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Proof of Oppenheim's area inequalities for triangles and quadrilaterals

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Abstract. Let a_1, b_1, c_1, A_1 and a_2, b_2, c_2, A_2 be the sides and areas of two triangles. If $a = (a_1^p + a_2^p)^{1/p}$, $b = (b_1^p + b_2^p)^{1/p}$, $c = (c_1^p + c_2^p)^{1/p}$, and $1 \le p \le 4$, then a, b, c are the sides of a triangle and its area satisfies $A^{p/2} \ge A_1^{p/2} + A_2^{p/2}$. If obtuse triangles are excluded, $p > 4$ is allowed. For convex cyclic quadrilaterals, a similar inequality holds. Also, let a, b, c, A be the sides and area of an acute or right triangle. If $f(x)$ satisfies certain conditions, $f(a)$, $f(b)$, $f(c)$ are the sides of a triangle having area A_f, which satisfies $(4A_f/\sqrt{3})^{1/2} \ge f((4A/\sqrt{3})^{1/2})$.

Introduction and results

The area of a triangle is a well known function of the lengths of its sides, and this function satisfies numerous inequalities [1]. Inequalities containing this function and another function appear in three conjectures published by Oppenheim [5] and [6]. The first of these conjectures is:

THEOREM 1. *If $1 \le p \le 4$ and if two triangles have sides a_1, b_1, c_1 and a_2, b_2, c_2 and areas A_1 and A_2, then $a = (a_1^p + a_2^p)^{1/p}$, $b = (b_1^p + b_2^p)^{1/p}$, $c = (c_1^p + c_2^p)^{1/p}$ are the sides of a triangle having area A, and*

$$A^{p/2} \ge A_1^{p/2} + A_2^{p/2}. \tag{1}$$

Apart from trivial cases with $p = 1$ and $A_1 = A_2 = 0$, equality holds if and only if

$$a_1/a_2 = b_1/b_2 = c_1/c_2. \tag{2}$$

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Oppenheim's paper [5] ends with an example using an obtuse triangle to show that $p \leq 4$ is a necessary condition. Larger values of p are included in:

THEOREM 2. *If* $p \geq 1$, *if the triangles having areas* A_1 *and* A_2 *are acute or right triangles, and if* a, b, c, A *are as in Theorem* 1, *then* (1) *holds, with equality iff* (2) *holds.*

Oppenheim's later paper [6] strongly suggests that a similar inequality holds for convex plane quadrilaterals. Steiner [8] showed that, if a quadrilateral has sides of fixed length, the area is maximum when the vertices lie on a circle. See also the proof given by Pólya [7].

THEOREM 3. *If* $1 \leq p \leq 4$ *and if two quadrilaterals have sides* a_1, b_1, c_1, d_1 *and* a_2, b_2, c_2, d_2, *then* $a = (a_1^p + a_2^p)^{1/p}, \ldots, d = (d_1^p + d_2^p)^{1/p}$ *are the sides of a quadrilateral, and the maximum areas satisfy* (1). *Equality holds iff the sets* a_1, b_1, c_1, d_1 *and* a_2, b_2, c_2, d_2 *are proportional*; *but there are trivial exceptions with* $p = 1$ *and* $A_1 = A_2 = 0$.

Oppenheim [5] and [6] showed that Theorems 1 and 3 hold when $p = 1, 2$, or 4. To prove Theorems 1, 2, and 3 when $p > 1$, we shall consider changes of $a_1, a_2, b_1, b_2, \ldots$ such that a, b, c or a, b, c, d are constant. We shall consider boundary values of $A^{p/2} - A_1^{p/2} - A_2^{p/2}$, and show that this function is stationary iff the triangles or quadrilaterals are similar.

Oppenheim's third conjecture [6] is that "if $f(x)$ is a non-negative, non-decreasing sub-additive function on $x > 0$" and if

$$G(a,b,c) = \left(\frac{4A}{\sqrt{3}}\right)^{\frac{1}{2}} = \left[\frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{3}\right]^{\frac{1}{4}},$$

then

$$G(f(a), f(b), f(c)) \geq f(G(a,b,c)). \tag{3}$$

He showed that (3) holds if $f(x) = \sum_n a_n x^{p_n}$, where $a_n > 0$, $0 \leq p_n \leq 1$. But let $a = b = 1$, $c = 1.9$,

$$f(x) = x \qquad \text{when } x \geq 1,$$

$$f(x) = x \exp\left[-\frac{\log x}{6} \exp\left(\frac{1}{\log x}\right)\right] \qquad \text{when } 0 < x < 1.$$

Then (3) does not hold. In both the example and the counterexample, the arbitrary function satisfies

$$f(x)>0, \qquad 0\le x\frac{d}{dx}\log f(x)\le 1, \quad \text{and} \quad 0\le\left(x\frac{d}{dx}\right)^2\log f(x)\le\frac{1}{4}.$$

We are led to exclude obtuse triangles.

