Semigroup Forum Vol. 14 (1977) 247-263

RESEARCH ARTICLE

EMBEDDING SEMIGROUPS IN SEMIBANDS

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Communicated by G. Lallement

1. INTRODUCTION

We shall use the notations and terminology of [3]. A semiband is an idempotent-generated semigroup. If n is any non-zero cardinal number, we shall say that a semiband is of type n if a minimal set of idempotent generators of the semiband has cardinality n [1].

In [8] Howie shows that any semigroup can be embedded in a semiband (see also [1]). We shall give a more easy embedding theorem here. We shall also show that any semigroup can be embedded in a simple semiband and in a bisimple semiband. Furthermore, we show that any completely semisimple semigroup can be embedded in a completely semisimple semiband, and that any completely regular semigroup can be embedded in a completely regular semiband. We shall make some remarks concerning semibands of type 3.

2. THE IDEMPOTENT-GENERATED HULL OF A SEMIGROUP

Let S be any semigroup. Let Y be a set such that $Y \cap S = \Box$ and such that for some $\bar{h} \in Y$ we have $|S| = |Y \setminus \{\bar{h}\}|$. Let $S \to Y \setminus \{\bar{h}\}$, $x \to \bar{x}$ be a one-to-one mapping of S onto $Y \setminus \{\bar{h}\}$. Let F be the semigroup which is generated by the elements of Y, subject to the defining relations $\bar{h}^2 = \bar{h}$, and $\bar{x}^2 = \bar{x}$ for all $x \in S$. Let K be the subsemigroup of F which is generated by the elements of $Z = \{\bar{h}\bar{x}\bar{h} \parallel x \in S\}$. It must be clear that K is a free semigroup which is freely generated by the elements of Z. Hence, there exists a homomorphism ϕ of K onto S which extends the mapping $Z \to S$, $\bar{h}\bar{x}\bar{h} \to x$. Putting $\beta = \phi\circ\phi^{-1}$, we have $K/_\beta \cong S$. Let β generate α on F. We call $F/_\alpha = \mathcal{I}(S)$ the idempotent-generated hull of semigroup S. $\mathcal{I}(S)$ is a semiband which is generated by the elements of Y, subject to the defining relations

$$\bar{h}^2 = \bar{h},$$
$$\bar{x}^2 = \bar{x} \qquad \text{for all } x\in S,$$
$$\bar{h}\bar{x}\bar{h}\bar{y}\bar{h} = \bar{h}\overline{xy}\bar{h} \qquad \text{for all } x,y\in S.$$

LEMMA 2.1. $K\alpha = \{w\in F \parallel (v,w)\in\alpha \text{ for some } v\in K\} = K$ and $\beta = \alpha\cap K\times K$.

PROOF. Since α is the congruence on F which is generated by β, we have $\beta\subseteq\alpha\cap K\times K$. Let us suppose $v\in K$, and $(v,w)\in\alpha$. Then there exist $n \geqslant 1$, elements $p_i\in K$, $i = 1,\ldots,2n$, elements $u_j, v_j \in F^1$, $j = 1,\ldots,n$, such that

$$(p_{2j-1}, p_{2j})\in\beta \qquad , j = 1,\ldots,n,$$
$$v = u_1 p_1 v_1 \qquad , w = u_n p_{2n} v_n ,$$
$$u_j p_{2j} v_j = u_{j+1} p_{2j+1} v_{j+1}, \quad j = 1,\ldots,n-1.$$

We have $(p_1, p_2)\in\beta$, and $p_1, p_2, u_1 p_1 v_1 = v\in K$. Let $v = \bar{h}\bar{x}_1\bar{h}\ldots \ldots\bar{h}\bar{x}_k\bar{h}$; consequently $p_1 = \bar{h}\bar{x}_m\bar{h}\ldots\bar{h}\bar{x}_{m+q}\bar{h}$, $1\leqslant m\leqslant k$, $0\leqslant q\leqslant k-m$. From this we have

$$u_1 = 1 \text{ or } \bar{h} \qquad \text{if } m = 1,$$
$$u_1 = \bar{h}\bar{x}_1\bar{h}\ldots\bar{h}\bar{x}_{m-1}\bar{h} \text{ or } \bar{h}\bar{x}_1\bar{h}\ldots\bar{h}\bar{x}_{m-1} \text{ otherwise},$$

and

$$v_1 = 1 \text{ or } \bar{h} \qquad \text{if } q = k-m,$$
$$v_2 = \bar{h}\bar{x}_{m+q+1}\bar{h}\ldots\bar{h}\bar{x}_k\bar{h} \text{ or } \bar{x}_{m+q+1}\bar{h}\ldots\bar{h}\bar{x}_k\bar{h} \text{ otherwise}.$$

Let $u_1' = 1$ if $m = 1$, and $u_1' = u_1\bar{h}$ otherwise; let $v_1' = 1$ if $q = k-m$, and $v_1' = \bar{h}v_1$ otherwise. Since $p_1, p_2\in K$ we have $\bar{h}p_1 = p_1\bar{h} = \bar{h}p_1\bar{h} = p_1$ and $\bar{h}p_2 = p_2\bar{h} = \bar{h}p_2\bar{h} = p_2$, and thus we have $u_1 p_1 v_1 = u_1'p_1 v_1'$ and $u_1 p_2 v_1 = u_1'p_2 v_1'$. Since u_1', $v_1'\in K^1$ we have $u_1'p_2 v_1' = u_2 p_3 v_2\in K$; furthermore $(p_1, p_2)\in\beta$ implies $(u_1'p_1 v_1', u_1'p_2 v_2')\in\beta$, and thus $(u_1 p_1 v_1, u_1 p_2 v_1)\in\beta$. By induction we can show that $u_j p_{2j-1} v_j$, $u_j p_{2j} v_j \in K$, and $(u_j p_{2j-1} v_j, u_j p_{2j} v_j)\in\beta$ for all $j = 1,\ldots,n$. Consequently $w\in K$ and $(v,w)\in\beta$. We conclude that $K\alpha = K$ and $\beta = \alpha\cap K\times K$.

THEOREM 2.2. *Any semigroup S can be embedded in the semiband* $\mathcal{A}(S)$

PROOF. By the preceding lemma we know that $K/_\beta \cong S$ a subsemigroup of the semiband $F/_\alpha = \mathcal{A}(S)$.

THEOREM 2.3. *Any countable semigroup can be embedded in a semiband of type* $n \leqslant 3$.

PROOF. Let S be a semigroup generated by the two elements a and b. Let $\bar{h}\bar{x}\bar{h}$ be any element of $K/_\beta$; if $x = a^{k_1} b^{k_2} \ldots$ $\ldots b^{k_m}$ in S we have $\bar{h}\bar{x}\bar{h} = (\bar{h}\bar{a}\bar{h})^{k_1} (\bar{h}\bar{b}\bar{h})^{k_2} \ldots (\bar{h}\bar{b}\bar{h})^{k_m}$ in $\mathcal{A}(S)$. We conclude that $S \cong K/_\beta$ can be embedded in the subsemigroup of $\mathcal{A}(S)$ which is generated by the three idempotents $\bar{h}, \bar{a}$ and $\bar{b}$. Since any countable semigroup can be embedded in a semigroup generated by two elements (theorem II of [5]; see also §9.1 of [3]), it follows that any countable semigroup can be embedded in a semiband of type $n \leqslant 3$.

In this paragraph from now on we shall suppose that S is a monoid with identity e. Let $X = S \cup (Y \setminus \{\bar{h}\})$, and let us consider the following elements of the full transformation semigroup $\mathcal{T}_X$:

$$\tilde{h} : X \to X, \quad x \to x$$
$$\bar{x} \to x \quad \text{for all } x \in S,$$

and for all $s \in S$

$$\tilde{s} : X \to X, \quad x \to \overline{xs}$$
$$\bar{x} \to \bar{x} \quad \text{for all } x \in S.$$

The subsemigroup of $\mathcal{T}_X$ generated by the elements $\tilde{h}$, $\tilde{s}$ ($s \in S$), will be denoted by $\widetilde{\mathcal{A}(S)}$. This semigroup $\widetilde{\mathcal{A}(S)}$ has been mentioned in [1].

