Numerical Algorithms 10(1995)187–202

Gaussian integration of Chebyshev polynomials and analytic functions

Knut Petras

Institut für Angewandte Mathematik, Technische Universität Braunschweig, Pockelsstr. 14, D-38106 Braunschweig, Germany

E-mail: k.petras@tu-bs.de

Dedicated to Luigi Gatteschi on the occasion of his 70th birthday

Explicit bounds for the quadrature error of the nth Gauss-Legendre quadrature rule applied to the mth Chebyshev polynomial are derived. They are precise up to the order $O(m^4n^{-6})$. As an application, error constants for classes of functions which are analytic in the interior of an ellipse are estimated. The location of the maxima of the corresponding kernel function is investigated.

Keywords: Gaussian integration, Chebyshev polynomials, error bounds, analytic functions.

AMS subject classification: 65D30, 41A55.

1. Introduction

Chebyshev expansions are very useful tools for numerical analysis. Their convergence is guaranteed under rather general conditions, they often converge fast compared with other polynomial expansions, and each summand of the series may easily be estimated. Considering functionals on certain function spaces it is therefore important to know, how they operate on the Chebyshev polynomials T_m of the first kind, or on the Chebyshev polynomials U_m of the second kind.

In this paper, we analyse the errors of the Gauss-Legendre quadrature formula Q_n^G,

$$Q_n^G[f] = \sum_{\nu=1}^{n} a_\nu f(x_\nu).$$

This formula is the quadrature formula with n nodes x_ν and weights a_ν, which is defined uniquely by having the error

$$R_n^G[p] = \int_{-1}^{1} p(x)\,dx - Q_n^G[p] = 0$$

for all polynomials p of degree less than $2n$. Its error for Chebyshev polynomials has been considered by several authors. Explicit expressions for the first non-vanishing errors, i.e. $R_n^G[T_{2n}]$, $R_n^G[T_{2n+2}]$ and $R_n^G[T_{2n+4}]$, as well as an asymptotic result for $R_n[T_{2n+k}]$, where k is arbitrary but fixed, have already been presented by Nicholson et al. [16]. We recall some refinements of these early results. The first nonvanishing errors $R_n^G[T_{2n+2l}]$ are known explicitly for $l = 0, 1, \ldots, 5$ (cf. Stegen [18, p. 107]), where

$$R_n^G[T_{2n}] = d_n = \frac{2^{4n} n!^4}{(2n)!^2(2n+1)} = \frac{\pi}{2}\left(1 - \frac{1}{4n} + O(n^{-2})\right) \tag{1}$$

and

$$R_n^G[T_{2n+2}] = -d_n\left(1 + \frac{2n+1}{(2n-1)(2n+3)}\right). \tag{2}$$

If $l \geqslant 2$ is fixed and n is increasing, then,

$$R_n^G[T_{2n+2l}] = \frac{\pi}{16n^2}(1 + O(n^{-1}))$$

(see Stegen [18, p. 50]). Curtis and Rabinowitz [7] pointed out that for fixed k,

$$R_n^G[T_{(4n+2)k\pm 2j}] = \begin{cases} \dfrac{2(-1)^k}{4j^2-1} + O(n^{-1}\ln n) & \text{for } j = 1, \ldots, n-1 \text{ and} \\[2ex] (-1)^k \dfrac{\pi}{2} + O(n^{-1}\ln n) & \text{for } j = n. \end{cases} \tag{3}$$

Of course, if we have an odd function, then the Gaussian error is zero, i.e., $R_n^G[T_{2m-1}] = R_n^G[U_{2m-1}] = 0$. Furthermore, we have the simple rough bound

$$|R_n^G[T_m]| \leqslant 2 + \frac{2}{m^2-1} \quad \text{for all } m \geqslant 2$$

(cf. Brass [4, p. 161]).

We will add explicit error bounds, which are of asymptotic precision $O(m^4 n^{-6})$, i.e., which allow an appropriate estimation at least for Chebyshev polynomials of degree $o(n^{3/2})$. Some consequences are the inequality $|R_n^G[T_m]| \leqslant 3n^{-2}$ for $2n+4 \leqslant m \leqslant 3n+1$ as well as a further asymptotic term in equation (3) including explicit bounds.

The method for obtaining these results is based on inequalities for the nodes and weights of the Gaussian formulae derived by Gatteschi [10] as well as Förster and Petras [9].

There are several possibilities to use the obtained bounds. We may apply them directly to the Chebyshev expansion of the integrand (see Davis and Rabinowitz [8, pp. 335 f.]), or we may approximate Peano kernels via a method of Brass and Förster [5] and therefore determine error constants for functions having derivatives of certain orders. In this paper, we consider the class of those functions, which are analytic in the interior of an ellipse with foci -1 and 1, and continuous on its closure. Explicit asymptotically sharp bounds for the related kernel function are proved, yielding almost best possible bounds for the quadrature error.

2. The error of Gaussian quadrature for Chebyshev polynomials

We give an estimate, which is simpler than that of the underlying lemma 2 below, but nevertheless almost as sharp as the lemma.

Theorem 1
Let R_n^G denote the error functional of the Gauss-Legendre rule involving n nodes. Furthermore, let $m = k(4n+2) + 2j$ with $|j| < n$ and an appropriate integer $k \geqslant 1$. Then,

$$R_n^G[T_m] = \frac{2}{m^2-1} + \frac{2(-1)^k}{4j^2-1}\left(1 + \frac{mj}{4N^2}\right) + \rho_{n,m}, \tag{5}$$

where $N = n + 1/2$ and

$$|\rho_{n,m}| \leqslant (0.08 + 0.004 \ln N)\frac{m^4}{N^6}.$$

If $m = (2k-1)(2n+1) \pm 1$, we have to add

$$\pm(-1)^k \frac{\pi}{2}\left(1 \pm \frac{m}{8N^2} + \frac{m^2}{128N^4}\right)$$

on the right-hand side of (5).

