JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 80, No. 1, JANUARY 1994

# Direct Relaxation of Optimal Layout Problems for Plates[1]

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**Abstract.** This paper suggests an application of a direct procedure initiated in Ref. 1 to problems of optimal layout for plates. Optimal microstructures are explicitly indicated for a number of special cases, particularly for the case where the original and conjugate strain tensors are coaxial.

**Key Words.** Direct relaxation, optimal microstructures, necessary conditions.

## 1. Introduction

In this paper, we consider non-self-adjoint optimization problems for thin anisotropic plates subjected to transverse load. The state of equilibrium of a plate $\Sigma$ is described by the equation

$$\nabla\nabla \cdot\cdot\, \mathcal{D} \cdot\cdot\, \nabla\nabla w = q, \qquad (x, y) \in \Sigma, \tag{1}$$

where $w$ denotes the normal displacement, $\mathcal{D}$ the tensor of stiffness, $q$ the transverse load density, and the symbol $\cdot\cdot$ denotes a double convolution. The symbol $\nabla$ is traditionally defined as $i\partial/\partial x + j\partial/\partial y$. The boundary $\partial\Sigma$ of a plate will be assumed clamped, this property being expressed by the boundary conditions

$$w|_{\partial\Sigma} = \partial w/\partial n|_{\partial\Sigma} = 0. \tag{2}$$

The 4th rank tensor $\mathcal{D} = \mathcal{D}(x, y)$ will play the role of control; it may take one of two admissible values $\mathcal{D}_1$ or $\mathcal{D}_2$ at each point of the plate. The

[1]This paper is dedicated to Professor Frithiof I. Niordson on the occasion of his 70th birthday. The research has been supported by AFOSR Grant No. 90-0268 and NSF Grant DMS-93-058040. The author acknowledges fruitful discussions with Andrei V. Cherkaev, Leonid V. Gibiansky, and Robert P. Lipton.

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0022-3239/94/0100-0093$07.00/0 

materials 1 and 2, with stiffness tensors $\mathscr{D}_1$ and $\mathscr{D}_2$, will both be assumed isotropic, i.e.,

$$\mathscr{D}_i = k_i a_1 a_1 + \mu_i (a_2 a_2 + a_3 a_3), \qquad i = 1, 2. \tag{3}$$

Here and below, $a_1, a_2, a_3$ represent an orthonormal basis in the space of 2nd rank symmetric tensors in the plane, i.e.,

$$a_1 = (1/\sqrt{2})(ii + jj), \qquad a_2 = (1/\sqrt{2})(ii - jj), \qquad a_3 = (1/\sqrt{2})(ij + ij). \tag{4}$$

Introduce the characteristic function $\chi_1(x, y)$ of the domain occupied by material 1, with stiffness tensor $\mathscr{D}_1$, and a similar function $\chi_2(x, y)$ for material 2; obviously, $\chi_1 + \chi_2 = 1$. It is required to find the distribution

$$\mathscr{D}(x, y) = \chi_1(x, y)\mathscr{D}_1 + \chi_2(x, y)\mathscr{D}_2 \tag{5}$$

of the stiffness tensor throughout $\Sigma$ which maximizes some weakly continuous functional $I(w)$ of solution to the boundary-value problem (1), (2). Weak continuity is assumed to be with respect to $W_2^2(\Sigma)$, this space naturally associated with (1), (2). Specifically, as a typical example, we will consider the functional

$$I(w) = -\int_\Sigma [w(x, y) - w_0(x, y)]^2 \, dx \, dy,$$

where $w_0(x, y) \in L_2(\Sigma)$.

This and similar optimization problems are known to be ill-posed and therefore requiring relaxation, i.e., the construction of an appropriate minimal extension of the initial set $U = \{\mathscr{D}_1, \mathscr{D}_2\}$ of admissible controls. Such an extension is currently proposed on the basis of a precise knowledge of the G-closure of $U$, i.e., the set GU of invariants of the effective stiffness tensors $\mathscr{D}_0$ of all composites assembled from the elements of $U$ (Ref. 2). However, the G-closures are known only for a few particular examples (Ref. 3), and the plate problem is not among them. Yet for these selected examples, the construction of GU presents difficulties, and for the plate problem these difficulties are still not overcome.

At the same time, for many applications we do not need to know the GU-set in full. Instead, it is often enough to specify some linear combination of components of $\mathscr{D}_0$; for our problem, this is the combination $\mathscr{D}_0 \cdot\cdot \nabla\nabla w$ which only matters in view of Hooke's law. To determine this combination, we apply a direct approach, free from any reference to the G-closure.

Similar problems for the 2nd order equation $\nabla \cdot \mathscr{D} \cdot \nabla w = f$ have been discussed in Refs. 1, 4, 5.

## 2. Reduction to a sup inf Problem

We first reduce the problem to a convenient sup inf form. Introduce the Lagrange multiplier $\lambda$ and consider the augmented functional

$$J = J(w, \lambda) = I(w) + \int_\Sigma \lambda(\nabla\nabla \cdot\cdot\, \mathscr{D} \cdot\cdot\, \nabla\nabla w - q)\, dx\, dy, \tag{6}$$

the second member on the right-hand side taking into account Eq. (1).

Equating to zero the first variation of (6) with respect to $w$ and bearing (2) in mind, we arrive at the conjugate equation

$$\nabla\nabla \cdot\cdot\, \mathscr{D} \cdot\cdot\, \nabla\nabla\lambda = 2(w - w_0) \tag{7}$$

and the boundary conditions

$$\lambda|_{\partial\Sigma} = \partial\lambda/\partial n|_{\partial\Sigma} = 0. \tag{8}$$

After integration by parts with the boundary conditions (8), the functional (6) takes on the form

$$J = I + \int_\Sigma (\nabla\nabla\lambda \cdot\cdot\, \mathscr{D} \cdot\cdot\, \nabla\nabla w - \lambda q)\, dx\, dy, \tag{9}$$

convenient for subsequent use.

The problem

$$\sup_{\mathscr{D}, w} I, \tag{10a}$$

subject to (1), (2), is equivalent to

$$\sup_{\mathscr{D}, w} \inf_\lambda J, \tag{10b}$$

subject to (2), (8). This is because, by (6),

$$\inf_\lambda J = I + \inf_\lambda \int_\Sigma \lambda(\nabla\nabla \cdot\cdot\, \mathscr{D} \cdot\cdot\, \nabla\nabla w - q)\, dx\, dy$$

$$= I + \begin{cases} 0, & \text{if } \nabla\nabla \cdot\cdot\, \mathscr{D} \cdot\cdot\, \nabla\nabla w = q, \\ \infty, & \text{otherwise.} \end{cases}$$

We observe that Eq. (1) appears as a necessary condition for a minimum in $\lambda$. Bearing (8) in mind, we may assume that $J$ in (10b) has the form (9). We have finally for (10b)

$$\sup_{\mathscr{D}, w} \inf_\lambda \left\{ I + \int_\Sigma (\nabla\nabla\lambda \cdot\cdot\, \mathscr{D} \cdot\cdot\, \nabla\nabla w - \lambda q)\, dx\, dy \right\}, \tag{11}$$

where

$$\mathscr{D} \in U = \{\mathscr{D}_1, \mathscr{D}_2\},$$

and $w$ and $\lambda$ satisfy, respectively, Eqs. (2) and (8).

In the sequel, we will establish the upper and lower bounds for the functional (11). An upper bound will be constructed analytically through an appropriate mathematical construction, and the lower bound will be generated by a specially chosen composite assembled from the original constituents. Both bounds will be shown to coincide, and the desired relaxation will thus be achieved.

## 3. Upper Bound for $\sup_{D,w} \inf_{\lambda} J$

This functional possesses the following upper bound:

$$\begin{aligned} \sup_{\mathscr{D},w} \inf_{\lambda} J &= \sup_{w} \sup_{\mathscr{D}} \inf_{\lambda} J \leq \sup_{w} \inf_{\lambda} \sup_{\mathscr{D}} J \\ &= \sup_{w} \inf_{\lambda} \left[ -\int_{\Sigma} (w - w_0)^2 \, dx \, dy - \int_{\Sigma} \lambda q \, dx \, dy \right. \\ &\qquad \left. + \int_{\Sigma} G(\nabla\nabla w, \nabla\nabla\lambda) \, dx \, dy \right], \end{aligned} \tag{12}$$

where

$$G(\xi, \eta) = \begin{cases} \xi \cdot\cdot\, \mathscr{D}_1 \cdot\cdot\, \eta, & \xi \cdot\cdot\, \mathscr{D}_1 \cdot\cdot\, \eta \geq \xi \cdot\cdot\, \mathscr{D}_2 \cdot\cdot\, \eta, \\ \xi \cdot\cdot\, \mathscr{D}_2 \cdot\cdot\, \eta, & \xi \cdot\cdot\, \mathscr{D}_1 \cdot\cdot\, \eta \leq \xi \cdot\cdot\, \mathscr{D}_2 \cdot\cdot\, \eta. \end{cases} \tag{13}$$

The notation

$$\xi = \nabla\nabla w, \qquad \eta = \nabla\nabla\lambda$$

will be used below. The function $G(\xi, \eta)$ is convex with respect to any of its arguments, but nonconvex with respect to their union.

