JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 85, No. 1, pp. 97-124, APRIL 1995

On the Stability of Projected Dynamical Systems[1]

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Communicated by G. Leitmann

Abstract. A class of projected dynamical systems (PDS), whose stationary points solve the corresponding variational inequality problem (VIP), was recently studied by Dupuis and Nagurney (Ref. 1). This paper initiates the study of the stability of such PDS around their stationary points and thus gives rise to the study of the dynamical stability of VIP solutions. Examples are constructed showing that such a study can be quite distinct from the classical stability study for dynamical systems (DS). We give the definition of a regular solution to a VIP and introduce the concept of a minimal face flow induced by a PDS, which is a standard DS of a lower dimension. We then show that, at the regular solutions of the VIP, the local stability of the PDS is essentially the same as that of its minimal face flow. Hence, we reduce the problem, in this case, to one of the classical stability study of DS, a more developed discipline. In a more direct way, we then establish a series of local and global stability results of the PDS, under various conditions of monotonicity.

Key Words. Projected dynamical systems, variational inequalities, stability theory, minimal face flows.

1. Introduction

The last decade has witnessed the successful employment of finite-dimensional variational inequality (VI) theory in the formulation, qualitative analysis, and computation of various perfectly and imperfectly competitive equilibrium problems [see, e.g., Nagurney (Ref. 2 and references therein)].

[1]This research was supported by the National Science Foundation under Grant DMS-9024071 under the Faculty Awards for Women Program. This support is gratefully acknowledged.
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However, the majority of the works in this realm have focused on the static study of the equilibrium, while most of these equilibrium problems originate from the study of some competitive mechanisms or adjustment processes of time evolution. This inability of VI methods to capture the dynamic setting of equilibrium problems has not received enough attention from the literature until recently.

Dupuis and Nagurney (Ref. 1) introduced a class of projected dynamical systems (PDS) for studying the dynamic evolution of the competitive systems underlying many equilibrium problems. These systems are characterized by a polyhedral constraint set, and their stationary points are the solutions to the corresponding variational inequality problems (VIP). In that paper, they studied the theoretical aspects of these PDS in regard to existence, uniqueness, and the continuous dependence of the solutions on the initial values chosen. Moreover, they investigated prospective applications in dynamic models of oligopolistic market equilibrium, general economic equilibrium, and traffic network equilibrium. A general iterative numerical scheme was also proposed to approximate the equilibrium points through time discretization of the PDS. The iterative scheme induces such well-known algorithms in dynamical systems as the Euler method, Heun method, and Runge–Kutta method. Following these lines, Nagurney, Takayama, and Zhang (Ref. 3) developed a dynamical model for the spatial price equilibrium problem and implemented the proposed Euler-type method on a massively parallel architecture for the computation of the equilibrium.

This paper proposes to study the stability of such projected dynamical systems around their stationary points. It thus raises the subject of the dynamical stability of VIP solutions, in contrast to parametric perturbative stability [cf. Dafermos (Ref. 4)]. Theoretically, since the associated ordinary differential equations (ODE) have discontinuous right-hand sides, due to the projection operator on the boundary of the constraint set, the stability study of PDS lies outside of the scope of standard dynamical systems (DS) in a continuous vector field (or more often in a C^1-vector field; cf. Refs. 5 and 6). However, as shall be established in this paper, it does naturally generalize the stability theory of dynamical systems with no constraints. Particularly, we note that, around the equilibrium points that lie in the interior of the constraint set, the stability of projected dynamical systems is identical to that of the dynamical systems in the same vector field.

Besides the theoretical issues, we are motivated in conducting this line of research for the following reasons. First, it will serve as a mathematical machinery to analyze the stability of the equilibrium in many social and economic competitive systems with natural constraints, including those mentioned above. For different applications, there has already been some work with such an orientation. For example, in the scenario of transportation, Smith (Ref. 7) investigated the stability of Wardropean equilibrium under a dynamical system constructed to describe the drivers' behavior in changing their route choices. However, the approach that we propose is distinguished from earlier ones in that we present a more general mathematical approach not yet confined to or derived from a certain application scenario.

Secondly, the equilibrium points were calculated as VIP solutions. Yet, we have no way of identifying which among the solutions (precisely, the multiple solutions) is a true equilibrium. The dynamical stability study that we initialize here will thus facilitate a discrimination between those VIP solutions having good stability as equilibria that are anticipated to occur realistically, and those with bad stability to be eliminated for consideration in applications.

Finally, the dynamical stability study undertaken here contains the local and global convergence of continuous-time algorithms for the computation of stationary points, and thus fertilizes the convergence study of the iterative scheme introduced in Dupuis and Nagurney (Ref. 1) that tracks the continuous trajectory of the PDS.

The paper is organized as follows. In Section 2, we give a brief review of projected dynamical systems and present the definitions and notations needed in addressing the study of stability. We also illustrate the difference between stability of standard dynamical systems and that of projected dynamical systems through some examples. In Section 3, we introduce the definition of a regular solution to a variational inequality problem and the concept of a minimal face flow around the equilibrium points, induced by the projected dynamical system. At those equilibrium points which solve the variational inequality problem regularly, we show that the projected dynamical system inherits many stability properties from its induced mimimal face flow. Since the latter is a standard dynamical system of a lower dimension, the stability of the projected dynamical systems can thus be exploited through the classical stability theory of dynamical systems, which is now more developed. In Section 4, more local and global results are established under various monotonicity conditions. It turns out that the stronger the monotonicity condition that is imposed, the better the stability enjoyed by the equilibrium. We close this paper with Section 5, which contains the conclusions and suggestions for future research.

2. Background and Preliminaries

In this section, we recall the projected dynamical system (PDS) with a constraint set K, studied by Dupuis and Nagurney (Ref. 1), and its associated variational inequality problem (VIP). We also review various stability concepts that are traditional in the stability theory of dynamical systems (DS;

cf. Refs. 5 and 6). We close this section with some examples showing that, in the same vector field, the stability of the PDS may be entirely different from that of the usually induced DS without constraints.

Let K be a closed convex subset of R^n, and let F be a mapping from K to R^n. The variational inequality problem $VI(F, K)$ is to find an x^* in K such that

$$\langle F(x^*), x-x^*\rangle \geq 0, \qquad \forall x \in K, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n.