The logarithmic summand in the estimate for $\rho_{n,m}$ may be omitted if we add the logarithmic term of the formula (10) below on the right-hand side of (5).

Note that under the assumptions of this theorem, the errors of the Gaussian rules are similar to those of the Clenshaw-Curtis rules (cf. Brass [4, p. 145]). Furthermore, we see that the theorem is an extension of the result of Curtis and Rabinowitz (3).

More helpful for rapidly convergent Chebyshev expansions is the following result.

Theorem 2
For $2n + 4 \leqslant m \leqslant 3n + 1$, we have that

$$|R_n^G[T_m]| \leqslant 3n^{-2}. \tag{6}$$

Now, we are in the position to give bounds for the quadrature errors for Chebyshev polynomials U_{2m} of the second kind. We use the relation

$$U_{2m} = 2\sum_{l=0}^{m} T_{2l} - 1$$

(cf. Abramowitz and Stegun [1, eq. 22.12.2]) and directly obtain the following corollary from the relations (1), (2) and (6). Various further estimates may be derived from theorem 1.

Corollary 1
Let $1 \leqslant j \leqslant (n+1)/2$, then,

$$R_n^G[U_{2n+2j}] = -\frac{(4n+2)d_n}{(2n-1)(2n+3)} + u_j,$$

where $|u_j| \leqslant 6(j-1)n^{-2}$.

3. The error of Gaussian quadrature for analytic functions

We want to apply theorem 2 to error constants for analytic functions. Let therefore C_r denote the interior of the ellipse with foci at ± 1 and with the lengths $(r+r^{-1})/2$ of its semi-axes, where $r > 1$. Let furthermore $\mathcal{A}_r^p$ be the class of all functions, which are analytic in C_r, continuous on its closure $\bar{C}_r$, and have there the norm

$$\left\{\int_{\partial C_r} |f(z)|^p |dz|\right\}^{1/p} \leqslant 1, \quad \text{where } 1 \leqslant p \leqslant \infty.$$

We define the error constant of a quadrature formula Q_n on a class $\mathcal{A}$ by

$$\varrho(Q_n, \mathcal{A}) = \sup\{|R_n[f]| : f \in \mathcal{A}\}.$$

From the representation

$$R_n^G[f] = \frac{1}{2\pi i}\int_{\partial C_r} f(z) k_n(z)\, dz \quad \text{where } k_n(z) = R_n^G\left[\frac{1}{z - \cdot}\right],$$

which holds for all $f \in \mathcal{A}_r^p$, we obtain the following estimate for the error constants,

$$\varrho(Q_n^G, \mathcal{A}_r^r) \leqslant \left\{\frac{1}{2\pi}\int_{\partial C_r} |k_n(z)|^q |dz|\right\}^{1/q} =: \tilde{\varrho}(Q_n^G, \mathcal{A}_r^p), \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

It is therefore convenient to analyse the behaviour of the kernel function k_n on the boundary of the ellipse C_r. The argument of R_n^G in the definition k_n is a function, which has a simple Chebyshev expansion. The results of section 2 may therefore be applied usefully.

Theorem 3
Let $2z = w + w^{-1}$, let d_n be defined as in (1) and let $c_n = (2n+1)/[(2n-1)(2n+3)]$.

(a)

$$k_n(z) = \frac{4d_n}{w^{2n+1}}\left(1 - \frac{c_n}{w^2 - 1} + \delta_n(z)\right),$$

where

$$|\delta_n(z)| \leqslant \frac{9n^{-2} + 6r^{2-n}}{4|w^2 - 1|(r^2 - 1)}.$$

(b) If $c_n \leqslant r^4 - 1$, we have

$$\tilde{\varrho}(Q_n^G, \mathcal{A}_r^1) = \frac{2d_n}{\pi r^{2n+1}}\left(1 + \frac{c_n}{r^2+1} + \gamma_n\right) < \frac{1}{r^{2n+1}}\left(1 + \frac{c_n}{r^2+1} + \gamma_n\right),$$

where

$$|\gamma_n| \leqslant \frac{9n^{-2} + 6r^{2-n}}{4(r^2 - 1)^2}.$$

Remark
Note that the sequence of the values d_n is monotonically increasing and we have

$$d_n = \frac{\pi}{2}\left(1 - \frac{1}{4n} + O(n^{-2})\right) \quad \text{and} \quad \frac{\pi}{2}\left(1 - \frac{1}{4n}\right) < d_n < \frac{\pi}{2}.$$

Gautschi and Varga [11] investigated the location of the points $z_{n,r} \in \partial\mathcal{C}_r$, where the modulus of the kernel function k_n attains its maximum on the boundary of the ellipse $\mathcal{C}_r$. Their numerical calculations indicated that $z_{n,r}$ is located on, or in some cases at least near the imaginary axis. The bounds calculated above cannot yield that $z_{n,r}$ lies exactly on the imaginary axis. However, the following corollary gives a slightly weaker result.

