Semigroup Forum Vol. 9 (1975), 294-309

PRÜFER AND DEDEKIND MONOIDS

M. P. Dorofeeva, V. L. Mannepalli and
M. Satyanarayana

Communicated by B. M. Schein

Dedicated to Professor E. S. Ljapin on the occasion of his 60th birthday.

The aim of this paper is to provide many equivalent characterizations of Prüfer monoids. Throughout this paper we consider only commutative cancellative monoids, which are not groups. Then a monoid S has a unique maximal ideal M denoted by $M(S)$. S admits a quotient group $G = \{a/b: a,b \in S\}$ and $a/b = c/d$ if and only if $ad = bc$. S can be considered as a subsemigroup of G by the natural map $a \to a/1$, $a \in S$. With respect to a prime ideal P of S one can define a semigroup $S_P = \{a/b: a,b \in S, b \notin P\}$. Then S_P may be identified as a subsemigroup of G. Let S be a submonoid of a monoid T. An element $t \in T$ is said to be an integral element over S if there exists an integer n such that $t^n \in S$. The overmonoid T is called integral over S if all elements of T are integral over S. The set of all integral elements of T over S forms a submonoid of T, which is called the integral closure of S in T. The integral closure of S in its group of quotients is called the integral closure of S and is denoted by S^*. S is called integrally closed if $S = S^*$. If $t \in T$, $S[t]$ denotes a submonoid of T generated by S and t. A monoid T predominates over S if T is an overmonoid of S and $M(S) = S \cap M(T)$. If A and B are ideals in a monoid S, we denote $A: B = \{x \in S | Bx \subseteq A\}$. In particular if $x, y \in S$, then $(x: y) = \{s \in S: (yS)s \subseteq xS\}$. All S-systems considered in this paper are unitary left S-systems. An S-system M is called torsion free if for all $s \in S$ and $m_1, m_2 \in M$, $m_1 = m_2$ whenever $sm_1 = sm_2$. An S-system M is called strong torsion free if it is torsion free and for all $s_1, s_2 \in S$ and $m \in M$, $s_1 = s_2$ whenever $s_1m = s_2m$. We refer the reader to [2] for the definitions of projective and free S-systems. When S is a commutative cancellative monoid the concepts of projective S-system and free S-system coincide. A fractional ideal A of S is an S-subsystem of G (the group of quotients of S) such that $sA \subseteq S$ for some $s \in S$. It is obvious that every finitely generated S-subsystem of G is a fractional ideal and G is not a fractional ideal if $G \neq S$. A fractional ideal A is called principal if $A = aS$ for some $a \in G$. A fractional ideal A is said to be invertible if there exists a fractional ideal B such that $AB = S$. We denote the inverse of A by A^{-1}. As in [2], commutative cancellative hereditary and semi-hereditary monoids are called Dedekind and Prüfer monoids respectively. According to Stenström [4] an equivalent characterization of weakly flat S-system is: an S-system M is weakly flat if and only if $sx = ty$ with $x, y \in M$ and $s, t \in S \Rightarrow$ there exists a $z \in M$ and $s', t' \in S$ such that $x = s'z$ and $y = t'z$ and $ss' = tt'$.

THEOREM 1. *Let* S *be a commutative cancellative monoid with a quotient group* G. *Suppose that* T *is a overmonoid of* S *such that* $T \subset G$. *Then the following statements are equivalent:*

i) T *is a weakly flat* S-*system.*

ii) *For every prime ideal* P *in* S, *either* $PT = T$ *or* $T \subseteq S_P$.

iii) $(y: x)T = T$ *for all* $x/y \in T$.

Proof: (i) $\Rightarrow$ (ii). By the definition of weakly flat S-system, since $y(x/y) = 1(x)$, there exist $z \in T$, $s', t' \in S$ such that

$$x/y = s'z, \quad 1 = t'z \quad \text{and} \quad yx' = xt'.$$

Now let P be a prime ideal of S. If $t' \in P$, then $T = PT$, since $1 = zt'$. If $t' \notin P$, then clearly $(y: x) \not\subseteq P$. Thus we have $PT = T$ or $(y: x) \not\subseteq P$ for $x/y \in T$. Consider the latter case. Let $s \in (y: x)$ and $s \notin P$. Then $sx = ay$ for some $a \in S$, which implies $x/y \in S_P$. Thus we have $T \subseteq S_P$.