Let ∂K and int K denote, respectively, the boundary and the interior of K. Given $x \in \partial K$, we define the inward normals to K at x by

$$n(x) = \{\gamma : \|\gamma\| = 1, \langle \gamma, x-y\rangle \leq 0, \forall y \in K\}. \tag{2}$$

For any closed and convex subset S of R^n, let $P_S : R^n \mapsto S$ denote the projection map defined by

$$P_S(x) = \operatorname*{argmin}_{x \in S} \|x-z\|. \tag{3}$$

For simplicity, we often write P for P_K when K is the feasible set for the $VI(F, K)$.

In this paper, we restrict ourselves to the case where K is a convex polyhedron, as do Dupuis and Nagurney (Ref. 1). This is the case of most applications that have been formulated as variational inequality problems, including among others traffic network equilibrium problems, spatial price equilibrium problems, oligopolistic market equilibrium problems, and general financial equilibrium problems (cf. Ref. 2 and references therein).

Given $x \in K$ and $v \in R^n$, the projection of the vector v at x is defined by

$$\pi_K(x, v) = \lim_{\delta \downarrow 0} [P_K(x+\delta v) - x]/\delta. \tag{4}$$

We recall the following result from Dupuis (Ref. 8), which we will use freely later in this paper.

Lemma 2.1.

(i) If $x \in \text{int } K$, then

$$\pi_K(x, v) = v. \tag{5}$$

(ii) If $x \in \partial K$, then

$$\pi_K(x, v) = v + \beta(x) n^*(x), \tag{6}$$

where

$$n^*(x) = \underset{n \in n(x)}{\operatorname{argmax}} \langle v, -n \rangle, \tag{7}$$

$$\beta(x) = \max\{0, \langle v, -n^*(x) \rangle\}. \tag{8}$$

Remark 2.1. It is obvious from Lemma 2.1 that

$$\|\pi_K(x, v)\| \leq \|v\|. \tag{9}$$

Dupuis and Nagurney (Ref. 1) proposed the following nonclassical ordinary differential equation (ODE):

$$\dot{x} = \pi_K(x, -F(x)), \qquad x(0) = x_0 \in K, \tag{10}$$

whose right-hand side is associated with a projection and hence is discontinuous on the boundary of K. It is clear from the definition that the solution to the ODE (10) always stays in the constraint set K. In that reference, the basic qualitative results of the solution to (10), such as the existence, uniqueness, and continuous dependence on the initial value, can be found. We also cite the following lemma from that paper, which established the connection between the $VI(F, K)$ and the ODE (10).

Lemma 2.2. Assume that K is a convex polyhedron. Then the stationary points of the ODE (10) coincide with the solutions of the $VI(F, K)$.

Through this lemma, we see that the PDS, defined as the solution to the ODE (10), depicts the dynamical behavior of a competitive system, whose equilibrium was previously formulated as the solution to the VIP. One major advantage of taking such an approach is that it enables one to study the stability of the equilibrium of the PDS, while such a topic is not raised in the static theory of the VIP. For a variational inequality approach to the study of stability in the framework of specific applications (in particular, traffic network and spatial price equilibrium problems), see Dafermos and Nagurney (Refs. 9 and 10). For a general approach to parametric perturbative stability of VIP, see Dafermos (Ref. 4). The stability study of the PDS is what we begin to address in this paper. In particular, we are interested in questions such as:

(a) If a competitive behavior starts near an equilibrium, will it stay close to it forever?

(b) Given the current state of a competitive system, will it asymptotically approach an equilibrium?

To make these questions clear, we recall the following definitions and notation that are classical in the stability study of dynamical systems (cf. Refs. 5 and 6).

Definition 2.1. Define the projected dynamical system (PDS) $x_0(t): K \times R \mapsto K$ as the family of solutions to the initial-value problem (IVP) (10) for all $x_0 \in K$.

It is clear by the definition that $x_0(0) = x_0$. For convenience, we will sometimes write $x_0 \cdot t$ for $x_0(t)$ and say interchangeably that x^* is an equilibrium or stationary point of the PDS, or x^* solves $VI(F, K)$, by virtue of Lemma 2.2.

Definition 2.2. For any subset A of R^n, the ω-limit set of A is defined by

$$\omega(A) = \{y: \exists x_n \in A,\ t_n \to \infty, \text{ such that } x_n \cdot t_n \to y, \text{ as } n \to \infty\}.$$

We will use $B(x, r)$, hereafter, to denote the open ball with radius r and center x.

Definition 2.3. An equilibrium point x^* is stable if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $x \in B(x^*, \delta)$ and $t \geq 0$,

$$x \cdot t \in B(x^*, \epsilon).$$

The equilibrium point x^* is unstable, if it is not stable.

Definition 2.4. An equilibrium point x^* is asymptotically stable if it is stable and there exists a $\delta > 0$ such that, for all $x \in B(x^*, \delta)$,

$$\lim_{t \to \infty} x \cdot t \to x^*. \tag{11}$$

Definition 2.5. An equilibrium point x^* is exponentially stable if there exists a neighborhood $N(x^*)$ of x^* and constants $B > 0$ and $\mu > 0$ such that

$$\|x_0 \cdot t - x^*\| \leq B \|x_0 - x^*\| \exp(-\mu t), \qquad \forall t \geq 0,\ \forall x_0 \in N(x^*); \tag{12}$$

x^* is globally exponentially stable if (12) holds true for all $x_0 \in K$.

Definition 2.6. An equilibrium point x^* is a monotone attractor, if there exists a $\delta > 0$ such that, for all $x \in B(x^*, \delta)$,

$$d(x, t) = \|x \cdot t - x^*\| \tag{13}$$

is a nonincreasing function of t; x^* is a global monotone attractor if $d(x, t)$ is decreasing in t for all $x \in K$.

An equilibrium point x^* is a strictly monotone attractor if there exists a $\delta > 0$ such that, for all $x \in B(x^*, \delta)$, $d(x, t)$ is monotonically decreasing to zero in t; x^* is a strictly global monotone attractor if $d(x, t)$ is monotonically decreasing to zero in t for all $x \in K$.

Definition 2.7. An equilibrium point x^* is a finite-time attractor if there is a $\delta > 0$ such that, for any $x \in B(x^*, \delta)$, there exists some $T(x) < \infty$ such that

$$x \cdot t = x^*, \qquad \text{when } t \geq T(x).$$

The following two examples show how greatly the stability of the PDS can differ from the stability of a classical DS in the same vector field.

