omega) + F_{\varepsilon_h}(u_\sigma, \Omega' \setminus K_\eta)] + M \| v_h - u_\sigma \|_{L^p(\Omega; \mathbf{R}^m)} + \eta$$

for a suitable constant M independent of h. Since $w_h = u_\sigma = \phi$ on $\Omega' \setminus \bar{\Omega}$, we have

$$F_{\varepsilon_h}(w_h, \Omega \cup \gamma) \leqq (1 + \eta)[F_{\varepsilon_h}(v_h, \Omega) + F_{\varepsilon_h}(u_\sigma, (\Omega \cup \gamma) \setminus K_\eta] + \eta F_{\varepsilon_h}(\phi, \Omega' \setminus \bar{\Omega})$$
$$+ M \| v_h - u_\sigma \|_{L^p(\Omega; \mathbf{R}^m)} + \eta,$$

and, denoting by I the right-hand side of (8.11), by (8.12) we have

$$I \leqq \limsup_{h \to +\infty} F_{\varepsilon_h}(w_h, \Omega \cup \gamma) \leqq (1 + \eta)(F_{\text{hom}}(u_\sigma, \Omega) + c\eta) + \eta \left(\limsup_{h \to +\infty} F_{\varepsilon_h}(\phi, \Omega' \setminus \bar{\Omega}) + 1 \right),$$

for a suitable constant c depending only on $\| u_\sigma \|_{L^\infty(\Omega'; \mathbf{R}^m)}$. We can now let η tend to 0, obtaining $I \leqq F_{\text{hom}}(u_\sigma, \Omega) \leqq F_{\text{hom}}(u_\sigma, \Omega \cup \gamma)$; i.e., (8.11).

Finally, consider the general case; $u \in \mathscr{D}_\phi$. For every $R > 0$ let $\varphi_R : \mathbf{R}^m \to \mathbf{R}^m$ be a Lipschitz function with compact support and Lipschitz constant less than or equal to 1, and such that $\varphi_R(y) = y$ when $|y| \leqq R$. We have $\lim_{R \to +\infty} F_{\text{hom}}(\varphi_R(u), \Omega \cup \gamma) = F_{\text{hom}}(u, \Omega \cup \gamma)$ (recall (7.19)); moreover, if $R \geqq \| \phi \|_{L^\infty}$, then $\varphi_R(u) \in \mathscr{D}_\phi$. An easy diagonalization argument yields the conclusion. □

We turn now to the application of the above results to functionals whose surface energy densities satisfy Griffith's growth conditions. More precisely, we consider a Borel function $g : \mathbf{R}^n \times \mathbf{R}^m \times S^{n-1} \to [0, +\infty[$ such that

(i) $g(x, s, v) = g(x, -s, -v)$ for every $(x, s, v) \in \mathbf{R}^n \times \mathbf{R}^m \times S^{n-1}$,
(ii) $g(\cdot, s, v)$ is 1-periodic for every $(s, v) \in \mathbf{R}^m \times S^{n-1}$,
(iii) there exists a function $\omega : [0, +\infty[\to [0, +\infty[$ continuous and non-decreasing, such that $\omega(0) = 0$ and

$$|g(x, s, v) - g(x, t, v)| \leqq \omega(|s - t|) \quad \text{for every } x \in \mathbf{R}^n,\, s, t \in \mathbf{R}^m,\, v \in S^{n-1},$$

(iv) there exist two positive constants α and β such that

$$\alpha \leqq g(x, s, \nu) \leqq \beta \quad \text{for every } (x, s, \nu) \in \mathbf{R}^n \times \mathbf{R}^m \times S^{n-1}.$$

Note that these assumptions on g differ from those of Section 2 only in the growth conditions.