(ii) $\Rightarrow$ (iii). Suppose $(y: x)T \neq T$ for some $x/y \in T$. Apply Zorn's lemma to the collection of ideals A containing $(y: x)$ in S such that $AT \neq T$. Then there exists a maximal element in this collection, which can be proved to be a prime ideal. Thus we have a prime ideal P in S such that $(y: x) \subseteq P$ and $PT \neq T$. If (ii) holds in S, then $T \subseteq S_P$ and so

$x/y \in S_P$. Thus $x/y = a/s$, where $s \notin P$; hence $xs = ay$. This implies that $s \in (y\colon x) \subseteq P$, a contradiction.

(iii) $\Rightarrow$ (i). Let $sx = ty$, where $x, y \in T$ and $s, t \in S$. Write $x = x_1/x_2$ and $y = y_1/y_2$ where x_1, x_2, y_1 and $y_2 \in S$. Then we have $(x_2\colon x_1)T = T$ and $(y_2\colon y_1)T = T$. Therefore $p_1t_1 = 1$ and $p_2t_2 = 1$, where $t_1, t_2 \in T$, $p_1p_2 \in S$, $p_1x_1 = a_1x_2$ and $p_2y_1 = a_2y_2$ for some $a_1, a_2 \in S$. Taking $z = 1/p_1p_2$, $s' = a_1p_2$ and $t' = a_2p_1$, it can be verified easily that $ss' = tt'$. Thus T is weakly flat.

COROLLARY. *If* S, G *and* T *satisfy the hypothesis of Theorem 1 and* T *is any weakly flat (considered as a S-system) oversemigroup properly containing* S, *then* T *is not integral over* S.

Proof: Let T be integral over S and A an ideal of S. If $AT = T$, then $at = 1$ for some $a \in A$ and $t \in T$. Since t is an integral element over S, $t^n = s \in S$. Hence $sa^n = t^na^n = 1$, i.e., $1 \in A$. Thus $AT \neq T$ for all proper ideals A of S. Then by (ii) of Theorem 1 we have $T \subseteq S_{M(S)} = S$.

A direct verification along with the Corollary 1 of Theorem 1 of [2], yields the following two lemmas:

LEMMA 2. *Let* S *be a Prüfer monoid. Then a* S-*system* A *is weakly flat if and only if* A *is torsion free.*

LEMMA 3. *Let* S *be a submonoid of a monoid* T *and let* T *be integral over* S. *Then* T *is a group if and only if* S *is a group.*

LEMMA 4. Let S be a submonoid of a monoid T with T integral over S. If P' is a prime ideal of T and $P = S \cap P'$, then P is the maximal ideal of S if and only if P' is the maximal ideal of T.

Proof: Evidently P is a prime ideal of S. So $V = T \setminus P'$ and $U = S \setminus P$ are monoids. If $v \in V$, then $v^n \in S$ for some n. As P' is a prime ideal, $v^n \notin P'$. Hence $v^n \in S \setminus P = U$, i.e., V is integral over U. By Lemma 3 V is a group if and only if U is a group.

LEMMA 5. Let S be a submonoid of a monoid T. Then the following statements are equivalent:

i) $M(S) \subseteq M(T)$.

ii) T predominates over S.

iii) $T \cdot M(S) \neq T$.

The proof of Lemma 5 is analogous to the proof of Proposition 1 of [1; 375].

LEMMA 6. A fractional ideal A of a cancellative monoid S is invertible if and only if it is a principal fractional ideal.

Proof: If $AB = S$ for some fractional ideal B, then $1 = ab$, $a \in A$, $b \in B$. Let $x \in A$. Then $xb = s \in S$ and $x = xba = sa$. The other part is evident.