Corollary 2
Let r be fixed and let $|k_n(z_{n,r})| = \sup_{z \in \partial\mathcal{C}_r} |k_n(z)|$ with $z_{n,r} \in \partial\mathcal{C}_r$, then,

$$\operatorname{Re} z_{n,r} = O(n^{-1/2}).$$

The behaviour of $\varrho(Q_n^G, \mathcal{A}_r^\infty)$ has been studied more intensively. The simple universal bound

$$\varrho(Q_n^G, \mathcal{A}_r^\infty) \leqslant \frac{32}{\pi r^{2n}}$$

(cf. Stenger [17]) is a ready consequence of a result of Achieser [2, sect. 111] on the approximation error for analytic functions. Kambo's bound [13],

$$\varrho(Q_n^G, \mathcal{A}_r^\infty) \leqslant \frac{d_n}{r^{2n}} \frac{r^2+1}{r^2-2} \quad \text{for } r > 2$$

gives a better estimate if $r > (64 + \pi^2)^{1/2}(32 - \pi^2)^{-1/2} = 1.8269\ldots$. A result of Chawla and Jain [6] says that for each r and each $\epsilon > 0$,

$$\varrho(Q_n^G, \mathcal{A}_r^\infty) \leqslant \frac{2l(\partial\mathcal{C}_r)}{\pi(r - r^{-1})r^{2n}}\left(1 - \frac{\epsilon}{r}\right)^{-n},$$

if we choose n sufficiently large. Here, we denote by $l(\partial C_r)$ the length of the boundary of the ellipse C_r. It is not useful to compate this result with the previous bounds, since the factor $(1-\epsilon/r)^{-n}$ may become arbitrarily large.

Theorem 4

Let d_n be defined as in (1) and set $c_n = (2n+1)/((2n-1)(2n+3))$. Denote by $l(\partial C_r) \in (\pi r, \min\{4r, \pi(r+r^{-1})\})$ the length of the boundary of C_r, then,

$$\varrho(Q_n^G, \mathcal{A}_r^\infty) \leqslant \frac{2d_n l(\partial C_r)}{\pi r^{2n+1}} \left(1 + \frac{3}{2nr^2} + \frac{5}{r^{n+1}}\right), \tag{7}$$

$$\begin{aligned}\varrho(Q_n^G, \mathcal{A}_r^\infty) &\leqslant \frac{l(\partial C_r)}{r^{2n+1}} \left(1 + \frac{3}{2nr^2} + \frac{9}{2r^{n+1}}\right)\\ &\leqslant \frac{\pi}{r^{2n}} \left(1 + \frac{1}{r^2}\right)\left(1 + \frac{3}{2nr^2} + \frac{9}{2r^{n+1}}\right).\end{aligned}$$

$$\varrho(Q_n^G, \mathcal{A}_r^\infty) \leqslant \frac{4}{r^{2n}} \left(1 + \frac{3}{2nr^2} + \frac{4}{r^{n+1}}\right). \tag{8}$$

I thank Helmut Brass for a useful hint, which enabled an upper bound without any factor $(r^2-1)^{-1}$. This factor would have destroyed the uniform quality of the bound.

Remark

(a) If $n \geqslant 3$ and if Stenger's bound is less than $(1/2\pi^3)(16-3\pi)^2 = 0.6971\ldots$, then the bound (8) improves upon Stenger's. For $n \geqslant 5$ and arbitrary parameter r, the bound (7) is smaller than Kambo's.

(b) A lower bound for the error constant is given by

$$\varrho(Q_n^G, \mathcal{A}_r^\infty) \geqslant R_n^G\left[\frac{2r^{2n}}{r^{4n}+1} T_{2n}\right] = \frac{2d_n}{r^{2n}(1+r^{-4n})} \geqslant \frac{\pi(1-(4n)^{-1})}{r^{2n}(1+r^{-4n})}.$$

(c) By theorem 3a, the modulus of the kernel function is almost constant on the boundary of each ellipse C_r. This implies that theorem 3b gives better results than theorem 4 except for those functions whose modulus is closer to a constant function on the boundary of the ellipse C_r than $|k_n|$ is. As an example, consider the function $f(x) = (d^2+x^2)^{-1}$ with a real parameter $d \neq 0$. We have

$$\max_{z\in\partial C_r} |f(z)| = m(d,r) := \left(d^2 - \frac{1}{4}\left(r - \frac{1}{r}\right)^2\right)^{-1},$$

while

$$\int_{z\in\partial C_r} |f(z)|\,|dz| = O(\ln(m(d,r))),$$

which is asymptotically an essential difference when $i \cdot d$ approaches the ellipse C_r.

The same remark is valid for all spaces $\mathcal{A}_r^p$, where $1 < p < \infty$. Usually, the estimate with the error constant $\tilde{\varrho}(Q_n^G, \mathcal{A}_r^1)$ should asymptotically be superior.

A further observation concerns more general regions than ellipses. It is an intermediate consequence of theorem 3.

Corollary 3

Let $\mathcal{C}_r$ be a subset of the region Ω and let each ellipse $\mathcal{C}_\rho$ with $\rho > r$ contain inner points of the complement of Ω. Denote by $\mathcal{A}^p(\Omega)$ the space of functions being analytic on Ω and continuous on its closure, endowed with the L_p-norm taken on its boundary curve. Then,

$$\lim_{n \to \infty} \sqrt[n]{\varrho(Q_n^G, \mathcal{A}^p(\Omega))} = \frac{1}{r^2}.$$

The upper bound r^{-2} for this limit follows from theorem 3. To prove the lower bound, we just have to consider errors for the functions $(z - \cdot)^{-1}$ with z in the complement of Ω and arbitrarily close to the ellipse $\mathcal{C}_r$.

