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Propagation of long waves in straits*

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Abstract — The non-linear effects induced when a long wave propagates in a strait of arbitrary cross-section are investigated. Comparative analysis of the wave parameters for straits with different geometries is carried out. The relationships between the wave amplitudinal characteristics, the non-linearity scales, and the parameters of the strait's cross-section are considered.

The entry of sea waves into straits, river mouths, and canals is accompanied, as a rule, by the intensification of non-linear effects. It is sufficient to mention the perfectly illustrated formation of a bore during the entry of a tidal wave into Fuchuntsiang River, China, described by Stoker [1]. Therefore it makes sense to provide an analysis of the non-linear properties of a sea wave in straits depending on the form of their cross-section. One should bear in mind that this problem has much in common with the hydraulic problem of wave propagation in an open canal or river bed [2]. However, the canal's form is usually assumed to be parabolic or trapezoidal, and the total slope of the bed is considered.

In this paper, the solution of the motion equations for long waves in a strait of constant arbitrary cross-section, when its width is comparable to depth, is obtained in a general form. In this case, we use the known differential equations derived in the hydraulic approximation which describe the motion of long waves in open canals. The solution is obtained in the form of a Rieman wave, and the non-linearity parameters are estimated.

1. Let us consider the motion of long waves in straits of arbitrary cross-section. All assumptions of the hydraulic theory for longitudinal waves [3] are realized for such motion. Hence the use of a quasi-one-dimensional approximation is possible, and the assumption about infinitesimal wave amplitude is not necessary. Let the fluid be inviscid and incompressible; the x-axis is directed along the strait; the plane $z = h_0$ coincides with the undisturbed surface, which is assumed to be horizontal in every cross-section; and the function $z(y)$ describes the form of the strait's bottom. In this case, the equations of motion of an ideal fluid in a hydraulic approximation (for long waves) have the following form:

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(AU) = 0, \tag{1}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial H}{\partial x} = 0, \tag{2}$$

where $A(x,t)$ denotes the disturbed area of the strait's cross-section, $u(x,t)$ is the longitudinal velocity of flow (assumed to be constant in the cross-section), $H(x,t)$ is the total

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length of the basin at the strait's axis, and g is the acceleration of gravity. Obviously, this system is closed by a relation of type $A = A(H)$, which describes the relationship between the area of the strait's cross-section and the depth.

It should be noted that system (1)–(2) is entirely equivalent to the equations describing the one-dimensional motion of a gas:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho U) = 0, \qquad p = p(\rho), \qquad \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

if we consider that $\rho = A$, $p = g \int A\,\mathrm{d}H$, where ρ and p are the density and pressure of the equivalent gas. In particular, if the gas is polytropic, i.e. $p \sim \rho^\gamma$ with constant γ, and in this case the analogous approximation is used for the area of the strait's cross-section $A \sim H^\alpha$, the constants α and γ are interrelated: $\alpha = 1/(\gamma - 1)$. In addition, if the strait's bottom is assumed to be 'parabolic',

$$z(y) = a|y|^m \tag{3}$$

with a constant value m (e.g. these canals are treated in ref. 6), all the coefficients can be determined through each other: $\alpha = 1 + 1/m$, $\gamma = (2m + 1)/(m + 1)$. Obviously $0 < m < \infty$; in this case, $1 < \gamma \leq 2$. From this, in particular, a well-known conclusion derived by Stoker follows in which $\gamma = 2$ for waves in a canal of rectangular cross-section (or in a completely one-dimensional problem), since this limiting transition is obtained at $m \to \infty$ [1]. Thus, the mathematical relationships which describe the geometry of tides correspond to the equations of state for different gases, both ideal and real (in the last case, the $z(y)$ relation is more complicated).

The use of gas hydrodynamic analogy allows us to derive numerous results as the particular cases of the respective solutions of the equations of gas dynamics [4]. In particular, the Rieman wave, or simple wave, is described by the following equation:

$$\frac{\partial H}{\partial t} + V(H)\frac{\partial H}{\partial x} = 0, \qquad V(H) = C(H) + U(H),$$

$$C = \sqrt{gA\frac{\partial H}{\partial A}}, \qquad U = \int_{H_0}^{H}\sqrt{\frac{g}{A}\frac{\mathrm{d}A}{\mathrm{d}H}}\,\mathrm{d}H,$$

and the solution to this equation exists in a closed form:

$$H(x,t) = H_0\left(t - \frac{x}{V(H)}\right), \tag{4}$$

where $H_0(0,t)$ denotes the sea-level disturbance at the entry to the strait.