Example 2.1. In Fig. 1, let the constraint set K be the square $ABCD$. Let

$$F(x_1, x_2) = \phi(x_1 + x_2 - 4)[(x_1 - 4x_1/(x_1 + x_2), x_2 - 4x_2/(x_1 + x_2)],$$

where $\phi(w) = -1, 0, 1$, when w is respectively negative, zero, or positive, and let $1 \leq x_1 \leq 3$, $1 \leq x_2 \leq 3$. Let $\tilde{F}$ be any continuous extension of F to the whole space R^2. Then, $B(2, 2)$ is an equilibrium point of the PDS solving

$$\dot{x} = \pi_K(x, -\tilde{F}(x)),$$

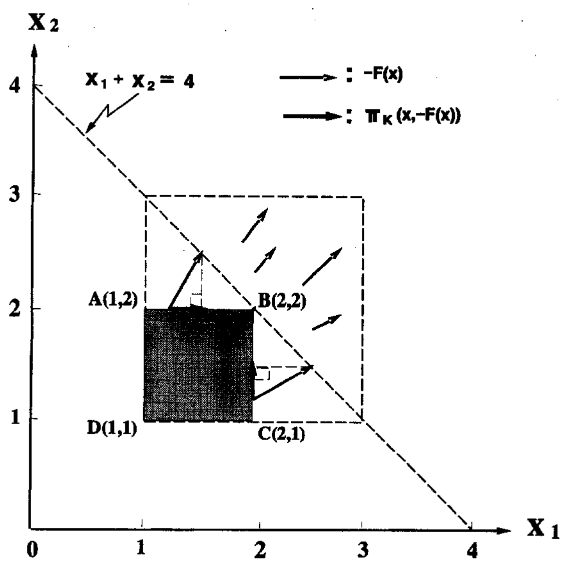

Fig. 1. B is asymptotically stable for $\dot{x} = \pi_k(x, -F(x))$, but unstable for $\dot{x} = -F(x)$.

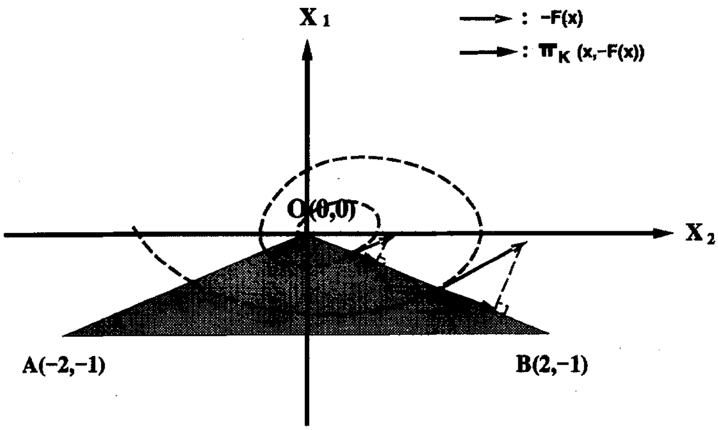

Fig. 2. O is asymptotically stable for $\dot{x}=-F(x)$, but unstable for $\dot{x}=\pi_k(x, -F(x))$.

and it is asymptotically stable. B is also an equilibrium point of the usual DS solving

$$\dot{x}=-\tilde{F}(x),$$

but it is not even stable there.

Example 2.2. Let the vector field

$$F(x_1, x_2)=(x_1+4x_2, -x_1)$$

and the constraint set K be the triangle AOB in Fig. 2. Then, the origin $O(0, 0)$ is the only equilibrium point for both the PDS solving

$$\dot{x}=\pi_K(x, -F(x))$$

and the usual DS solving

$$\dot{x}=-F(x).$$

Although O is asymptotically stable for the linear system $\dot{x}=-F(x)$, it is not stable for the PDS, as we easily observe along the edge OB in Fig. 2.

3. Local Properties under Regularity

Having witnessed the substantial difference between the stability of DS and the stability of PDS, the following question can naturally be raised: Is it possible to study the stability of PDS via the more developed theory for classical DS? As we will establish in this section, the answer is affirmative, at least under some condition of regularity at the equilibrium points.

For the $VI(F, K)$ given in (1), let us specify the convex polyhedron K by

$$K=\{x\in R^n: Bx\leq b\}, \tag{14}$$

where B is an $m\times n$ matrix, with rows B_{i-}, $i=1, 2, \ldots, m$, and b is an n-dimensional column vector.

A convex subset E of K is a face of K, if E is the intersection of K and a number of hyperplanes that support K. So the intersection of any faces of K is again a face of K. For any $x\in K$, denote by $\mathscr{F}(x)$ the collection of all faces of K that contains x. The minimal face of K containing x, denoted by $E(x)$, is defined as the intersection of all the faces of K that contain x, namely,

$$E(x):=\bigcap_{E\in\mathscr{F}(x)} E. \tag{15}$$

For any $x\in K$, let

$$I(x):=\{i: B_{i-}x=b_i\}, \qquad I^c(x):=\{i: B_{i-}x<b_i\},$$

so that

$$I(x)\cup I^c(x)=\{1, 2, \ldots, m\}.$$

It is apparently true that

$$\begin{aligned} E(x)&=\{x: B_{i-}x=b_i,\, i\in I(x);\, B_{j-}x<b_j, j\in I^c(x)\}\\ &=\{S(x)+x\}\cap K, \end{aligned} \tag{16}$$

where

$$S(x)=\{z\in R^n: B_{i-}z=0,\ \forall i\in I(x)\} \tag{17}$$

is a subspace of R^n and

$$S(x)+x=\{z+x: z\in S(x)\}=\{y\in R^n: B_{i-}y=b_i,\ \forall i\in I(x)\} \tag{18}$$

is an affine manifold translated from the subspace $S(x)$ by x, which is itself a supporting hyperplane at x. For completeness, we assume that

$$S(x)=S(x)+x=R^n, \qquad E(x)=R^n\cap K=K, \qquad \text{if } I(x)=\varnothing.$$

Let x^* be any solution to $VI(F, K)$. R^n can then be expressed as the direct sum of the subspace $S(x^*)$ and its orthogonal complement $S^{\perp}(x^*)$. For any $x\in R^n$, if we project $x-x^*$ onto $S(x^*)$ and $S^{\perp}(x^*)$, respectively, and write

$$z_1(x) := P_{S(x^*)}(x - x^*), \qquad z_2(x) := P_{S^\perp(x^*)}(x - x^*), \tag{19}$$

then $z_1(x)$ and $z_2(x)$ comprise the unique orthogonal decomposition of $x - x^*$.