THEOREM 7. Let S be a submonoid of a group G. Then the following conditions are equivalent:

1) S is a maximal element in the set of all submonoids of G ordered by predominance relation.

2) There exists a divisible group H and a semigroup homomorphism f of the monoid S in $H \cup 0$, which is maximal in the set of all semigroup homomorphisms of submonoids of G in $H \cup 0$ ordered by the extension relation.

3) If $z \in G \setminus S$, then $z^{-1} \in S$.

If, furthermore, G is a group of quotients of S, then the statements (1) - (3) are equivalent to each one of the following statements:

4) The set of all principal ideals of S is linearly ordered.

5) The set of all ideals of S is linearly ordered.

6) $A(B \cap C) = AB \cap AC$ for all ideals A, B, C, of S.

7) $(A \cup B)(A \cap B) = AB$ for all ideals A, B of S.

8) Ideals of S generated by two elements are invertible.

9) Every finitely generated ideal of S is invertible.

10) All finitely generated fractional ideals of S form a group.

11) For all $z \in G$, the fractional ideal $S \cup Sz$ is invertible.

12) If C is a finitely generated ideal and A is an ideal of S such that $A \subseteq C$, then $A = BC$ for some ideal B.

13) All finitely generated fractional ideals of S form a cancellative monoid.

14) S *is integrally closed and for any* $a,b \in S$, *there exists an integer* $n > 1$ *such that*

$$(aS \cup bS)^n = a^nS \cup b^nS \quad .$$

15) S *is integrally closed and for any* $a,b \in S$, *there exists an integer* $n > 1$ *such that* $a^{n-1}b \in a^nS \cup b^nS$.

16) $(A \cup B): C = (A: C) \cup (B: C)$ *for all ideals* A, B, C *of* S, *with* C *finitely generated*.

17) $C: (A \cap B) = (C: A) \cup (C: B)$ *for all ideals* A, B, C *of* S *with* A *and* B *finitely generated*.

18) *Every monoid* T *such that* $S \subseteq T \subsetneq G$ *is integrally closed*.

19) *Every monoid* T *such that* $S \subseteq T \subsetneq G$ *is weakly flat in the sense of Stenström*.

20) *All finitely generated strong torsion free* S-*systems are free*.

21) *All finitely generated fractional ideals of* S *are free*.

22) S *is a Prüfer monoid*.

Proof: (1) $\Rightarrow$ (2). Let H be an arbitrary divisible group containing the group $S \setminus M(S)$. Let $f: S \to H \cup 0$ be a mapping such that $f(s) = s$ if $s \in S \setminus M(S)$, and $f(s) = 0$ if $s \in M(S)$. Then f is a semigroup homomorphism. Let T be a submonoid of $G, T \supsetneq S$ and $f': T \to H \cup 0$, be a semigroup homomorphism such that f' is an extension of f. Then $P = \{t \in T | f'(t) = 0\}$ is a prime ideal of T and $P \cap S = M(S)$. Since $P = M(T_P)$, T_P predominates over S. By assumption $T_P = S$. Hence $T = S$ since $T_P \supsetneq T$.

(2) $\Rightarrow$ (3). Let $z \in G$. If z is not an integral element over S, then $z \notin S[z^{-1}]$. Hence $z^{-1} \in M(S[z^{-1}])$ and so $S[z^{-1}] \setminus M(S[z^{-1}]) = S \setminus M(S)$. Then we can easily extend the mapping f given in (2) onto $S[z^{-1}]$. Then, by (2), $S[z^{-1}] = S$, i.e., $z^{-1} \in S$. Let now z be an integral element over S. If $z^r \in M(S)$ for some natural number r, then consider $f': S[z] \to H \cup 0$, where $f'(s) = f(s)$ and $f'(sz^k) = 0$ for $s \in S$. If $f(M(S)) \subseteq H$, then we can extend f onto the quotient group of S. Hence $f^{-1}(0) \neq \square$. Then $f^{-1}(0) = P$ is a prime ideal of S. If $P \neq M(S)$, then we can extend f onto S_P. Thus $f(M(S)) = 0$. By virtue of this f' is a semigroup homomorphism extending f onto $S[z]$ by an application of Lemma 4. Let n be the minimal number such that $z^n \in S$ and $z^n \notin M(S)$. Then $f(z^n) = h \in H$. Since H is divisible, there exists $h_1 \in H$ such that $h_1^n = h$. Then $f': S[z] \to H \cup 0$, such that $f'(sz^k) = f(s)h_1^k$ for all k, is a homomorphism and f' extends f. Hence by (2), $S[z] = S$, i.e., $z \in S$.