LEMMA 2.4. *In the semigroup* $\widetilde{\mathcal{A}(S)}$ *the following equalities must hold* :

$$\tilde{h}^2 = \tilde{h},$$
$$\tilde{x}^2 = \tilde{x} \quad \text{for all } x \in S,$$
$$\tilde{h}\tilde{x}\tilde{h}\tilde{y}\tilde{h} = \tilde{h}\widetilde{xy}\tilde{h} \quad \text{for all } x,y \in S.$$

There exists a homomorphism ψ *of* $\mathcal{A}(S)$ *onto* $\widetilde{\mathcal{A}(S)}$ *such that* $\bar{h}\psi = \tilde{h}$ *and* $\bar{x}\psi = \tilde{x}$ *for all* $x \in S$. *The restriction of* ψ *to* $K/_\beta \cong S$ *is an isomorphism*.

PROOF. The first part of the lemma is straightforward. Let us suppose that for some $\bar h\bar x\bar h$, $\bar h\bar y\bar h \in K/_\beta$ we have $(\bar h\bar x\bar h)\psi = (\bar h\bar y\bar h)\psi$. Then $\tilde h\tilde x\tilde h = \tilde h\tilde y\tilde h$, and thus $x = e\tilde h\tilde x\tilde h = e\tilde h\tilde y\tilde h = y$; consequently $\bar h\bar x\bar h = \bar h\bar y\bar h$. We conclude that ψ maps $K/_\beta \cong S$ isomorphically onto the subsemigroup of $\widetilde{\mathcal{K}}(S)$ consisting of the elements $\tilde h\tilde x\tilde h$ $(x \in S)$.

THEOREM 2.5. *Any monoid S can be embedded in the semiband* $\widetilde{\mathcal{K}}(S)$.

LEMMA 2.6. *In the semigroup* $\widetilde{\mathcal{K}}(S)$ *the following equalities must hold* :

$$\tilde h\tilde e = \tilde e,$$
$$\tilde e\tilde h = \tilde h,$$
$$\tilde s\tilde t = \tilde s \quad \textit{for all } s,t \in S,$$
$$\tilde h\tilde s\tilde h\tilde t = \tilde h\widetilde{st} \quad \textit{for all } s,t \in S.$$

The elements of $\widetilde{\mathcal{K}}(S)$ *are* :

$\tilde h, \tilde e, \tilde s, \tilde h\tilde s, \tilde s\tilde h, \tilde h\tilde s\tilde h, \tilde s\tilde h\tilde t, \tilde s\tilde h\tilde t\tilde h$, *with* $s,t \in S\setminus\{e\}$.

These elements are all different except for the following cases :

$$\tilde s\tilde h\tilde t = \tilde h\tilde t \iff \tilde s\tilde h\tilde t\tilde h = \tilde h\tilde t\tilde h \iff t = st$$
$$\tilde s\tilde h\tilde t = \tilde v\tilde h\tilde t \iff \tilde s\tilde h\tilde t\tilde h = \tilde v\tilde h\tilde t\tilde h \iff st = vt \quad ,$$
with $v,s,t \in S\setminus\{e\}$.

LEMMA 2.7. (i) *If* L *is an* $\mathcal{L}$*-class of* $(K/_\beta)\psi \cong S$, *then* $\bigcup_{s\in S}\tilde sL$ *is the* $\mathcal{L}$*-class of* $\widetilde{\mathcal{K}}(S)$ *containing* L.

(ii) *If* R *is an* $\mathcal{R}$*-class of* $(K/_\beta)\psi \cong S$, *then* $R \cup R\tilde e$ *is the* $\mathcal{R}$*-class of* $\widetilde{\mathcal{K}}(S)$ *containing* R.

(iii) *If* D *is a* $\mathcal{D}$*-class of* $(K/_\beta)\psi \cong S$, *then* $(\bigcup_{s\in S}\tilde sD) \cup (\bigcup_{s\in S}\tilde sD\tilde e)$ *is the* $\mathcal{D}$*-class of* $\widetilde{\mathcal{K}}(S)$ *containing* D.

PROOF. (i) Let $\tilde h\tilde x\tilde h$ be any element of L. The $\mathcal{L}$-class of $\widetilde{\mathcal{K}}(S)$ containing this element will be denoted by $L_{\tilde h\tilde x\tilde h}$. We must show that $L_{\tilde h\tilde x\tilde h} = \bigcup_{s\in S}\tilde sL$. Consider $\tilde h\tilde y\tilde h \in L$ and any $s \in S$; then $\tilde e(\tilde s\tilde h\tilde y\tilde h) = \tilde e\tilde h\tilde y\tilde h = \tilde h\tilde y\tilde h$, and consequently $\tilde s\tilde h\tilde y\tilde h$ and $\tilde h\tilde y\tilde h$ will be $\mathcal{L}$-related in $\widetilde{\mathcal{K}}(S)$; since $\tilde h\tilde x\tilde h$ and $\tilde h\tilde y\tilde h$ are $\mathcal{L}$-related in $(K/_\beta)\psi$, they must also be $\mathcal{L}$-related in $\widetilde{\mathcal{K}}(S)$, and we can conclude that $\tilde s\tilde h\tilde y\tilde h$ is $\mathcal{L}$-related with $\tilde h\tilde x\tilde h$ in $\widetilde{\mathcal{K}}(S)$. We have shown that $\bigcup_{s\in S}\tilde sL \subseteq L_{\tilde h\tilde x\tilde h}$ (observe that $\tilde eL = L$, and thus $L \subseteq \bigcup_{s\in S}\tilde sL$).

All elements of $L_{\tilde{h}\tilde{x}\tilde{h}}$ must be $\mathcal{L}$-related with $\tilde{h}\tilde{x}\tilde{h}$ in $\mathcal{T}_X$, i.e. they must have the same range as $\tilde{h}\tilde{x}\tilde{h}$ (see lemma 2.5 of [3]). Therefore the elements $\tilde{t}$, $\tilde{h}\tilde{t}$, $\tilde{t}\tilde{h}\tilde{v}$, with $t,v\in S$, cannot belong to $L_{\tilde{h}\tilde{x}\tilde{h}}$. Let us investigate the range of elements $\tilde{h}\tilde{x}\tilde{h}$, $\tilde{t}\tilde{h}$, $\tilde{h}\tilde{t}\tilde{h}$, $\tilde{t}\tilde{h}\tilde{v}\tilde{h}$, with $t,v\in S$:

$$X\,\tilde{h}\tilde{x}\tilde{h} = Sx,$$
$$X\,\tilde{t}\tilde{h} = S,$$
$$X\,\tilde{h}\tilde{t}\tilde{h} = St,$$
$$X\,\tilde{t}\tilde{h}\tilde{v}\tilde{h} = Sv.$$

If $\tilde{h}\tilde{x}\tilde{h}$ and $\tilde{t}\tilde{h}$ are $\mathcal{L}$-related in $\widetilde{\mathcal{A}}(S)$ for some $t\in S$, then $Sx = S = Se$, and in this case we will have that x and e are $\mathcal{L}$-related in S; this implies that $\tilde{h} = \tilde{h}\tilde{e}\tilde{h}$ belongs to L, and consequently $\tilde{t}\tilde{h}\in \bigcup_{s\in S} \tilde{s}L$. If $\tilde{h}\tilde{x}\tilde{h}$ and $\tilde{h}\tilde{t}\tilde{h}$ are $\mathcal{L}$-related in $\widetilde{\mathcal{A}}(S)$ for some $t\in S$, then $Sx = St$, and in this case x and t are $\mathcal{L}$-related in S; this implies that $\tilde{h}\tilde{t}\tilde{h}$ belongs to L, and consequently $\tilde{h}\tilde{t}\tilde{h} \in \bigcup_{s\in S}\tilde{s}L$. If $\tilde{h}\tilde{x}\tilde{h}$ and $\tilde{t}\tilde{h}\tilde{v}\tilde{h}$ are $\mathcal{L}$-related in $\widetilde{\mathcal{A}}(S)$ for some $t,v\in S$, then $Sx = Sv$, and in this case we will have that x and v are $\mathcal{L}$-related in S; this implies that $\tilde{h}\tilde{v}\tilde{h}$ belongs to L, and consequently $\tilde{t}\tilde{h}\tilde{v}\tilde{h} \in \bigcup_{s\in S}\tilde{s}L$. Hence we can conclude that $L_{\tilde{h}\tilde{x}\tilde{h}} = \bigcup_{s\in S}\tilde{s}L$.