We can see immediately that, for any PDS $x(t)$ which solves IVP(10), $z_1(t) = z_1(x(t))$ given by (19) is a standard DS in the subspace $S(x^*)$, due to the fact that $P_{S(x^*)}$ is continuous. In fact, it is this DS whose stability we are going to examine and exploit for studying the stability of the PDS $x(t)$. The induced DS $z_1(t)$ will be referred to as a minimal face flow, since it is isomorphic to $z_1(t) + x^*$, which with reference to (16) is a DS in the minimal face $E(x^*)$. In what follows, we will spell this out through mathematical rigor.

Definition 3.1. Suppose that $x^* \in K$ with $\dim S(x^*) \geq 1$ and that there is a corresponding $\delta = \delta(x^*) > 0$ such that

$$z_1 + x^* \in (S(x^*) + x^*) \cap K, \qquad \forall z_1 \in S(x^*) \cap B(0, \delta).$$

Define the induced governing function about x^*,

$$g(\cdot) = g(\cdot, x^*) : S(x^*) \cap B(0, \delta) \mapsto S(x^*),$$

as

$$g(z_1) = g(z_1, x^*) := P_{S(x^*)} F(z_1 + x^*). \tag{20}$$

Note that, in the above definition, since $P_{S(x^*)}$ is linear, continuous and nonexpansive,

(i) g is continuous when F is;
(ii) g is linear when F is;
(iii) when F is Lipschitz continuous with constant L, so is g with constant $L_g \leq L$;
(iv) when F is differentiable, g is differentiable and $\nabla g(z_1) = P_{S(x^*)}(\nabla F(z_1 + x^*))$, so g is continuously differentiable when F is.

Definition 3.2. Suppose that F is continuously differentiable. From the results above, the induced governing function g is also continuously differentiable in a neighborhood about the origin of the subspace $S(x^*)$. It follows from Peano's theorem (see, e.g., Ref. 6) that, for any $z_1^0 \in S(x^*) \cap B(0, \delta)$, there exists a $\theta = \theta(z_1^0) > 0$ such that the IVP

$$\dot{z}_1 = -g(z_1), \tag{21}$$

$$z_1(0) = z_1^0, \tag{22}$$

has a unique solution $z_1(t)$ on the interval $(-\theta, \theta)$. We call the solutions to the IVP (21)–(22) for all $z_1^0 \in S(x^*) \cap B(0, \delta)$ the minimal face flow (MFF) about x^*.

The first result following these definitions is

Theorem 3.1. If x^* is an equilibrium point of a PDS, then $0 \in S(x^*)$ is an equilibrium point of its induced MFF.

Proof. For any $z_1 \in S(x^*)$ and $\lambda \in R$ small, $\lambda z_1 + x^*$ is always in K. Since x^* solves the $VI(F, K)$,

$$\langle F(x^*), (\lambda z_1 + x^*) - x^*) \rangle = \lambda F(x^*)^T z_1 \geq 0,$$

for λ small. Therefore, $F(x^*)^T z_1 = 0$, and hence $F(x^*) \in S^\perp(x^*)$. Consequently,

$$-g(0) = P_{S(x^*)}(-F(x^*)) = 0;$$

i.e., 0 is an equilibrium point of the MFF, by Definition 3.2. □

Notice that a MFF is a standard DS in a subspace, for whose stability study there is a relatively mature theory available. Hence, it is appealing to explore the relationship of stability between a PDS and its MFF. This is done, in this section, under some regularity condition on a VIP solution, which we now introduce.

For any $x \in K$, the normal cone of K at x is defined by

$$C(x) := \{y \in R^n : y^T(x' - x) \leq 0, \forall x' \in K\}.$$

Therefore, $C(x) = \{0\}$, when $x \in \text{int } K$.

It is direct from the definition that the necessary and sufficient condition of x^* being a solution to the $VI(F, K)$ is

$$-F(x^*) \in C(x^*). \tag{23}$$

Also, it is easy to see that the normal cone $C(x^*)$ is contained in the subspace $S^\perp(x^*)$, i.e.,

$$C(x^*) \subset S^\perp(x^*). \tag{24}$$

For any subset V of R^n, let $L(V)$ denote the linear subspace spanned by V; then,

$$S(x^*) = L^\perp(B_{i-}, i \in I(x^*)), \tag{25}$$

$$S^\perp(x) = L^{\perp\perp}(B_{i-}, i \in I(x^*)) = L(B_{i-}, i \in I(x^*)). \tag{26}$$

Recall that the relative interior of $C(x^*)$, denoted by ri $C(x^*)$, is the interior of $C(x^*)$ when it is regarded as the subset of $L(C(x^*))$ with respect to the induced metric topology on it; the relative boundary of $C(x^*)$, denoted by rb $C(x^*)$, is defined accordingly (Ref. 11).

In view of the necessary and sufficient condition (23) for a solution to the $VI(F, K)$, it is natural to bring in the following definition.

Definition 3.3. A variational inequality solution x^* of the $VI(F, K)$ is regular, or x^* solves the $VI(F, K)$ regularly, if

$$-F(x^*)\in \text{ri } C(x^*), \qquad \text{when } x^*\in\partial K, \tag{27}$$

$$F(x^*)=0, \qquad \text{when } x^*\in \text{int } K. \tag{28}$$

We point out that the above-defined regularity condition is not a stern restriction for VIP solutions. In fact, any interior solution is regular and any boundary solution x^* is regular, if $-F(x^*)$ is in the relative interior of the convex cone $C(x^*)$. However, it is not regular, if $-F(x^*)$ is on the relative boundary of $C(x^*)$. In particular, when x^* is a solution to the $VI(F, K)$ that lies on an $(n-1)$-dimensional face of K, it is regular if and only if $F(x^*)\neq 0$. Hence, the regularity condition only excludes a few solutions, many less than those retained by it, measured in dimension.

We aim to show that, around a regular solution to the $VI(F, K)$, the PDS inherits many stability properties from its MFF, and thus the problem can be reduced to a classical stability study of DS. The next two lemmas are provided for establishing our major results in this section. In particular, Lemma 3.1 below presents an analytic characterization of ri $C(x^*)$.