We also consider a function $f: \mathbf{R}^n \times M^{m \times n} \to [0, +\infty[$, which represents the bulk energy density and satisfies the same properties as in Section 2. The function f gives rise to the function f_{hom} according to Proposition 2.1. Moreover, under the present growth conditions on g, Steps 1 through 5 in the proof of Proposition 2.2 remain unchanged and Step 7 still works with minor modifications. Thus, for every $(z, \nu) \in \mathbf{R}^m \times S^{n-1}$ there exists

$$g_{\text{hom}}(z, \nu) = \liminf_{\varepsilon \to 0} \left\{ \int_{S_u \cap Q_\nu} g(x/\varepsilon, u^+ - u^-, \nu_u) \, d\mathcal{H}^{n-1} : \right.$$
$$\left. u \in SBV(Q_\nu; \mathbf{R}^m), \nabla u = 0, \text{ a.e.}, u = u_{z,\nu} \text{ on } \partial Q_\nu \right\},$$

where Q_ν is any unit cube in $\mathbf{R}^n$ with centre at the origin and one face orthogonal to ν, and

$$\alpha \leqq g_{\text{hom}}(z, \nu) \leqq \beta.$$

For every $u \in \mathcal{D}_\phi$ and $\varepsilon > 0$ define

$$F_\varepsilon(u) = \int_\Omega f(x/\varepsilon, \nabla u) \, dx + \int_{S_u \cap (\Omega \cup \gamma)} g(x/\varepsilon, u^+ - u^-, \nu_u) \, d\mathcal{H}^{n-1},$$

$$F_{\text{hom}}(u) = \int_\Omega f_{\text{hom}}(\nabla u) \, dx + \int_{S_u \cap (\Omega \cup \gamma)} g_{\text{hom}}(u^+ - u^-, \nu_u) \, d\mathcal{H}^{n-1}.$$

Theorem 8.4. *Let (ε_h) be a sequence of positive number tending to* 0. *Assume that there exists a sequence (u_h) in $\mathcal{D}_\phi$ such that*

$$\lim_{h \to +\infty} \left(F_{\varepsilon_h}(u_h) - \inf_{\mathcal{D}_\phi} F_{\varepsilon_h} \right) = 0, \quad \sup_{h \in \mathbf{N}} \| u_h \|_{L^\infty(\Omega; \mathbf{R}^m)} < +\infty.$$

Then the functional F_{hom} attains its infimum on $\mathcal{D}_\phi$ and $\min_{\mathcal{D}_\phi} F_{\text{hom}} = \lim_{h \to +\infty} \inf_{\mathcal{D}_\phi} F_{\varepsilon_h}$. Moreover, there exists a subsequence $(u_{\sigma(h)})$ of (u_h) which converges in $L^1(\Omega; \mathbf{R}^m)$ to a minimizer of F_{hom} on $\mathcal{D}_\phi$.

Proof. For every $\varepsilon > 0$, $j \in \mathbf{N}$ define

$$F_\varepsilon(u, B) = \int_B f(x/\varepsilon, \nabla u) \, dx + \int_{S_u \cap B} g(x/\varepsilon, u^+ - u^-, \nu_u) \, d\mathcal{H}^{n-1},$$

$$F_\varepsilon^j(u, B) = F_\varepsilon(u, B) + \int_{S_u \cap B} (|u^+ - u^-| - 2j) \vee 0 \; d\mathcal{H}^{n-1},$$

$$F_{\text{hom}}^j(u, B) = \int_B f_{\text{hom}}(\nabla u) \, dx + \int_{S_u \cap B} g_{\text{hom}}^j(u^+ - u^-, \nu_u) \, d\mathcal{H}^{n-1}$$

whenever $A \in \mathscr{A}$, $u \in SBV^p(A; \mathbf{R}^m)$ and $B \in \mathscr{B}(A)$, where (see Proposition 2.2)

$$g^j_{\mathrm{hom}}(z, v) = \lim_{\varepsilon \to 0} \inf \left\{ \int_{S_u \cap Q_v} (g(x/\varepsilon, u^+ - u^-, v_u) + (|u^+ - u^-| - 2j) \vee 0) \, d\mathscr{H}^{n-1} : u \in SBV(Q_v; \mathbf{R}^m), \nabla u = 0 \text{ a.e.}, u = u_{z,v} \text{ on } \partial Q_v \right\}$$

for every $(z, v) \in \mathbf{R}^m \times S^{n-1}$.