As in Theorem 1 of [1; 376], one can show that (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(5) $\Rightarrow$ (6). Trivial.

(6) $\Rightarrow$ (7). If (6) holds, we have for all ideals A and B of S

$$(A \cup B)(A \cap B) = (A \cup B)A \cap (A \cup B)B \supseteq AB$$

and the reverse inclusion always holds.

(7) $\Rightarrow$ (8). Let $A = a_1S \cup a_2S$ be an ideal of S. Then by (7), $(a_1S \cup a_2S)(a_1S \cap a_2S) = a_1a_2S$, which is invertible.

(8) $\Rightarrow$ (9). Let $A = a_1S \cup a_2S \cup \ldots \cup a_nS$ be a finitely generated ideal of S. By (8), $a_1S \cup a_2S$ is invertible. Hence by lemma 6 $a_1S \cup a_2S = bS$ for some $b \in S$. So $A = bS \cup a_3S \cup \ldots \cup a_nS$. Then by induction A is invertible.

(9) $\Rightarrow$ (10). It is easy to see that (9) $\Rightarrow$ (4). Let now $A = \bigcup_{i=1}^{n} Sz_i$, $z_i \in G$ be a fractional ideal. Since (4) $\Leftrightarrow$ (3), $Sz \supset S$ if $z \notin S$. Hence we can assume that $z_1, z_2, \ldots, z_n \in S$ or $z_1, z_2, \ldots, z_n \notin S$. In the first case A is invertible by (9). If $z_1, z_2, \ldots, z_n \notin S$, then $z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1} \in S$. By (4) we have for some m $Sz_m^{-1} \subseteq Sz_k^{-1}$, $k = 1,2,\ldots,n$. Then $Sz_k \subseteq Sz_m$ and $A = Sz_m$ i.e. A is invertible.

(10) $\Rightarrow$ (11). Trivial.

(11) $\Rightarrow$ (3). Let $z \in G \setminus S$ and $S \cup zS$ be invertible. Then $S \cup zS = uS$, $u \in G$. Hence $1 = us$ and $u = \lambda z$, λ, $s \in S$. So $\lambda zs = us = 1$, which implies $z^{-1} \in S$.

Thus we have established the equivalence of (1) through (11).

(4) $\Rightarrow$ (12). Let A and C be ideals of S with C finitely generated and such that $A \subseteq C$. Since principal ideals are linearly ordered and C is finitely generated, C is principal and so $A_1C = S$ for some fractional ideal A_1 of S. Hence $BC = A$, where $B = AA_1$ is an ideal of S.

(12) $\Rightarrow$ (9). Let A be a finitely generated ideal of S. Then $aS \subseteq A$ for $a \in A$. By hypothesis, $aS = AB$ for some ideal B of S. Since $(aS)(a^{-1}S) = S$, we

have $S = AB(a^{-1}S)$, and thus A is invertible.

(10) $\Rightarrow$ (13). Trivial.

(13) $\Rightarrow$ (3). Since $(S \cup Sz \cup Sz^2)(S \cup Sz)$ $= (S \cup Sz^2)(S \cup Sz)$, it follows $z \in Sz^2 \cup S$. Hence $z \in S$ or $z = sz^2$ for some $s \in S$, i.e., $z^{-1} \in S$.

(14) $\Rightarrow$ (15). Trivial.