(ii) Let $\tilde{h}\tilde{x}\tilde{h}$ be any element of R. The $\mathcal{R}$-class of $\widetilde{\mathcal{A}}(S)$ containing this element will be denoted by $R_{\tilde{h}\tilde{x}\tilde{h}}$. We must show that $R_{\tilde{h}\tilde{x}\tilde{h}} = R\cup R\tilde{e}$. Consider any element $\tilde{h}\tilde{z}\tilde{h}\in R$; then $(\tilde{h}\tilde{z}\tilde{h})\tilde{e} = \tilde{h}\tilde{z}\tilde{e} = \tilde{h}\tilde{z}$, and consequently $\tilde{h}\tilde{z}$ and $\tilde{h}\tilde{z}\tilde{h}$ will be $\mathcal{R}$-related in $\widetilde{\mathcal{A}}(S)$; since $\tilde{h}\tilde{x}\tilde{h}$ and $\tilde{h}\tilde{z}\tilde{h}$ are $\mathcal{R}$-related in $(K/_\beta)\psi$, they must also be $\mathcal{R}$-related in $\widetilde{\mathcal{A}}(S)$, and we can conclude that $\tilde{h}\tilde{z} = (\tilde{h}\tilde{z}\tilde{h})\tilde{e}$ is $\mathcal{R}$-related with $\tilde{h}\tilde{x}\tilde{h}$ in $\widetilde{\mathcal{A}}(S)$. We have shown that $R\cup R\tilde{e}\subseteq R_{\tilde{h}\tilde{x}\tilde{h}}$.

If $\tilde{h}\tilde{x}\tilde{h}$ and $\tilde{h}\tilde{v}\tilde{h}$ are $\mathcal{R}$-related in $\widetilde{\mathcal{A}}(S)$ for some $v\in S$, we must have $\tilde{h}\tilde{x}\tilde{h}\in \tilde{h}\tilde{v}\tilde{h}\widetilde{\mathcal{A}}(S)$. Since for all $s,t\in S$

$$X\tilde{h}\tilde{x}\tilde{h} = Sx, \quad X(\tilde{h}\tilde{v}\tilde{h})\tilde{s}\subsetneq Y, \quad X(\tilde{h}\tilde{v}\tilde{h})(\tilde{h}\tilde{s})\subseteq Y, \quad X(\tilde{h}\tilde{v}\tilde{h})(\tilde{s}\tilde{h}\tilde{t})\subseteq Y,$$

$\tilde{h}\tilde{x}\tilde{h}$ cannot be equal to $(\tilde{h}\tilde{v}\tilde{h})\tilde{s}$ or $(\tilde{h}\tilde{v}\tilde{h})(\tilde{h}\tilde{s})$ or $(\tilde{h}\tilde{v}\tilde{h})(\tilde{s}\tilde{h}\tilde{t})$ for some $s,t\in S$. If $\tilde{h}\tilde{x}\tilde{h} = (\tilde{h}\tilde{v}\tilde{h})(\tilde{h}\tilde{s}\tilde{h})$ for some $s\in S$, then $\tilde{h}\tilde{x}\tilde{h} = \tilde{h}\widetilde{vs}\tilde{h}$; if $\tilde{h}\tilde{x}\tilde{h} = (\tilde{h}\tilde{v}\tilde{h})(\tilde{s}\tilde{h})$ for some $s\in S$, then again $\tilde{h}\tilde{x}\tilde{h} = \tilde{h}\widetilde{vs}\tilde{h}$; if $\tilde{h}\tilde{x}\tilde{h} = (\tilde{h}\tilde{v}\tilde{h})(\tilde{s}\tilde{h}\tilde{t}\tilde{h})$ for some $s,t\in S$, then $\tilde{h}\tilde{x}\tilde{h} = \tilde{h}\widetilde{vs}\tilde{t}\tilde{h}$; in all cases we can conclude that $x\in vS$ by lemma 2.6. Analogously we will have $v \in xS$. This implies that x and v are $\mathcal{R}$-related in S, and consequently $\tilde{h}\tilde{v}\tilde{h} \in R$.

If $\tilde{h}\tilde{x}\tilde{h}$ and $\tilde{h}\tilde{v}$ are $\mathcal{R}$-related in $\widetilde{\mathcal{T}(S)}$ for some $v \in S$, then $(\tilde{h}\tilde{v}\tilde{h})\tilde{e} = \tilde{h}\tilde{v}\tilde{e} = \tilde{h}\tilde{v}$ shows that $\tilde{h}\tilde{v}\tilde{h}$ will be $\mathcal{R}$-related with $\tilde{h}\tilde{x}\tilde{h}$ in $\widetilde{\mathcal{T}(S)}$. By the foregoing this implies $\tilde{h}\tilde{v}\tilde{h} \in R$, and thus $\tilde{h}\tilde{v} \in R\tilde{e}$.

If $\tilde{h}\tilde{x}\tilde{h}$ and $\tilde{s}$ are $\mathcal{R}$-related in $\widetilde{\mathcal{T}(S)}$ for some $s \in S$ we would have $\tilde{h}\tilde{s} = \tilde{s}$ since $\tilde{h}(\tilde{h}\tilde{x}\tilde{h}) = \tilde{h}\tilde{x}\tilde{h}$; we have $\tilde{h}\tilde{s} = \tilde{s}$ if and only if $\tilde{s} = \tilde{e}$; but $\tilde{e}$ is $\mathcal{R}$-related with $\tilde{h} = \tilde{h}\tilde{e}\tilde{h}$ in $\widetilde{\mathcal{T}(S)}$. In this case $\tilde{h}\tilde{x}\tilde{h}$ and $\tilde{h}\tilde{e}\tilde{h}$ are $\mathcal{R}$-related in $\widetilde{\mathcal{T}(S)}$, and by the foregoing this implies that $\tilde{h} = \tilde{h}\tilde{e}\tilde{h} \in R$, and thus $\tilde{s} = \tilde{e} \in R\tilde{e}$.

If $\tilde{h}\tilde{x}\tilde{h}$ and $\tilde{s}\tilde{h}$ are $\mathcal{R}$-related in $\widetilde{\mathcal{T}(S)}$ for some $s \in S$, then $(\tilde{s}\tilde{h})\tilde{e} = \tilde{s}\tilde{e} = \tilde{s}$ shows that $\tilde{s}$ will be $\mathcal{R}$-related with $\tilde{h}\tilde{x}\tilde{h}$ in S. By the foregoing this implies $\tilde{s} = \tilde{e} \in R\tilde{e}$, and thus $\tilde{h} = \tilde{s}\tilde{h} \in R$.

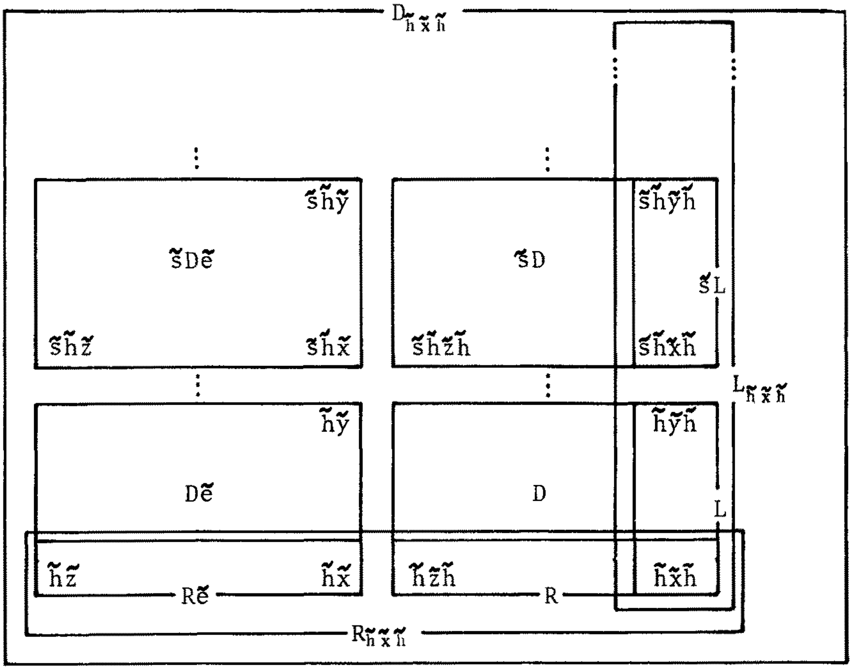
If $\tilde{h}\tilde{x}\tilde{h}$ and $\tilde{s}\tilde{h}\tilde{t}$ are $\mathcal{R}$-related in $\widetilde{\mathcal{T}(S)}$ for some $s,t \in S$, then they must also be $\mathcal{R}$-related in $\mathcal{I}_X$, i.e. $(\tilde{h}\tilde{x}\tilde{h}) \circ (\tilde{h}\tilde{x}\tilde{h})^{-1} = (\tilde{s}\tilde{h}\tilde{t}) \circ (\tilde{s}\tilde{h}\tilde{t})^{-1}$ (see lemma 2.6. of [3]). Clearly $e\tilde{h}\tilde{x}\tilde{h} = \bar{e}\tilde{h}\tilde{x}\tilde{h}$, and thus $\bar{s}\bar{t} = e\tilde{s}\tilde{h}\tilde{t} = \bar{e}\tilde{s}\tilde{h}\tilde{t} = \bar{t}$; consequently $st = t$, and $\tilde{s}\tilde{h}\tilde{t} = \tilde{h}\tilde{t}$ by lemma 2.6. By the foregoing we then have $\tilde{s}\tilde{h}\tilde{t} = \tilde{h}\tilde{t} \in R\tilde{e}$.

