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Sign Reversing and Matrix Classes[1]

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Abstract. The concept of sign reversing is a useful tool to characterize certain matrix classes in linear complementarity problems. In this paper, we characterize the sign-reversal set of an arbitrary square matrix M in terms of the null spaces of the matrices $I-\Lambda+\Lambda M$, where Λ is a diagonal matrix such that $0\leq\Lambda\leq I$. These matrices are used to characterize the membership of M in the classes P_0, P, and the class of column-sufficient matrices. A simple proof of the Gale and Nikaido characterization theorem for the membership in P is presented.

We also study the class of diagonally semistable matrices. We prove that this class is contained properly in the class of sufficient matrices. We show that to characterize the diagonally semistable property is equivalent to solving a concave Lagrangian dual problem. For 2×2 matrices, there is no duality gap between a primal problem and its Lagrangian problem. Such a primal problem is motivated by the definition of column sufficiency.

Key Words. Linear complementarity problems, matrix classes, sufficient matrices, diagonally semistable matrices, Lagrangian dual problems.

1. Introduction

This paper concerns several classes of matrices that have arisen in connection with the linear complementarity problem (Refs. 1 and 2), namely P_0, column sufficient matrices, and P. When all the principal minors of a real square matrix are nonnegative [positive], the matrix belongs to the class

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P_0 [P]. A matrix $M \in R^{n \times n}$ is column sufficient (CSU) if, for all $z \in R^n$,

$$z_i(Mz)_i \leq 0, \qquad \forall i \Rightarrow z_i(Mz)_i = 0, \qquad \forall i;$$

and M is said to be row sufficient (RSU) if M^T is column sufficient. A matrix that is both row and column sufficient is simply called sufficient (SU). The classes P_0, P and their subclasses have been studied with some intensity for at least thirty years, largely because of their prevalence in scientific computing (Refs. 3 and 4), the theory of piecewise-linear electrical networks (Ref. 5), matrix theory (Refs. 6 and 7), and the theoretical foundations of the linear complementarity problem (Refs. 8 and 9).

The linear complementarity problem (LCP) is defined as follows: given $q \in R^n$ and $M \in R^{n \times n}$, find $z \in R^n$ which satisfies the conditions

$$\text{(i) } z \geq 0, \qquad \text{(ii) } q + Mz \geq 0, \qquad \text{(iii) } z^T(q + Mz) = 0,$$

or show that no such z exists. An LCP with data q and M is denoted by (q, M). If there exists a z satisfying (i) and (ii), then (q, M) is termed feasible. In general, feasibility is not enough to guarantee the existence of a solution of (q, M).

The most essential role played by P in the LCP concerns the existence and uniqueness of a solution. To be precise, (q, M) has a unique solution for any given q if and only if M is a P-matrix (Refs. 10–12). Although P_0 can be thought of as a natural generalization of P, there are major differences in terms of LCPs. For instance, the number of solutions of (q, M) with $M \in P_0$ is quite different from that for the case of $M \in P$. In fact, if $M \in P_0$, then (q, M) may have no solution for some q, even if it is feasible. On the other hand, it is easy to construct examples in which an infinite number of solutions exist for the LCP (q, M) with given q and $M \in P_0$.

Interest in the classes CSU and RSU stems from the study of the linear complementarity problem (Ref. 13). In that paper, it was proven that the solution set (which may be empty) of (q, M) is convex for every q if and only if $M \in$ CSU. Furthermore, all minima of the quadratic program

$$\min \quad z^T(q + Mz), \tag{1a}$$

$$\text{s.t.} \quad q + Mz \geq 0, z \geq 0, \tag{1b}$$

are solutions of (q, M) for every q if and only if $M \in$ RSU. These matrix classes have algorithmic significance for (q, M) as well. Column sufficiency is useful in justifying the least-index degeneracy resolution scheme in connection with the principal pivoting method (Ref. 14). The interior-point method for solving LCPs is also connected with column and row sufficiency (Refs. 15 and 16).