(15) $\Rightarrow$ (16). Let $a,b \in S$. Then $a^{n-1}b \in a^nS \cup b^nS$, for some $n > 1$. If $a^{n-1}b \in a^nS$, then $b \in aS$ and thus we get $bS \subseteq aS$. If $a^{n-1}b \in b^nS$, then $(a/b)^{n-1} \in S$. Since S is integrally closed, this case shows that $a/b \in S$ and hence we get $aS \subseteq bS$. Thus principal ideals are linearly ordered and so the ideals of S are linearly ordered by the implication (4) $\Rightarrow$ (5). Then (15) is evident.

(16) $\Rightarrow$ (17). Let $a,b \in S$. Since $S = (aS \cup bS): (aS \cup bS)$ by (16), we have

$$S = [aS: (aS \cup bS)] \cup [bS: (aS \cup bS)]$$
$$= (aS: bS) \cup (bS: aS).$$

It follows then $bS \subseteq aS$ or $aS \subseteq bS$. Since (4) $\Rightarrow$ (5), the set of all ideals of S is linearly ordered and hence (17) is evident.

(17) $\Rightarrow$ (18). By a similar argument as above, we can show that the ideals are linearly ordered and hence (18) can be seen evident by combining Theorems 1 and 5 of [2].

(18) $\Rightarrow$ (19). Again by combining Theorems 1 and 5 of [2], the principal ideals and hence all the ideals of S are linearly ordered. Then from Theorem 1 of [2] and Lemma 2, (19) follows.

(19) $\Rightarrow$ (3). Let $z = x/y \notin S$ and $T = S[z]$. Since

T is weakly flat, by Theorem 1, we have $(y: x)T = T$. But $(y: x) \neq S$. So $1 = sz^n$, $s \in (y: x)$. Hence $z^{-n} = s \in S$, i.e., z^{-1} is an integral element over S. Then $S[z^{-1}]$ is weakly flat and is integral over S. This implies that, by Corollary of Theorem 1, $S[z^{-1}] = S$, i.e., $z^{-1} \in S$.

(3) $\Rightarrow$ (14). Let $x \in G \setminus S$ and $x^n \in S$ for some natural number n. Then by (3), $x^{-1} \in S$ and so $x = x^n(x^{-1})^{n-1} \in S$. Thus S is integrally closed. The last part of (14) is evident since (3) $\Rightarrow$ (4).

(20) $\Rightarrow$ (21) $\Rightarrow$ (22). Trivial.

(22) $\Rightarrow$ (4). Follows from Corollary 1 of Theorem 2 of [2].

(4) $\Rightarrow$ (20). Let M be a finitely generated strong torsion free S-system. Then $M = \bigcup_{i=1}^{n} Sm_i$ and $m_i \notin Sm_j$ if $i \neq j$. We claim that $Sm_i \cap Sm_j = \varnothing$ if $i \neq j$. If not, let $s_1m_i = s_2m_j$. By (4), we can write for definiteness $s_2 = \lambda s_1$, $\lambda \in S$. Then, since M is strong torsion free $m_i = \lambda m_j$, which is impossible. Since M is strong torsion free $s_1m_i = s_2m_i$ if and only if $s_1 = s_2$. Hence Sm_i is isomorphic to S, i.e., M is a free S-system.

PROPOSITION 8. Let G be a group and f be a homomorphism of a submonoid S of G in $H \cup 0$, where H is a divisible group and f maps 1 into 1 and $f(S) \neq 0$. Then there exists a monoid $T \subset G$, which is a group or a Prüfer monoid and a homomorphism $F: T \to H \cup 0$ such that $T \supset S$, F extends f and $F^{-1}(0) = \varnothing$ or $F^{-1}(0) = M(T)$.

Proof: Let $\mathcal{H}$ be the set of all homomorphisms of submonoids of G in $H \cup 0$, ordered by the extension relation. By analogy with Theorem 2 [1; 378] we can show that there exists a maximal element F of $\mathcal{H}$, which extends f. Let T be the domain of the map F. If $F^{-1}(0) = \square$, then F can be extended to a homomorphism of the group of quotients of T into $H \cup 0$. Hence in this case T is a group. If $F^{-1}(0) \neq \square$, then $F^{-1}(0) = P$ is a prime ideal. Hence by Theorem 7, T is a Prüfer monoid. Then F may be extended to a homomorphism of T_P into $H \cup 0$. Hence $T_P = T$ and $P = M(T)$.