THEOREM 4. *Suppose a, b, c are the sides of an acute or right triangle, $f(x)>0$, $\log f(x)$ is a convex function of $\log x$, and*

$$0<\log[f(x)/f(y)]/\log(x/y)<1, \tag{4}$$

where x and y are distinct positive numbers. Then (3) *holds, with equality iff $a=b=c$.*

Proof of Theorem 1

Minkowski's inequality [4, p. 115] can be used to show that a, b, c are the sides of a triangle. If triangles of zero area are allowed, $a_1\le b_1+c_1$ and $a_2\le b_2+c_2$. Hence,

$$a\le[(b_1+c_1)^p+(b_2+c_2)^p]^{1/p}\le b+c.$$

If $a=b+c$, then $A_1=A_2=0$; the converse is not true. Similarly, $b\le a+c$ and $c\le a+b$.

If $p=1$, Oppenheim [5] notes that

$$2A^{1/2}=[(a+b+c)(-a+b+c)(a-b+c)(a+b-c)]^{1/4},$$

where

$$a+b+c=(a_1+b_1+c_1)+(a_2+b_2+c_2)$$

and so forth. A known inequality [4, p. 117] gives $2A^{1/2}\ge 2A_1^{1/2}+2A_2^{1/2}$, with equality only if triangles 1 and 2 are similar or $A_1=A_2=0$. Thus, we may assume $p>1$. We shall minimize $A^{p/2}-A_1^{p/2}-A_2^{p/2}$. The minimum value of this function is not positive, because it vanishes when triangles 1 and 2 are similar. Suppose that a_1 is variable and $a_1^p+a_2^p$ is constant. Then the maximum and minimum values of a_1

are such that $A_1A_2=0$. The following lemma implies that $A^{p/2}-A_1^{p/2}-A_2^{p/2}$ is positive when a_1 is at either end of its range, except in a degenerate case.

LEMMA 1. *If* $p>1$ *and* A_1, A_2, A *are defined as in Theorem* 1, *then* $A \geq A_1$ *and* $A \geq A_2$. *If* $A = A_1$ *or* $A = A_2$, *then* $A_1 = A_2 = 0$ *and the sets* a_1, b_1, c_1 *and* a_2, b_2, c_2 *are proportional.*

If $A_1A_2=0$, this lemma is equivalent to the theorem. If $A_1A_2\neq 0$, we may vary a_1 and a_2. Since $\partial a_2/\partial a_1 = -(a_1/a_2)^{p-1}$,

$$(\partial/\partial a_1)(A^{p/2}-A_1^{p/2}-A_2^{p/2})$$
$$=(pa_1^{p-1}/16)[-A_1^{(p-4)/2}a_1^{2-p}(-a_1^2+b_1^2+c_1^2)+A_2^{(p-4)/2}a_2^{2-p}(-a_2^2+b_2^2+c_2^2)]. \quad (5)$$

This quantity vanishes at the minimum of $A^{p/2}-A_1^{p/2}-A_2^{p/2}$. Let α_1, β_1, γ_1, R_1 and α_2, β_2, γ_2, R_2 be the internal angles and circumradii of the first and second triangles. Then

$$-a_1^2+b_1^2+c_1^2=2b_1c_1\cos\alpha_1, \qquad a_1b_1c_1=4A_1R_1,$$

and (5) vanishes only if

$$(A_1^{1/2}/R_1)^{p-2}(\cos\alpha_1)\sin^{1-p}\alpha_1=(A_2^{1/2}/R_2)^{p-2}(\cos\alpha_2)\sin^{1-p}\alpha_2. \quad (6)$$

Variation of b_1 and c_1 gives similar relations between β_1 and β_2 and between γ_1 and γ_2. To prove that $\alpha_1=\alpha_2$, $\beta_1=\beta_2$, and $\gamma_1=\gamma_2$, it suffices to show that α, β, γ are uniquely determined by $\alpha+\beta+\gamma=\pi$ and the ratios

$$\frac{\cos\alpha\,\sin^{p-1}\gamma}{\cos\gamma\,\sin^{p-1}\alpha}=r_1, \qquad \frac{\cos\beta\,\sin^{p-1}\gamma}{\cos\gamma\,\sin^{p-1}\beta}=r_2. \quad (7)$$