The problem

$$\sup_{w} \inf_{\lambda} \left[ -\int_{\Sigma} (w - w_0)^2 \, dx \, dy - \int_{\Sigma} \lambda q \, dx \, dy + \int_{\Sigma} G(\nabla\nabla w, \nabla\nabla\lambda) \, dx \, dy \right] \tag{14}$$

is still ill-posed. It would be well-posed if the integrand $G(\xi, \eta)$ were a saddle function, i.e., concave in $\xi$ for fixed $\eta$ and convex in $\eta$ for fixed $\xi$. Then the solution would exist and the operations sup and inf would commute. For our problem, this is obviously not the case. However, the requirement that the function $G(\xi, \eta)$ be a saddle is too restrictive now that $\xi$ and $\eta$ are gradients; to ensure the existence of the sup inf for this case, it is enough to require that this function be only a quasisaddle (Ref. 1). The quasisaddle envelope $G^{**}(\xi, \eta)$ of $G(\xi, \eta)$ will be constructed applying the so called polysaddlification transform introduced in Ref. 1. This transform plays the same role for sup inf problems as the polyconvexification transform (Refs. 6–8) plays for the minimum problems. For the problem considered, the polysaddlification transform is given by the formula

$$G^{**}(\xi, \eta) = \sup_{\omega,d} \sup_{b} \inf_{a} \{a \cdot\cdot \xi + b \cdot\cdot \eta + \omega \cdot\cdot (\xi \times \eta) + d\xi \cdot\cdot T \cdot\cdot \eta - \inf_{\xi} \sup_{\eta} [a \cdot\cdot \xi + b \cdot\cdot \eta + \omega \cdot\cdot (\xi \times \eta) + d\xi \cdot\cdot T \cdot\cdot \eta - G(\xi, \eta)]\}. \tag{15}$$

Here, we introduced the notation $T$ for a tensor,

$$T = a_1 a_1 - a_2 a_2 - a_3 a_3; \tag{16}$$

the terms $\omega \cdot\cdot \xi \times \eta$ and $d\xi \cdot\cdot T \cdot\cdot \eta$ represent the null Lagrangians

$$\xi \times \eta = (\xi_2 \eta_3 - \xi_3 \eta_2) a_1 + (\xi_3 \eta_1 - \xi_1 \eta_3) a_2 + (\xi_1 \eta_2 - \xi_2 \eta_1) a_3$$

and $\xi \cdot\cdot T \cdot\cdot \eta$ (Refs. 3 and 6–8) taken into account with the aid of Lagrange multipliers $\omega$ and $d$. The symbols $\xi_1, \ldots, \eta_3$ denote the components of $\xi$ and $\eta$ in the basis $a_1, a_2, a_3$.

The transform $G^{**}(\xi, \eta)$ defined by (15) satisfies the inequality

$$G^{**}(\xi, \eta) \geq G(\xi, \eta), \tag{17}$$

for any $G(\xi, \eta)$ convex in $\eta$ and arbitrary in $\xi$ (Ref. 1). Applying $G^{**}(\xi, \eta)$ instead of $G(\xi, \eta)$, we arrive at the upper bound

$$\sup_{w} \inf_{\lambda} \left[ -\int_{\Sigma} (w - w_0)^2 \, dx \, dy - \int_{\Sigma} \lambda q \, dx \, dy + \int_{\Sigma} G^{**}(\xi, \eta) \, dx \, dy \right] \tag{18}$$

for (14), and consequently for the original functional (10b).

## 4. Computation of $G^{**}(\xi, \eta)$

We first compute

$$\bar{h}(\xi, b) = \sup_{\eta} [b \cdot\cdot \eta - H(\xi, \eta)],$$

with

$$H(\xi, \eta) = -\omega \cdot\cdot (\xi \times \eta) - d\xi \cdot\cdot T \cdot\cdot \eta + G(\xi, \eta),$$

and obtain

$$b \cdot\cdot \eta - H(\xi, \eta) = \begin{cases} c^1 \cdot\cdot \eta, & \text{if } \eta \in \xi \cdot\cdot (\mathscr{D}_1 - \mathscr{D}_2) \cdot\cdot \eta \geq 0, \\ c^2 \cdot\cdot \eta, & \text{if } \eta \in \xi \cdot\cdot (\mathscr{D}_1 - \mathscr{D}_2) \cdot\cdot \eta \leq 0. \end{cases}$$

With $\operatorname{dev} \xi = \xi_2 a_2 + \xi_3 a_3$, the tensors $c^1, c^2$ are defined as

$$c^1 = b + (d - k_1)\xi_1 a_1 - (d + \mu_1) \operatorname{dev} \xi + \omega \times \xi, \tag{19a}$$

$$c^2 = b + (d - k_2)\xi_1 a_1 - (d + \mu_2) \operatorname{dev} \xi + \omega \times \xi. \tag{19b}$$

By an argument similar to that described in Ref. 1, we arrive at the formula

$$\bar{h}(\xi, b) = \sup_\eta [b \cdot\cdot \eta - H(\xi, \eta)] = \begin{cases} 0, & \text{if } b = \langle S \rangle \cdot\cdot \xi, \\ +\infty, & \text{otherwise.} \end{cases} \tag{20}$$

In (20), the matrix $\langle S \rangle$ is defined as the convex hull

$$\langle S \rangle = t_1 S_1 + t_2 S_2, \qquad t_1, t_2 \geq 0, \qquad t_1 + t_2 = 1, \tag{21}$$

of matrices

$$S_i = \Delta_i + \omega \cdot\cdot \epsilon, \qquad \Delta_i = \mathscr{D}_i - dT, \qquad i = 1, 2, \tag{22}$$

where the matrix

$$\epsilon = -E \times E \tag{23}$$

defines the Levi-Civita tensor of the 6th rank acting in the linear space of $2 \times 2$ symmetric tensors. The unit tensor $E$ in this space can be defined as

$$E = a_1 a_1 + a_2 a_2 + a_3 a_3 \tag{24}$$

in the basis (4), and by a similar formula in any other orthonormal basis.

Here, we note the formulas (Ref. 9)

$$\epsilon = -E \times E = -a_s a_s \times a_k a_k = -a_s a_t a_k \epsilon^{skt} = a_s a_t a_k \epsilon^{stk}, \tag{25}$$

where

$$\epsilon^{skt} = a_s \cdot\cdot (a_k \times a_t) \tag{26}$$

are Levi-Civita symbols,

$$\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = 1,$$

$$\epsilon^{132} = \epsilon^{213} = \epsilon^{321} = -1,$$

$$\epsilon^{stk} = 0, \qquad \text{otherwise;}$$

also,

$$\omega \cdot\cdot \epsilon = -\omega \cdot\cdot E \times E = -\omega \times E = -E \times \omega = \epsilon \cdot\cdot \omega. \tag{27}$$

Geometrically, the function $\bar{h}(\xi, b)$ of $\xi$ for fixed $b$ is equal to positive infinity everywhere, except for points of the set

$$b = \langle S \rangle \cdot\cdot \xi, \qquad t_1, t_2 \in (21). \tag{28}$$

Equation (28) can be inverted to express $\xi$ in terms of $b$. To this end, we introduce symmetric tensors of the 4th rank [see (22)],

$$\Delta_1 = \mathscr{D}_1 - dT, \qquad \Delta_2 = \mathscr{D}_2 - dT, \qquad \langle \Delta \rangle = t_1 \Delta_1 + t_2 \Delta_2, \tag{29}$$

and compute the inverse matrix

$$\langle S \rangle^{-1} = [\langle \Delta \rangle + \omega \cdot\cdot \epsilon]^{-1} = [\langle \Delta \rangle - \omega \times E]^{-1}.$$

We obtain by direct calculation

$$\begin{aligned}\langle S \rangle^{-1} &= [1/(\det\langle \Delta \rangle + \omega \cdot\cdot \langle \Delta \rangle \cdot\cdot \omega)] \\ &\quad \times \{(\det\langle \Delta \rangle)\langle \Delta \rangle^{-1} + \omega\omega + (\omega \cdot\cdot \langle \Delta \rangle) \times E\} = \delta + \Omega \times E,\end{aligned} \tag{30}$$

where

$$\delta = [1/(\det\langle \Delta \rangle + \omega \cdot\cdot \langle \Delta \rangle \cdot\cdot \omega)]\{(\det\langle \Delta \rangle)\langle \Delta \rangle^{-1} + \omega\omega\} \tag{31}$$

denotes the symmetric part of $\langle S \rangle^{-1}$ and

$$\Omega = [1/(\det\langle \Delta \rangle + \omega \cdot\cdot \langle \Delta \rangle \cdot\cdot \omega)](\omega \cdot\cdot \langle \Delta \rangle) \tag{32}$$

denotes the $2 \times 2$ tensor associated with its skew-symmetric part.