Sign reversing (defined in Section 2) is a useful tool in characterizing certain matrix classes in LCPs. Gale and Nikaido (Ref. 17) characterized the class of P-matrices in terms of the sign-reversal set. Later, Eaves (Ref. 9) characterized the class of column adequate matrices in a similar manner. In Section 2, we present a characterization of the sign-reversal set of M in terms of the null spaces of the matrices $I-\Lambda+\Lambda M$, where Λ is a diagonal matrix such that $0\leq\Lambda\leq I$. In Section 3, matrices of the form $I-\Lambda+\Lambda M$ are employed to further characterize the classes P_0, CSU, and P. Membership in P_0 and P is characterized by the sign of the determinants of $I-\Lambda+\Lambda M$, while $M\in$CSU if and only if $I-\Lambda+\Lambda M\in$CSU, $\forall\Lambda$. Based on these matrices, we provide a simple proof to the Gale and Nikaido characterization of membership in P.

In Section 4, we define the notion of diagonally semistable matrices and establish some properties of these matrices. The diagonally semistable matrices are a natural generalization of diagonally stable matrices (see, for example, Ref. 2). We show that the class of diagonally semistable matrices is contained properly in the class SU.

As in linear programming, there exists the Lagrangian dual problem, which is related closely to a given nonlinear programming problem (the primal problem). Under certain convexity assumptions and constraint qualifications, the primal and dual problems have equal optimal objective values. Thus, it is possible for the dual problem to generate a solution to the primal problem. Based on the definition of column sufficiency, which is related closely to the concept of sign reversing, we consider a nonlinear program (the primal problem) whose constraints are given directly by the sign reversal set and the normalization condition. By the property of Lagrangian dual problems (see Section 6.3 in Ref. 18), we know that the dual problem is concave. We show that characterizing the diagonally semistable property is equivalent to solving such a concave dual problem. In the final section, we show that $M\in R^{2\times 2}$ is diagonally semistable if and only if $M\in$SU. Moreover, for 2×2 matrices, $M\in$SU$\backslash$P if and only if strong duality exists between the primal and dual problems. Conventional assumptions for the absence of a duality gap (Ref. 18) include the Slater constraint qualification and convexity in the objective function, constraints, and domain of the primal problem. Note that, for $M\in$SU$\backslash$P, the Slater condition will never hold. Furthermore, the domain is not convex, and the objective function is not convex in general. We present examples that violate conventional assumptions, yet produce strong duality.

A few words about notations are needed. $M_{\alpha\beta}$ denotes the submatrix of M containing rows α and columns β; the empty set is denoted by $\varnothing$; $\bar{\alpha}$ denotes the complement of index set $\alpha\subset\{1, 2, \ldots, n\}$; I denotes the identity matrix; matrix inequalities of the type $A\leq B$ are componentwise.

2. Sign Reversing Revisited

Definition 2.1. The matrix $M\in R^{n\times n}$ reverses the sign of the vector $z\in R^n$ if $z_i(Mz)_i\leq 0$, for all $i=1, 2, \ldots, n$.

The term "sign reversing" was coined by Gale and Nikaido (Ref. 17). For convenience, we write

$$\text{rev } M=\{z: z_i(Mz)_i\leq 0, i=1, 2, \ldots, n\},$$

where $M\in R^{n\times n}$. Notice that rev M is a cone containing ker M, the nullspace of M, and hence is always nonempty. In particular, the zero vector belongs always to rev M.

Sign reversing is a useful tool in characterizing certain matrix classes. One of such characterization pertains to the class of P-matrices as follows.

Proposition 2.1. Let M be an $n\times n$ matrix. Then,

$$M\in \text{P} \qquad \text{if and only if rev } M=\{0\}. \tag{2}$$

As stated, Proposition 2.1 was proven by Gale and Nikaido (Ref. 17). It had earlier been established in an equivalent form by Fiedler and Pták (Ref. 19). In Section 3, we give an alternative proof of this result.