Lemma 3.1. For any $y\in$ ri $C(x^*)$, there exists an $\alpha>0$ such that, for all $x\in R^n$,

$$\langle y, z_2(x)\rangle \leq -\alpha\|z_2(x)\|. \tag{29}$$

Proof. For any $w\in C(x^*)$, we have, by definition,

$$w^Tx\leq w^Tx^*, \tag{30}$$

whenever x solves

$$Bx\leq b. \tag{31}$$

It follows from the nonhomogeneous Farkas lemma that there exists a $\lambda\in R^m_+$ such that

$$\text{either } \lambda^TB=w^T \text{ and } \lambda^Tb\leq w^Tx^*, \tag{32}$$

$$\text{or } \lambda^TB=0 \text{ and } \lambda^Tb<0. \tag{33}$$

However, (33) cannot hold because, for any $x \in K$, it gives

$$0 = \lambda^T Bx \leq \lambda^T b < 0, \tag{34}$$

which is a contradiction. Hence, (32) is true. In other words,

$$C(x^*) \subset \left\{ \sum_{i=1}^{m} \lambda_i B_i, \lambda_i \geq 0, i = 1, \ldots, m \right\}. \tag{35}$$

In view of (24) and (26), it follows that

$$C(x^*) \subset \left\{ \sum_{i \in I(x^*)} \lambda_i B_{i-}, \lambda_i \geq 0, i \in I(x^*) \right\}. \tag{36}$$

For any subset S of R^n, denote its polar by S^*, defined as

$$S^* = \{ y \in R^n : x^T y \leq 0, \forall x \in S \}.$$

Then, $S_1 \subset S_2$ implies $S_1^* \supset S_2^*$. If C is a closed and convex cone, then $C^{**} = C$ (Ref. 11).

Since $\{ \sum_{i \in I(x^*)} \lambda_i B_{i-}, \lambda_i \geq 0 \}$ is the closed convex cone generated by $\{ B_{i-}, i \in I(x^*) \}$, which we will denote by $\mathrm{Con}(\{ B_{i-}, i \in I(x^*) \})$, its polar is given by (Ref. 11)

$$\mathrm{Con}^*(\{ B_{i-}, i \in I(x^*) \}) = \{ y \in R^n : B_{i-} y \leq 0, i \in I(x^*) \}. \tag{37}$$

It follows from (24) that

$$\{ y \in R^n : B_{i-} y \leq 0, i \in I(x^*) \} \subset C^*(x^*). \tag{38}$$

On the other hand, for all $x \in K$ and $i \in I(x^*)$, we have

$$B_{i-} z_2(x) = B_{i-} z_1(x) + B_{i-} z_2(x) = B_{i-}(x - x^*) \leq 0. \tag{39}$$

Therefore, it follows from (38) that

$$z_2(x) \in C^*(x^*), \qquad \forall x \in K,$$

and

$$z_2(x) \in C^*(x^*) \cap S^\perp(x^*). \tag{40}$$

Since $y \in \mathrm{ri}\, C(x^*)$, so

$$y^T w \leq 0, \qquad \forall w \in C^*(x^*).$$

We claim that, for all $w \in C^*(x^*) \cap S^\perp(x^*)$ and $w \neq 0$,

$$y^T \cdot w < 0. \tag{41}$$

If it is not true, then there is $w' \in C^*(x^*) \cap S^\perp(x^*)$, $w' \neq 0$, such that $y^T \cdot w' = 0$. Now, $w' \in S^\perp(x^*)$ and $y \in \text{ri } C(x^*)$ imply that there is a sufficiently small $\epsilon > 0$ such that

$$y + \epsilon w' \in C(x^*).$$

Hence,

$$(y + \epsilon w')^T w' = \epsilon \|w'\|^2 > 0, \tag{42}$$

which contradicts the fact that $y + \epsilon w' \in C(x^*)$ and $w' \in C^*(x^*)$. Therefore, (41) is true.

$C^*(x^*) \cap S^\perp(x^*)$ is a closed convex cone. Denote its intersection with the unit ball $B(0, 1)$ by W. Then, W is compact and contains an element w_0 which maximizes the linear functional $\langle y, \cdot \rangle$ on W. Namely,

$$y^T w_0 = \max\{y^T w \colon \|w\| = 1, w \in C^*(x^*) \cap S^\perp(x^*)\}. \tag{43}$$

Therefore, letting $\alpha = -y^T w_0 > 0$, from (41) we have

$$y^T w / \|w\| \leq -\alpha, \tag{44}$$

for all $w \in C^*(x^*) \cap S^\perp(x^*)$, from which the result of the lemma follows directly. □

Lemma 3.2. Suppose that x^* is a regular solution of the $VI(F, K)$ and that $x(\cdot)$ solves the IVP (10). Let $z_i = z_i(x)$, $i = 1, 2$, be as in (19). For any $\epsilon > 0$ and small enough, there exists a neighborhood $N(x^*)$ of x^* and some $T(x_0, \epsilon)$, $0 \leq T(x_0, \epsilon) \leq 2\alpha^{-1}\|z_2(x_0)\|$, such that, when $x_0 \in N(x^*)$,

$$x(T(x_0, \epsilon)) \in B(0, \epsilon) \cap S(x^*) + x^*, \tag{45}$$

where α is the constant provided by Lemma 3.1.

Proof. Let $x(t)$ be the solution to the IVP (10). Let $T_0 = 2\alpha^{-1}\|z_2(x_0)\|$ and $z_i(t) = z_i(x(t))$, $i = 1, 2$. By definition, and since $P_{S(x^*)}$ is linear and continuous, we have

$$\begin{aligned} \dot{z}_1(t) &= (d/dt)(P_{S(x^*)}(x(t) - x^*)) \\ &= P_{S(x^*)}(\dot{x}(t)) = P_{S(x^*)}(\pi(x(t), \ -F(x(t)))). \end{aligned} \tag{46}$$

By Lemma 2.1,

$$\pi(x, -F(x)) = -F(x) + \beta(x) n^*(x), \tag{47}$$

where $\beta(x) \geq 0$ and $n^*(x)$ is some inward normal at x. It is clear that $I(x) \subset I(x^*)$, when x is in some neighborhood $N_1(x^*)$ of x^*, which implies that

$$S(x^*) \subset S(x). \tag{48}$$

Using this fact and (47), it follows from (46) that, for $x(t) \in N_1(x^*)$,

$$\begin{aligned} \dot{z}_1(t) &= P_{S(x^*)}(-F(x) + \beta(x)n^*(x)) \\ &= P_{S(x^*)}P_{S(x)}(-F(x) + \beta(x)n^*(x)) \\ &= P_{S(x^*)}(P_{S(x)}(-F(x)) + P_{S(x)}(\beta(x)n^*(x))) \\ &= P_{S(x^*)}(P_{S(x)}(-F(x))) = P_{S(x^*)}(-F(x)). \end{aligned} \tag{49}$$