If $\tilde{h}\tilde{x}\tilde{h}$ and $\tilde{s}\tilde{h}\tilde{t}\tilde{h}$ are $\mathcal{R}$-related in $\widetilde{\mathcal{T}(S)}$ for some $s,t \in S$, then $(\tilde{s}\tilde{h}\tilde{t}\tilde{h})\tilde{e} = \tilde{s}\tilde{h}\tilde{t}\tilde{e} = \tilde{s}\tilde{h}\tilde{t}$ shows that $\tilde{s}\tilde{h}\tilde{t}$ will be $\mathcal{R}$-related with $\tilde{h}\tilde{x}\tilde{h}$ in $\widetilde{\mathcal{T}(S)}$. By the foregoing this implies $\tilde{s}\tilde{h}\tilde{t} = \tilde{h}\tilde{t} \in R\tilde{e}$, and thus $\tilde{s}\tilde{h}\tilde{t}\tilde{h} = \tilde{h}\tilde{t}\tilde{h} \in R$.

Hence we can conclude that $R_{\tilde{h}\tilde{x}\tilde{h}} = R \cup R\tilde{e}$.

(iii) is an immediate consequence of (i) and (ii).

<u>REMARK</u> 2.8. The situation described in lemma 2.7 is made clear by the following picture of the $\mathcal{D}$-class of an element $\tilde{h}\tilde{x}\tilde{h}$ of $\widetilde{\mathcal{T}(S)}$; we suppose $\tilde{h}\tilde{y}\tilde{h} \in L$ and $\tilde{h}\tilde{z}\tilde{h} \in R$.



Remark that, if $\tilde{x} = \tilde{e}$, we have $\tilde{h}\tilde{x}\tilde{h} = \tilde{h}$, $\tilde{h}\tilde{x} = \tilde{e}$, $\tilde{s}\tilde{h}\tilde{x} = \tilde{s}$.

LEMMA 2.9. Every $\mathcal{D}$-class of $\widetilde{\mathcal{A}(S)}$ meets $(K/_{\beta})\psi \cong S$ in exactly one $\mathcal{D}$-class of $(K/_{\beta})\psi$, and $\widetilde{\mathcal{A}(S)} = \bigcup_{x \in S} D_{\tilde{h}\tilde{x}\tilde{h}}$.

PROOF. $\tilde{h}, \tilde{e}, \tilde{s}, \tilde{s}\tilde{h}$ all belong to the $\mathcal{D}$-class of $\widetilde{\mathcal{A}(S)}$ which contains $\tilde{h} = \tilde{h}\tilde{e}\tilde{h} \in (K/_{\beta})\psi$ for all $s \in S \setminus \{e\}$, by lemma 2.7 and remark 2.8. $\tilde{h}\tilde{t}$, $\tilde{s}\tilde{h}\tilde{t}$, $\tilde{s}\tilde{h}\tilde{t}\tilde{h}$ all belong to the $\mathcal{D}$-class of $\widetilde{\mathcal{A}(S)}$ which contains $\tilde{h}\tilde{t}\tilde{h} \in (K/_{\beta})\psi$ for all $s, t \in S \setminus \{e\}$. We conclude that every $\mathcal{D}$-class of $\widetilde{\mathcal{A}(S)}$ has a non-void intersection with $(K/_{\beta})\psi \cong S$, and consequently $\widetilde{\mathcal{A}(S)} = \bigcup_{x \in S} D_{\tilde{h}\tilde{x}\tilde{h}}$

Let $\tilde{h}\tilde{s}\tilde{h}$ and $\tilde{h}\tilde{t}\tilde{h}$ be any elements of $(K/_{\beta})\psi$ that are $\mathcal{D}$-related in $\widetilde{\mathcal{A}(S)}$. Then $R_{\tilde{h}\tilde{s}\tilde{h}} \cap L_{\tilde{h}\tilde{t}\tilde{h}}$ is non-void, and by lemma 2.7 there exists an element $v \in S$, and an element $w \in S$ which is $\mathcal{L}$-related with t in S, such that $\tilde{v}\tilde{h}\tilde{w}\tilde{h} \in R_{\tilde{h}\tilde{s}\tilde{h}} \cap L_{\tilde{h}\tilde{t}\tilde{h}}$. Since $\tilde{h}\,(\tilde{h}\tilde{s}\tilde{h}) = \tilde{h}\tilde{s}\tilde{h}$, we have $\tilde{h}\tilde{v}\tilde{w}\tilde{h} = \tilde{h}(\tilde{v}\tilde{h}\tilde{w}\tilde{h}) = \tilde{v}\tilde{h}\tilde{w}\tilde{h}$. By lemma 2.6, we must have $\tilde{v}\tilde{h}\tilde{w}\tilde{h} = \tilde{h}\tilde{w}\tilde{h}$, and we can conclude that $\tilde{h}\tilde{w}\tilde{h} \in R_{\tilde{h}\tilde{s}\tilde{h}} \cap L_{\tilde{h}\tilde{t}\tilde{h}}$. From the proof of lemma

2.7 we know that this implies that w and s are $\mathcal{R}$-related in S. We conclude that s and t are $\mathcal{D}$-related in S, and consequently, $\tilde{h}\tilde{s}\tilde{h}$ and $\tilde{h}\tilde{t}\tilde{h}$ will be $\mathcal{D}$-related in $(K/_{\beta})\psi \cong S$.

THEOREM 2.10. *Any monoid S can be embedded in the semiband $\mathcal{A}(S)$ [resp. $\widetilde{\mathcal{A}}(S)$] in such a way that the restrictions to S of Green's equivalence relations on $\mathcal{A}(S)$ [resp. $\widetilde{\mathcal{A}}(S)$] are exactly the corresponding Green's equivalence relations on S.*

PROOF. The restriction to $(K/_{\beta})\psi \cong S$ of the $\mathcal{L}$- [resp. $\mathcal{R}$-, $\mathcal{H}$-, $\mathcal{D}$-] relation on $\widetilde{\mathcal{A}}(S)$ is exactly the $\mathcal{L}$-[resp. $\mathcal{R}$-, $\mathcal{H}$-, $\mathcal{D}$-] relation on $(K/_{\beta})\psi \cong S$: this follows immediately from the proof of lemma 2.7 and from lemma 2.9. Let $\tilde{h}\tilde{s}\tilde{h}$ and $\tilde{h}\tilde{t}\tilde{h}$ be any elements of $(K/_{\beta})\psi$ which are $\mathcal{J}$-related in $\widetilde{\mathcal{A}}(S)$. We have $\tilde{h}\tilde{s}\tilde{h} \in \widetilde{\mathcal{A}}(S)(\tilde{h}\tilde{t}\tilde{h})\widetilde{\mathcal{A}}(S)$; in fact $\tilde{h}\tilde{s}\tilde{h} \in (\tilde{h}\widetilde{\mathcal{A}}(S)\tilde{h})(\tilde{h}\tilde{t}\tilde{h})(\tilde{h}\widetilde{\mathcal{A}}(S)\tilde{h})$. Since $\tilde{h}\widetilde{\mathcal{A}}(S)\tilde{h} = (K/_{\beta})\psi$, the foregoing implies that there exist elements $\tilde{h}\tilde{v}\tilde{h}$, $\tilde{h}\tilde{w}\tilde{h} \in (K/_{\beta})\psi$, such that $\tilde{h}\tilde{s}\tilde{h} = (\tilde{h}\tilde{v}\tilde{h})(\tilde{h}\tilde{t}\tilde{h})(\tilde{h}\tilde{w}\tilde{h}) = \tilde{h}\widetilde{vtw}\tilde{h}$. From lemma 2.6 we conclude that $s = vtw \in StS$. Analogously we can show $t \in SsS$. We conclude that s and t are $\mathcal{J}$-related in S, and consequently $\tilde{h}\tilde{s}\tilde{h}$ and $\tilde{h}\tilde{t}\tilde{h}$ will be $\mathcal{J}$-related in $(K/_{\beta})\psi$. Thus the restriction to $(K/_{\beta})\psi \cong S$ of the $\mathcal{J}$-relation on $\widetilde{\mathcal{A}}(S)$ is exactly the $\mathcal{J}$-relation on $(K/_{\beta})\psi$.