The preceding proposition shows that rev M may be expressed in terms of ker M if M is in some matrix class; note that, if $M\in$P, then ker $M=\{0\}$. We now present a characterization of rev M for any matrix M in a similar fashion.

Lemma 2.1. Let $M\in R^{n\times n}$, and let $x\in R^n$ be nonzero. If $x\in\text{rev } M$, then there exists a diagonal matrix Λ, with $0\leq\Lambda\leq I$, such that $x\in\ker(I-\Lambda+\Lambda M)$.[3]

Proof. The problem is to define the appropriate diagonal elements Λ_i of Λ, and can be done as follows.

Case 1. Suppose that $x_i(Mx)_i=0$. If both x_i and $(Mx)_i$ are zero, then any $\Lambda_i\in[0, 1]$ will do. Otherwise, there are two possibilities.

Case 1a. If $x_i=0$, take $\Lambda_i=0$.

Case 1b. If $x_i\neq 0$, and hence $(Mx)_i=0$, take $\Lambda_i=1$.

[3]Matrices of this form appeared, for example, in Ref. 20.

Case 2. Suppose that $x_i(Mx)_i<0$. Then, x_i and $(Mx)_i$ are nonzero and of opposite sign. Take

$$\Lambda_i=x_i/[x_i-(Mx)_i]=x_i^2/[x_i^2-x_i(Mx)_i].$$

In each case, we have

$$0\leq\Lambda_i\leq 1 \quad \text{and} \quad (1-\Lambda_i)x_i+\Lambda_i(Mx)_i=0, \qquad \text{for all } i.$$

This completes the proof. □

Theorem 2.1. Let M be an $n\times n$ matrix. Then,

$$\text{rev}\, M=\bigcup_{0\leq\Lambda\leq I}\ker(I-\Lambda+\Lambda M). \tag{3}$$

Proof. By Lemma 2.1, we have immediately

$$\text{rev}\, M\subseteq\bigcup_{0\leq\Lambda\leq I}\ker(I-\Lambda+\Lambda M).$$

Conversely, given a diagonal matrix Λ such that $0\leq\Lambda\leq I$, consider $x\in\ker(I-\Lambda+\Lambda M)$. We must show that $x\in\text{rev}\, M$. For each $i\in\{1, 2, \ldots, n\}$, we have

$$(1-\Lambda_i)x_i+\Lambda_i(Mx)_i=0, \tag{4}$$

and consequently,

$$(1-\Lambda_i)x_i^2+\Lambda_i x_i(Mx)_i=0. \tag{5}$$

From the latter, we deduce that:

(a) $\Lambda_i=0 \Rightarrow x_i=0 \Rightarrow x_i(Mx)_i=0$;
(b) $\Lambda_i=1 \Rightarrow x_i(Mx)_i=0$;
(c) $0<\Lambda_i<1 \Rightarrow x_i(Mx)_i=-(1-\Lambda_i)x_i^2/\Lambda_i\leq 0$.

Hence, $x\in\text{rev}\, M$. □

3. Characterization of Matrices

We have seen that the matrices $I-\Lambda+\Lambda M$ play an important role in deriving an equivalent condition for the sign-reversal set of a given matrix M. In this section, we employ the matrices $I-\Lambda+\Lambda M$ to characterize the classes of matrices P_0, CSU, and P.