This corollary shows that ellipses $\mathcal{C}_r$ are the most appropriate complex regions for the error estimation of the Gauss-Legendre rule for analytic functions. An extension of the region of analyticity makes only sense with respect to error constants, if the new region contains a larger ellipse. This makes clear that regions such as small circles enclosing the basic intervals are not optimal for error estimation for the Gauss-Legendre rule (cf. Kowalski et al. [14]). It is an obvious conjecture that each quadrature formula has particularly favourable regions of analyticity. These might be the regions; where the modulus of the kernel function is almost constant. Note that, for example, the Gaussian rules are close to optimality on ellipses $\mathcal{C}_r$ with arbitrary parameter $r > 1$ (see theorem 4 and Bakhvalov [3]).

4. Proof of the results on Chebyshev polynomials

First, recall some known estimates for the nodes and weights of the Gauss-Legendre rule.

Defining $\phi_\nu = (\nu - 1/4)(\pi/N)$, $\psi_\nu = \phi_\nu + (\cot\phi_\nu)/(8N^2)$ and $x_\nu = -\cos\theta_\nu$, we have that

$$a_\nu = \frac{\pi}{N}\sin\phi_\nu\left(1 - \frac{1}{8N^2}\right)(1 + \epsilon_\nu),$$

where

$$-\frac{\cos^2\phi_\nu}{16N^4\sin^4\phi_\nu} \leqslant \epsilon_\nu \leqslant \frac{1}{2N^4\sin^4\phi_\nu},$$

and for $\nu \leqslant \lfloor (n+1)/2 \rfloor$ that

$$\theta_\nu = \psi_\nu - \delta_\nu, \quad \text{where } 0 \leqslant \delta_\nu \leqslant \frac{11\cos\phi_\nu}{128N^4\sin^3\phi_\nu}$$

(see Gatteschi [10] and Förster and Petras [9]).

In the following, we set $y_\nu = -\cos\phi_\nu$ and first prove two lemmata.

Lemma 1
For even $m \geqslant 2n$, we have

$$\begin{aligned} Q_n^G[T_m] = {} & \frac{\pi}{N}\sum_{\nu=1}^{n} \sin\phi_\nu \cos m\phi_\nu - \frac{\pi m}{8N^3}\sum_{\nu=1}^{n} \cos\phi_\nu \sin m\phi_\nu \\ & - \frac{\pi m^2}{128N^5}\sum_{\nu=1}^{n} \frac{\cos^2\phi_\nu \cos m\phi_\nu}{\sin\phi_\nu} + \varrho_{n,m}^{(1)}, \end{aligned}$$

where

$$N^2|\varrho_{n,m}^{(1)}| \leqslant \frac{2}{3} + \frac{m}{7N} + \frac{m^3}{1871N^3} + \frac{m^4}{180313N^4}.$$

Proof
We prove the lemma for $n \geqslant 2$. For $n = 1$, it may be verified easily. Since m is even, we have

$$\begin{aligned} & \sum_{\nu=1}^{n}\left(a_\nu T_m(x_\nu) - \frac{\pi}{N}\sin\phi_\nu T_m(y_\nu)\right) \\ & = \sum_{\nu=1}^{n}\left(a_\nu \cos m\theta_\nu - \frac{\pi}{N}\sin\phi_\nu \cos m\phi_\nu\right) \\ & = \sum_{\nu=1}^{n}\left(a_\nu - \frac{\pi}{N}\sin\phi_\nu\right)\cos m\theta_\nu + \frac{\pi}{N}\sum_{\nu=1}^{n}\sin\phi_\nu(\cos m\theta_\nu - \cos m\psi_\nu) \\ & \quad + \frac{\pi}{N}\sum_{\nu=1}^{n}\sin\phi_\nu(\cos m\psi_\nu - \cos m\phi_\nu) \\ & = I + II + III. \end{aligned}$$

We consider the three sums I, II and III on the right-hand side separately.

First sum: From

$$-\frac{\pi}{N}\sin\phi_\nu\left\{\frac{1}{8N^2} + \frac{\cos^2\phi_\nu}{16N^4\sin^4\phi_\nu}\right\} < a_\nu - \frac{\pi}{N}\sin\phi_\nu < \frac{\pi}{2N^5\sin^3\phi_\nu}$$

we obtain

$$\left| a_\nu - \frac{\pi}{N}\sin\phi_\nu \right| \leqslant \frac{\pi}{N}\sin\phi_\nu \left\{ \frac{1}{8N^2} + \frac{1}{2N^4\sin^4\phi_\nu} \right\}.$$

We have

$$\sum_{\nu=1}^{n} \sin\phi_\nu = \tfrac{1}{2}\csc\frac{\pi}{4N}$$

and, by the convexity of $\csc^3$,

$$\begin{aligned}\frac{\pi}{N}\sum_{\nu=1}^{n}\frac{1}{\sin^3\phi_\nu} &\leqslant \frac{2\pi}{N\sin^3\phi_1} + \int_{5\pi/(4N)}^{\pi-5\pi/(4N)} \frac{dx}{\sin^3 x}\\ &= \frac{2\pi}{N}\csc^3\frac{3\pi}{4N} + \cos\frac{5\pi}{4N}\csc^2\frac{5\pi}{4N} + \ln\cot\frac{5\pi}{8N}.\end{aligned}$$

Hence, we obtain the following upper bound,

$$\begin{aligned}|I| &< \frac{\pi}{N}\sum_{\nu=1}^{n}\sin\phi_\nu\left\{\frac{1}{8N^2} + \frac{1}{2N^4\sin^4\phi_\nu}\right\}\\ &< \frac{1}{N^2}\left\{\frac{\pi}{16N}\csc\frac{\pi}{4N} + \frac{\pi}{N^3}\csc^3\frac{3\pi}{4N} + \frac{1}{2N^2}\cos\frac{5\pi}{4N}\csc^2\frac{5\pi}{4N} + \frac{1}{2N^2}\ln\cot\frac{5\pi}{8N}\right\}.\end{aligned}$$