Since ψ is a homomorphism of $\mathcal{A}(S)$ onto $\widetilde{\mathcal{A}}(S)$, the corresponding statement for $\mathcal{A}(S)$ will hold as well.

3. MAIN RESULTS

THEOREM 3.1. *Let S be any monoid; then :*

(i) *S is bisimple if and only if $\widetilde{\mathcal{A}}(S)$ is bisimple.*
(ii) *S is simple if and only if $\widetilde{\mathcal{A}}(S)$ is simple.*

PROOF : immediate from lemma 2.9 and theorem 2.10.

COROLLARY 3.2. (i) *Any semigroup can be embedded in a simple semiband.*
(ii) *Any semigroup can be embedded in a bisimple semiband.*

PROOF. (i) From theorem 8.3. of [2] (see also §8.5 of [3]), we know that any semigroup S can be embedded in a simple monoid $\mathfrak{C}(S)$. Hence, S can be embedded in the simple semiband $\widetilde{\mathcal{A}(\mathfrak{C}(S))}$.

(ii) From a result of [9] (see also §8.6 of [3], we know that any semigroup S can be embedded in a bisimple monoid T. Hence, S can be embedded in the bisimple semiband $\widetilde{\mathcal{A}}(T)$.

REMARK 3.3. Corollary 3.2. (ii) contradicts a conjecture of [4].

THEOREM 3.4. *Let S be any monoid; then* :

(i) *S is regular if and only if $\widetilde{\mathcal{A}}(S)$ is regular.*

(ii) *S is completely semisimple if and only if $\widetilde{\mathcal{A}}(S)$ is completely semisimple.*

PROOF. (i) If S is regular, then every $\mathcal{D}$-class of S contains an idempotent; hence every $\mathcal{D}$-class of $(K/_\beta)\psi \cong S$ contains an idempotent. By lemma 2.9, we can conclude that every $\mathcal{D}$-class of $\widetilde{\mathcal{A}}(S)$ contains an idempotent, and consequently, $\widetilde{\mathcal{A}}(S)$ will be regular.

If $\widetilde{\mathcal{A}}(S)$ is regular, then $\widetilde{h}\widetilde{x}\widetilde{h} \in (\widetilde{h}\widetilde{x}\widetilde{h})\widetilde{\mathcal{A}}(S)(\widetilde{h}\widetilde{x}\widetilde{h})$ for all $x \in S$. Since $(\widetilde{h}\widetilde{x}\widetilde{h})\widetilde{\mathcal{A}}(S)(\widetilde{h}\widetilde{x}\widetilde{h}) = (\widetilde{h}\widetilde{x}\widetilde{h})(\widetilde{h}\widetilde{\mathcal{A}}(S)\widetilde{h})(\widetilde{h}\widetilde{x}\widetilde{h}) = (\widetilde{h}\widetilde{x}\widetilde{h})((K/_\beta)\psi)(\widetilde{h}\widetilde{x}\widetilde{h})$ this shows that $\widetilde{h}\widetilde{x}\widetilde{h}$ is a regular element of $(K/_\beta)\psi$ for all $x \in S$. We conclude that $(K/_\beta)\psi \cong S$ is regular.

(ii) Let S be a completely semisimple monoid. Let $\widetilde{h}\widetilde{x}\widetilde{h}$ be any idempotent of $(K/_\beta)\psi$. If $D_{\widetilde{h}\widetilde{x}\widetilde{h}}$, the $\mathcal{D}$-class of $\widetilde{h}\widetilde{x}\widetilde{h}$ in $\widetilde{\mathcal{A}}(S)$, would contain a pair of distinct comparable idempotents, then $D_{\widetilde{h}\widetilde{x}\widetilde{h}}$ contains a bicyclic subsemigroup having $\widetilde{h}\widetilde{x}\widetilde{h}$ as identity element; this would imply that $D_{\widetilde{h}\widetilde{x}\widetilde{h}} \cap (\widetilde{h}\widetilde{x}\widetilde{h})\widetilde{\mathcal{A}}(S)(\widetilde{h}\widetilde{x}\widetilde{h}) \subsetneq D_{\widetilde{h}\widetilde{x}\widetilde{h}} \cap (K/_\beta)\psi$ contains an idempotent which is different from $\widetilde{h}\widetilde{x}\widetilde{h}$; by lemma 2.9 this would mean that the $\mathcal{D}$-class of $\widetilde{h}\widetilde{x}\widetilde{h}$ in $(K/_\beta)\psi \cong S$ would contain a pair of distinct comparable idempotents, and this is impossible since S is completely semisimple. We conclude that for any $x \in S$, $D_{\widetilde{h}\widetilde{x}\widetilde{h}}$ contains no pair of distinct comparable idempotents. Since then no pair of distinct comparable idempotents are $\mathcal{D}$-related in $\widetilde{\mathcal{A}}(S)$, $\widetilde{\mathcal{A}}(S)$ must be completely semisimple by result 6 of [6] .

Let $\widetilde{\mathcal{T}(S)}$ be completely semisimple. No pair of distinct idempotents of $\widetilde{\mathcal{T}(S)}$ are $\mathcal{D}$-related in $\widetilde{\mathcal{T}(S)}$, and consequently, no pair of distinct idempotents of $(K/_\beta)\psi \cong S$ are $\mathcal{D}$-related in $(K/_\beta)\psi \cong S$. Again by result 6 of [6] this implies that S must be completely semisimple.

THEOREM 3.5. *Any completely regular semigroup can be embedded in a completely regular semiband.*

PROOF. If semigroup S is completely regular, then S is a semilattice Y of completely simple semigroups D_α, $\alpha \in Y$. We can suppose that S is a monoid with identity e; if the original completely regular semigroup has no identity element, we can always add the identity e.
We shall consider a subset T of $\widetilde{\mathcal{T}(S)}$:

$$T = \{\tilde{y}\tilde{h}\tilde{x}\tilde{h},\ \tilde{y}\tilde{h}\tilde{x} \parallel x \in D_\mu,\ y \in D_\nu,\ \mu,\nu \in Y,\ \nu \geq \mu\}.$$

T contains $(K/_\beta)\psi \cong S$ since for any $x \in S$, $\tilde{h}\tilde{x}\tilde{h} = \tilde{e}\tilde{h}\tilde{x}\tilde{h} \in T$. We now proceed to show that the product of any two elements of T must belong to T. Therefore, let $x \in D_\mu$, $y \in D_\nu$, $s \in D_\kappa$, $t \in D_\lambda$, with $\kappa,\lambda,\mu,\nu \in Y$, and $\nu \geq \mu$, $\lambda \geq \kappa$. Then the elements $\tilde{y}\tilde{h}\tilde{x}\tilde{h}$, $\tilde{y}\tilde{h}\tilde{x}$, $\tilde{t}\tilde{h}\tilde{s}\tilde{h}$ and $\tilde{t}\tilde{h}\tilde{s}$ belong to T.

$$(\tilde{y}\tilde{h}\tilde{x}\tilde{h})(\tilde{t}\tilde{h}\tilde{s}\tilde{h}) = \tilde{y}\tilde{h}\widetilde{xts}\tilde{h}$$

and

$$(\tilde{y}\tilde{h}\tilde{x}\tilde{h})(\tilde{t}\tilde{h}\tilde{s}) = \tilde{y}\tilde{h}\widetilde{xts}$$

both belong to T since $y \in D_\nu$, $xts \in D_\gamma$ with $\nu \geq \mu \geq \kappa\wedge\mu = \gamma$.

$$(\tilde{y}\tilde{h}\tilde{x})(\tilde{t}\tilde{h}\tilde{s}\tilde{h}) = \tilde{y}\tilde{h}\widetilde{xs}\tilde{h}$$

and

$$(\tilde{y}\tilde{h}\tilde{x})(\tilde{t}\tilde{h}\tilde{s}) = \tilde{y}\tilde{h}\widetilde{xs}$$

both belong to T since $y \in D_\nu$, $xs \in D_\gamma$ with $\nu \geq \mu \geq \kappa\wedge\mu = \gamma$.
We conclude that T is a subsemigroup of $\widetilde{\mathcal{T}(S)}$, and that $(K/_\beta)\psi \cong S$ is subsemigroup of T.