Here we have assumed $\cos\gamma\neq 0$, because two of $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$ are positive, and $\cos\gamma_1>0$ implies $\cos\gamma_2>0$. We may assume $\cos\gamma>0$ and $r_2>0$. Let $g(x)$ be a function with range $(0,\pi/2)$, defined by

$$x=\log[\cos g(x)]-(p-1)\log[\sin g(x)]. \quad (8)$$

If $r_1>0$, the angles are acute, and the solution of

$$g(x)+g(x+\log r_1)+g(x+\log r_2)=\pi$$

is unique, because $dg/dx = -(\tan g)/(p-1+\tan^2 g) < 0$; hence, $\gamma = g(x)$ is uniquely determined. If $r_1 = 0$, then $\alpha = \pi/2$ and the solution of $g(x)+g(x+\log r_2) = \pi/2$ is unique. If $r_1 < 0$, then $\alpha > \pi/2$, $\pi - \alpha = \beta + \gamma$, and

$$\frac{g(x)}{g(x+\log|r_1|)} + \frac{g(x+\log r_2)}{g(x+\log|r_1|)} = 1. \tag{9}$$

Since $\pi - \alpha > \gamma$, $\log|r_1| < 0$. Since $\pi - \alpha > \beta$, $\log|r_1| - \log r_2 < 0$. The following lemma implies that the solution of (9) is unique. Hence, $A^{p/2} - A_1^{p/2} - A_2^{p/2}$ is stationary only if triangles 1 and 2 are similar.

LEMMA 2. *Let $g(x)$ be defined as in* (8). *If* $1 < p \leq 4$ *and* $l < 0$, *then* $g(x)/g(x+l)$ *is a decreasing function of* x.

Proof of lemmas. The proof of Lemma 1 is similar to that of Lemma 3, which appears below. To prove Lemma 2, we write

$$\log g(x) - \log g(x+l) = \int_{x+l}^{x} \left[\frac{d}{dt}\log g(t)\right] dt.$$

This is a decreasing function of x if

$$\frac{d^2}{dx^2}\log g(x) = \frac{-\sin g}{g\cos^3 g\,(p-1+\tan^2 g)^3}\left[(\tan^2 g)\left(1+\frac{\sin 2g}{2g}\right) - (p-1)\left(1-\frac{\sin 2g}{2g}\right)\right]$$

is negative. It suffices to show that the expression in square brackets is positive, or that

$$(\tan^2 \theta/2)\left(1+\frac{\sin\theta}{\theta}\right) - 3\left(1-\frac{\sin\theta}{\theta}\right)$$

is positive when $0 < \theta < \pi$. We shall show that

$$3\left(1-\frac{\sin\theta}{\theta}\right) < \frac{3(1+\cos\theta)(\tan^2\theta/2)}{2+\cos\theta} < (\tan^2\theta/2)\left(1+\frac{\sin\theta}{\theta}\right). \tag{10}$$

Since $2(\theta - \sin\theta)\sin\theta$ is positive, $h(\theta) = 2\sin\theta + \sin\theta\cos\theta - \theta - 2\theta\cos\theta$ is increasing when $0 < \theta < \pi$. Since $h(0) = 0$, $(\sin\theta)(2+\cos\theta) > \theta(1+2\cos\theta)$,

$$1+\frac{\sin\theta}{\theta} > \frac{3(1+\cos\theta)}{2+\cos\theta}, \quad \text{and} \quad 1-\frac{\sin\theta}{\theta} < \frac{1-\cos\theta}{2+\cos\theta}$$

hold when $0 < \theta < \pi$. This proves (10).