Let us show that for every $(z, v) \in \mathbf{R}^m \times S^{n-1}$,

$$\lim_{j \to +\infty} g^j_{\mathrm{hom}}(z, v) = g_{\mathrm{hom}}(z, v). \tag{8.13}$$

Let $(z, v) \in \mathbf{R}^m \times S^{n-1}$ be fixed. For every $\sigma > 0$ there exists $h_\sigma \in \mathbf{N}$ and $u_\sigma \in SBV(Q_v; \mathbf{R}^m)$ such that $\nabla u_\sigma = 0$ a.e., $u_\sigma = u_{z,v}$ on ∂Q_v and

$$\int_{S_{u_\sigma} \cap Q_v} g(h_\sigma x, u_\sigma^+ - u_\sigma^-, v_{u_\sigma}) \, d\mathscr{H}^{n-1} < g_{\mathrm{hom}}(z, v) + \sigma. \tag{8.14}$$

Let $\varphi_R : \mathbf{R}^m \to \mathbf{R}^m$ be a Lipschitz function with compact support and such that $\varphi_R(y) = y$ if $|y| \leqq R$. Then (8.14) still holds with u_σ replaced by $\varphi(u_\sigma)$, provided R is sufficiently large. Thus we may assume that u_σ is bounded.

Consider u_σ extended with value $u_{z,v}$ on $\{x \in \mathbf{R}^n : |\langle x, v \rangle| \geqq \frac{1}{2}\}$ and by periodicity on $\{x \in \mathbf{R}^n : |\langle x, v \rangle| < \frac{1}{2}\}$. Then it is possible to define

$$u^\sigma_k(x) = u_\sigma(kx), \quad x \in \mathbf{R}^n.$$

Clearly $u^\sigma_k \in SBV(Q_v; \mathbf{R}^m)$, $\nabla u^\sigma_k = 0$ a.e. on Q_v, $u^\sigma_k = u_{z,v}$ on ∂Q_v and $\| u^\sigma_k \|_{L^\infty(Q_v; \mathbf{R}^m)} = \| u_\sigma \|_{L^\infty(Q_v; \mathbf{R}^m)}$. Moreover, by a change of variable, we have

$$\int_{S_{u^\sigma_k} \cap Q_v} g(kh_\sigma x, (u^\sigma_k)^+ - (u^\sigma_k)^-, v_{u^\sigma_k}) d\mathscr{H}^{n-1} = \int_{S_{u_\sigma} \cap Q_v} g(h_\sigma x, u_\sigma^+ - u_\sigma^-, v_{u_\sigma}) d\mathscr{H}^{n-1}.$$

Therefore, for every $j \geqq \| u_\sigma \|_{L^\infty(Q_v; \mathbf{R}^m)}$,

$$\begin{aligned} g^j_{\mathrm{hom}}(z, v) &= \lim_{k \to +\infty} \inf \left\{ \int_{S_u \cap Q_v} (g(kh_\sigma x, u^+ - u^-, v_u) + (|u^+ - u^-| - 2j) \vee 0) \, d\mathscr{H}^{n-1} : u \in SBV(Q_v; \mathbf{R}^m), \nabla u = 0 \text{ a.e.}, u = u_{z,v} \text{ on } \partial Q_v \right\} \\ &\leqq \limsup_{k \to +\infty} \int_{S_{u^\sigma_k}} g(kh_\sigma x, (u^\sigma_k)^+ - (u^\sigma_k)^-, v_{u^\sigma_k}) \, d\mathscr{H}^{n-1} \leqq g_{\mathrm{hom}}(z, v) + \sigma. \end{aligned}$$

Since $g^j_{\mathrm{hom}} \geqq g_{\mathrm{hom}}$, we obtain (8.13).