Let us now consider any elements $\tilde{y}\tilde{h}\tilde{x}\tilde{h}$ and $\tilde{y}\tilde{h}\tilde{x}$ of T, with $x \in D_\mu$, $y \in D_\nu$, $\mu,\nu \in Y$, $\nu \geq \mu$. Then $(\tilde{y}\tilde{h}\tilde{x}\tilde{h})^2 = \tilde{y}\tilde{h}\widetilde{xyx}\tilde{h}$, and $(\tilde{y}\tilde{h}\tilde{x})^2 = \tilde{y}\tilde{h}\widetilde{xx}$. Since S is a completely regular semigroup, x, xyx and x^2 belong to a same $\mathcal{H}$-class of S, and consequently $\tilde{h}\tilde{x}\tilde{h}$, $\tilde{h}\widetilde{xyx}\tilde{h}$ and $\tilde{h}\widetilde{xx}\tilde{h}$ belong to a same $\mathcal{H}$-class of T. Let g be the identity of the maximal subgroup of S containing x, then $\tilde{y}\tilde{h}\tilde{g}$ belongs to T; $\tilde{e}$ belongs to T since

$\tilde{e} = \tilde{e}\tilde{h}\tilde{e}$. We have $(\tilde{y}\tilde{h}\tilde{g})(\tilde{h}\tilde{x}\tilde{h}) = \tilde{y}\tilde{h}\widetilde{gx}\tilde{h} = \tilde{y}\tilde{h}\tilde{x}\tilde{h}$, and $\tilde{e}(\tilde{y}\tilde{h}\tilde{x}\tilde{h}) = \tilde{e}\tilde{h}\tilde{x}\tilde{h} = \tilde{h}\tilde{x}\tilde{h}$. By Green's lemma (lemma 2.2 of [3]) $(\tilde{y}\tilde{h}\tilde{g})(\tilde{h}\tilde{x}\tilde{h}) = \tilde{y}\tilde{h}\tilde{x}\tilde{h}$, $(\tilde{y}\tilde{h}\tilde{g})(\tilde{h}\widetilde{xyx}\tilde{h}) = \tilde{y}\tilde{h}\widetilde{xyx}\tilde{h}$ and $(\tilde{y}\tilde{h}\tilde{g})(\tilde{h}\widetilde{xx}\tilde{h}) = \tilde{y}\tilde{h}\widetilde{xx}\tilde{h}$ must belong to a same $\mathcal{H}$-class of T. We have $(\tilde{y}\tilde{h}\tilde{x}\tilde{h})\tilde{e} = \tilde{y}\tilde{h}\tilde{x}$ and $(\tilde{y}\tilde{h}\tilde{x})\tilde{h} = \tilde{y}\tilde{h}\tilde{x}\tilde{h}$, and so, by Green's lemma, $(\tilde{y}\tilde{h}\tilde{x}\tilde{h})\tilde{e} = \tilde{y}\tilde{h}\tilde{x}$ and $(\tilde{y}\tilde{h}\widetilde{xx}\tilde{h})\tilde{e} = \tilde{y}\tilde{h}\widetilde{xx}$ must belong to a same $\mathcal{H}$-class of T. We conclude that in T any element and its square belong to a same $\mathcal{H}$-class of T, and consequently T is a union of groups by lemma 2.16 of [3]. Since $(\tilde{y}\tilde{h}\tilde{g})^2 = \tilde{y}\tilde{h}\tilde{g}$, $(\tilde{x}\tilde{h}\tilde{g})^2 = \tilde{x}\tilde{h}\tilde{g}$, and $\tilde{y}\tilde{h}\tilde{x} = (\tilde{y}\tilde{h}\tilde{g})\tilde{h}(\tilde{x}\tilde{h}\tilde{g})$, we conclude that $\tilde{x}\tilde{h}\tilde{g}$ is the product of the three idempotents $\tilde{y}\tilde{h}\tilde{g}$, $\tilde{h}$ and $\tilde{x}\tilde{h}\tilde{g}$ of T; $\tilde{y}\tilde{h}\tilde{x}\tilde{h}$ will then be the product of the idempotents $\tilde{y}\tilde{h}\tilde{g}$, $\tilde{h}$, $\tilde{x}\tilde{h}\tilde{g}$ and $\tilde{h}$ of T. Consequently T is a completely regular semiband which contains $(K/_{\beta})\psi \cong S$.

THEOREM 3.6. *Let S be any monoid. S contains a kernel V if and only if $\widetilde{\mathcal{A}(S)}$ contains a kernel; if this is the case V is embeddable in the kernel of $\widetilde{\mathcal{A}(S)}$. If V is regular, the kernel of $\widetilde{\mathcal{A}(S)}$ is a semiband.*

PROOF. By lemma 2.9 and theorem 2.10 every $\mathcal{J}$-class of $\widetilde{\mathcal{A}(S)}$ meets $(K/_{\beta})\psi \cong S$ in exactly one $\mathcal{J}$-class of $(K/_{\beta})\psi$, and $\widetilde{\mathcal{A}(S)} = \bigcup_{x\in S} J_{\tilde{h}\tilde{x}\tilde{h}}$, $J_{\tilde{h}\tilde{x}\tilde{h}}$ being the $\mathcal{J}$-class of $\tilde{h}\tilde{x}\tilde{h}$ in $\widetilde{\mathcal{A}(S)}$ for any $x \in S$. From this we have that there exists an order preserving one-to-one mapping of $\widetilde{\mathcal{A}(S)}/\mathcal{J}$ onto $(K/_{\beta})\psi/\mathcal{J}$. Hence, there exists a minimum $\mathcal{J}$-class in $\widetilde{\mathcal{A}(S)}$ if and only if there exists a minimum $\mathcal{J}$-class in $(K/_{\beta})\psi \cong S$. Since a minimum $\mathcal{J}$-class of a semigroup clearly is the kernel of that semigroup, the first part of the theorem follows. If V is the kernel of S, $\bigcup_{x\in V} D_{\tilde{h}\tilde{x}\tilde{h}}$ will be the minimum $\mathcal{J}$-class of $\widetilde{\mathcal{A}(S)}$; since V is isomorphic with the subsemigroup $\{\tilde{h}\tilde{x}\tilde{h} \parallel x \in V\}$ of $(K/_{\beta})\psi$, we can conclude that V is embeddable in the kernel of $\widetilde{\mathcal{A}(S)}$. If V is regular, $D_{\tilde{h}\tilde{x}\tilde{h}}$ is a regular $\mathcal{D}$-class of $\widetilde{\mathcal{A}(S)}$ for all $x \in V$; by lemma 1 of [6] any regular element of a semiband is a product of idempotents of its $\mathcal{D}$-class; thus, in our case $\bigcup_{x\in V} D_{\tilde{h}\tilde{x}\tilde{h}}$ is a regular semiband.

COROLLARY 3.7. *Let S be any regular simple semigroup, and let* $S\cup\{e\}$ *be a monoid with identity e. Then S can be embedded in the kernel of* $\widetilde{\mathcal{A}}(S\cup\{e\})$ *which is a simple regular semiband. The kernel of* $\widetilde{\mathcal{A}}(S\cup\{e\})$ *is completely simple if and only if S is completely simple.*

PROOF. Since S is a regular kernel of $S\cup\{e\}$, S can be embedded in the kernel of $\widetilde{\mathcal{A}}(S\cup\{e\})$ by theorem 3.6; this kernel of $\widetilde{\mathcal{A}}(S\cup\{e\})$ is of course a regular simple semiband. We can now use the same arguments as in the proof of 3.4 (ii) : no pair of distinct comparable idempotents are $\mathcal{D}$-related in S if and only if no pair of distinct comparable idempotents are $\mathcal{D}$-related in the regular simple semiband $\bigcup_{x\in S} D_{\tilde{h}\tilde{x}\tilde{h}}$: this implies that S is completely simple if and only if $\bigcup_{x\in S} D_{\tilde{h}\tilde{x}\tilde{h}}$ is completely simple.