Proof of Theorem 2

Since Theorem 1 has been proved, we may assume $p > 2$. Again, a_1, b_1, c_1 are varied to find the minimum of $A^{p/2} - A_1^{p/2} - A_2^{p/2}$. In this variation, a, b, c are constant and obtuse triangles are excluded. The maximum value of a_1 is such that

$$A_2(-a_1^2 + b_1^2 + c_1^2)(a_2^2 - b_2^2 + c_2^2)(a_2^2 + b_2^2 - c_2^2) = 0.$$

If $A_2 = 0$, then $a_2 = 0$ and $b_2 = c_2$, because obtuse triangles cannot occur as a_2 decreases; hence, Lemma 1 gives $A > A_1$. If $a_1^2 = b_1^2 + c_1^2$, then (5) is positive (unless $a_2^2 = b_2^2 + c_2^2$), and the minimum of $A^{p/2} - A_1^{p/2} - A_2^{p/2}$ lies at a smaller value of a_1. If $a_2^2 - b_2^2 + c_2^2 = 0$, a similar calculation shows that the minimum lies at a smaller value of b_2, unless $a_1^2 - b_1^2 + c_1^2 = 0$. Similar reasoning applies to $a_2^2 + b_2^2 - c_2^2 = 0$ and to the minimum value of a_1. Inside these boundaries, $A^{p/2} - A_1^{p/2} - A_2^{p/2}$ is stationary only for similar triangles. The proof of this is similar to the previous calculation, except that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are positive and Lemma 2 is not used. The exceptional cases involving two right triangles remain to be considered.

We may assume $a_1^2 = b_1^2 + c_1^2$ and $a_2^2 = b_2^2 + c_2^2$. If $c_1/b_1 = c_2/b_2$, then $A^{p/2} - A_1^{p/2} - A_2^{p/2} = 0$. If $c_1/b_1 \neq c_2/b_2$, appropriate small changes will increase $-a_1^2 + b_1^2 + c_1^2$ and $-a_2^2 + b_2^2 + c_2^2$ and decrease $A^{p/2} - A_1^{p/2} - A_2^{p/2}$. Let a_1 and a_2 be constant. Formulae similar to (5) give

$$(\partial/\partial b_1)(A^{p/2} - A_1^{p/2} - A_2^{p/2}) = 2^{-1-p/2} p b_1^{p-1}[(c_2/b_2)^{p/2} - (c_1/b_1)^{p/2}]$$

and

$$(\partial/\partial c_1)(A^{p/2} - A_1^{p/2} - A_2^{p/2}) = 2^{-1-p/2} p c_1^{p-1}[(b_2/c_2)^{p/2} - (b_1/c_1)^{p/2}].$$

Let $dc_1/db_1 = -(b_2/c_2)(b_1c_2/b_2c_1)^{p/2}$. Then

$$(c_1/b_1)dc_1/db_1 = -(b_1c_2/b_2c_1)^{(p-2)/2}, \quad (c_2/b_2)dc_2/db_2 = -(b_2c_1/b_1c_2)^{(p-2)/2},$$

and

$$(d/db_1)(A^{p/2} - A_1^{p/2} - A_2^{p/2}) = 2^{-1-p/2} p b_1^p (b_1^{-1} + b_2/c_1c_2)[(c_2/b_2)^{p/2} - (c_1/b_1)^{p/2}].$$

If $c_1/b_1 > c_2/b_2$, then $p > 2$ implies

$$(c_2/b_2)dc_2/db_2 < -1 < (c_1/b_1)dc_1/db_1;$$

hence a sufficiently small increase in b_1 will increase $b_1^2+c_1^2$ and $b_2^2+c_2^2$ and decrease $A^{p/2}-A_1^{p/2}-A_2^{p/2}$. If $c_1/b_1<c_2/b_2$, a small decrease in b_1 is used. Thus, $A^{p/2}-A_1^{p/2}-A_2^{p/2}$ is not minimized at this boundary point unless $c_1/b_1=c_2/b_2$.

Proof of Theorem 3

Minkowski's inequality can be used to show that a, b, c, d are the sides of a quadrilateral. The maximum area of this quadrilateral is

$$A=\tfrac{1}{4}[(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)]^{1/2},$$

and Oppenheim's method of proof can be used when $p=1$. We may assume $p>1$ and minimize $A^{p/2}-A_1^{p/2}-A_2^{p/2}$, using

$$2^pA_1^{p/2}=[-a_1^4-b_1^4-c_1^4-d_1^4+2(a_1^2b_1^2+\cdots+c_1^2d_1^2)+8a_1b_1c_1d_1]^{p/4}$$

and a similar formula for $A_2^{p/2}$. If $a_1=a_2=0$, all three quadrilaterals degenerate to triangles, and Theorem 1 is applicable. We may assume a, b, c, d are positive. Suppose a_1 is variable and $a_1^p+a_2^p$ is constant. The maximum value of a_1 is such that $A_1A_2a_2=0$. If $a_2=0$ and $A_1A_2\neq 0$ at this point, we may use

$$\frac{\partial}{\partial a_1}(A^{p/2}-A_1^{p/2}-A_2^{p/2})=\frac{pb_2c_2d_2A_2^{(p-4)/2}}{8}\left(\frac{a_1}{a_2}\right)^{p-1}+\cdots \tag{11}$$

to show that this point is not a minimum of $A^{p/2}-A_1^{p/2}-A_2^{p/2}$; the terms not shown explicitly are negligible when $a_2\to 0^+$. If $A_1A_2=0$ at the maximum of a_1, we use:

LEMMA 3. *If* $1<p\leq 4$ *and* A_1, A_2, A *are the maximum areas of the quadrilaterals in Theorem* 3, *then* $A\geq A_1$ *and* $A\geq A_2$. *If* $A=A_1$ *or* $A=A_2$, *then* $A_1=A_2=0$ *and the sets* a_1, b_1, c_1, d_1 *and* a_2, b_2, c_2, d_2 *are proportional.*

The minimum value of a_1 can be treated similarly. The minimum of $A^{p/2}-A_1^{p/2}-A_2^{p/2}$ is attained at an intermediate value of a_1, except in degenerate cases covered by Lemma 3. The condition for $A^{p/2}-A_1^{p/2}-A_2^{p/2}$ to be stationary is

$$\frac{A_1^{(p-4)/2}(-a_1^3+a_1b_1^2+a_1c_1^2+a_1d_1^2+2b_1c_1d_1)}{a_1^{p-1}}$$
$$=\frac{A_2^{(p-4)/2}(-a_2^3+a_2b_2^2+a_2c_2^2+a_2d_2^2+2b_2c_2d_2)}{a_2^{p-1}}.$$

Since A_1 is the maximum area, the corresponding quadrilateral has a circumradius R_1. At the centre of the circumcircle, the sides a_1, b_1, c_1, d_1 subtend angles $2\alpha_1$, $2\beta_1$, $2\gamma_1$, $2\delta_1$. Then $a_1 = 2R_1 \sin\alpha_1, \dots, d_1 = 2R_1 \sin\delta_1$, and $\alpha_1+\beta_1+\gamma_1+\delta_1 = \pi$. Hence,

$$\begin{aligned} b_1c_1d_1 &= 2R_1^3[\sin(-\beta_1+\gamma_1+\delta_1)+\sin(\beta_1-\gamma_1+\delta_1) \\ &\qquad +\sin(\beta_1+\gamma_1-\delta_1)-\sin(\beta_1+\gamma_1+\delta_1)] \\ &= 2R_1^3(\sin\alpha_1)(\cos 2\beta_1+\cos 2\gamma_1+\cos 2\delta_1-1) \\ &\quad +2R_1^3(\cos\alpha_1)(\sin 2\beta_1+\sin 2\gamma_1+\sin 2\delta_1). \end{aligned}$$

The condition for a minimum becomes

$$\begin{aligned} &\left(\frac{A_1^{1/2}}{R_1}\right)^{p-4}\frac{(\cos\alpha_1)(\sin 2\alpha_1+\sin 2\beta_1+\sin 2\gamma_1+\sin 2\delta_1)}{(\sin^{p-1}\alpha_1)} \\ &\quad =\left(\frac{A_2^{1/2}}{R_2}\right)^{p-4}\frac{(\cos\alpha_2)(\sin 2\alpha_2+\sin 2\beta_2+\sin 2\gamma_2+\sin 2\delta_2)}{(\sin^{p-1}\alpha_2)}. \end{aligned}$$

Since $\sin 2\alpha_1+\sin 2\beta_1+\sin 2\gamma_1+\sin 2\delta_1 = 2A_1/R_1^2$, we obtain (6). Variation of b_1, c_1, and d_1 gives three other equations. To prove that $\alpha_1=\alpha_2$, $\beta_1=\beta_2$, $\gamma_1=\gamma_2$, and $\delta_1=\delta_2$, it suffices to show that α, β, γ, δ are uniquely determined by $\alpha+\beta+\gamma+\delta=\pi$, the ratios (7), and $(\cos\delta\sin^{p-1}\gamma)/(\cos\gamma\sin^{p-1}\delta)=r_3$. We have assumed $\cos\gamma>0$, as we may. The remainder of the proof is as for triangles.