<u>COROLLARY 9. Every submonoid S of a group G is predominated by at least one Prüfer submonoid of G.</u>

Proof: Let H be a divisible group containing the group $S \setminus M(S)$ and let f be the natural homomorphism from S into $H \cup 0$. Then by Proposition 8, there exists a Prüfer monoid which predominates over S.

<u>THEOREM 10. Let S be a submonoid of a group G. Then the integral closure S' of S in G is the intersection of all Prüfer submonoids of G which predominate over S.</u>

Proof: Let $x \in S'$ and T be a Prüfer submonoid of G which predominates over S. Since x is integral over T, by Lemma 4, we have $M(T[x]) \cap T = M(T)$, i.e., T[x] predominates over T. Then by Theorem 7, $x \in T$. Conversely, let $z \in G$ and z is not an integral element over S. Then $z^{-1} \in M(S[z^{-1}])$. Let T be a Prüfer submonoid of G which predominates over $S[z^{-1}]$. Since $z^{-1} \in M(T)$,

$z \notin T$. But $M(S) \subseteq M(S[z^{-1}])$. Hence T predominates over S.

COROLLARY 11. A cancellative monoid is integrally closed if and only if it is an intersection of Prüfer submonoids of its group of quotients.

THEOREM 12. Let S be a commutative cancellative monoid and G be its group of quotients. Then the following conditions are equivalent:

1) S is a Dedekind monoid.

2) $S = F \times N$ or $S = F$, where F is a group and N is the additive monoid of non-negative integers.

3) All ideals of S are principal.

4) S is a group or a Noetherian integrally closed monoid with unique prime ideal different from S.

5) Every ideal in S is a product of prime ideals.

6) Every fractional ideal of S is free.

7) If $A \subseteq B$, A, B ideals of S, then there is an ideal C of S such that $A = BC$.

8) Every strong torsion free finitely generated S-system is free and a strong torsion free S-system which does not have a finite system of generators, is a union of disjoint S-systems, each of which is isomorphic to S or G.

9) All fractional ideals of S form a group.

10) All fractional ideals of S form a cancellative monoid.

Proof: The equivalence of (1), (2), (3), (4) and (5) was proved in Theorem 4 of [2].

(2) $\Rightarrow$ (6) $\Rightarrow$ (1). Trivial.

(3) $\Leftrightarrow$ (7). Evident from the proof of Theorem 7.

(2) $\Rightarrow$ (8). Since (1) $\Leftrightarrow$ (2), every finitely generated strong torsion free S-system is free. Suppose that M is a strong torsion free S-system which does not have finite system of generators and $L = \{m_\alpha\}$ be the system of generators of M. If $Sm_\alpha \cap Sm_\beta \neq \emptyset$, then $Sm_\alpha \subseteq Sm_\beta$ or $Sm_\beta \subseteq Sm_\alpha$ since M is strong torsion free. Let $Sm_\alpha \cap Sm_\beta \neq \emptyset$ for all $m_\alpha, m_\beta \in L$. Then we can so number L that $Sm_1 \subset Sm_2 \subset \dots \subset Sm_\alpha \subset \dots$ By (2), we can consider that $m_1 = a^{r_2} m_2 = a^{r_3} m_3 = \dots$, where a is a generator of the monoid N and $r_2 < r_3 < \dots$ Hence L is a countable set. Since M is strong torsion free Sm_α is isomorphic to S for all $m_\alpha \in L$. Hence M is isomorphic to $\bigcup_{i=1}^{\infty} Sa^{-r_i} = G(r_1 = 0)$. Let now $Sm_\alpha \cap Sm_\beta = \emptyset$ for some $m_\alpha, m_\beta \in L$. By Zorn's lemma there is a maximal subsystem of L, $L' = \{m'_\gamma\}$ such that $Sm'_\gamma \cap Sm'_\delta = \emptyset$ for all $m'_\gamma \neq m'_\delta$ of L'. For every $m'_\gamma \in L'$ consider the set $K_\gamma = \{m_\alpha \in L: Sm_\alpha \supseteq Sm'_\gamma\}$. As it was proved above $\cup K_\gamma$ is a system of generators of M, sybsystems SK_γ are disjoint and each SK_γ is isomorphic to S or G.