We now prove that (u_h) has a subsequence which converges in L^1 to a minimizer of F_{hom}. Note that

$$\begin{aligned} \limsup_{h \to +\infty} F_{\varepsilon_h}(u_h, \Omega') &\leqq \limsup_{h \to +\infty} (F_{\varepsilon_h}(u_h, \Omega \cup \gamma) + F_{\varepsilon_h}(\phi, \Omega' \setminus \bar{\Omega})) \\ &\leqq \limsup_{h \to +\infty} \left(\inf_{\mathscr{D}_\phi} F_{\varepsilon_h}(\cdot, \Omega \cup \gamma) + F_{\varepsilon_h}(\phi, \Omega' \setminus \bar{\Omega}) \right) \\ &\leqq \limsup_{h \to +\infty} F_{\varepsilon_h}(\phi, \Omega') \leqq K \left(\int_{\Omega'} (1 + |\nabla \phi|^p) \, dx + \mathscr{H}^{n-1}(S_\phi \cap \Omega') \right) \end{aligned}$$

for a suitable constant K. Hence, from the growth conditions of f and g from below we obtain the boundedness of the sequence $(\int_{\Omega'} |\nabla u_h|^p \, dx + \mathscr{H}^{n-1}(S_{u_h} \cap \Omega'))_h$. Since (u_h) is equibounded, we can apply AMBROSIO's compactness and lower semicontinuity theorems ([4, 6]), deducing the existence of a subsequence $(u_{\sigma(h)})$ of (u_h) which converges in $L^1(\Omega'; \mathbf{R}^m)$ to a function $u_0 \in SBV^p(\Omega'; \mathbf{R}^m)$. Clearly $u_0 \in \mathscr{D}_\phi$. Moreover, for every $j \geqq \sup_h \| u_h \|_{L^\infty(\Omega; \mathbf{R}^m)} \vee \| \phi \|_{L^\infty(\Omega'; \mathbf{R}^m)}$

$$\int_{\Omega'} f_{\mathrm{hom}}(\nabla u_0) \, dx + \int_{S_u \cap \Omega'} g_{\mathrm{hom}}(u_0^+ - u_0^-, \nu_{u_0}) \, d\mathscr{H}^{n-1}$$
$$\leqq F^j_{\mathrm{hom}}(u_0, \Omega') \leqq \liminf_{h \to +\infty} F_{\varepsilon_{\sigma(h)}}(u_{\varepsilon_{\sigma(h)}}, \Omega').$$

Therefore, if we set

$$c_{\Omega'} = \int_{\Omega' \setminus \bar{\Omega}} f_{\mathrm{hom}}(\nabla \phi) \, dx + \int_{S_\phi \cap (\Omega' \setminus \bar{\Omega})} g_{\mathrm{hom}}(\phi^+ - \phi^-, \nu_\phi) \, d\mathscr{H}^{n-1} - \limsup_{h \to +\infty} F_{\varepsilon_h}(\phi, \Omega' \setminus \bar{\Omega}),$$

we obtain

$$F_{\mathrm{hom}}(u_0) + c_{\Omega'} \leqq \liminf_{h \to +\infty} \inf_{\mathscr{D}_\phi} F_{\varepsilon_{\sigma(h)}}.$$

Note that for every $\varepsilon > 0$ the value $\inf_{\mathscr{D}_\phi} F_\varepsilon$ is clearly independent of the choice of Ω', provided $\Omega' \cap \partial\Omega$ remains unchanged. By means of a diagonalization argument we can make the subsequence $(\varepsilon_{\sigma(h)})$ independent of Ω' if the latter is taken in a sequence (Ω'_n) which shrinks to Ω. Thus we obtain

$$F_{\mathrm{hom}}(u_0) \leqq \liminf_{h \to +\infty} \inf_{\mathscr{D}_\phi} F_{\varepsilon(h)}. \tag{8.15}$$