Second sum: The inequalities

$$|\cos m\theta_\nu - \cos m\psi_\nu| \leqslant m|\delta_\nu| \leqslant \frac{11m|\cos\phi_\nu|}{128N^4\sin^3\phi_\nu}$$

and the convexity of $|\cos|\cdot\csc^2$ yield

$$\begin{aligned}|II| &\leqslant \frac{11m}{128N^4}\left\{\frac{2\pi}{N}\frac{\cos\phi_1}{\sin^2\phi_1} + \frac{\pi}{N}\sum_{\nu=2}^{n-1}\frac{|\cos\phi_\nu|}{\sin^2\phi_\nu}\right\}\\ &\leqslant \frac{11m}{128N^4}\left\{\frac{2\pi}{N}\frac{\cos\phi_1}{\sin^2\phi_1} + 2\int_{5\pi/(4N)}^{\pi/2}\frac{\cos x}{\sin^2 x}dx\right\}\\ &= \frac{1}{N^2}\left\{\frac{11m}{64N}\left[\frac{\pi}{N^2}\cos\frac{3\pi}{4N}\csc^2\frac{3\pi}{4N} + \frac{1}{N}\csc\frac{5\pi}{4N} - \frac{1}{N}\right]\right\}.\end{aligned}$$

Third sum:

$$\cos m\psi_\nu - \cos m\phi_\nu = -\frac{m}{8N^2}\cot\phi_\nu \sin m\phi_\nu - \frac{m^2}{128N^4}\cot^2\phi_\nu \cos m\phi_\nu$$
$$+\left\{\frac{m}{8N^2}\cot\phi_\nu - \sin\left(\frac{m}{8N^2}\cot\phi_\nu\right)\right\}\sin m\phi_\nu$$
$$+\left\{\cos\left(\frac{m}{8N^2}\cot\phi_\nu\right) - 1 + \frac{1}{2}\left(\frac{m}{8N^2}\cot\phi_\nu\right)^2\right\}\cos m\phi_\nu$$
$$= -\frac{m}{8N^2}\cot\phi_\nu \sin m\phi_\nu - \frac{m^2}{128N^4}\cot^2\phi_\nu\cos m\phi_\nu + r_\nu,$$

where

$$|r_\nu| \leqslant \frac{1}{6}\left|\frac{m}{8N^2}\cot\phi_\nu\right|^3 + \frac{1}{24}\left(\frac{m}{8N^2}\cot\phi_\nu\right)^4.$$

Applying the same method as for the first sum, i.e., for the sum over the $\csc^3$-terms, we obtain

$$\left|\frac{\pi}{N}\sum_{\nu=1}^{n} r_\nu \sin\phi_\nu\right| \leqslant \frac{1}{N^2}\left\{\frac{m^3}{1536N^3}\left[\frac{\pi\cos^3\phi_1}{N^2\sin^2\phi_1} + \frac{1}{N}\left(\csc\frac{5\pi}{4N} + \sin\frac{5\pi}{4N} - 2\right)\right]\right\}$$
$$+\frac{1}{N^2}\left\{\frac{m^4}{49152N^4}\left[\frac{\pi\cos^4\phi_1}{N^3\sin^3\phi_1} + \frac{1}{2N^2}\cos\frac{5\pi}{4N}\csc^2\frac{5\pi}{4N}\right]\right\},$$

whence lemma 1. □

Lemma 2
Using the notation of theorem 1, we have

$$Q_n^G[T_m] = -\frac{\pi}{2N}\left(1 + \frac{m^2}{128N^4}\right)\frac{\sin\dfrac{\pi}{4N}\cos\dfrac{m\pi}{4N}}{\cos^2\dfrac{\pi}{4N} - \cos^2\dfrac{m\pi}{4N}}$$
$$-\frac{\pi m}{16N^3}\frac{\cos\dfrac{\pi}{4N}\sin\dfrac{m\pi}{4N}}{\cos^2\dfrac{\pi}{4N} - \cos^2\dfrac{m\pi}{4N}} + \frac{(-1)^k m^2}{64N^4}\ln\left(\tan\frac{2|j|+1}{8N}\pi\right) + \varrho_{n,m}^{(2)},$$

where $|\varrho_{n,m}^{(2)}| \leqslant |\varrho_{n,m}^{(1)}| + m^2/(18N^4)$. If $m = (2k-1)(2n+1) \pm 1$, we have to add

$$\mp(-1)^k\frac{\pi}{2}\left(1 \pm \frac{m}{8N^2} + \frac{m^2}{128N^4}\right)$$

on the right-hand side of (8).

Proof
The lemma is readily verified for $n = 1$. For $n \geqslant 2$, we have to estimate the three

sums occurring in lemma 1. The first two sums are calculated readily by using the equations

$$\sum_{\nu=1}^{n} \sin\phi_\nu \cos m\phi_\nu = \frac{1}{2}\sum_{\nu=1}^{n}(\sin(m+1)\phi_\nu - \sin(m-1)\phi_\nu), \tag{9}$$

$$\sum_{\nu=1}^{n} \cos\phi_\nu \sin m\phi_\nu = \frac{1}{2}\sum_{\nu=1}^{n}(\sin(m+1)\phi_\nu + \sin(m-1)\phi_\nu)$$

and

$$\sum_{\nu=1}^{n} \sin\tau\phi_\nu = \begin{cases} 0 & \text{if } \tau \text{ is even,} \\ \frac{1}{2}\csc\dfrac{\tau\pi}{4N} & \text{if } \tau \text{ is odd and } \tau \not\equiv 0 \ (\mathrm{mod}\, 2n+1) \text{ and} \\ (-1)^k n & \text{if } \tau = (2k-1)(2n+1) \end{cases}$$