On the other hand, it follows from Lemma 3.1, and Definition 3.3 that, since x^* is a regular solution,

$$\langle -F(x^*), z_2(x') \rangle \leq -\alpha \| z_2(x') \|, \qquad \forall x' \in K. \tag{50}$$

Because $F(x)$ is continuous, there exists a neighborhood $N_2(x^*)$ of x^* such that

$$\| F(x) - F(x^*) \| < \alpha/2, \qquad \text{when } x \in N_2(x^*).$$

Therefore,

$$\begin{aligned} &\langle -F(x), z_2(x')/\|z_2(x')\| \rangle \\ &= \langle -F(x^*), z_2(x')/\|z_2(x')\| \rangle + \langle F(x^*) - F(x'), z_2(x')/\|z_2(x')\| \rangle \\ &\leq -\alpha + \| F(x^*) - F(x) \| \leq -\alpha/2, \qquad \forall x' \in K. \end{aligned} \tag{51}$$

Choose δ small enough so that the neighborhood

$$N_3(x^*) = \{x \colon \|z_1(x)\| < \delta,\ \|z_2(x)\| < \delta\}$$

is contained in $N_1(x^*) \cap N_2(x^*)$. Therefore, when $x \in N_3(x^*)$, (49) and (51) always hold true. Let M be large so that

$$M \geq \max\{|F(x)|, x \in \overline{N_3(x^*)}\} \quad \text{and} \quad M \geq \alpha,$$

where $\overline{N_3(x^*)}$ denotes the closure of $N_3(x^*)$. For any $\epsilon > 0$, $\epsilon < \delta$, define

$$N_4(x^*) = \{x \colon \|z_1(x)\| < \epsilon/2,\ \|z_2(x)\| < (\alpha\delta/4M)\}.$$

We claim that, for any $w \in [0, T_0]$, when $x_0 \in N_4(x^*)$ and $x(t) \notin S(x^*)$, $\forall t \in [0, w]$, we have

$$x(t) \in N_3(x^*), \qquad \forall t \in [0, w].$$

In fact, if this is not true, there must be some v, $v \in [0, w]$, such that $x(t) \in N_3(x^*)$, $\forall t \in [0, v)$, but $x(v) \notin N_3(x^*)$, and $x(t) \notin S(x^*)$, $\forall t \in [0, v)$. This is because $x(0) = x_0 \in N_4(x^*) \subset N_3(x^*)$ and $x(t)$ is continuous.

Notice that, for all $t \in [0, v)$,

$$\begin{aligned}
&(d/dt)(\|z_1(t)+z_2(t)\|^2/2)\\
&=(d/dt)(\|x(t)-x^*\|^2/2)\\
&=\langle x(t)-x^*, \pi(x(t), -F(x(t)))\rangle\\
&=\langle x(t)-x^*, -F(x(t))\rangle+\langle x(t)-x^*, \beta(x)n^*(x(t))\rangle\\
&\leq\langle x(t)-x^*, -F(x(t))\rangle\\
&=\langle z_1(t), -F(x(t))\rangle+\langle z_2(t), -F(x(t))\rangle. \qquad (52)
\end{aligned}$$

The first item on the right-hand side of (52) can be rewritten as

$$\begin{aligned}
&\langle z_1(t), -F(x(t))\rangle\\
&=\langle z_1(t), P_{S(x^*)}(-F(x(t))\rangle+\langle z_1(t), P_{S^\perp(x^*)}(-F(x(t)))\rangle\\
&=\langle z_1(t), P_{S(x^*)}(-F(x(t)))\rangle=\langle z_1(t), \dot{z}_1(t)\rangle, \qquad (53)
\end{aligned}$$

where the last equality follows from (49).

The second expression on the right-hand side of (52) can be estimated from (51), namely,

$$\langle z_2(t), -F(x(t))\rangle\leq-\alpha\|z_2(t)\|/2. \qquad (54)$$

Substituting (53) and (54) into the right-hand side of (52) gives

$$\begin{aligned}
&\langle z_1(t), \dot{z}_1(t)\rangle+\|z_2(t)\|(d/dt)(\|z_2(t)\|)\\
&=(d/dt)(\|z_1(t)\|^2/2)+(d/dt)(\|z_2(t)\|^2/2)\\
&=(d/dt)(\|z_1(t)+z_2(t)\|^2/2)\\
&\leq\langle z_1(t), \dot{z}_1(t)\rangle-\alpha\|z_2(t)\|/2. \qquad (55)
\end{aligned}$$

Since $\|z_2(t)\|>0$, $\forall t\in[0, v]$, we have directly from the above that

$$(d/dt)(\|z_2(t)\|)\leq-\alpha/2, \qquad \forall t\in[0, v], \qquad (56)$$

which means that $\|z_2(t)\|$ is strictly decreasing. However, it follows from (49) that

$$\begin{aligned}
\|z_1(v)\|&\leq\|z_1(0)\|+\int_0^v \|P_{S(x^*)}(-F(x(t)))\|\,dt\\
&\leq\|z_1(0)\|+vM\\
&\leq\|z_1(0)\|+2\alpha^{-1}\|z_2(0)\|M<\epsilon/2+2\alpha^{-1}(\epsilon\alpha/4M)M\\
&=\epsilon. \qquad (57)
\end{aligned}$$

Combining (56) and (57), we conclude that $x(v)\in N_3(x^*)$, which is a contradiction to the definition of v. Hence, the claim is correct.

We now turn to proving the lemma. For any $x_0\in N_4(x^*)$, there must exist some $u\in[0, T_0]$ such that $z_2(u)=0$, or equivalently $x(u)\in S(x^*)$. Otherwise, it follows from the claim that $x(t)\in N_3(x^*)$, when $t\in[0, T_0]$. So, (49) and (51) are valid. Notice that they are the only two conditions needed in the derivation of (56), in addition to $x(t)\in S(x^*)$, $\forall t\in[0, T_0]$. Therefore, applying (56), we have

$$\begin{aligned}\|z_2(T_0)\| &= \|z_2(0)\| + \int_0^{T_0} ((d/dt)\|z_2(t)\|)\, dt\\ &\leq \|z_2(0)\| - (\alpha/2)(2\alpha^{-1}\|z_2(0)\|) = 0,\end{aligned} \tag{58}$$

which is a contradiction.