The set (28) is a segment of the curve in $\xi$-space traced as $t_1$ varies between 0 and 1. This segment connects the points $\xi^{(1)}$ and $\xi^{(2)}$ corresponding, respectively, to $t_1 = 1$ and $t_1 = 0$,

$$\xi^{(1)} = S_1^{-1} \cdot\cdot b, \qquad \xi^{(2)} = S_2^{-1} \cdot\cdot b. \tag{33}$$

We now compute the result of the operation

$$\inf_a \left\{ a \cdot\cdot \xi - \inf_\xi [a \cdot\cdot \xi - (-\bar{h}(\xi, b))] \right\}, \tag{34}$$

which comes second in the sequence (15). This one is known to put into correspondence with any given function $-\bar{h}(\xi, b)$ its concave $\xi$-envelope, i.e., the least concave function of $\xi$ greater than or equal to $-\bar{h}(\xi, b)$. In particular, if $-\bar{h}(\xi, b)$ is itself concave in $\xi$, then the operation (34) leaves this function intact.

In our special circumstances, this is obviously not the case. The concave envelope of $-\bar{h}(\xi, b)$ appears to be the function defined as negative infinity everywhere except for points of the convex hull $\Xi$ of the curvilinear segment (28) where this envelope is equal to zero,

$$\inf_a \left\{ a \cdot\cdot \xi - \inf_\xi [a \cdot\cdot \xi - (-\bar{h}(\xi, b))] \right\} = \begin{cases} 0, & \xi \in \Xi, \\ -\infty, & \xi \notin \Xi. \end{cases} \quad (35)$$

The hull $\Xi$ is a convex body in the $\xi$-space. We will assume that the curvilinear segment (28) and a line segment

$$[\xi_1 - \xi_1^{(2)}]/[\xi_1^{(1)} - \xi_1^{(2)}] = [\xi_2 - \xi_2^{(2)}]/[\xi_2^{(1)} - \xi_2^{(2)}]$$
$$= [\xi_3 - \xi_3^{(2)}]/[\xi_3^{(1)} - \xi_3^{(2)}], \quad (36)$$

connecting the endpoints $\xi^{(1)}$ and $\xi^{(2)}$ [see (33)], both belong to the boundary $\partial\Xi$ of $\Xi$.

For our future purposes, we need to know the left-hand side of (35) as a function of $b$ for fixed $\xi$. This function can be defined as equal to negative infinity everywhere in the $b$-space, except for the body $\mathscr{B}$ which appears as the $b$-image of $\Xi$; specifically, the boundary $\partial\mathscr{B}$ of $\mathscr{B}$ is described by the same equation as that of $\partial\Xi$; this time, however, $\xi$ should be kept fixed whereas $b$ should be considered variable. Obviously, the set (28), which is perceived as a curvilinear segment in the $\xi$-space, appears as a line segment in the $b$-space, and in this capacity belongs to $\partial\mathscr{B}$. Also, the set (36), which represents a line segment in the $\xi$-space, appears as a curvilinear segment in the $b$-space, and this segment also belongs to $\partial\mathscr{B}$. Summarizing these results, we arrive at the following: the transform (15) reduces to a single operation,

$$\sup_{\omega,d,b} [b \cdot\cdot \eta + \omega \cdot\cdot (\xi \times \eta) + d\xi \cdot\cdot T \cdot\cdot \eta], \quad (37)$$

subjected to the constraint $b \in \mathscr{B}$. Note that the set $\mathscr{B}$ itself depends on $\omega$ and $d$.

The curvilinear segment (36) in the $b$-space obviously represents a rib on $\partial\mathscr{B}$. The calculation (37) of the supremum with respect to $b$ will include among others the possibility that the supremum is attained at points belonging to this segment. In the sequel, we investigate this possibility and show that it generates the desired upper bound. Equation (36) can be represented in the equivalent form [see (33)]

$$\xi = (m_1 S_1^{-1} + m_2 S_2^{-1}) \cdot\cdot b = \langle S^{-1} \rangle \cdot\cdot b. \quad (38)$$

Here, $m_1, m_2 \geq 0, m_1 + m_2 = 1$.

This relationship will be taken into account with the aid of the Lagrange multiplier $\Lambda$ in the course of the maximization operation (37). We will examine the stationary points of the function

$$\phi = b \cdot\cdot \eta + \omega \cdot\cdot (\xi \times \eta) + d\xi \cdot\cdot T \cdot\cdot \eta + \Lambda \cdot\cdot (\xi - \langle S^{-1} \rangle \cdot\cdot b), \quad (39)$$

viewed as the function of $b$, $\omega$, $d$, $m_1$.

A routine calculation shows that

$$\phi_b = \eta - \Lambda \cdot\cdot \langle S^{-1} \rangle = 0,$$

which means that

$$\Lambda = \eta \cdot\cdot \langle S^{-1} \rangle^{-1}. \tag{40}$$

With Eqs. (38) and (40) in mind, the function $\phi$ becomes

$$\phi = \eta \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \xi + \omega \cdot\cdot (\xi \times \eta) + d\xi \cdot\cdot T \cdot\cdot \eta. \tag{41}$$

It can be shown (cf. Ref. 9) that

$$\begin{aligned} \phi_\omega &= -(\Lambda \cdot\cdot \langle S^{-1} \rangle \cdot\cdot b)_\omega + \xi \times \eta \\ &= m_1(\Lambda \cdot\cdot S_1^{-1}) \times (S_1^{-1} \cdot\cdot b) + m_2(\Lambda \cdot\cdot S_2^{-1}) \times (S_2^{-1} \cdot\cdot b) + \xi \times \eta. \end{aligned}$$

This expression can be rewritten in either of two forms,

$$\begin{aligned} \phi_\omega &= m_1(\Lambda \cdot\cdot S_1^{-1}) \times (S_1^{-1} \cdot\cdot b) + m_2(\Lambda \cdot\cdot S_2^{-1}) \times (S_2^{-1} \cdot\cdot b) \\ &\quad + ((m_1 S_1^{-1} + m_2 S_2^{-1}) \cdot\cdot b) \times (\Lambda \cdot\cdot (m_1 S_1^{-1} + m_2 S_2^{-1})) \\ &= -m_1 m_2 (\Delta S^{-1} \cdot\cdot b) \times (\Lambda \cdot\cdot \Delta S^{-1}), \qquad \Delta S^{-1} = S_2^{-1} - S_1^{-1}, \end{aligned} \tag{42}$$

or

$$\phi_\omega = \eta \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot [m_1 S_1^{-1} \times S_1^{-1} + m_2 S_2^{-1} \times S_2^{-1}] \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \xi + \xi \times \eta. \tag{43}$$

The stationary condition $\phi_\omega = 0$ can now be written as

$$\begin{aligned} \Delta S^{-1} \cdot\cdot b &= (\Delta S^{-1}) \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \xi \\ &= \kappa \Lambda \cdot\cdot \Delta S^{-1} \\ &= \kappa \eta \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot (\Delta S^{-1}), \end{aligned} \tag{44}$$

where $\kappa$ is a scalar multiplier. An equivalent representation is associated with Eq. (43),

$$\eta \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot [m_1 S_1^{-1} \times S_1^{-1} + m_2 S_2^{-1} \times S_2^{-1}] \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \xi + \xi \times \eta = 0. \tag{45}$$

The condition $\phi_d = 0$ reduces to

$$\begin{aligned} \phi_d &= -m_1 m_2 (\Delta S^{-1} \cdot\cdot b) \cdot\cdot T \cdot\cdot (\Lambda \cdot\cdot \Delta S^{-1}) \\ &= -m_1 m_2 \kappa^{-1} (\Delta S^{-1} \cdot\cdot b) \cdot\cdot T \cdot\cdot (\Delta S^{-1} \cdot\cdot b) \\ &= 0, \end{aligned} \tag{46}$$

or equivalently,

$$\phi_d = -\eta \cdot\cdot \langle S^{-1}\rangle^{-1} \cdot\cdot [m_1 S_1^{-1} \cdot\cdot T \cdot\cdot S_1^{-1} + m_2 S_2^{-1} \cdot\cdot T \cdot\cdot S_2^{-1}] \cdot\cdot \langle S^{-1}\rangle^{-1} \cdot\cdot \xi + \xi \cdot\cdot T \cdot\cdot \eta = 0. \tag{47}$$

Note that the stationarity condition (46) applies as the necessary condition for a maximum if the corresponding root $d$ is such that the function $\phi$ defined by (41) is concave in $d$ for all $\omega$. To guarantee this, we must require that

$$\det S_i \geq 0, \qquad i = 1, 2,$$

i.e., that

$$\det \Delta_i + \omega \cdot\cdot \Delta_i \cdot\cdot \omega \geq 0, \qquad i = 1, 2. \tag{48}$$

These inequalities should be considered as additional constraints influencing the $d$-maximization.

Computing the expression (41) for $\phi$ at the stationary values of $\omega$ and $d$, we have to maximize it with regard to $m_1$. Before we do so, we investigate this expression in terms of its attainability with the aid of special microstructures. This is the right time for such investigation, since the aforementioned construction depends explicitly on $m_1$, this dependence being very special for a number of popular microstructures.