(cf. Hansen [12, eq. (14.1.1)]). From the third sum, a sum of the form (9) may be extracted. Let the primes at the sums below denote that the last summand has to be halved if n is odd. Then, the remaining sum may be estimated as follows,

$$\begin{aligned}\sum_{\nu=1}^{n}\frac{\cos m\phi_\nu}{\sin\phi_\nu} &= \sum_{\nu=1}^{n}\left(\frac{1}{\sin\phi_\nu} - 2\sum_{\mu=1}^{m/2}\sin(2\mu-1)\phi_\nu\right) \\ &= 2\sum_{\nu=1}^{\lfloor (n+1)/2\rfloor}{}' \frac{1}{\sin\phi_\nu} - \sum_{\mu=1}^{m/2}\csc(2\mu-1)\frac{\pi}{4N} \\ &\quad + \sum_{\substack{2\mu-1=(2k-1)(2n+1)\\ 1\leqslant\mu\leqslant m/2}}\left(\csc(2\mu-1)\frac{\pi}{4N} - 2(-1)^k n\right) \\ &= \sum_{\mu=1}^{n^*}(-1)^\mu\csc(2\mu-1)\frac{\pi}{4N} - \sum_{\mu=n+2}^{m/2}\csc(2\mu-1)\frac{\pi}{4N} \\ &\quad + N(1-(-1)^{\lfloor (m+2n)/4N\rfloor}),\end{aligned}$$

where n^* is the least odd number greater or equal to n. We obtain

$$\sum_{\mu=1}^{n^*}(-1)^\mu\csc(2\mu-1)\frac{\pi}{4N} = (-1)^{n+1}\sum_{\nu=1}^{n}\sec\frac{\nu\pi}{N} + \frac{1-(-1)^n}{2} = -n$$

(for the last equation, see Hansen [12, eq. (26.1.2)]). The periodicity and symmetry of the cosecant yield

$$\frac{1}{2} + \sum_{\mu=n+2}^{m/2}\csc(2\mu-1)\frac{\pi}{4N} = -(-1)^k\left(\frac{1}{2} + \sum_{\mu=n+2}^{2n+1-|j|}\csc(2\mu-1)\frac{\pi}{4N}\right).$$

Since the cosecant and its second derivative are convex, we may apply theorem 100 in Brass [4] to this sum and obtain

$$0 \leqslant \frac{1}{2} + \sum_{\mu=n+2}^{2n+1-|j|} \csc(2\mu-1)\frac{\pi}{4N} - \left(\frac{2N}{\pi}\ln\cot(2|j|+1)\frac{\pi}{8N} + \frac{1}{2}\csc(2|j|+1)\frac{\pi}{4N}\right)$$

$$\leqslant \frac{\pi}{24N}\cos(2|j|-1)\frac{\pi}{4N}\csc^2(2|j|-1)\frac{\pi}{4N}.$$

Lemma 2 follows. □

A slight simplification of lemma 2 is the relation

$$Q_n^G[T_m] = -\frac{\pi^2}{8N^2}\frac{\cos\frac{m\pi}{4N} + \frac{m}{2\pi N}\sin\frac{m\pi}{4N}}{\cos^2\frac{\pi}{4N} - \cos^2\frac{m\pi}{4N}} + \frac{(-1)^k m^2}{64N^4}\ln\left(\tan\frac{2|j|+1}{8N}\pi\right) + \sigma_{n,m}, \tag{10}$$

where

$$N^2|\sigma_{n,m}| \leqslant \frac{2}{3} + \frac{m}{7N} + \frac{m^2}{10N^2} + \frac{m^3}{1871N^3} + \frac{m^4}{180313N^4}.$$

If $n=1$, we readily verify this result. If $n>1$, we just have to calculate the error when omitting the term $m^2/(128N^4)$ in the first summand on the right-hand side of (8) and when replacing $\sin\pi/(4N)$ by $\pi/(4N)$ in the first and $\cos\pi/(4N)$ by 1 in the second summand on the right-hand side of (8). The calculations are elementary.

Proof of theorem 1

For the main term on the right-hand side of (10), we may write

$$-\frac{\pi^2}{8N^2}\frac{\cos\frac{m\pi}{4N} + \frac{m}{2\pi N}\sin\frac{m\pi}{4N}}{\cos^2\frac{\pi}{4N} - \cos^2\frac{m\pi}{4N}} = (-1)^k\frac{\pi^2}{8N^2}\frac{\cos\frac{j\pi}{2N} + \frac{m}{2\pi N}\sin\frac{j\pi}{2N}}{\cos^2\frac{j\pi}{2N} - \cos^2\frac{\pi}{4N}}.$$

Setting $\alpha = \pi/(4N)$ and $x = j\pi/(2N)$, we have $x=0$ or $2\alpha \leqslant x \leqslant \pi/2-\alpha$, and we may prove with some analytical effort that

$$-\frac{2}{3} \leqslant \frac{\cos x}{\cos^2\alpha - \cos^2 x} - \frac{1}{x^2-\alpha^2} \leqslant 0,$$

as well as

$$0 \leqslant \frac{\sin x}{\cos^2\alpha - \cos^2 x} - \frac{x}{x^2-\alpha^2} \leqslant \tfrac{3}{4}x.$$