THEOREM 3.8. *Let S be any monoid. Then* $\widetilde{\mathcal{A}}(S)$ *is completely simple if and only if S is a group.*

PROOF. Let $\widetilde{\mathcal{A}}(S)$ be completely simple; by theorem 3.4 S must be completely semisimple, and by lemma 2.9 S must be bisimple. Since S is a monoid, we conclude that S must be a group. Conversely, if S is a group, $\widetilde{\mathcal{A}}(S)$ must be completely semisimple by theorem 3.4, and $\widetilde{\mathcal{A}}(S)$ must be bisimple by lemma 2.9; hence, $\widetilde{\mathcal{A}}(S)$ must be completely simple.

COROLLARY 3.9. *Any countable group can be embedded in a completely simple semiband generated by 5 idempotents.*

PROOF. Let S be a group generated by two elements a and b. It must be clear that $(K/_{\beta})\psi \cong S$ will be a subgroup of the subsemigroup of $\widetilde{\mathcal{A}}(S)$ which is generated by the idempotents $\tilde{h}$, $\tilde{a}$, $\tilde{b}$, $\widetilde{a^{-1}}$, $\widetilde{b^{-1}}$. This subsemigroup of $\widetilde{\mathcal{A}}(S)$ has the following elements : $\tilde{h}\tilde{x}\tilde{h}$, $\tilde{h}\tilde{x}$, $\tilde{a}\tilde{h}\tilde{x}\tilde{h}$, $\tilde{a}\tilde{h}\tilde{x}$, $\widetilde{a^{-1}}\tilde{h}\tilde{x}\tilde{h}$, $\widetilde{a^{-1}}\tilde{h}\tilde{x}$, $\tilde{b}\tilde{h}\tilde{x}\tilde{h}$, $\tilde{b}\tilde{h}\tilde{x}$, $\widetilde{b^{-1}}\tilde{h}\tilde{x}\tilde{h}$, $\widetilde{b^{-1}}\tilde{h}\tilde{x}$, for all $x\in S$; hence this subsemigroup of $\widetilde{\mathcal{A}}(S)$ is completely simple and is a union of 10 copies of S. By a result of [7] any countable group can be embedded in a group generated by 2 elements; the result then follows.

REMARK 3.10. If S is any monoid, every element of $\widetilde{\mathcal{A}}(S)$ is a product of at most 4 idempotents. If S is completely regular, then every element of the completely regular semiband T considered in theorem 3.5 is a product of at most 4 idempotents. If S has a regular kernel, then every element of the kernel of $\widetilde{\mathcal{A}}(S)$ is a product of at most 4 idempotents of this kernel of $\widetilde{\mathcal{A}}(S)$ (by lemma 1 of [6]); hence, by corollary 7 any completely simple semigroup is embeddable in a completely simple semiband in which every element is a product of at most 4 idempotents.

4. AN EXAMPLE : $\widetilde{\mathcal{A}}(\mathcal{C})$

We shall consider the bicyclic semigroup $\mathcal{C}$ generated by the two-element set $\{a,b\}$ subject to the defining relation ab = e, e being the identity element of $\mathcal{C}$. Since $\mathcal{C}$ is bisimple but not completely simple, $\widetilde{\mathcal{A}}(\mathcal{C})$ must be a bisimple semiband which is not completely simple, by theorem 3.8.

We shall look for the idempotents of $\widetilde{\mathcal{A}}(\mathcal{C})$. $\tilde{h}$ and $\widetilde{b^i a^j}$ (i,j non-negative integers) clearly are idempotents of $\widetilde{\mathcal{A}}(\mathcal{C})$. If for some non-negative integers i,j, $\tilde{h}\widetilde{b^i a^j}$ is an idempotent of $\widetilde{\mathcal{A}}(\mathcal{C})$, then

$$\tilde{h}\widetilde{b^i a^j} = (\tilde{h}\widetilde{b^i a^j})^2 = \tilde{h}\widetilde{(b^i a^j)^2},$$

and by lemma 2.6 this implies $b^i a^j = (b^i a^j)^2$ in $\mathcal{C}$; hence $\tilde{h}\widetilde{b^i a^j}$ is an idempotent of $\widetilde{\mathcal{A}}(\mathcal{C})$ if and only if i = j. Analogously, for any non-negative integers i,j,m,n, $\tilde{h}\widetilde{b^i a^j}\tilde{h}$ and $\widetilde{b^m a^n}\tilde{h}\widetilde{b^i a^j}$ will be idempotents of $\widetilde{\mathcal{A}}(\mathcal{C})$ if and only if i = j. If for some non-negative integers i,j,m,n $\widetilde{b^m a^n}\tilde{h}\widetilde{b^i a^j}\tilde{h}$ is an idempotent of $\widetilde{\mathcal{A}}(\mathcal{C})$, then

$$\widetilde{b^m a^n}\tilde{h}\widetilde{b^i a^j}\tilde{h} = (\widetilde{b^m a^n}\tilde{h}\widetilde{b^i a^j}\tilde{h})^2 = \widetilde{b^m a^n}\tilde{h}\widetilde{b^i a^j b^m a^n b^i a^j}\tilde{h},$$

and by lemma 2.6 this implies $b^i a^j = b^i a^j b^m a^n b^i a^j$ in $\mathcal{C}$; from this we have $m \leq j$ and $j-m = i-n$; conversely, for any non-negative integers i,j,m,n with $m \leq j$ and $j-m = i-n$ $\widetilde{b^m a^n}\tilde{h}\widetilde{b^i a^j}\tilde{h}$ must be an idempotent of $\widetilde{\mathcal{A}}(\mathcal{C})$.

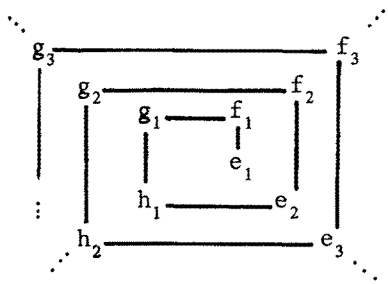
The idempotents of $\widetilde{\mathcal{A}}(\mathcal{C})$ have been marked in the table on next page; in this table the rows are the $\mathcal{R}$-classes and the columms are the $\mathcal{L}$-classes of $\widetilde{\mathcal{S}(\mathfrak{C})}$.

$\widetilde{\mathcal{S}(\mathfrak{C})}$ contains a copy of the spiral semigroup : $\tilde{a}\,\widetilde{\mathcal{S}(\mathfrak{C})} \cup \tilde{e}\,\widetilde{\mathcal{S}(\mathfrak{C})}$ is a bisimple subsemigroup of $\widetilde{\mathcal{S}(\mathfrak{C})}$ generated by the 4 idempotents $\tilde{a}$, $\tilde{e}$, $\tilde{h}$ and $\tilde{a}\tilde{h}\tilde{b}\tilde{h}$. It is easy to see that this subsemiband of $\widetilde{\mathcal{S}(\mathfrak{C})}$ is isomorphic with the semigroup generated by the elements of $\{e_i, f_i, g_i, h_i \parallel i$ non-negative integer$\}$ subject to the defining relations

$$e_i^2 = e_i\ ,\ f_i^2 = f_i\ ,\ g_i^2 = g_i\ ,\ h_i^2 = h_i\ ,$$

$$e_i f_i = e_i\ ,\ f_i e_i = f_i = g_i f_i\ ,\ f_i g_i = g_i = g_i h_i\ ,$$
$$h_i g_i = h_i\ ,\ h_i e_{i+1} = e_{i+1}\ ,\ e_{i+1} h_i = h_i\ ,$$
$$e_i e_{i+1} = e_{i+1} e_i = e_{i+1}\ ,\ f_i f_{i+1} = f_{i+1} f_i = f_{i+1}\ ,$$
$$g_i g_{i+1} = g_{i+1} g_i = g_{i+1}\ ,\ h_i h_{i+1} = h_{i+1} h_i = h_{i+1}$$

for any non-negative integer i.