Proof of Lemma 3. It suffices to prove $A \ge A_1$ and find when equality holds. Let $a=(a_1^p+\lambda a_2^p)^{1/p}, \dots, d=(d_1^p+\lambda d_2^p)^{1/p}$, where $a_1, a_2, \dots, d_1, d_2$ are constant and λ is variable. As the case of $a_1=a_2=0$ is covered by Lemma 1, we may assume that a, b, c, d are positive when $0<\lambda\le 1$. We shall show that the right side of

$$\begin{aligned} dA^2/d\lambda &= (4p)^{-1}[(-a^4+a^2b^2+a^2c^2+a^2d^2+2abcd)(a_2/a)^p \\ &\qquad +(-b^4+a^2b^2+b^2c^2+b^2d^2+2abcd)(b_2/b)^p \\ &\qquad +\cdots] \end{aligned}$$

is non-negative. We may treat λ, a, b, c, d as constants and a_2, b_2, c_2, d_2 as variables. We may assume $a\ge b$, $a \ge c$, and $a\ge d$. Then the coefficients of $(b_2/b)^p$, $(c_2/c)^p$, and $(d_2/d)^p$ are positive. If

$$-a^4+a^2b^2+a^2c^2+a^2d^2+2abcd \ge 0,$$

then $dA^2/d\lambda>0$. Thus, we may assume

$$a^4 - a^2b^2 - a^2c^2 - a^2d^2 - 2abcd > 0. \tag{12}$$

This implies $a > b$, $a > c$, $a > d$, and $dA^2/d\lambda \geq (4p)^{-1}F(b_2, c_2, d_2)$, where

$$\begin{aligned} F(b_2, c_2, d_2) = &(-a^4 + a^2b^2 + a^2c^2 + a^2d^2 + 2abcd)[(b_2 + c_2 + d_2)/a]^p \\ &+(-b^4 + a^2b^2 + b^2c^2 + b^2d^2 + 2abcd)(b_2/b)^p \\ &+(-c^4 + a^2c^2 + b^2c^2 + c^2d^2 + 2abcd)(c_2/c)^p \\ &+(-d^4 + a^2d^2 + b^2d^2 + c^2d^2 + 2abcd)(d_2/d)^p. \end{aligned}$$

We may exclude the trivial case of $b_2 = c_2 = d_2 = 0$. If $\partial F/\partial b_2 = \partial F/\partial c_2 = \partial F/\partial d_2 = 0$, then

$$\begin{aligned} &(-b^4 + a^2b^2 + b^2c^2 + b^2d^2 + 2abcd)(b_2/b)^p \\ &\quad = \frac{b_2}{b_2 + c_2 + d_2}(a^4 - a^2b^2 - a^2c^2 - a^2d^2 - 2abcd)\left(\frac{b_2 + c_2 + d_2}{a}\right)^p \end{aligned}$$

and so forth; hence $F(b_2, c_2, d_2) = 0$. To show that this is the minimum of F, we consider the boundaries, where $b_2c_2d_2 = 0$. If $b_2 = 0$ and $c_2d_2 \neq 0$, we set $\partial F/\partial c_2 = \partial F/\partial d_2 = 0$ and again obtain $F(b_2, c_2, d_2) = 0$. The cases of $c_2 = 0$, $b_2d_2 \neq 0$ and $d_2 = 0$, $b_2c_2 \neq 0$ are similar. Since $(b_2/a)^{p-1} < (b_2/b)^{p-1}$,

$$F(b_2, 0, 0) > (a + b)(-a + b + c + d)(a - b + c + d)(b_2/b)^{p-1}b_2 \geq 0.$$

The cases of $b_2 = c_2 = 0$ and $b_2 = d_2 = 0$ are similar. We conclude that $dA^2/d\lambda \geq 0$, with equality only if (12) holds and $a_2 = b_2 + c_2 + d_2$.

If $b_2 = 0$, $c_2d_2 \neq 0$, and $a_2 = c_2 + d_2$, then $b_1 \neq 0$, because the three-triangle case has been excluded. Let $b_2 \to 0$ and $A_2 \to 0$ while $b_1^p + b_2^p$ is constant. Since $p \leq 4$, a formula similar to (11) shows that $A > A_1$ in the limit. The cases of $c_2 = 0$, $b_2d_2 \neq 0$ and $d_2 = 0$, $b_2c_2 \neq 0$ are similar.