One can demonstrate that $\mathrm{d}V/\mathrm{d}H \neq 0$, i.e. the propagation rate of the preset point at the wave profile depends on the level. This results in increasing steepness of the wave front. Thus, if the disturbance is smooth at the entry $H = H_0(0,t)$, then at a fixed distance

$$x_* = \max\left\{V^2(H)\Big/\left(\frac{\mathrm{d}V}{\mathrm{d}H}\frac{\mathrm{d}H_0}{\mathrm{d}t}\right)\right\} \tag{5}$$

in solution (4) a gradient catastrophe occurs ($\mathrm{d}H/\mathrm{d}x$ or $\mathrm{d}H/\mathrm{d}t$ vanishes). Within the framework of a hydrodynamic model, the wave has to break. The value x_* defines the length of the non-linearity. It is of importance for the estimation of non-linearity in practical problems.

We will assume for certainty that outside the channel's axis ($y = 0$) $z(y)$ is a rather smooth function and the wave has a relatively low amplitude: $H(x,t) = H_0 + \eta(x,t)$, $|\eta| \ll H_0$. In this case, expansion of $V(H)$ into a Taylor series results in the expression

$$V = C_0\left(1 + \varkappa\frac{\eta}{H_0}\right), \qquad C_0 = \sqrt{\frac{gA_0}{A}}, \qquad \varkappa = \frac{H_0}{2}\frac{A'}{A_0^3}\left(\frac{A^3}{A'}\right)', \tag{6}$$

where the prime denotes differentiation with respect to H at the point H_0. The dimensionless coefficient $\varkappa$ defines the non-linearity of this problem.

The wave profile deformation was considered above. Knowledge of not only the wave form, but also its spectrum is crucial for many practical problems. To derive the simple wave harmonic composition, we rewrite solution (4), with respect to (6), in the following form:

$$\eta(x,t) = \eta_0\left(t - \frac{x}{C_0} + \frac{\alpha}{C_0^2}x\eta\right), \qquad \alpha = \frac{C_0\varkappa}{H_0}, \tag{7}$$

where $\eta_0(0,t)$ denotes the sea-level disturbance at the entry to the strait. As is known, a temporal Fourier spectrum is determined as follows:

$$S(\omega,t) = \int_{-\infty}^{+\infty} \eta(x,t)\,\exp(-i\omega t)\,\mathrm{d}t\,.$$

In our problem, this is the integral of the implicit function. However, by means of several substitutions [4], one can derive an explicit equation for the temporal spectrum:

$$S(\omega,t) = -\frac{iC_0^2}{\alpha x\omega}\int_{-\infty}^{+\infty} \exp(-i\omega\xi)\left[\exp\left\{i\omega\frac{\alpha}{C_0^2}x\eta_0(\xi)\right\} - 1\right]\mathrm{d}\xi\,.$$

In particular, if the disturbance at the entry is monochromatic $\eta_0(t) = \eta_0\sin(\omega_0 t)$, then using the well-known expansions from the theory of Bessel functions [5], we derive an equation for the discrete spectrum, which allows us to rewrite (7) in the form of a Fourier series:

$$\begin{aligned} \eta(x,t) &= \eta_0\sum_{n=1}^{\infty} A_n(x)\,\sin(n\omega_0 t)\,, \\ A_n(x) &= \frac{2I_n(ny)}{ny}\,, \qquad y = \frac{\alpha}{C_0^2}x\eta_0\omega_0\,. \end{aligned} \tag{8}$$

Owing to non-linearity, the amplitude of the first harmonic decreases and the amplitude of the rest increases. It should be noted that at the moment of breaking, the relative amplitudes of the harmonics do not depend on the parameters of the strait cross-section, and they are related in the following way: $A_1(t_*) = 0.88$, $A_2(t_*) = 0.353$, $A_3(t_*) = 0.206$, i.e. the energy of the highest harmonic is relatively low.