Lemma 3.1. Let M be an $n \times n$ matrix, and let Λ be an $n \times n$ diagonal matrix. Then, M is a P_0-matrix if and only if

$$\det(I-\Lambda+\Lambda M)\geq 0, \qquad \forall 0\leq\Lambda\leq I. \tag{6}$$

Proof. We first observe that

$$\begin{aligned}\det(I-\Lambda+\Lambda M)&=\sum_\alpha \det((I-\Lambda)_{\alpha\alpha})\det((\Lambda M)_{\bar\alpha\bar\alpha})\\&=\sum_\alpha \Pi_{i\in\alpha}(1-\Lambda_i)(\Pi_{j\in\bar\alpha}\Lambda_j)\det(M_{\bar\alpha\bar\alpha}),\end{aligned} \tag{7}$$

where α runs over the index subsets (including $\varnothing$) of $\{1, 2, \ldots, n\}$. Furthermore, since $0\leq\Lambda\leq I$, we have

$$0\leq\Lambda_i\leq 1, \qquad i=1, 2, \ldots, n.$$

If $M\in P_0$, then (6) follows from (7) as each summand is nonnegative. Conversely, suppose that (6) holds. For any fixed α, choose

$$\Lambda_j=\begin{cases}1, & \forall j\in\bar\alpha,\\ 0, & \forall j\in\alpha\end{cases}$$

Then, by (7),

$$\det(I-\Lambda+\Lambda M)=\det(M_{\bar\alpha\bar\alpha}).$$

It now follows from (6) that M is a P_0-matrix. □

We can obtain easily a similar characterization of P-matrices as expressed in the following theorem.

Lemma 3.2. Let M be an $n \times n$ matrix. Then, M is a P-matrix if and only if

$$\det(I-\Lambda+\Lambda M)>0, \qquad \forall 0\leq\Lambda\leq I. \tag{8}$$

Remark 3.1. The characterizations given above can be viewed of as extensions of a result of Aganagić (Ref. 20) that $M\in P$ if and only if $\det(I-\Lambda+\Lambda M)\neq 0$ for all diagonal matrices Λ with $0\leq\Lambda\leq I$.

Corollary 3.1. Let M be an $n \times n$ matrix. Then,

(i) $M\in P_0$ if and only if $I-\Lambda+\Lambda M\in P_0$, for all $0\leq\Lambda\leq I$.
(ii) $M\in P$ if and only if $I-\Lambda+\Lambda M\in P$, for all $0\leq\Lambda\leq I$ (Ref. 20).

We are ready to give a simple proof of Proposition 2.1.

Proof of Proposition 2.1. By Corollary 3.1, if $M \in P$, then so is $I - \Lambda + \Lambda M$ for all $0 \le \Lambda \le I$. Hence, the kernel of each such matrix is $\{0\}$, which by Theorem 2.1 implies rev $M = \{0\}$. Conversely, if rev $M = \{0\}$, then

$$\ker(I - \Lambda + \Lambda M) = \{0\}, \qquad \text{for all } 0 \le \Lambda \le I.$$

By a continuity argument, the determinants of all such matrices are positive. By Lemma 3.2, it follows that $M \in P$. □

From Theorem 2.1, we have the following definition.

Definition 3.1. Let M be a real square matrix, and let $x \in \operatorname{rev} M$. If $0 \le \Lambda \le I$ and $(I - \Lambda + \Lambda M)x = 0$, Λ annihilates x.

Theorem 3.1. If $M \in R^{n \times n}$, then the following statements are equivalent:

(a) M is column sufficient.
(b) For each $x \in \operatorname{rev} M$ and its annihilator Λ, if $0 < \Lambda_i < 1$, then $x_i = 0$.

Proof. Suppose that $M \in \mathrm{CSU}$ and there exists an $x \in \operatorname{rev} M$ and its annihilator Λ with $0 < \Lambda_i < 1$ such that $x_i \ne 0$. Since $(1 - \Lambda_i)x_i^2 + x_i \Lambda_i (Mx)_i = 0$, we have $x_i (Mx)_i < 0$, which is impossible since $M \in \mathrm{CSU}$.