On the other hand, for every fixed $u \in \mathscr{D}_\phi$, by Theorem 8.1 applied to the sequence $(F^j_{\varepsilon_h})_h$, there exists a sequence (v_h) in $\mathscr{D}_\phi$ such that

$$\lim_{h \to +\infty} F^j_{\varepsilon_h}(v_h, \Omega \cup \gamma) = F^j_{\mathrm{hom}}(u, \Omega \cup \gamma).$$

Therefore

$$F^j_{\mathrm{hom}}(u, \Omega \cup \gamma) \geqq \limsup_{h \to +\infty} F_{\varepsilon_h}(v_h, \Omega \cup \gamma) \geqq \limsup_{h \to +\infty} \mathrm{lin}_{\mathscr{D}_\phi} F_{\varepsilon_h}.$$

Let j tend to $+\infty$ and apply (8.13); then

$$F_{\mathrm{hom}}(u) \geqq \limsup_{h \to +\infty} \inf_{\mathscr{D}_\phi} F_{\varepsilon_h}.$$

This, together with (8.15), immediately implies that u_0 is a minimizer of F_{hom} and that the subsequence $(\inf_{\mathscr{D}_\phi} F_{\varepsilon_{\sigma(h)}})$, and hence the whole sequence $(\inf_{\mathscr{D}_\phi} F_{\varepsilon_{\sigma(h)}})$, converges to $\min_{\mathscr{D}_\phi} F_{\mathrm{hom}}$. □

Example 8.5 (Homogenization of a composite medium with a chessboard structure). We illustrate the homogenization process by a simple example, which shows how the bulk and surface energy behaviours can differ. Let $m = 1$, $n = 2$, and let $a:\mathbf{R}^2 \to [0, +\infty[$ be the 1-periodic function defined on $[0, 1[^2$ by

$$a(x_1, x_2) = \begin{cases} 1 & \text{if } |x_1| \vee |x_2| \leqq \frac{1}{2} \text{ or } |x_1| \wedge |x_2| \geqq \frac{1}{2}, \\ 2 & \text{otherwise.} \end{cases}$$

Consider the functionals

$$F_\varepsilon(u, \Omega) = \int_\Omega a\left(\frac{x}{\varepsilon}\right) |\nabla u|^2 \, dx + \int_{\Omega \cap S_u} a\left(\frac{x}{\varepsilon}\right) d\mathscr{H}^{n-1},$$

i.e., we take $f(x, \xi) = a(x)|\xi|^2$, and $g(x, s, v) = a(x)$. The functional F_ε represents a composite, with a chessboard structure of mesh size ε, of two isotropic materials whose local energies are

$$j\left(\int_A |\nabla u|^2 \, dx + \mathscr{H}^{n-1}(A \cap S_u)\right)$$

with $j = 1$ and 2, respectively. The homogenized functional of (F_ε) is

$$F_{\text{hom}}(u, \Omega) = \sqrt{2} \int_\Omega |\nabla u|^2 \, dx + \int_{\Omega \cap S_u} \psi(v) \, d\mathscr{H}^{n-1},$$

where the function $\psi: S^1 \to [0, +\infty)$ is given by

$$\psi(v) = (\sqrt{2} - 1) \min\{|v_1|, |v_2|\} + \max\{|v_1|, |v_2|\}.$$

In fact, it is proved in [41] (see also [45 Proposition 3.1]) that the homogenization of $\int_\Omega a(x/\varepsilon)|\nabla u|^2 \, dx$ in $W^{1,2}(\Omega)$ gives $\sqrt{2}\int_\Omega |\nabla u|^2 \, dx$, while the formula for ψ is obtained from [22, Example 5.2], once we note that in the definition of g_{hom} we can restrict to the case of u taking values in $\{0, 1\}$. Note the loss of isotropy in the surface term.

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