2. To discuss the effect of the strait's form on the non-linear effects, we consider first straits with the parabolic profile (3). For these, relation (6) assumes the form

$$C_0 = \sqrt{\frac{gH_0 m}{m+1}}, \qquad \varkappa = \frac{3}{2} + \frac{1}{m},$$

from which, in particular, we derive the known formula for the wave propagation velocity in a rectangular canal at $m \to \infty$ [1]:

$$V \cong \sqrt{gH_0}\left(1 + \frac{3\eta}{2H_0}\right).$$

In the formulae given above, the strait's geometry is determined by two parameters: the depth at the axis H_0 and 'the coefficient of form' $\varkappa$. Therefore when comparing straits of various geometries we can use two approaches. We call the first approach hydraulic since it stems from the conservation of fluid transport at a similar width of the strait. Then it is expedient to reduce the straits of various profiles to a single equivalent strait of rectangular cross-section with constant mean depth $H_m = H_0 m/(m+1)$. As a result, formula (6) assumes the following form:

$$V = C\left(1 + \delta \frac{\eta}{H_m}\right), \qquad C = \sqrt{gH_m}, \qquad \delta = \frac{3m+2}{2m+2},$$

and the coefficient δ coincides with $(\gamma + 1)/2$, as follows from gas hydrodynamic analogy for a polytropic gas. The value of δ varies within the limits from $3/2$ ($m = \infty$) to 1 ($m = 0$). Thus, variation in the canal's form (in the class of power functions) only weakly affects the non-linear correction to the wave propagation velocity.

Formula (5) allows the estimation of the non-linearity length for a sinusoidal disturbance at the entry $\eta(0,t) = \eta_0 \sin(\omega_0 t)$, i.e. the distance at which the wave breaks

$$L_* = \frac{\lambda_0}{2\pi\delta}\frac{H_m}{\eta_0} = L_{\text{rec}}\frac{m+1}{m+2/3}, \qquad \lambda_0 = \frac{2\pi C}{\omega_0},$$

where $L_{\text{rec}} = \lambda_0 H_0/(3\pi\eta_0)$ is the non-linearity length corresponding to the wave breaking in a strait of rectangular cross-section. Thus, the non-linearity length decreases with an increase of the exponent of the function which describes the strait's geometry, but not more than by 33% (Fig. 1, curve 1).

Now we consider another approach which assumes that straits with equal depth H_0 and equal undisturbed width of the water mirror are comparable. In this case, at different fluid transports, the non-linearity length for a sinusoidal disturbance at the entry has the following form: $L_* = L_{\text{rec}} m/(m + 2/3)$. It is obvious that the non-linearity length increases with an increase of m (Fig. 1, curve 2). It should be noted that in

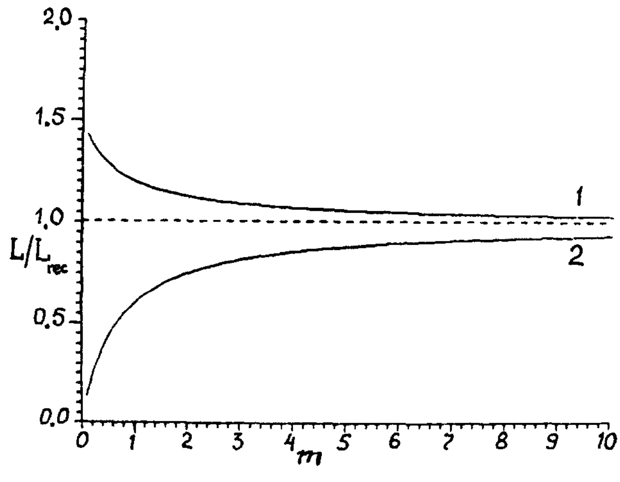

Figure 1. Relationship between the relative length of the non-linearity and the factor of the strait's form: (1) equal transport; (2) equal depth.