We obtain

$$\left| \frac{\cos x + \frac{m}{2\pi N}\sin x}{\cos^2\alpha - \cos^2 x} - \frac{1 + m/(2\pi N)x}{x^2 - \alpha^2} \right| \leqslant \frac{3m}{16N}$$

and therefore inequality (5) with

$$N^2|\rho_{n,m}| \leqslant \frac{2}{3} + \frac{3m}{8N} + \frac{m^2}{128N^2}(15 + 2\ln N) + \frac{m^3}{1871N^3} + \frac{m^4}{180313N^4}.$$

The theorem now follows from $2N < m$. □

Proof of theorem 2

Nothing has to be proved for $n \leqslant 2$. Let therefore be $n \geqslant 3$ fixed. Then, consider $m/N \in [2, 3]$ as a variable in the expression on the right-hand side of (8) times N^2. This expression is a combination of monotonic terms. We divide the interval $[2, 3]$ into 100 subintervals of equal length and estimate each monotonic term in (8) in each of these subintervals from above and from below. The theorem follows. □

5. Proof of the results on analytic functions

Proof of theorem 3

Following Lether [15], we set $2z = w + w^{-1}$, so that

$$\sum_{\nu=0}^{\infty} \frac{4\cos \nu x}{w^{\nu}} = 2 + \frac{w^2 - 1}{w(z - \cos x)}$$

(see Hansen [12, eq. (17.17.2)]) and therefore

$$k_n(z) = \sum_{\nu=n}^{\infty} \frac{4}{w^{2\nu-1}(w^2-1)} R_n^G[T_{2\nu}]$$

$$= \frac{4}{w^{2n+1}} \left(d_n - \frac{(2n+1)d_n}{(2n-1)(2n+3)(w^2-1)} + \frac{w^2}{w^2-1} \sum_{\nu=2}^{\infty} \frac{1}{w^{2\nu}} R_n^G[T_{2n+2\nu}] \right).$$

On the boundary of the ellipse C_r, the variable w had modulus r, so that theorem 2 and the estimate (4) yield

$$|\delta_n(z)| \leqslant \frac{r^2}{|w^2-1|} \cdot \frac{3}{n^2} \sum_{\nu=2}^{\infty} \frac{1}{r^{2\nu}} + \frac{r^2}{|w^2-1|} \cdot \left(2 + \frac{2}{9n^2} - \frac{3}{n^2}\right) \sum_{\nu=\lfloor (n+3)/2 \rfloor}^{\infty} \frac{1}{r^{2\nu}},$$

which gives part (a) of the theorem. Now consider the function $\tilde{k}_n$ defined by $\tilde{k}_n(z) = 1 - c_n(w^2-1)^{-1}$. Using $w = re^{i\varphi}$ with $\varphi \in [0, 2\pi)$, we have

$$|\tilde{k}_n(z)| = \sqrt{1 - c_n \frac{2r^2 \cos 2\varphi - 2 - c_n}{r^4 - 2r^2 \cos 2\varphi + 1}}.$$

For sufficiently large n, namely for $c_n \leqslant r^4 - 1$, the maximum of $|\tilde{k}_n|$ is attained for $\varphi = \pi/2$ and $\varphi = 3\pi/2$, i.e., for $w = \pm ir$ and $z = \pm(i/2)(r - r^{-1})$. (If $c_n > r^4 - 1$, the maximum is attained for $\varphi = 0$ and $\varphi = \pi$.) Part (b) of the theorem now follows from part (a). □

Proof of corollary 2

We set $2z = w + w^{-1}$ and $z = z(t) = \frac{1}{2}(r + r^{-1}) \cos t + (i/2)(r - r^{-1}) \sin t$, $t \in [0, 2\pi)$. Theorem 3a and some calculations show that

$$\frac{r^{2n+1}}{4d_n} |k_n(z)| = \left| 1 - \frac{1}{2n(w^2 - 1)} + O(n^{-2}) \right| = 1 + \frac{1}{2n} c(t) + O(n^{-2}),$$

where

$$c(t) = \frac{1 - r^2 \cos 2t}{r^4 + 1 - 2r^2 \cos 2t}.$$

To obtain a maximum of $|k_n|$, the value $c(t)$ may differ from the maximum $c(\pi/2) = c(3\pi/2)$ at most by the amount $O(n^{-1})$, which yields the corollary. □

Proof of theorem 4

Define $m = \lfloor (3n + 1)/2 \rfloor$ and

$$L_n[f] = \frac{2}{\pi} \sum_{\nu=n}^{m} R_n^G[T_{2\nu}] \int_{-1}^{1} \frac{f(x) T_{2\nu}(x)}{\sqrt{1 - x^2}} dx.$$

Since the functional $R_n^G - L_n$ vanishes for all polynomials of degree $2m$, we obtain

$$|R_n^G[f]| \leqslant \|R_n^G - L_n\| \cdot E_{2m}[f] + |L_n[f]|,$$

where $E_s[f]$ denotes the error of the best approximation of f from the space of polynomials of degree at most s. Achieser [2, sect. 111] proved the bound $E_s[f] \leqslant 8r^{-s-1}/\pi$. The norm

$$\|R_n^G - L_n\| = 2 + \int_{-1}^{1} \left| 1 - \frac{2}{\pi} \sum_{\nu=n}^{m} R_n^G[T_{2\nu}] \frac{T_{2\nu}(x)}{\sqrt{1 - x^2}} \right| dx$$

may be calculated explicitly for $n = 1$ and $n = 2$. For $n \geqslant 11$, we estimate