This work has shown that $A \geq A_1$. If equality holds, then $a_2b_2c_2d_2 \neq 0$ and $dA^2/d\lambda = 0$ for $0 < \lambda < 1$. For a fixed value of λ, we assumed $a \geq b$, $a \geq c$, $a \geq d$. This gives (12), $a_2 = b_2 + c_2 + d_2$,

$$\frac{-b^4 + a^2b^2 + b^2c^2 + b^2d^2 + 2abcd}{a^4 - a^2b^2 - a^2c^2 - a^2d^2 - 2abcd} = \left(\frac{a_2}{b_2}\right)^{p-1}\frac{b^p}{a^p},$$

$$\frac{-c^4 + a^2c^2 + b^2c^2 + c^2d^2 + 2abcd}{a^4 - a^2b^2 - a^2c^2 - a^2d^2 - 2abcd} = \left(\frac{a_2}{c_2}\right)^{p-1}\frac{c^p}{a^p},$$

and

$$\frac{-d^4+a^2d^2+b^2d^2+c^2d^2+2abcd}{a^4-a^2b^2-a^2c^2-a^2d^2-2abcd}=\left(\frac{a_2}{d_2}\right)^{p-1}\frac{d^p}{a^p}.$$

These equations give

$$\frac{(a^2-b^2)(a^2+b^2-c^2-d^2)}{a^4-a^2b^2-a^2c^2-a^2d^2-2abcd}=1+\left(\frac{a_2}{b_2}\right)^{p-1}\frac{b^p}{a^p}, \tag{13}$$

$$\frac{(c^2-d^2)(a^2+b^2-c^2-d^2)}{a^4-a^2b^2-a^2c^2-a^2d^2-2abcd}=\left(\frac{a_2}{c_2}\right)^{p-1}\frac{c^p}{a^p}-\left(\frac{a_2}{d_2}\right)^{p-1}\frac{d^p}{a^p}, \tag{14}$$

and four other equations. Here, a, b, c, d are functions of λ and a_2, b_2, c_2, d_2 are positive constants. Equations (13) and (14) hold when $0<\lambda<1$, or at least for a subinterval where $a>b$, $a>c$, $a>d$. These two equations can be continued analytically to complex and negative values of λ. The right sides have the form

$$(\text{linear function of } \lambda)/(a_1^p+\lambda a_2^p),$$

and the left sides have branch points at $\lambda=-a_1^p/a_2^p$, $-b_1^p/b_2^p$, $-c_1^p/c_2^p$, $-d_1^p/d_2^p$ unless cancellations occur.

The case of $p=2$ demands separate treatment. Neither a^2-b^2 nor $a^2+b^2-c^2-d^2$ can vanish for all λ. The four branch points must coincide in pairs, for otherwise $abcd$ and the left side of (13) would have branch points. We may assume $a_1/a_2=b_1/b_2$ and $c_1/c_2=d_1/d_2$. Then $(a^2-b^2)/a^2$ and the right side of (13) are independent of λ. Hence,

$$\frac{a^2+b^2-c^2-d^2}{a^2-b^2-c^2-d^2-2bcd/a}$$

and

$$\frac{b^2+bcd/a}{a^2+b^2-c^2-d^2}=\frac{(b_2^2/a_2^2)a^2+(b_2d_2/a_2c_2)c^2}{(1+b_2^2/a_2^2)a^2-(1+d_2^2/c_2^2)c^2}$$

are independent of λ. Since a_2, b_2, c_2, d_2 are positive, this is possible only if

$$a_1/a_2=b_1/b_2=c_1/c_2=d_1/d_2. \tag{15}$$

Since $a_2=b_2+c_2+d_2$, we have $A_1=A_2=0$.

We may now assume $p\neq 2$. Then a^2, b^2, c^2, d^2 have branch points. The ratio of (14) to (13), $(c^2-d^2)/(a^2-b^2)$, is a rational function of λ. Similarly, $(b^2-d^2)/(a^2-c^2)$ and $(b^2-c^2)/(a^2-d^2)$ are rational functions. If $b-d$ and $c-d$ vanish for all λ, (13) becomes

$$1+\frac{b^2}{a(a-2b)}=1+\left(\frac{a_2}{b_2}\right)^{p-1}\frac{b^p}{a^p},$$

and the left side is rational only if (15) holds. If $c-d$ vanishes identically and $b-d$ does not, $(b^2-d^2)/(a^2-c^2)$ is rational only if $a_1/a_2=b_1/b_2$, and (13) becomes

$$\frac{(a^2-b^2)(a^2+b^2-2c^2)}{a^2(a^2-b^2-2c^2)-2abc^2}=1+\left(\frac{a_2}{b_2}\right)^{p-1}\frac{b^p}{a^p}.$$