Let

$$\underline{u} = \min\{u\in[0, T_0] : x(u)\in S(x^*)\}. \tag{59}$$

If $\underline{u}=0$, then we are done, because $x_0=x(0)\in N_4(x^*)$ and by definition $\|z_1(0)\| = \|z_1(x_0)\| < \epsilon/2$. If $\underline{u}>0$, then $x(t)\notin S(x^*)$, $\forall t\in[0, \underline{u})$. Using again the claim for $w=\underline{u}$, we conclude that $x(t)$ always lies in $N_3(x^*)$ when $0\leq t\leq \underline{u}$. Therefore, by (49),

$$\begin{aligned}\|z_1(\underline{u})\| &\leq \|z_1(0)\| + \int_0^{\underline{u}} \|P_{S(x^*)}(-F(x(t)))\|\, dt\\ &\leq \epsilon/2 + \underline{u}M\\ &\leq \epsilon/2 + T_0 M\\ &= \epsilon/2 + 2\alpha^{-1}\|z_2(0)\|M < \epsilon/2 + 2\alpha^{-1}(\epsilon\alpha/4M)M\\ &= \epsilon.\end{aligned} \tag{60}$$

So,

$$x(\underline{u})\in B(0, \epsilon)\cap S(x^*) \quad\text{and}\quad \underline{u}\leq T_0.$$

Let $T(x_0, \epsilon)=\underline{u}$. We have completed the proof of the lemma. □

The major results of this section are now ready for presentation. First, it is pointed out by the following theorem that a PDS has the best stability around its regular solutions to the $VI(F, K)$ when they are extreme points of the feasible set K.

Theorem 3.2. When $S(x^*)=\{0\}$, any regular solution x^* to the $VI(F, K)$ is a finite-time attractor for the PDS. In particular, there exists a neighborhood $N(x^*)$ of x^* such that, when $x(0)\in N(x^*)$,

$$(d/dt)\|x(t)-x^*\|\leq -\alpha/2, \qquad \forall x(t)\neq x^*, \tag{61}$$

where α is the constant prescribed in Lemma 3.1.

Proof. It follows directly from Lemma 3.2 that, when x^* solves the $VI(F, K)$ regularly, x^* is a finite-time attractor. For the inequality (61), since $S(x^*)=\{0\}$, $z_1(x)=P_{S(x^*)}=0$, so $z_2(x)=x-x^*$. For a neighborhood $N(x^*)$ of x^* such that

$$\|F(x)-F(x^*)\|\leq \alpha/2, \qquad \forall x\in N(x^*),$$

applying Lemma 3.1, we have

$$\begin{aligned}
&(d/dt)(\|x(t)-x^*\|^2/2)\\
&=(d/dt)(\|z_2(t)\|^2/2)\\
&=\langle x(t)-x^*, \pi(x(t), -F(x(t)))\rangle\\
&\leq\langle x(t)-x^*, -F(x(t))\rangle\\
&=\langle z_2(t), -F(x(t))\rangle\\
&=\langle z_2(t), -F(x^*)\rangle+\langle z_2(t), F(x^*)-F(x(t))\rangle\\
&\leq -\alpha\|z_2(t)\|/2+\|z_2(t)\|\,\|F(x^*)-F(x(t))\|\\
&\leq -\alpha\|z_2(t)\|/2.
\end{aligned} \tag{62}$$

Therefore, for $z_2(t)\neq 0$,

$$(d/dt)\|z_2(t)\|\leq -\alpha/2.$$

The proof is complete. □

The next theorem illuminates the fact that, around a regular solution x^*, some neighborhood on its minimal face is relatively invariant for the PDS.

Theorem 3.3. Suppose that x^* is a regular solution to the $VI(F, K)$ and $\dim S(x^*)\geq 1$. Then, there exists an $\epsilon>0$ such that $x^*+B(0, \epsilon)\cap S(x^*)\subset K$ and, for any $x\in x^*+B(0, \epsilon)\cap S(x^*)$,

$$\pi(x, -F(x))=-g(z_1). \tag{63}$$

In other words, starting from any initial point x_0 in the neighborhood $x^*+B(0, \epsilon)\cap S(x^*)$, the PDS is identical to the MFF by a translation from the origin to x^*.

Proof. Denote the relative neighborhood $x^* + B(0, \epsilon) \cap S(x^*)$ by $Nr(\epsilon)$, and denote $S(x^*)$ by S. For a sufficiently small $\epsilon > 0$ chosen, we have

$$S(x) = S(x^*) = S, \qquad \forall x \in Nr(\epsilon), \tag{64}$$

$$Nr(\epsilon) \subset K, \tag{65}$$

and

$$\langle -F(x), z_2(x')\rangle \leq -\alpha \|z_2(x')\|/2, \qquad \forall x \in Nr(\epsilon), \forall x' \in K, \tag{66a}$$

or equivalently,

$$\begin{aligned}&\langle P_{S^\perp}(-F(x)), z_2(x')\rangle\\ &\leq -\alpha \|z_2(x')\|/2, \qquad \forall x \in Nr(\epsilon), \forall x' \in K,\end{aligned} \tag{66b}$$

where (64) is direct from the definition, (65) follows from $\dim S \geq 1$, and (66) has been proved earlier [cf. (51)].

Since $x = x^* + z_1(x)$, when $x \in Nr(\epsilon)$, so for any $x' \in K$, it follows from

$$z_2(x') = P_{S^\perp}(x' - x^* - z_1(x)) = P_{S^\perp}(x' - x)$$

and (66b) that

$$\begin{aligned}\langle x' - x, P_{S^\perp}(-F(x))\rangle &= \langle P_{S^\perp}(x' - x), P_{S^\perp}(-F(x))\rangle\\ &= \langle z_2(x'), P_{S^\perp}(-F(x))\rangle\\ &\leq -\alpha \|z_2(x')\|/2 \leq 0.\end{aligned} \tag{67}$$

This implies that

$$P_{S^\perp}(F(x))/\|P_{S^\perp}(F(x))\| \in n(x),$$

and hence,

$$P_{S^\perp}(F(x))/\|P_{S^\perp}(F(x))\| = n^*(x). \tag{68}$$

Therefore, by Lemma 2.1 and expression (68),

$$\begin{aligned}\pi(x, -F(x)) &= -F(x) + \beta(x) n^*(x)\\ &= -F(x) + \max\{0, \langle -F(x), -P_{S^\perp}(F(x))/\|P_{S^\perp}(F(x))\|\rangle\}\\ &\quad \cdot P_{S^\perp}(F(x))/\|P_{S^\perp}(F(x))\|\\ &= -F(x) + P_{S^\perp}(F(x)) = P_S(-F(x^* + z_1))\\ &= -g(z_1).\end{aligned} \tag{69}$$

The proof is complete. □

Finally, we summarize the main results in the general case by the following theorem.