$$\begin{aligned}
\|R_n^G - L_n\| &\leqslant 2 + \int_{-1}^{1}\left|1 - \frac{2d_n}{\pi\sqrt{1-x^2}}(T_{2n}(x) - T_{2n+2}(x))\right| dx \\
&\quad + \frac{2d_nc_n}{\pi}\int_{-1}^{1}\frac{|T_{2n+2}(x)|}{\sqrt{1-x^2}}dx + \frac{3}{n^2}\sum_{\nu=n+2}^{m}\int_{-1}^{1}\frac{|T_{2\nu}(x)|}{\sqrt{1-x^2}}dx \\
&\leqslant 2 + \int_0^{\pi}\left|\sin x - \frac{d_n}{\pi}(\cos 2nx + \cos(2n+2)x)\right| dx + \frac{4d_nc_n}{\pi} + \frac{3n-3}{n^2} \\
&\leqslant 2 + \int_0^{\pi}|1 - 2\sin(2n+1)x|\sin x\, dx + \frac{4d_nc_n}{\pi} + \frac{3n-3}{n^2}
\end{aligned}$$

and, setting again $N = n + 1/2$,

$$\begin{aligned}
\int_0^{\pi}|1 - 2\sin(2n+1)x|\sin x\, dx &= 2 + \frac{2n+1}{2n(n+1)}\sin\frac{\pi}{6N}\csc\frac{\pi}{4N}\left(\sqrt{3}\cot\frac{\pi}{6N} - 2N\right) \\
&\leqslant 2 + \frac{1}{3\pi}\left(12\sqrt{3} - 4\pi + \frac{6\sqrt{3}-\pi}{n(n+1)}\right).
\end{aligned}$$

If $3 \leqslant n \leqslant 10$, we use the explicit expressions of Stegen [18, p. 107] for $R_n^G[T_{2n}], \ldots, R_n^G[T_{2m}]$ instead of the upper bound $3n^{-2}$ as above. We obtain $\|R_n^G - L_n\| \leqslant 5.5$ for all n and therefore, from Achieser's bound for the approximation error,

$$\|R_n^G - L_n[f]\| \cdot E_{2m}[f] \leqslant \frac{44}{\pi r^{3n+1}}.$$

Now consider the kernel function of the functional L_n,

$$\begin{aligned}
k(z) := L_n\left[\frac{1}{z - \cdot}\right] &= \sum_{\nu=n}^{m}\frac{4R_n^G[T_{2\nu}]}{w^{2\nu-1}(w^2-1)} \\
&= \frac{4d_n}{w^{2n+1}}\left(1 - \frac{1}{w^2-1}\left(c_n + \frac{w^2}{d_n}\sum_{\nu=2}^{m-n}\frac{R_n^G[T_{2n+2\nu}]}{w^{2\nu}}\right)\right) = \frac{4d_n}{w^{2n+1}}(1 + \delta(z)),
\end{aligned}$$

where

$$|\delta(z)| \leqslant \frac{1}{|w^2-1|}\left(c_n + \frac{3n-3}{2d_nn^2}\right) \leqslant \frac{\pi+6}{2\pi n|w^2-1|}.$$

The theorem now follows from

$$\int_{\partial C_r}\frac{|dz|}{|w^2-1|} = \frac{\pi}{r}.$$

□

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, 1964).

[2] N.L. Achieser, *Vorlesungen über Appoximationstheorie* (German translation) (Akademie-Verlag, Berlin, 1967).

[3] N.S. Bakhvalov, On the optimal speed of integrating analytic functions (in Russian), Zh. Vychisl. Mat. Mat. Fiz. 7 (1967) 1011–1020 [English transl.: USSR Comput. Math. Math. Phys. 7 (1967) 63–75.]

[4] H. Brass, *Quadraturverfahren* (Vandenhoeck & Ruprecht, Göttingen, 1977).

[5] H. Brass and K.-J. Förster, On the estimation of linear functionals, Analysis 7 (1987) 237–258.

[6] M.M. Chawla and M.K. Jain, Error estimates for Gauss quadrature formulas for analytic functions, Math. Comp. 22 (1968) 82–90.

[7] A.R. Curtis and P. Rabinowitz, On the Gaussian integration of Chebyshev polynomials, Math. Comp. 26 (1972) 207–211.

[8] P.J. Davis and P. Rabinowitz, *Methods of Numerical Integration*, 2nd ed. (Academic Press, Orlando, 1984).

[9] K.-J. Förster and K. Petras, Inequalities for the zeros of ultraspherical polynomials and Bessel functions, Z. Angew. Math. Mech. 73 (1993) 232–236.

[10] L. Gatteschi, New inequalities for the zeros of Jacobi polynomials, SIAM J. Math. Anal. 18 (1987) 1549–1562.

[11] W. Gautschi and R.S. Varga, Error bounds for Gaussian quadrature of analytic functions, SIAM J. Numer. Anal. 20 (1983) 1170–1186.

[12] E.R. Hansen, *A Table of Series and Products* (Prentice-Hall, Englewood Cliffs, NJ, 1975).

[13] N.S. Kambo, Error of Newton-Cotes and Gauss-Legendre quadrature formulas, Math. Comp. 24 (1970) 261–269.

[14] M.A. Kowalski, A.G. Werschulz and H. Woźniakowski, Is Gauss quadrature optimal for analytic functions?, Numer. Math. 47 (1985) 89–98.

[15] F.G. Lether, Some observations on Gauss-Legendre quadrature error estimates for analytic functions, J. Comp. Appl. Math. 7 (1981) 63–66.

[16] D. Nicholson, P. Rabinowitz, N. Richter and D. Zeilberger, On the error in the numerical integration of Chebyshev polynomials, Math. Comp. 25 (1971) 79–86.

[17] F. Stenger, Bounds on the error of Gauss-type quadratures, Numer. Math. 8 (1966) 150–160.

[18] R. Stegen, Fehlerasymptotik beim Gaussschen Quadraturverfahren, Dissertation, Braunschweig (1981).