Conversely, suppose that M is not column sufficient. Then, there exists an $x \in \operatorname{rev} M$ and an index set α such that $\bar{\alpha} \ne \emptyset$ and

$$x_i (Mx)_i = 0, \qquad i \in \alpha,$$

$$x_i (Mx)_i < 0, \qquad i \in \bar{\alpha}.$$

Define $\Lambda = \operatorname{diag}(\Lambda_1, \ldots, \Lambda_n)$, where

$$\Lambda_i = \begin{cases} 1, & \text{if } i \in \alpha, \\ x_i^2 / [x_i^2 - x_i (Mx)_i], & \text{if } i \in \bar{\alpha}. \end{cases}$$

Then,

$$(I - \Lambda + \Lambda M)x = 0,$$

so Λ annihilates x. However, $0 < \Lambda_i < 1$ and $x_i \ne 0$, for each $i \in \bar{\alpha}$, a contradiction. □

Theorem 3.2. $M \in \mathrm{CSU}$ if and only if $I - \Lambda + \Lambda M \in \mathrm{CSU}$, for all $0 \le \Lambda \le I$.

Proof. Assume that $M \in \mathrm{CSU}$. Let $x \in R^n$ be a vector such that

$$x_i[(I-\Lambda+\Lambda M)x]_i \leq 0, \qquad \forall i.$$

In other words,

$$x_i^2(1-\Lambda_i)+\Lambda_i x_i(Mx)_i \leq 0.$$

It follows that

$$x_i(Mx)_i \leq 0, \qquad \text{for all } i.$$

Since M is column sufficient,

$$x_i(Mx)_i = 0, \qquad \text{for all } i.$$

This implies

$$x_i[(I-\Lambda+\Lambda M)x]_i = 0, \qquad \forall i.$$

Hence, $I-\Lambda+\Lambda M$ is column sufficient for any Λ with $0 \leq \Lambda \leq I$. The other direction is obvious. □

Corollary 3.2. $M \in \mathrm{SU}$ if and only if $I-\Lambda+\Lambda M \in \mathrm{SU}$ for all $0 \leq \Lambda \leq I$.

4. Duality Problem and Diagonally Semistable Matrices

In this section, we consider the following nonlinear programming problem:

$$\min \quad x^T M x, \tag{9a}$$

$$\text{s.t.} \quad x_i(Mx)_i \leq 0, \qquad i=1,\ldots,n, \tag{9b}$$

$$\|x\|_2 = 1. \tag{9c}$$

The corresponding Lagrangian dual problem can be written as

$$\max \quad \min \left\{ \sum_1^n (1+u_i)x_i(Mx)_i \colon \|x\|_2 = 1 \right\}, \tag{10a}$$

$$\text{s.t.} \quad u \geq 0. \tag{10b}$$

Note that the objective function of the dual problem can be further reduced. Given a nonnegative $u \in R^n$ and letting

$$D = \mathrm{diag}(1+u_1, \ldots, 1+u_n),$$

the dual objective function becomes

$$\begin{aligned}
&\min\left\{\sum_{1}^{n}(1+u_i)x_i(Mx)_i:\ \|x\|_2=1\right\}\\
&=\min\{x^T(DM)x:\ \|x\|_2=1\}\\
&=\min\{x^T[(DM+M^TD)/2]x:\ \|x\|_2=1\}\\
&=\lambda(u)/2,
\end{aligned} \tag{11}$$

where $\lambda(u)$ is the minimal eigenvalue of the matrix $DM+M^TD$ (by the Rayleigh principle, see Section 6.4 of Ref. 21). Thus, the dual problem becomes

$$\max\{\lambda(u)/2:\ u\geq 0\}.$$

It follows from the properties of the dual function (see Chapter 6 in Ref. 18) that $\lambda(u)/2$ is concave; hence, a local optimum is also a global optimum. This makes the maximization of the dual function an attractive task. However, in this paper, we shall study the theoretical aspects of the dual problem only.

In the sequel, we shall define the notion of diagonally semistable matrix as an extension of positive semidefiniteness. A vector z solves (q, M) if and only if $E^{-1}z$ solves (Dq, DME), where D and E are arbitrary diagonal matrices with positive diagonal elements. If the matrix DME is positive semidefinite, the LCP (Dq, DME) [thus (q, M)] is equivalent to a convex quadratic program. Such an equivalent formulation raises the question of whether it is possible to transform M into a positive-semidefinite matrix by two-sided positive rescaling. This leads to the following definition.