After maximization in $m_1$, the expression (41) should produce a final construction (37) for $G^{**}(\xi, \eta)$. This program is elaborate in its entirety, and we consider here a special case when the tensors $\xi, \eta$ are coaxial. The case when these terms are proportional has been considered earlier by Gibiansky and Cherkaev in Ref. 10. This latter case is self-adjoint and can therefore be handled with the aid of the $G$-closure technique. Contrary to that, in the more general situation when $\xi$, $\eta$ are merely coaxial, this technique does not apply, and we have to address the direct relaxation.

## 5. Case Where Tensors $\xi$, $\eta$ Are Arbitrary: General Analysis

Introduce, without any loss of generality, the unit vectors $i, j$ and the associated basis $a_1$, $a_2$, $a_3$ [see Eqs. (4)] so as to make the tensor $a_3$ proportional to dev $\omega$,

$$\omega = \omega_1 a_1 + \omega_3 a_3. \tag{49}$$

Then, applying the general representation for $\xi$, $\eta$,

$$\xi = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \tag{50a}$$

$$\eta = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, \tag{50b}$$

we compute the tensors $\Delta S^{-1}\cdot\cdot\langle S^{-1}\rangle^{-1}\cdot\cdot\xi$ and $\eta\cdot\cdot\langle S^{-1}\rangle\cdot\cdot\Delta S^{-1}$ participating in the necessary conditions. After calculations, the following relations appear:

$$\tilde{q}\Delta S^{-1}\cdot\cdot\langle S^{-1}\rangle^{-1}\cdot\cdot\xi = g_1 a_1 + g_2 a_2 + g_3 a_3, \tag{51a}$$

$$\tilde{q}\eta\cdot\cdot\langle S^{-1}\rangle^{-1}\cdot\cdot\Delta S^{-1} = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3. \tag{51b}$$

Here, the symbols $g_1, \ldots, g_3$ are defined as

$$g_1 = A\xi_1 + B\xi_2 + E\xi_3, \qquad \gamma_1 = A\eta_1 - B\eta_2 + E\eta_3, \tag{52a}$$

$$g_2 = C\xi_1 + D\xi_2 + F\xi_3, \qquad \gamma_2 = -C\eta_1 + D\eta_2 - F\eta_3, \tag{52b}$$

$$g_3 = G\xi_1 - F\xi_2 + L\xi_3, \qquad \gamma_3 = G\eta_1 + F\eta_2 + L\eta_3, \tag{52c}$$

and the coefficients $A, \ldots, L$ and $\tilde{q}$ are given by the formulas

$$A = -(\tilde{M}^2 + \omega_1^2)\Delta k, \qquad B = \omega_3 \tilde{M}\Delta\mu, \tag{53a}$$

$$C = -\omega_3 \tilde{M}\Delta k, \qquad D = -\tilde{K}\tilde{M}\Delta\mu, \tag{53b}$$

$$E = -\omega_1\omega_3\Delta\mu, \qquad F = \omega_1\tilde{K}\Delta\mu, \tag{53c}$$

$$G = -\omega_1\omega_3\Delta k, \qquad L = -(\tilde{K}\tilde{M} + \omega_3^2)\Delta\mu, \tag{53d}$$

$$\tilde{q} = \tilde{K}\tilde{M}^2 + \tilde{K}\omega_1^2 + \tilde{M}\omega_3^2, \tag{53e}$$

with

$$K_i = k_i - d, \qquad M_i = \mu_i + d, \quad i = 1, 2, \tag{54a}$$

$$\tilde{K} = m_1 K_2 + m_2 K_1 = \tilde{k} - d, \qquad \tilde{M} = m_1 M_2 + m_2 M_1 = \tilde{\mu} + d, \tag{54b}$$

$$\Delta k = K_2 - K_1 = k_2 - k_1, \qquad \Delta\mu = M_2 - M_1 = \mu_2 - \mu_1. \tag{54c}$$

We now consider different situations that arise depending on whether or not the variations $\delta d, \delta\omega$ are free or linked through the constraints expressed by the requirements that $\det S_1 = 0$ or/and $\det S_2 = 0$ [see (48)].

## 6. Case of Free Variations $\delta d$, $\delta\omega$: Upper Bound

We apply the necessary conditions (44) and (46). The first of them reduces to

$$g_1/\gamma_1 = g_2/\gamma_2 = g_3/\gamma_3, \tag{55}$$

whereas the second, by virtue of (44), is rewritten as

$$g_1^2 = g_2^2 + g_3^2. \tag{56}$$

Equations (55) and (56) are equivalent to the system of four equations [see (52)]

$$C\xi_1 + \mathscr{D}\xi_2 + F\xi_3 = (A\xi_1 + B\xi_2 + E\xi_3)\cos\chi, \tag{57a}$$

$$G\xi_1 - F\xi_2 + L\xi_3 = (A\xi_1 + B\xi_2 + E\xi_3)\sin\chi, \tag{57b}$$

$$-C\eta_1 + \mathscr{D}\eta_2 - F\eta_3 = (A\eta_1 - B\eta_2 + E\eta_3)\cos\chi, \tag{57c}$$

$$G\eta_1 + F\eta_2 + L\eta_3 = (A\eta_1 - B\eta_2 + E\eta_3)\sin\chi, \tag{57d}$$

containing an auxiliary parameter $\chi$. The system (57) should determine the parameters $\omega_1, \omega_3, d, \chi$ along with the orientation of the basis $a_1, \ldots, a_3$ with respect to the main axes of $\xi, \eta$. We thus arrive at four equations for five unknowns, which reserves some additional freedom. We shall see below that this freedom is substantial.

Applying Eqs. (53), we obtain the following solution to Eqs. (57):

$$\zeta_2 = \xi_2/\xi_1 = (\Delta k/\Delta\mu)(\tilde{M}\cos\chi + \omega_1\sin\chi - \omega_3)/(\tilde{K} + \omega_3\cos\chi), \tag{58a}$$

$$\zeta_3 = \xi_3/\xi_1 = (\Delta k/\Delta\mu)(\tilde{M}\sin\chi - \omega_1\cos\chi)/(\tilde{K} + \omega_3\cos\chi), \tag{58b}$$

$$\sigma_2 = \eta_2/\eta_1 = (\Delta k/\Delta\mu)(\tilde{M}\cos\chi - \omega_1\sin\chi + \omega_3)/(\tilde{K} - \omega_3\cos\chi), \tag{58c}$$

$$\sigma_3 = \eta_3/\eta_1 = (\Delta k/\Delta\mu)(\tilde{M}\sin\chi + \omega_1\cos\chi)/(\tilde{K} - \omega_3\cos\chi). \tag{58d}$$

Introduce now a system of mutually perpendicular unit vectors $\bar{i}, \bar{j}$ making the angle $\chi/2$ with the system $i, j$ participating in the basis (4) linked with $\omega$ through (49). This system of vectors is shown in Fig. 1. On the same figure, the unit vectors $e_1$, $l_1$ of the main axes of tensors $\xi$, $\eta$ are exposed, these vectors making angles $\varphi$, $\psi$ with $i$. If $\lambda_1, \lambda_2$ denote the principal values

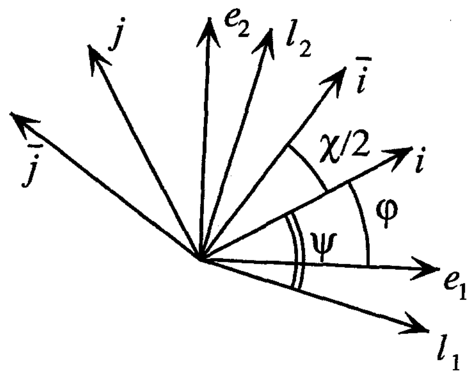


Fig. 1. Mutual orientation of orthonormal vector bases.

of $\xi$ and if $\nu_1, \nu_2$ denote the same values of $\eta$, then the following formulas apply:

$$\xi = \lambda_1 e_1 e_1 + \lambda_2 e_2 e_2, \qquad \eta = \nu_1 l_1 l_1 + \nu_2 l_2 l_2; \tag{59}$$

for the components $\xi_1, \ldots, \eta_3$ of $\xi$, $\eta$ in the $(a_1, a_2, a_3)$-basis, we obtain the expressions

$$\xi_1 = 2^{-1/2}(\lambda_1 + \lambda_2), \qquad \eta_1 = 2^{-1/2}(\nu_1 + \nu_2), \tag{60a}$$

$$\xi_2 = 2^{-1/2}(\lambda_1 - \lambda_2) \cos 2\varphi, \qquad \eta_2 = 2^{-1/2}(\nu_1 - \nu_2) \cos 2\psi, \tag{60b}$$

$$\xi_3 = 2^{-1/2}(\lambda_2 - \lambda_1) \sin 2\varphi, \qquad \eta_3 = 2^{-1/2}(\nu_2 - \nu_1) \sin 2\psi. \tag{60c}$$