We also have

$$\frac{(a^2-b^2)(a^2-c^2)}{a^2(a^2-b^2-2c^2)-2abc^2}=1+\left(\frac{a_2}{c_2}\right)^{p-1}\frac{c^p}{a^p}.$$

The ratio is $(a^2+b^2-2c^2)/(a^2-c^2)$, a rational function. Since neither a^2-b^2 nor $a^2+b^2-2c^2$ can vanish identically, this ratio is rational only if (15) holds. Similarly, (15) holds if $b-d$ or $b-c$ vanishes identically. We may assume that neither b^2-c^2 nor b^2-d^2 nor c^2-d^2 vanishes identically. Since $(c^2-d^2)/(a^2-b^2)$ is rational, we have $a_1/a_2=c_1/c_2$, $b_1/b_2=d_1/d_2$ or $a_1/a_2=d_1/d_2$, $b_1/b_2=c_1/c_2$. In the first case, $(b^2-d^2)/(a^2-c^2)$ is rational only if (15) holds. In the second case, $(b^2-c^2)/(a^2-d^2)$ is rational only if (15) holds. Hence, (15) must hold and $A_1=A_2=0$.

Proof of Theorem 4

Let $A=f(a)$, $B=f(b)$, and $C=f(c)$. We may assume $a\geq b\geq c$. Then (4) gives $A\geq B\geq C$ and $A/a\leq B/b\leq C/c$. Since obtuse triangles are excluded, $a^2\leq b^2+c^2$, which gives

$$A^2\leq(b^2+c^2)(A/a)^2\leq B^2+C^2.$$

Since equality cannot hold in both places, $A^2<B^2+C^2$. A, B, C are the sides of an acute triangle and $G(A,B,C)$ is an increasing function of each argument. Let us exclude the case of $a=b=c$. Then we can show

$$c < G(a, b, c) < a. \tag{16}$$

Since the area of an acute triangle is an increasing function of each side,

$$G(a, b, c) \geq G(b, b, c) = [(4b^2 - c^2)c^2/3]^{1/4} \geq c.$$

Equality cannot hold in both places. The arithmetic-geometric inequality gives

$$(-a + b + c)(a - b + c)(a + b - c) < [(a + b + c)/3]^3,$$

which gives the second part of (16). Using (16), we may distinguish two cases.

If $b \leq G(a, b, c) < a$, Jensen's inequality for convex functions [3] gives an upper bound for $f(G(a, b, c))$, because $\log f(x)$ is a continuous convex function of $\log x$. We have $f(G(a, b, c)) \leq A^\alpha B^{1-\alpha}$, where α depends on a, b, c and $0 \leq \alpha < 1$. It suffices to show that $G(A, B, C) > A^\alpha B^{1-\alpha}$. Since $G(A, B, C)$ is an increasing function of C, we replace C by its lower bound, which is also determined by the convexity of $\log f(x)$. These two steps amount to replacing $\log f(x)$ by a linear function of $\log x$. It suffices to show that

$$G(f(a), f(b), f(c)) > f(G(a, b, c)) \tag{17}$$

holds when $f(x) = kx^p$. We may set $k = 1$, and (4) gives $0 < p < 1$. This function is treated by Oppenheim [6].

In the other case, $c < G(a, b, c) < b$, and the convexity of $\log f(x)$ gives $f(G(a, b, c)) \leq B^\beta C^{1-\beta}$, where $0 < \beta < 1$. It suffices to show that $G(A, B, C) > B^\beta C^{1-\beta}$. Since $G(A, B, C)$ is an increasing function of A, we replace A by its lower bound, which is also determined by the convexity of $\log f(x)$. Again it suffices to show that (17) holds when $f(x) = kx^p$, or when $f(x) = x^p$.

Oppenheim [6] proves that, if a, b, c are sides of a triangle and $0 < p < 1$,

$$G(a^p, b^p, c^p) \geq [G(a, b, c)]^p.$$

Equality holds iff $a = b = c$. This result can also be derived from Jensen's inequality [3]. Since $\log G(1, 1, 1) = 0$, it suffices to show that $\log G(a^p, b^p, c^p)$ is a concave function of p, or that

$$(d^2/dp^2)\log[(a^p + b^p + c^p)(-a^p + b^p + c^p)(a^p - b^p + c^p)(a^p + b^p - c^p)] \leq 0,$$

with equality iff $a = b = c$. This last inequality is a modified form of lemma 2 of Carroll, Yang and Ahn [2].

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