Definition 4.1. The matrix M is said to be diagonally semistable if there exists a positive diagonal matrix D such that DM is positive semidefinite. The matrix M is called diagonally stable[4] if DM is positive definite.

Note that, if M is diagonally semistable, one can choose E equal to I and then solve (Dq, DM) to obtain a solution for (q, M). Moreover, it is easy to see that M is diagonally semistable [stable] if and only if $\max\{\lambda(u): u\geq 0\}\geq 0$ [>0].

[4]See, for example, Ref. 2.

Theorem 4.1. M is a P-matrix if and only if the primal problem is infeasible.

Proof. This follows directly from Proposition 2.1. □

Theorem 4.2. Weak Duality Theorem. If the primal problem (9) is feasible, then

$$\max\{\lambda(u)/2: u \geq 0\}$$
$$\leq \min\{x^T Mx: x_i(Mx)_i \leq 0, \text{ for all } i \text{ and } \|x\|_2 = 1\}. \tag{12}$$

Proof. See Theorem 6.2.1 in Ref. 18. □

Remark 4.1. A duality gap exists if strict inequality holds true in (12).

Theorem 4.3. If $\max\{\lambda(u): u \geq 0\} > 0$, then $M \in \text{P}$.

Proof. Suppose that M is not a P-matrix. By Theorem 4.1, the primal problem is feasible. By Theorem 4.2, we have the contradiction

$$\max\{\lambda(u)/2: u \geq 0\}$$
$$\leq \min\{x^T Mx: x_i(Mx)_i \leq 0, \text{ for all } i \text{ and } \|x\|_2 = 1\} \leq 0.$$ □

Theorem 4.4. The optimal value of the primal problem (9) is zero if and only if $M \in \text{CSU} \setminus \text{P}$.

Proof. This follows from Theorem 4.1 and the definition of column sufficiency. □

Theorem 4.5. If M is diagonally semistable, then M is sufficient.

Proof. Since M is diagonally semistable, there exist positive diagonal matrices D and I so that DMI is positive semidefinite, and hence column sufficient. Because column sufficiency is invariant under two-sided positive scaling (see Ref. 15), we then have

$$D^{-1}(DMI)I = M,$$

a column sufficiency. Similar arguments show that M^T is column sufficient. □

Theorem 4.6. The following statements are equivalent:

(a) M is diagonally semistable.
(b) There exist positive diagonal matrices S and E such that SME is positive semidefinite.
(c) There exists a positive diagonal matrix F such that $F^{-1}MF$ is positive semidefinite.

Moreover, if M is diagonally semistable, then all the eigenvalues of M have nonnegative real parts.

Proof. See Theorem 3.3.9 in Ref. 2. □

Example 4.1. Consider the matrix

$$M = \begin{bmatrix} 0 & -1 & 2 \\ 2 & 0 & -2 \\ -1 & 1 & 0 \end{bmatrix}.$$

Note that M is sufficient (see Exercise 3.12.16 in Ref. 2). The eigenvalues of M are 0.32748 and $-0.16374 \pm 2.46585i$. Thus, M is not diagonally semistable.

5. Matrices of Size 2×2

In this section, we examine the conditions for a 2×2 matrix to be diagonally semistable. Consider a 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & f \end{bmatrix}.$$

We need the following lemma.

Lemma 5.1. See Ref. 22. The matrix $M \in R^{2 \times 2}$ is sufficient if and only if, for every principal pivotal transform $\bar{M}$ of M,

(i) $\bar{m}_{ii} \geq 0$, $i = 1, 2$;
(ii) for $i = 1, 2$, if $\bar{m}_{ii} = 0$, then either $\bar{m}_{ij} = \bar{m}_{ji} = 0$ or $\bar{m}_{ij}\bar{m}_{ji} < 0$.