We will now compute the components $\bar{\xi}_1, \ldots, \bar{\eta}_3$ of $\xi$, $\nu$ with respect to the basis

$$\bar{a}_1 = 2^{-1/2}(\bar{i}\bar{i} + \bar{j}\bar{j}), \qquad \bar{a}_2 = 2^{-1/2}(\bar{i}\bar{i} - \bar{j}\bar{j}), \qquad \bar{a}_3 = 2^{-1/2}(\bar{i}\bar{j} + \bar{j}\bar{i}), \tag{61}$$

defined by the vectors $\bar{i}, \bar{j}$ in the same manner as the basis $a_1, a_2, a_3$ is defined by $i, j$ [see (4)]. The following formulas hold:

$$a_1 \cdot\cdot\, \bar{a}_1 = 1, \qquad a_1 \cdot\cdot\, \bar{a}_2 = a_1 \cdot\cdot\, \bar{a}_3 = 0, \tag{62a}$$

$$a_2 \cdot\cdot\, \bar{a}_2 = \cos \chi, \qquad a_2 \cdot\cdot\, \bar{a}_3 = -\sin \chi, \tag{62b}$$

$$a_3 \cdot\cdot\, \bar{a}_2 = \sin \chi, \qquad a_3 \cdot\cdot\, \bar{a}_3 = \cos \chi, \tag{62c}$$

$$\bar{\xi}_1 = \xi \cdot\cdot\, \bar{a}_1 = \xi_1, \tag{63a}$$

$$\begin{aligned}\bar{\xi}_2 &= \xi \cdot\cdot\, \bar{a}_2 = \xi_2 \cos \chi + \xi_3 \sin \chi \\ &= (\Delta k/\Delta\mu)\bar{\xi}_1(\tilde{M} - \omega_3 \cos \chi)/(\tilde{K} + \omega_3 \cos \chi),\end{aligned} \tag{63b}$$

$$\begin{aligned}\bar{\xi}_3 &= \xi \cdot\cdot\, \bar{a}_3 = -\xi_2 \sin \chi + \xi_3 \cos \chi \\ &= (\Delta k/\Delta\mu)\bar{\xi}_1(\omega_3 \sin \chi - \omega_1)/(\tilde{K} + \omega_3 \cos \chi),\end{aligned} \tag{63c}$$

$$\bar{\eta}_1 = \eta \cdot\cdot\, \bar{a}_1 = \eta_1, \tag{64a}$$

$$\bar{\eta}_2 = \eta \cdot\cdot\, \bar{a}_2 = (\Delta k/\Delta\mu)\bar{\eta}_1(\tilde{M} + \omega_3 \cos \chi)/(\tilde{K} - \omega_3 \cos \chi), \tag{64b}$$

$$\bar{\eta}_3 = \eta \cdot\cdot\, \bar{a}_3 = -(\Delta k/\Delta\mu)\bar{\eta}_1(\omega_3 \sin \chi - \omega_1)/(\tilde{K} - \omega_3 \cos \chi), \tag{64c}$$

or in terms of the components $\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3$ of $\omega$,

$$\bar{\omega}_1 = \omega \cdot\cdot\, \bar{a}_1 = \omega_1, \tag{65a}$$

$$\bar{\omega}_2 = \omega \cdot\cdot\, \bar{a}_2 = (\omega_1 a_1 + \omega_3 a_3) \cdot\cdot\, \bar{a}_2 = \omega_3 \sin \chi, \tag{65b}$$

$$\bar{\omega}_3 = \omega \cdot\cdot\, \bar{a}_3 = (\omega_1 a_1 + \omega_3 a_3) \cdot\cdot\, \bar{a}_3 = \omega_3 \cos \chi, \tag{65c}$$

$$\bar{\xi} = \bar{\xi}_1, \qquad \bar{\eta}_1 = \bar{\eta}_1, \tag{66a}$$

$$\bar{\xi}_2 = (\Delta k/\Delta\mu)\bar{\xi}_1(\tilde{M} - \bar{\omega}_3)/(\tilde{K} + \bar{\omega}_3), \tag{66b}$$

$$\bar{\eta}_2 = (\Delta k/\Delta\mu)\bar{\eta}_1(\tilde{M} + \bar{\omega}_3)/(\tilde{K} - \bar{\omega}_3), \tag{66c}$$

$$\bar{\xi}_3 = (\Delta k/\Delta\mu)\bar{\xi}_1(\bar{\omega}_2 - \bar{\omega}_1)/(\tilde{K} + \bar{\omega}_3), \tag{66d}$$

$$\bar{\eta}_3 = -(\Delta k/\Delta\mu)\bar{\eta}_1(\bar{\omega}_2 - \bar{\omega}_1)/(\tilde{K} - \bar{\omega}_3). \tag{66e}$$

Eliminating the parameters $\bar{\omega}_3, \bar{\omega}_2 - \bar{\omega}_1, d$ from these equations, we arrive at the relationship

$$\bar{\xi}_3(\Delta k\bar{\eta}_1 + \Delta\mu\bar{\eta}_2) + \bar{\eta}_3(\Delta k\bar{\xi}_1 + \Delta\mu\bar{\xi}_2) = 0. \tag{67}$$

This important formula determines the basis $\bar{a}_1, \bar{a}_2, \bar{a}_3$; it defines the unit vectors $\bar{i}, \bar{j}$ in terms of the tensors $\xi, \eta$, these tensors considered as primary entities. The deviatoric components $\bar{\xi}_2, \bar{\xi}_3, \bar{\eta}_2, \bar{\eta}_3$ of $\xi, \eta$ in this new basis depend on three fundamental parameters $(\bar{\omega}_3, \bar{\omega}_2 - \bar{\omega}_1, d)$. Observe that these parameters can only be defined by virtue of Eqs. (66) in terms of $\xi, \eta$; as for $\omega_1, \omega_3, \chi$, these parameters cannot be defined completely; we only know their combinations

$$\bar{\omega}_3 = \omega_3 \cos\chi, \qquad \bar{\omega}_2 - \bar{\omega}_1 = \omega_3 \sin\chi - \omega_1.$$

Now, it is possible to compute the expression for $\phi$ defined by (41). We will compute a related expression,

$$w = \phi - \eta \cdot\cdot \langle \mathscr{D} \rangle \cdot\cdot \xi = \eta \cdot\cdot (\langle S^{-1}\rangle^{-1} - \langle \mathscr{D} \rangle) \cdot\cdot \xi$$
$$+ \omega \cdot\cdot (\xi \times \eta) + d\xi \cdot\cdot T \cdot\cdot \eta. \tag{68}$$

In the basis (4), the matrix $\langle S^{-1}\rangle^{-1}$ is given by the following components:

$$\langle S^{-1}\rangle_{11}^{-1} = (\tilde{q})^{-1}[K_1K_2(\tilde{M}^2 + \omega_1^2) + \omega_3^2\tilde{M}\langle K\rangle], \tag{69a}$$

$$\langle S^{-1}\rangle_{22}^{-1} = (\tilde{q})^{-1}[\tilde{K}\tilde{M}M_1M_2 + (\tilde{K}\omega_1^2 + \tilde{M}\omega_3^2)\langle M\rangle], \tag{69b}$$

$$\langle S^{-1}\rangle_{33}^{-1} = (\tilde{q})^{-1}[\tilde{K}\tilde{M}M_1M_2 + \tilde{K}\langle M\rangle\omega_1^2 + M_1M_2\omega_3^2], \tag{69c}$$

$$\langle S^{-1}\rangle_{12}^{-1} = (\tilde{q})^{-1}\omega_3[\tilde{M}(KM) + \tilde{K}\omega_1^2 + \tilde{M}\omega_3^2] = -\langle S^{-1}\rangle_{21}^{-1}, \tag{69d}$$

$$\langle S^{-1}\rangle_{23}^{-1} = (\tilde{q})^{-1}\omega_1[\tilde{K}(M^2) + \tilde{K}\omega_1^2 + \tilde{M}\omega_3^2] = -\langle S^{-1}\rangle_{32}^{-1}, \tag{69e}$$

$$\langle S^{-1}\rangle_{31}^{-1} = -m_1m_2(\tilde{q})^{-1}\omega_1\omega_3\Delta k\Delta\mu = \langle S^{-1}\rangle_{13}^{-1}, \tag{69f}$$

with

$$(KM) = m_1K_2M_2 + m_2K_1M_1, \qquad (M^2) = m_1M_2^2 + m_2M_1^2.$$

These formulas allow us to compute $w$. We get

$$w = -m_1 m_2(\tilde{q})^{-1}[(\tilde{M}^2 + \omega_1^2)(\Delta k)^2 \xi_1 \eta_1 + \tilde{K}\tilde{M}(\Delta\mu)^2 \xi_2 \eta_2 + (\tilde{K}\tilde{M} + \omega_3^2)(\Delta\mu)^2 \xi_3 \eta_3 + \tilde{M}\omega_3 \Delta k \Delta\mu(\xi_1 \eta_2 - \xi_2 \eta_1) + \tilde{K}\omega_1(\Delta\mu)^2(\xi_2 \eta_3 - \xi_3 \eta_2) + \omega_1 \omega_3 \Delta k \Delta\mu(\xi_1 \eta_3 + \xi_3 \eta_1)]. \quad (70)$$