THEOREM 4.1. <u>There exists a countable infinite number of non-isomorphic bisimple semibands of type</u> 3 <u>which are not completely simple</u>.

<u>PROOF</u>. Let $m \neq 0$ be any non-negative integer. Since $(\tilde{h}\tilde{a})^m \tilde{h}\widetilde{b^m} = \tilde{h}\widetilde{a^m}\tilde{h}\widetilde{b^m} = \tilde{h}\widetilde{a^m b^m} = \tilde{h}\tilde{e} = \tilde{e}$, and $(\tilde{h}\tilde{a})^{m-1}\tilde{h}\widetilde{b^m} = \tilde{h}\widetilde{a^{m-1}}\tilde{h}\widetilde{b^m} = \tilde{h}\widetilde{a^{m-1}b^m} = \tilde{h}\tilde{b}$, the subsemiband of $\widetilde{\mathcal{S}(\mathfrak{C})}$ generated by the 3 idempotents $\tilde{h}$, $\tilde{a}$, $\widetilde{b^m}$ is exactly the bisimple semiband $\tilde{e}\,\widetilde{\mathcal{S}(\mathfrak{C})} \cup \tilde{a}\,\widetilde{\mathcal{S}(\mathfrak{C})} \cup \widetilde{b^m}\,\widetilde{\mathcal{S}(\mathfrak{C})}$. The $\mathcal{L}$-classes of this subsemiband $\langle \tilde{h}, \tilde{a}, \widetilde{b^m} \rangle$ of $\widetilde{\mathcal{S}(\mathfrak{C})}$ contain 2 or 3 idempotents. For any $0 \leq i < m$ the $\mathcal{L}$-class of $\tilde{a}\tilde{h}\widetilde{a^i}\tilde{h}$ contains the two idempotents $\tilde{h}\widetilde{b^i a^i}\tilde{h}$ and $\tilde{a}\tilde{h}\widetilde{b^{i+1}a^i}\tilde{h}$ (these idempotents are

TABLE of $\mathfrak{a}(\widetilde{\mathfrak{t}})$

…	$\widetilde{hb^2a^{m+1}h}$	$\widetilde{hb^2a^mh}$	… $\widetilde{hb^2ah}$	$\widetilde{hb^2h}$	$\widetilde{hb^2}$	$\widetilde{hb^2a}$	* $\widetilde{hb^2a^2}$	…
…	$\widetilde{hba^{m+1}h}$	$\widetilde{hba^mh}$	… * $\widetilde{hbah}$	$\widetilde{hbh}$	$\widetilde{hb}$	* $\widetilde{hba}$	$\widetilde{hba^2}$	…
…	$\widetilde{ha^{m+1}h}$	$\widetilde{ha^mh}$	… $\widetilde{hah}$	* $\widetilde{h}$	* $\widetilde{e}$	$\widetilde{ha}$	$\widetilde{ha^2}$	…
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
…	$\widetilde{ahb^2a^{m+1}h}$	$\widetilde{ahb^2a^mh}$	… * $\widetilde{ahb^2ah}$	$\widetilde{ahb^2h}$	$\widetilde{ahb^2}$	$\widetilde{ahb^2a}$	* $\widetilde{ahb^2a^2}$	…
…	$\widetilde{ahba^{m+1}h}$	$\widetilde{ahba^mh}$	… $\widetilde{ahbah}$	* $\widetilde{ahbh}$	$\widetilde{ahb}$	* $\widetilde{ahba}$	$\widetilde{ahba^2}$	…
…	$\widetilde{aha^{m+1}h}$	$\widetilde{aha^mh}$	… $\widetilde{ahah}$	$\widetilde{ah}$	* $\widetilde{a}$	$\widetilde{aha}$	$\widetilde{aha^2}$	…
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
…	$\widetilde{b^mhb^2a^{m+1}h}$	$\widetilde{b^mhb^2a^mh}$	… $\widetilde{b^mhb^2ah}$	$\widetilde{b^mhb^2h}$	$\widetilde{b^mhb^2}$	$\widetilde{b^mhb^2a}$	* $\widetilde{b^mhb^2a^2}$	…
…	* $\widetilde{b^mhba^{m+1}h}$	$\widetilde{b^mhba^mh}$	… $\widetilde{b^mhbah}$	$\widetilde{b^mhbh}$	$\widetilde{b^mhb}$	* $\widetilde{b^mhba}$	$\widetilde{b^mhba^2}$	…
…	$\widetilde{b^mha^{m+1}h}$	* $\widetilde{b^mha^mh}$	… $\widetilde{b^mhah}$	$\widetilde{b^mh}$	* $\widetilde{b^m}$	$\widetilde{b^mha}$	$\widetilde{b^mha^2}$	…
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
…	* $\widetilde{b^ma^nhb^{n+1}a^{m+1}h}$	$\widetilde{b^ma^nhb^{n+1}a^mh}$	… $\widetilde{b^ma^nhb^{n+1}ah}$	$\widetilde{b^ma^nhb^{n+1}h}$	$\widetilde{b^ma^nhb^{n+1}}$	$\widetilde{b^ma^nhb^{n+1}a}$	$\widetilde{b^ma^nhb^{n+1}a^2}$	…
…	$\widetilde{b^ma^nhb^na^{m+1}h}$	* $\widetilde{b^ma^nhb^na^mh}$	… $\widetilde{b^ma^nhb^nah}$	$\widetilde{b^ma^nhb^nh}$	$\widetilde{b^ma^nhb^n}$	$\widetilde{b^ma^nhb^na}$	$\widetilde{b^ma^nhb^na^2}$	…
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
…	$\widetilde{b^ma^nhb^2a^{m+1}h}$	$\widetilde{b^ma^nhb^2a^mh}$	… $\widetilde{b^ma^nhb^2ah}$	$\widetilde{b^ma^nhb^2h}$	$\widetilde{b^ma^nhb^2}$	$\widetilde{b^ma^nhb^2a}$	* $\widetilde{b^ma^nhb^2a^2}$	…
…	$\widetilde{b^ma^nhba^{m+1}h}$	$\widetilde{b^ma^nhba^mh}$	… $\widetilde{b^ma^nhbah}$	$\widetilde{b^ma^nhbh}$	$\widetilde{b^ma^nhb}$	* $\widetilde{b^ma^nhba}$	$\widetilde{b^ma^nhba^2}$	…
…	$\widetilde{b^ma^nha^{m+1}h}$	$\widetilde{b^ma^nha^mh}$	… $\widetilde{b^ma^nhah}$	$\widetilde{b^ma^nh}$	* $\widetilde{b^ma^n}$	$\widetilde{b^ma^nha}$	$\widetilde{b^ma^nha^2}$	…
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

different by lemma 2.6). For any non-negative integer i, with $m \leqslant i$, the $\mathcal{L}$-class of $\tilde{a}h\widetilde{a^i}\tilde{h}$ contains the three idempotents $\tilde{h}\widetilde{b^i a^i}\tilde{h}$, $\tilde{a}\tilde{h}\widetilde{b^{i+1} a^i}\tilde{h}$ and $\widetilde{b^m}\tilde{h}\widetilde{b^{i-m} a^i}\tilde{h}$ (these idempotents are different by lemma 2.6). For any non-negative integer i, the $\mathcal{L}$-class of $\tilde{h}\widetilde{a^i}$ contains the three idempotents $\tilde{h}\widetilde{b^i a^i}$, $\tilde{a}\tilde{h}\widetilde{b^i a^i}$ and $\widetilde{b^m}\tilde{h}\widetilde{b^i a^i}$ (these idempotents are different by lemma 2.6). We conclude that for any non-negative integer m, $\langle \tilde{h}, \tilde{a}, \widetilde{b^m} \rangle$ is a bisimple semiband of type 3 in which exactly m $\mathcal{L}$-classes contain only 2 idempotents; consequently, if $m_1 \neq 0$ and $m_2 \neq 0$ are any two different non-negative integers, $\langle \tilde{h}, \tilde{a}, \widetilde{b^{m_1}} \rangle$ and $\langle \tilde{h}, \tilde{a}, \widetilde{b^{m_2}} \rangle$ cannot be isomorphic.

AKNOWLEDGEMENTS

I am much indebted to G.J. Lallement for suggesting several improvements.

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Received February 25, 1977 and in final form May 13, 1977.