Theorem 5.1. Let $M \in R^{2 \times 2}$. The following statements hold:

(a) $M \in \mathrm{P}$ if and only if $\max\{\lambda(u): u \geq 0\} > 0$;
(b) $M \in \mathrm{SU} \setminus \mathrm{P}$ if and only if $\max\{\lambda(u): u \geq 0\} = 0$.

Proof. Recall that M is diagonally semistable if and only if

$$\max\{\lambda(u): u \geq 0\} \geq 0.$$

First, compute

$$\lambda(u) = y - \sqrt{y^2 + (cd_2 + bd_1)^2 - 4afd_1d_2},$$

where

$$y = ad_1 + fd_2 \quad \text{and} \quad d_i = 1 + u_i, \qquad i = 1, 2.$$

Case 1. If $af = 0$, then

$$\lambda(u) = y - \sqrt{y^2 + (cd_2 + bd_1)^2}.$$

Thus,

$\max\{\lambda(u): u \geq 0\} = 0$

$\Leftrightarrow$ there exist $d_1, d_2 > 0$ such that $cd_2 + bd_1 = 0$ and $y \geq 0$

$\Leftrightarrow$ either $bc < 0$ or $b = c = 0$

$\Leftrightarrow$ $M \in \mathrm{SU}$; see Lemma 5.1.

Case 2. If $af < 0$, then $\lambda(u) < 0$ for all $u \geq 0$. Moreover,

$$\max\{\lambda(u): u \geq 0\} < 0.$$

Case 3. If $af > 0$, then

$\max\{\lambda(u): u \geq 0\} \geq 0$

$\Leftrightarrow (cd_2 + bd_1)^2 \leq 4afd_1d_2$ and $y \geq 0$, (13)

for some positive d_1, d_2. It is easy to see that $y \geq 0$ can be replaced by $a > 0$ and $f > 0$ under the assumption of $af > 0$.

Case 3A. If $bc > 0$, then we need to have $af \geq bc$ for (13) to hold; this can be seen from

$$4bcd_1d_2 \leq (cd_2 + bd_1)^2.$$

Note that if $af > bc$, then

$$\max\{\lambda(u): u \geq 0\} > 0,$$

and M is a P-matrix.
If $af = bc$, then

$$\max\{\lambda(u): u \geq 0\} = 0.$$

Moreover, since $a>0$, $f>0$, and $bc>0$, a principal pivotal transform of M looks like

$$\begin{bmatrix} 1/a & -b/a \\ c/a & 0 \end{bmatrix},$$

and again is sufficient.

Case 3B. If $bc<0$, then $M \in \mathrm{P}$. Also, there exist positive d_1 and d_2 such that $cd_2+bd_1=0$. This shall make

$$\max\{\lambda(u): u \geq 0\} > 0.$$

Case 3C. If $bc=0$, then $M \in \mathrm{P}$. Without loss of generality, we assume $b=0$. Then,

$$(cd_2)^2 < 4afd_1d_2,$$

for some positive d_1 and d_2. This yields

$$\max\{\lambda(u): u \geq 0\} > 0. \qquad \square$$

Corollary 5.1. $M \in R^{2\times 2}$ is diagonally semistable if and only if it is sufficient.

It is interesting to note that the second statement of Theorem 5.1 can be rephrased as: $M \in \mathrm{SU} \setminus \mathrm{P}$ if and only if there is no duality gap (called strong duality) between the corresponding primal and dual problems. In general, for the absence of a duality gap, one needs to impose certain convexity assumptions and suitable constraint qualifications on the primal problem. For instance, one may impose the Slater constraint qualification and assume that the objective function, constraints, and domain are convex in the primal problem (see Ref. 18). Note that, for $M \in \mathrm{SU} \setminus \mathrm{P}$, the Slater condition will never hold. Also, the domain $\|x\|_2 = 1$ is not convex and the objective function is not convex in general. We have thus presented examples that violate conventional assumptions, yet produce strong duality.

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