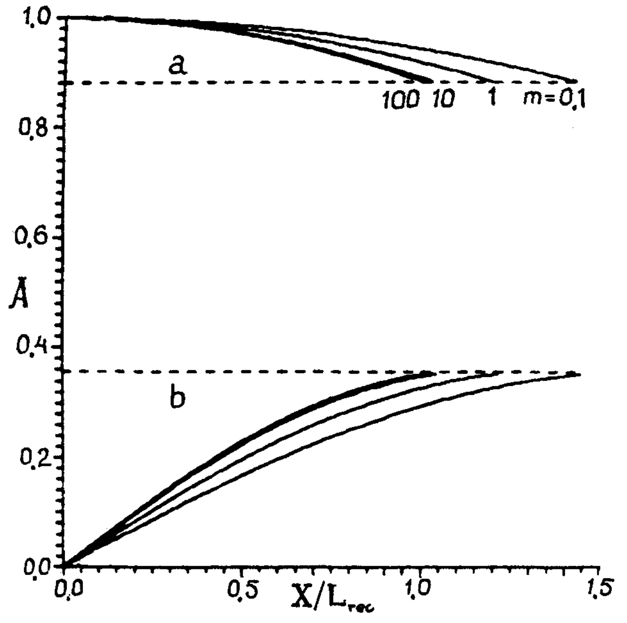

Figure 2. (a) Dependence of the relative amplitude of the first and (b) second harmonics on the normalized amplitude for different values of the strait's form factor in the case of equal transport.

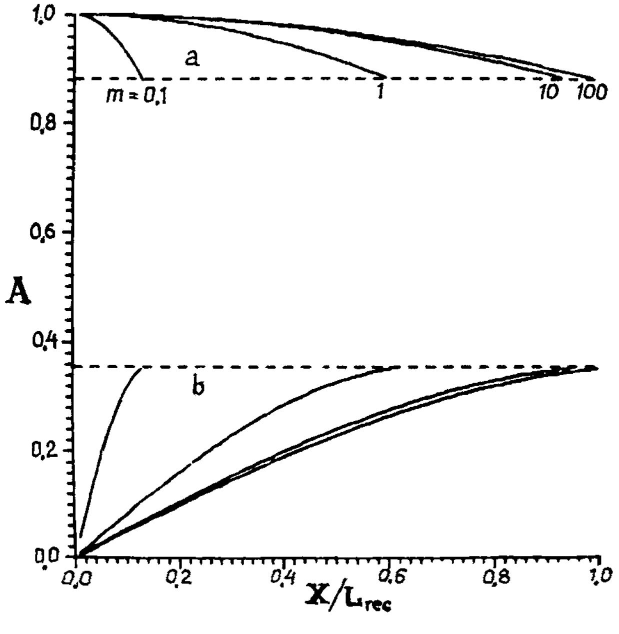

Figure 3. (a) Dependence of the relative amplitude of the first and (b) second harmonics on the normalized amplitude for different values of the strait's form factor in the case of equal depths of the strait.

this case, unlike the previous one, the variation in the non-linearity length is significant (from the zero value to L_{rec}), i.e. the non-linear effects are more pronounced at the given approach within the same interval of variation of the parameter m.

The dependences on the coordinate normalized for the wave breaking length in a strait with rectangular cross-section, of the relative amplitudes of the first and second harmonics for different values of the strait's form factor m are shown in Fig. 2, in the case of equal fluid transport, and in Fig. 3 at equal depths of the strait. These figures demonstrate that within the same interval of variation of the parameter m, in the case of equal depth in the strait, the periods of breaking of the harmonics are spaced apart, which supports stronger manifestation of the non-linear effects at the given problem's statement.

As another example, we consider briefly a symmetrical prism-like strait whose cross-section is described by the function

$$z(y) = \begin{cases} 0, & |y| < L/2, \\ b(y - L/2), & |y| > L/2, \end{cases}$$

where L is the strait's width in its base. In this case, the form factor of the strait has the form

$$\varkappa = \frac{5p^2 + 5p + 6}{2p^2 + 6p + 4}, \qquad p = \frac{2H_0}{bL}.$$

The limiting transition $p \to 0$ ($b \to \infty$) corresponds to a rectangular strait ($m = \infty$) and $p \to \infty$ ($L \to 0$) corresponds to a triangular strait ($m = 1$). Here, $\varkappa$ varies from 3/2 to 5/2, i.e. the peculiarities of the trapezoidal strait dynamics remain within the framework of the above non-linear characteristics of the 'parabolic' straits. Of interest is the case when $b < 0$. In this case, a minimum value of $\varkappa$ is attained if $p \cong -0.3$, and if $p \to -1$, $\varkappa$ increases infinitely (the limiting case $p = -1$ corresponds to a 'dead-end' basin). However, this geometry corresponds rather to canals, not to sea straits.

Thus, the form of a strait affects the rate of the non-linear effect accumulation. At equal transport of seawater through the strait, the rectangular strait turns out to be the most non-linear, and at similar depths the strait with minimum cross-section. Variation in the strait's cross-section results in variation of the coordinate of wave breaking and the rate of increase of non-linear distortions. This effect should be allowed for when plotting theoretical models.

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