With the aid of Eqs. (62)–(65), this expression is reduced to

$$w = -m_1 m_2(\tilde{q})^{-1}\{(\tilde{M}^2 + \bar{\omega}_1^2)(\Delta k)^2 \bar{\xi}_1 \bar{\eta}_1 + (\tilde{K}\tilde{M} + \bar{\omega}_2^2)(\Delta\mu)^2 \bar{\xi}_2 \bar{\eta}_2 + (\tilde{K}\tilde{M} + \bar{\omega}_3^2)(\Delta\mu)^2 \bar{\xi}_3 \bar{\eta}_3 + \bar{\omega}_2 \bar{\omega}_3 (\Delta\mu)^2(\bar{\xi}_2 \bar{\eta}_3 + \bar{\xi}_3 \bar{\eta}_2) + \tilde{M}\Delta k \Delta\mu[\bar{\omega}_3(\bar{\xi}_1 \bar{\eta}_2 + \bar{\xi}_2 \bar{\eta}_1) + \bar{\omega}_2(\bar{\xi}_3 \bar{\eta}_1 - \bar{\xi}_1 \bar{\eta}_3)] + \tilde{K}\bar{\omega}_1(\Delta\mu)^2(\bar{\xi}_2 \bar{\eta}_3 - \bar{\xi}_3 \bar{\eta}_2) + \bar{\omega}_1 \Delta k \Delta\mu[\bar{\omega}_2(\bar{\xi}_1 \bar{\eta}_2 + \bar{\xi}_2 \bar{\eta}_1) + \bar{\omega}_3(\bar{\xi}_1 \bar{\eta}_3 + \bar{\xi}_3 \bar{\eta}_1)]\}.$$

If we now apply Eqs. (66), then the expression for $w$ will reduce to

$$w = -m_1 m_2 (\Delta k)^2 \bar{\xi}_1 \bar{\eta}_1 (\tilde{k} + \tilde{\mu})/(\tilde{K}^2 - \bar{\omega}_3^2). \quad (71)$$

## 7. Case of Free Variations: Microstructure

We will show in this section that the value (71) of $w$ will be achieved if we compute it for a suitable microstructure with an effective tensor $\mathscr{D}_0$ of stiffness. This microstructure will be the rank 1 lamination with layers made of materials $\mathscr{D}_1$ and $\mathscr{D}_2$ and oriented along the unit vector $\bar{j}$ introduced above. This orientation of layers makes the quantity

$$\eta \cdot\cdot \mathscr{D}_0 \cdot\cdot \xi = \eta \cdot\cdot \langle \mathscr{D} \rangle \cdot\cdot \xi - m_1 m_2 [\xi \cdot\cdot (\mathscr{D}_2 - \mathscr{D}_1) \cdot\cdot nn] \times [\eta \cdot\cdot (\mathscr{D}_2 - \mathscr{D}_1) \cdot\cdot nn] / nn \cdot\cdot (m_1 \mathscr{D}_2 + m_2 \mathscr{D}_1) \cdot\cdot nn \quad (72)$$

stationary with respect to $n$, this stationary value being attained at $n = \bar{i}$. The stationarity condition is given by Eq. (67). With this result, it is easy to compute the expression (72).

Because $n = \bar{i}$ and the components of tensors $\xi, \eta$ are given by (66), it is easy to see that

$$\begin{aligned}
&[\xi \cdot\cdot (\mathscr{D}_2 - \mathscr{D}_1) \cdot\cdot nn][\eta \cdot\cdot (\mathscr{D}_2 - \mathscr{D}_1) \cdot\cdot nn] \\
&= (1/2)(\bar{\xi}_1 \Delta k + \bar{\xi}_2 \Delta\mu)(\bar{\eta}_1 \Delta k + \bar{\eta}_2 \Delta\eta) \\
&= (1/2)\bar{\xi}_1 \bar{\eta}_1 (\Delta k)^2 [1 + (\tilde{M} - \bar{\omega}_3)/(\tilde{K} + \bar{\omega}_3)][1 + (\tilde{M} + \bar{\omega}_3)/(\tilde{K} - \bar{\omega}_3)] \\
&= (1/2)\bar{\xi}_1 \bar{\eta}_1 (\Delta k)^2 (\tilde{k} + \tilde{\mu})^2/(\tilde{K}^2 - \bar{\omega}_3^2).
\end{aligned}$$

This together with the relationship

$$nn \cdot\cdot (m_1 \mathscr{D}_2 + m_2 \mathscr{D}_1) \cdot\cdot nn = (1/2)(\tilde{k} + \tilde{\mu})$$

shows that the second term in (72) coincides with (71).

The above argument illustrates the fundamental significance of the $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ basis: this one is defined by the stationary orientation of the layers in the optimal rank 1 laminate.

**Remark 7.1.** Equation (67) offers several stationary solutions for the basis $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ and several corresponding values $\phi_{1s}$ for $\phi$; every such solution will determine its own orientation $\bar{j}$ of layers in the rank 1 laminate. The choice of a solution will be dictated by the parameters characterizing the pair $\xi, \eta$.

**Remark 7.2.** The Lagrange multipliers $\omega, d$ may be computed as functions of the components $\bar{\xi}_1, \bar{\xi}_2, \bar{\eta}_1, \bar{\eta}_2$ and then eliminated from (71); the result will be the expression appearing on the right-hand side of (72) with $n = \bar{i}$.

The regime of free variations $\delta d$, $\delta\omega$ will be valid within the range of tensors $\xi, \eta$ defined by Ineqs. (48) together with Remark 7.2. Without this range, laminates of a higher rank will be applied to saturate the corresponding bounds.

## 8. Case Where the Variations $\delta d$, $\delta\omega$ Are Linked through One Constraint

We will assume in this section that the tensors $\xi, n$ are coaxial and

$$\omega = \omega_3 a_3, \qquad \xi = \xi_1 a_1 + \xi_2 a_2, \qquad \eta = \eta_1 a_1 + \eta_2 a_2. \tag{73}$$

The tensors $\Delta S^{-1} \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \xi$ and $\eta \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \Delta S^{-1}$ are now computed as [cf. (51)–(53)]

$$-(\tilde{K}\tilde{M} = \omega_3)^2 \Delta S^{-1} \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \xi = (\tilde{M}\Delta k \xi_1 - \omega_3 \Delta\mu \xi_2) a_1 + (\tilde{K}\Delta\mu\xi_2 + \omega_3 \Delta k \xi_1) a_2, \tag{74a}$$

$$-(\tilde{K}\tilde{M} = \omega_3)^2 \eta \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \Delta S^{-1} = (\tilde{M}\Delta k \eta_1 + \omega_3 \Delta\mu \eta_2) a_1 + (\tilde{K}\Delta\mu\eta_2 - \omega_3 \Delta k \eta_1) a_2. \tag{74b}$$

Consider for example the case

$$\det \Delta_2 + \omega \cdot\cdot \Delta_2 \cdot\cdot \omega = 0,$$

or in view of (73),

$$K_2 M_2 + \omega_3^2 = 0. \tag{75}$$

This is a manifold in the space $(\omega_3, d)$, and the variations $\delta\omega = a_3 \delta\omega_3$, $\delta d$ are therefore linked by the relationship

$$2d\delta d - (k_2 - \mu_2)\delta d - 2\omega_3 \delta\omega_3 = 0$$

as we move along this manifold. The latter relation can be rewritten as

$$\delta d = 2\omega_3 \delta\omega_3 / (M_2 - K_2), \tag{76}$$

and instead of two necessary conditions $\phi_\omega = \phi_d = 0$ [see (44) and (46)], we arrive at only one condition,

$$\begin{aligned}&(\Delta S^{-1} \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \xi) \times (\eta \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \Delta S^{-1}) \cdot\cdot a_3 \\ &+ [2\omega_3/(M_2 - K_2)](\Delta S^{-1} \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \xi) \cdot\cdot T \\ &\cdot\cdot (\eta \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \Delta S^{-1}) = 0.\end{aligned} \tag{77}$$

This condition should hold along with (75).