(8) $\Rightarrow$ (1). Each ideal of S is a strong torsion free S-system. Let I be an ideal of S and $\phi: G \to I$ be an isomorphism. If $\phi(1) = s$, then

$$s = \phi(1) = \phi(ss^{-1}) = s\phi(s^{-1})$$

which is impossible. Hence by (8), each ideal of S is isomorphic to S, which proves that S is a Dedekind monoid.

(2) $\Rightarrow$ (9) $\Rightarrow$ (10). Trivial.

(10) $\Rightarrow$ (3). By Theorem 7, S is a Prüfer monoid. If there exist non-invertible $a,b \in S$ such that $Sa^n \supset Sb$ for every n, then $I = \bigcup_{m=1}^{\infty} Sa^{-m}$ is a fractional ideal. Then $Sa^{-1}I = Sa^{-2}I$, which leads to a contradiction by virtue of (10). Consider now $\overline{S} = S/\theta$, where $s_1\theta s_2 \Leftrightarrow s_1 = \lambda s_2$, $\lambda \in S$, λ is invertible in S. Let $\overline{s}$ be an image of $s \in S$ in $\overline{S}$. $\overline{S}$ is naturally linearly ordered cancellative archimedean monoid and hence $\overline{S}$ is a submonoid of the additive monoid of the real numbers by Theorem 2 of [3; 165]. Suppose $\overline{S}$ does not satisfy the ascending chain condition for principal ideals. Then the maximal ideal of $\overline{S}$ is a union of an infinite ascending chain of principal ideals. Hence in $\overline{S}$ there exist $\overline{a},\overline{b},\overline{a}_1,\overline{a}_2,\ldots,\overline{a}_\alpha,\ldots,\overline{b}_1,\overline{b}_2,\ldots,\overline{b}_\beta,\ldots$ such that $\overline{Sa_1} \subset \overline{Sa_2} \subset \ldots \subset \overline{Sa_\alpha} \subset \ldots$, $\overline{Sa} \setminus \bigcup \overline{Sa_\alpha} = \overline{a}$, and $\overline{Sb_1} \subset \overline{Sb_2} \subset \ldots \subset \overline{Sb_\beta} \subset \ldots$, $\overline{Sb} \setminus \bigcup \overline{Sb_\beta} = \overline{b}$. Then

$$(\overline{Sa})(\bigcup \overline{Sb_\beta}) = (\bigcup \overline{Sa_\alpha})(\bigcup \overline{Sb_\beta}).$$

Hence it follows $Sa(\bigcup Sb_\beta) = (\bigcup Sa_\alpha)(\bigcup Sb_\beta)$. By condition (10), we have then $Sa = \bigcup Sa_\alpha$, which leads to a contradiction. Thus $\overline{S}$ and hence S satisfies ascending chain condition for principal ideals.

REFERENCES

1. Bourbaki, N., Elements of mathematics, Commutative Algebra, Addison-Wesley Publishing Company (1972) (in English).

2. Dorofeeva, M. P., Hereditary and semi-hereditary monoids, Semigroup Forum 4 (1972), 301-311.

3. Fuchs, L., Partially ordered algebraic systems, Pergamon Press (1963).

4. Stenström, Bo., Flatness and localization over monoids, Math. Nachr. 48 (1971), 315-334.

Chair of Algebra
Mechanics-Mathematical Faculty
Moscow University, U.S.S.R.

Department of Mathematics
Bowling Green State University
Bowling Green, Ohio, U.S.A.

Received February 6, 1974.