Equation (77) can be transformed with the aid of Eqs. (74) defining the matrices $\Delta S^{-1} \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \xi$ and $\eta \cdot\cdot \langle S^{-1} \rangle^{-1} \cdot\cdot \Delta S^{-1}$. We arrive at the relationship

$$\begin{aligned}&[\tilde{M}\tilde{K} - \omega_3^2 + 2\omega_3^2(\tilde{M} - \tilde{K})/(M_2 - K_2)](\sigma - \zeta) \\ &+ 2\omega_3 \{[(\tilde{M}^2 - K_2 M_2)/(M_2 - K_2) - \tilde{M}](\Delta k/\Delta\mu) \\ &- [(\tilde{K}^2 - K_2 M_2)/(M_2 - K_2) + \tilde{K}](\Delta\mu/\Delta k)\sigma\zeta\} = 0.\end{aligned} \tag{78}$$

The expressions in the square brackets can be transformed as we use (75) to eliminate $\omega_3^2$. After some algebra, we arrive at the relationships

$$\tilde{M}\tilde{K} - \omega_3^2 + 2\omega_3^2(\tilde{M} - \tilde{K})/(M_2 - K_2) = [m_2/(M_2 - K_2)](\beta d + \gamma), \tag{79a}$$

$$\begin{aligned}&[(\tilde{M}^2 - K_2 M_2)/(M_2 - K_2) - \tilde{M}](\Delta k/\Delta\mu) \\ &- [(\tilde{K}^2 - K_2 M_2)/(M_2 - K_2) + \tilde{K}](\Delta\mu/\Delta k)\sigma\zeta = -m_2 c/(M_2 - K_2).\end{aligned} \tag{79b}$$

Here, the symbols $\beta, \gamma, c$ are defined as

$$\beta = -(u + v), \qquad \gamma = k_2 v - \mu_2 u, \qquad c = u - v\sigma\zeta, \tag{80}$$

where

$$u = (k_2 + \tilde{\mu})\Delta k, \qquad v = (\tilde{k} + \mu_2)\Delta\mu. \tag{81}$$

Equation (78) now shows that

$$\omega_3 = (1/2c)(\beta d + \gamma)(\sigma - \zeta). \tag{82}$$

We now use this relation to eliminate $\omega_3$ from (75). The result will be a quadratic equation for $d$,

$$d^2[\beta^2(\sigma-\zeta)^2-4c^2]+2[\beta\gamma(\sigma-\zeta)^2+2c^2(k_2-\mu_2)]d$$
$$+\gamma^2(\sigma-\zeta)^2+4c^2k_2\mu_2=0. \tag{83}$$

The discriminant of this equation is equal to

$$4c^2\{(\sigma-\zeta)^2(\gamma+\beta k_2)(\gamma-\beta\mu_2)+c^2(k_2+\mu_2)^2\}.$$

From Eqs. (80) and (81), it follows that

$$(\gamma+\beta k_2)(\gamma-\beta\mu_2)=-uv(k_2+\mu_2)^2,$$

and the discriminant turns out to be

$$4c^2(k_2+\mu_2)^2[-(\sigma^2-2\sigma\zeta+\zeta^2)uv+u^2-2uv\sigma\zeta+v^2\sigma^2\zeta^2]$$
$$=4v^2c^2(k_2+\mu_2)^2(\sigma^2-u/v)(\zeta^2-u/v).$$

Equation (83) now shows that

$$d=-\{1/[\beta^2(\sigma-\zeta)^2-4c^2]\}[\beta\gamma(\sigma-\zeta)^2+2c^2(k_2-\mu_2)$$
$$\mp 2vc(k_2+\mu_2)\sqrt{(\sigma^2-u/v)(\zeta^2-u/v)}]. \tag{84}$$

The corresponding values of $\omega_3$ will be

$$\omega_3=\{(\sigma-\zeta)/[\beta^2(\sigma-\zeta)^2-4c^2]\}\{-\beta c(k_2-\mu_2)-2\gamma c$$
$$\pm\beta v(k_2+\mu_2)\sqrt{(\sigma^2-u/v)(\zeta^2-u/v)}\},$$

or in view of (80),

$$\omega_3=-\{(k_2+\mu_2)(\sigma-\zeta)/[\beta^2(\sigma-\zeta)^2-4c^2]\}\{c(v-u)$$
$$\pm v(u+v)\sqrt{(\sigma^2-u/v)(\zeta^2-u/v)}\}. \tag{85}$$

Now, it is easy to compute the bilinear form (41). After some algebra, we obtain

$$\phi/\xi_1\eta_1=[(K_2M_2-K_1M_1)/(K_2\Delta\mu+\tilde{M}\Delta k)][K_2+M_2\sigma\zeta-\omega_3(\sigma-\zeta)]$$
$$+\omega_3(\sigma-\zeta)+d(1-\sigma\zeta).$$

Making use of (50) and (68), we reduce this to the form

$$\phi/\xi_1\eta_1=k_2+\mu_2\sigma\zeta+\{m_1\Delta k\Delta\mu/[k_2\Delta\mu+\tilde{\mu}\Delta k+d(\Delta k-\Delta\mu)]\}$$
$$\times[-k_2+d+\omega_3(\sigma-\zeta)-(d+\mu_2)\sigma\zeta]. \tag{86}$$

With the aid of (84) and (85), one can show that

$$-k_2 + d + \omega_3(\sigma - \zeta) - (d + \mu_2)\sigma\zeta$$
$$= \{(k_2 + \mu_2)/[(u + v)^2(\sigma - \zeta)^2 - 4c^2]\}\mathscr{L},$$
$$k_2\Delta\mu + \tilde{\mu}\Delta k + d(\Delta k - \Delta\mu)$$
$$= \{1/[(u + v)^2(\sigma - \zeta)^2 - 4c^2]\}\mathscr{M},$$

where

$$\mathscr{L} = 2(1 + \sigma\zeta)[c^2 - uv(\sigma - \zeta)^2] \pm v[2c(1 - \sigma\zeta) - (u + v)(\sigma - \zeta)^2] \times \sqrt{(\sigma^2 - u/v)(\zeta^2 - u/v)},$$
$$\mathscr{M} = 2uv(u + v)(\sigma - \zeta)^2 - 2c[2cu - (u - v)(c + v\sqrt{(\sigma^2 - u/v)(\zeta^2 - u/v)})].$$

Now, it is easy to check by direct inspection that

$$\mathscr{L}/\mathscr{M} = -(1/2u)[u/v + \sigma\zeta \pm \sqrt{(\sigma^2 - u/v)(\zeta^2 - u/v)}],$$

and from (86) we obtain

$$\phi/\xi_1\eta_1 = \phi_{2\pm}/\xi_1\eta_1$$
$$= k_2 + \mu_2\sigma\zeta - [m_1\Delta k\Delta\mu(k_2 + \mu_2)/2] \times [1/v + (\sigma\zeta/u) \pm (1/u)\sqrt{(\sigma^2 - u/v)(\zeta^2 - u/v)}]. \tag{87}$$

## 9. Case of Constrained Variations: Microstructure

The values (87) of $\phi_{2\pm}$ are attained by the rank 2 lamination, with the material $\mathscr{D}_1$ being the core and the layers being parallel to the main axes of $\xi$ and $\eta$. To show this, consider the formula (Ref. 11)

$$\mathscr{D}_0 = \mathscr{D}_2 + m_1[(\mathscr{D}_1 - \mathscr{D}_2)^{-1} + [2m_2/(k_2 + \mu_2)](\alpha_1 nnnn + \alpha_2 tttt)]^{-1}$$
$$= \mathscr{D}_2 + m_1 A^{-1}, \tag{88}$$

for the effective tensor $\mathscr{D}_0$ of such a composite assembled from materials $\mathscr{D}_1$ and $\mathscr{D}_2$ taken with volume fractions $m_1$ and $m_2$, respectively. The parameters $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, are linked with the geometric parameters $f$, $\rho$ of the microstructure (see Fig. 2) by the formulas

$$\alpha_1 = f(1 - \rho)/m_2, \qquad \alpha_2 = \rho/m_2.$$

The matrix $A$ in (88) can be represented in the form

$$A = \pi_{11}a_1a_1 + \pi_{12}(a_1a_2 + a_2a_1) + \pi_{22}a_2a_2 + \pi_{33}a_3a_3,$$

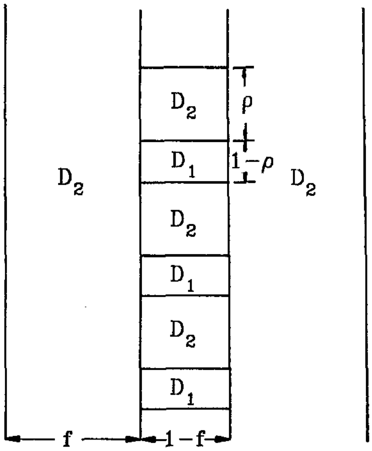


Fig. 2. Rank 2 laminates.

where

$$\pi_{11} = -(\tilde{k} + \mu_2)/[(k_2 + \mu_2)\Delta k]$$
$$= -v/[(k_2 + \mu_2)\Delta k \Delta\mu], \tag{89a}$$
$$\pi_{12} = m_2(2\alpha_1 - 1)/(k_2 + \mu_2), \tag{89b}$$
$$\pi_{22} = -(k_2 + \tilde{\mu})/[(k_2 + \mu_2)\Delta\mu]$$
$$= -u/[(k_2 + \mu_2)\Delta k \Delta\mu], \tag{89c}$$
$$\pi_{33} = -1/\Delta\mu, \tag{89d}$$

and the basis $a_1, a_2, a_3$ is chosen as suggested in (4) and (56), (57) with the unit vectors $i, j$ oriented along the main axes of $\xi$ and $\eta$.

The inverse matrix $A^{-1}$ is computed as

$$A^{-1} = (\pi_{22}/\chi)a_1 a_1 - (\pi_{12}/\chi)(a_1 a_2 + a_2 a_1)$$
$$+ (\pi_{11}/\chi)a_2 a_2 + (1/\pi_{33})a_3 a_3,$$

where $\chi$ is defined by the formula

$$\chi = \pi_{11}\pi_{22} - \pi_{12}^2.$$

The bilinear form $\xi \cdot\cdot\, \mathscr{D}_0 \cdot\cdot\, \eta$ depends obviously on $\alpha_1$; the extremal values of this parameter can be found from the relationship

$$(\xi \cdot\cdot\, A^{-1} \cdot\cdot\, \eta)_{\alpha_1} = \xi \cdot\cdot\, (A^{-1})_{\alpha_1} \cdot\cdot\, \eta = 0,$$

or equivalently from

$$\xi \cdot\cdot\, A^{-1} \cdot\cdot\, A_{\alpha_1} \cdot\cdot\, A^{-1} \cdot\cdot\, \eta = 0.$$

This one is easily reduced to

$$(\pi_{12}^2 + \pi_{11}\pi_{22})(\xi_1\eta_2 + \xi_2\eta_1) - 2\pi_{12}(\pi_{22}\xi_1\eta_1 + \pi_{11}\xi_2\eta_2) = 0,$$

and referring to (89) we obtain the extremal values of $\pi_{12}$,

$$\pi_{12} = [\pi_{11}/(\sigma + \zeta)][(u/v) + \sigma\zeta \mp \sqrt{(\sigma^2 - u/v)(\zeta^2 - u/v)}],$$

with $\xi = \xi_2/\xi_1$ and $\sigma = \eta_2/\eta_1$. With these values for $\pi_{12}$, it is easy to arrive at the following expression for the bilinear form:

$$\xi \cdot\cdot \mathscr{D}_0 \cdot\cdot \eta/\xi_1\eta_1 = k_2 + \mu_2\sigma\zeta - [m_1\Delta k\Delta\mu(k_2 + \mu_2)/2]$$
$$\times [(1/v) + \sigma\zeta/u \pm (1/u)\sqrt{(\sigma^2 - u/v)(\zeta^2 - u/v)}].$$

This expression is the same as (87), and the attainability of the latter bound is thereby proved. A result similar to (87) can be established if the condition

$$K_1 M_1 + \omega_3^2 = 0 \tag{90}$$

holds instead of (75). We then arrive at the formula

$$\begin{aligned}\phi/\xi_1\eta_1 &= \phi_{3\pm}/\xi_1\eta_1\\ &= k_1 + \mu_1\sigma\zeta + [m_2\Delta k\Delta\mu(k_1 + \mu_1)/2]\\ &\quad \times [(1/\bar{v}) + \sigma\zeta/\bar{u} \pm (1/\bar{u})\sqrt{(\sigma^2 - \bar{u}/\bar{v})(\zeta^2 - \bar{u}/\bar{v})}],\end{aligned} \tag{91}$$

with $\bar{u}$, $\bar{v}$ are defined as [cf. (81)]

$$\bar{u} = (k_1 + \tilde{\mu})\Delta k, \qquad \bar{v} = (\tilde{k} + \mu_1)\Delta\mu.$$

The values (91) are attained for the 2nd rank lamination, with material $\mathscr{D}_2$ being the core and layers parallel to the main axes of $\zeta$ and $\eta$.

To complete the classification of various ranges, mention should be made of the case where Eqs. (75) and (90) hold simultaneously. This question still remains open as well as that of the generality of the assumption $\omega = \omega_3 a_3$ in (49). Once these issues receive a solution and the corresponding additional formulas for $\phi$ are obtained, then the final operation of maximizing $\phi$ with respect to $m_1$ will be applied to construct the desired material pattern.

## 10. Appendix: Computation of $\langle S^{-1}\rangle^{-1}$

This procedure is similar to that applied to compute

$$\langle S\rangle^{-1} = [\langle\Delta\rangle - \omega \times E]^{-1};$$

cf. Eq. (30). We start with the expression

$$\langle S^{-1}\rangle = m_1 S_1^{-1} + m_2 S_2^{-1}$$
$$= m_1\delta_1 + m_2\delta_2 + (m_1\Omega_1 + m_2\Omega_2) \times E,$$

where $\delta_1(\delta_2)$ and $\Omega_1(\Omega_2)$ are defined, respectively, by Eqs. (31) and (32) in which we apply $\Delta_1(\Delta_2)$ instead of $\langle\Delta\rangle$.

Using the notation

$$\langle\delta\rangle = m_1\delta_1 + m_2\delta_2,$$
$$e\langle\Omega\rangle = m_1\Omega_1 + m_2\Omega_2,$$

we may now invert the matrix

$$\langle S^{-1}\rangle = \langle\delta\rangle + \langle\Omega\rangle \times E.$$

Referring to Eq. (30), we get

$$\langle S^{-1}\rangle^{-1} = [1/(\det\langle\delta\rangle + \langle\Omega\rangle \cdot\cdot \langle\delta\rangle \cdot\cdot \langle\Omega\rangle)]$$
$$\times \{\det\langle\delta\rangle\langle\delta\rangle^{-1} + \langle\Omega\rangle\langle\Omega\rangle - (\langle\Omega\rangle \cdot\cdot \langle\delta\rangle) \times E\},$$

where

$$\langle\delta\rangle = m_1\delta_1 + m_2\delta_2$$
$$= [m_1/(\det\Delta_1 + \omega \cdot\cdot \Delta_1 \cdot\cdot \omega)](\det\Delta_1 \cdot \Delta_1^{-1} + \omega\omega)$$
$$+ [m_2/(\det\Delta_2 + \omega \cdot\cdot \Delta_2 \cdot\cdot \omega)](\det\Delta_2 \cdot \Delta_2^{-1} + \omega\omega)$$
$$= \langle[\det\Delta/(\det\Delta + \omega \cdot\cdot \Delta \cdot\cdot \omega)]\Delta^{-1}\rangle$$
$$+ \langle 1/(\det\Delta + \omega \cdot\cdot \Delta \cdot\cdot \omega)\rangle\omega\omega$$
$$= \Phi + g\omega\omega,$$
$$\langle\Omega\rangle = m_1\Omega_1 + m_2\Omega_2$$
$$= [m_1/(\det\Delta_1 + \omega \cdot\cdot \Delta_1 \cdot\cdot \omega)]\omega \cdot\cdot \Delta_1$$
$$+ [m_2/(\det\Delta_2 + \omega \cdot\cdot \Delta_2 \cdot\cdot \omega)]\omega \cdot\cdot \Delta_2$$
$$= \omega \cdot\cdot \langle\Delta/(\det\Delta + \omega \cdot\cdot \Delta \cdot\cdot \omega)\rangle.$$

The matrix $\langle\delta\rangle = \Phi + g\omega\omega$ allows inversion,

$$\langle\delta\rangle^{-1} = (\Phi + g\omega\omega)^{-1}$$
$$= \Phi^{-1} - [g/(1 + g\omega \cdot\cdot \Phi^{-1} \cdot\cdot \omega)](\Phi^{-1} \cdot\cdot \omega)(\Phi^{-1} \cdot\cdot \omega).$$

We also compute $\det\langle\delta\rangle$,

$$\det\langle\delta\rangle = \det(\Phi + g\omega\omega)$$
$$= (\det\Phi)[1 + g(\omega \cdot\cdot \Phi^{-1} \cdot\cdot \omega)].$$

The final expression for $\langle S^{-1}\rangle^{-1}$ becomes

$$\begin{aligned}\langle S^{-1}\rangle^{-1} = (1/A)\{&\det \Phi[1 + g(\omega \cdot\cdot\ \Phi^{-1} \cdot\cdot\ \omega)]\Phi^{-1} \\ &- g \det \Phi(\phi^{-1} \cdot\cdot\ \omega)(\Phi^{-1} \cdot\cdot\ \omega) + \langle\Omega\rangle\langle\Omega\rangle \\ &- [\langle\Omega\rangle \cdot\cdot\ (\Phi + g \cdot\cdot\ )] \times E\},\end{aligned}$$

where

$$A = \det \Phi[1 + g(\omega \cdot\cdot\ \Phi^{-1} \cdot\cdot\ \omega)] + \langle\Omega\rangle \cdot\cdot\ (\Phi + g\omega\omega) \cdot\cdot\ \langle\Omega\rangle;$$

the matrices $\Phi$, $\langle\Omega\rangle$ and the scalar parameter $g$ are defined by the formulas

$$\Phi = \langle[\det \Delta/(\det \Delta + \omega \cdot\cdot\ \Delta \cdot\cdot\ \omega)]\Delta^{-1}\rangle,$$

$$\langle\Omega\rangle = \omega \cdot\cdot\ \langle\Delta/(\det \Delta + \omega \cdot\cdot\ \Delta \cdot\cdot\ \omega)\rangle,$$

$$g = \langle 1/(\det \Delta + \omega \cdot\cdot\ \Delta \cdot\cdot\ \omega)\rangle.$$

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