Theorem 3.4. Suppose that x^* is a regular solution to the *VI(F, K)*. We have the following relationships between the stability of the PDS and its induced MFF:

(i) if 0 is a stable equilibrium point of the MFF, then x^* is stable for the PDS;
(ii) if 0 is an asymptotically stable equilibrium point of the MFF, then x^* is asymptotically stable for the PDS;
(iii) if 0 is a finite-time attractor of the MFF, then x^* is also a finite-time attractor for the PDS.

Proof.

(i) For any $\epsilon > 0$, we want to show the existence of a $\delta > 0$ such that the solution $x(t)$ to the initial-value problem

$$\text{(IVP(A))} \quad \dot{x} = \pi(x, -F(x)), \qquad x(0) = x_0,$$

lies forever in the ϵ-neighborhood of x^*, whenever $\|x_0 - x^*\| < \delta$.

Since 0 is a stable equilibrium point of the MFF, there exists a $\delta_1 > 0$, such that, for the solution $z_1(t)$ to the initial-value problem

$$\text{(IVP(B))} \quad \dot{z}_1 = -g(z_1), \qquad z_1(0) = z_1^0,$$

we have

$$\|z_1(t)\| < \epsilon, \qquad \forall t \geq 0 \tag{70}$$

when $\|z_1^0\| < \delta_1$.

Let $r > 0$ be arbitrarily fixed, and let

$$M = \max\{\|F(x)\|, x \in \overline{B(x^*, 2r) \cap K}\}.$$

Choose $\delta > 0$ small so that

$$\delta < \min\{r, ra/(2M), \epsilon a/(a + 2M)\} \tag{71}$$

and $B(x^*, \delta)$ is contained in the neighborhood $N(x^*)$ specified in Lemma 3.2 for $\delta_1 > 0$.

For any $x(0) = x_0 \in B(x^*, \delta)$, it follows from Lemma 3.2 that there is a $T(x_0, \delta_1)$, $0 \leq T(x_0, \delta_1) \leq 2a^{-1}\|z_2(x_0)\|$, so that

$$x(T(x_0, \delta_1)) \in B(0, \delta_1) \cap S(x^*) + x^*, \tag{72}$$

where x solves IVP(A). Let

$$x(T(x_0, \delta_1)) = x^* + z_1(T(x_0, \delta_1)). \tag{73}$$

Then,

$$\|z_1(T(x_0, \delta_1))\| < \delta_1. \tag{74}$$

By the uniqueness of the solution to the IVP(A) (cf. Ref. 1), for $t \geq T(x_0, \delta_1)$,

$$x(t) = x_1(t - T(x_0, \delta_1)), \tag{75}$$

where $x_1(\tau)$ solves the initial-value problem.

$$\text{(IVP(C))} \quad \dot{x}_1 = \pi(x_1, -F(x_1)), \qquad x_1(0) = x(T(x_0, \delta_1)).$$

In view of Lemma 3.3, we have

$$x_1(\tau) = x^* + z_1(\tau), \tag{76}$$

where $z_1(\tau)$ solves IVP(B) with

$$z_1^0 = z_1(T(x_0, \delta_1)), \tag{77}$$

which combined with (70) gives

$$x_1(\tau) = x^* + z_1(\tau) \in x^* + S(x^*) \cap B(0, \epsilon), \qquad \forall \tau \geq 0, \tag{78}$$

or

$$x(t) \in B(x^*, \epsilon), \qquad \forall t \geq T(x_0, \delta_1). \tag{79}$$

It remains to show now that, during the finite time interval $[0, T(x_0, \delta_1)]$, $x(t)$ does not exit $B(x^*, \epsilon)$. First, we will show that

$$\|x(t) - x^*\| < 2r, \qquad \forall t \in [0, T(x_0, \delta_1)]. \tag{80}$$

If not, let

$$\underline{t} = \min\{t \in [0, T(x_0, \delta_1)] : \|x(t) - x^*\| \geq 2r\}. \tag{81}$$

Because of (70), $\|x(0) - x^*\| < r$, so $\underline{t} > 0$. Hence, we have

$$\begin{aligned} \|x(\underline{t}) - x^*\| &\leq \|x(0) - x^*\| + \int_0^{\underline{t}} \|\pi(x(t), -F(x(t)))\| \, dt \\ &\leq \|x(0) - x^*\| + \underline{t}M \leq r + T(x_0, \delta_1)M \\ &\leq r + 2\alpha^{-1}\|z_2(x_0)\|M \leq r + 2\alpha^{-1}M\delta. \end{aligned} \tag{82}$$

But this produces

$$2r \leq r + 2\alpha^{-1}M\delta,$$

a contradiction to (71). Hence, (80) is true; therefore, for $0 \leq t \leq T(x_0, \delta_1)$,

$$\begin{aligned}\|x(t)-x^*\| &\leq \|x(0)-x^*\| + \int_0^t \|\pi(-F(x(u)), x(u))\| \, du \\ &\leq \delta + T(x_0, \delta_1)M \leq \delta + 2\alpha^{-1}\|z_2(x_0)\|M \\ &\leq \delta(1+2\alpha^{-1}M) \\ &< \varepsilon. \end{aligned} \tag{83}$$

We have completed the proof for result (i).

(ii) Given result (i), it now suffices to prove that, for $\delta > 0$ chosen in the proof for result (i), we have

$$\lim_{t \to \infty} x_0 \cdot t = x^*, \tag{84}$$

when $\|x_0 - x^*\| < \delta$. But, by the assumption that 0 is asymptotically stable for the minimal face flow, it is direct from (75) and (76) that

$$\begin{aligned}\lim_{t\to\infty} x_0 \cdot t &= \lim_{t\to\infty} x(T(x_0, \delta_1)) \cdot (t - T(x_0, \delta_1)) \\ &= \lim_{t\to\infty} x_1(0) \cdot t \\ &= \lim_{t\to\infty} (x^* + z_1(t)) \\ &= x^* + \lim_{t\to\infty}, z_1(t) \\ &= x^*. \end{aligned} \tag{85}$$

(iii) Let T_1 be such that

$$z_1(T(x_0, \delta_1)) \cdot t = 0, \qquad \forall t \geq T_1. \tag{86}$$

We have, following (75) and (76), for $t \geq T_1 + T(x_0, \delta_1)$,

$$\begin{aligned} x_0 \cdot t &= x(T(x_0, \delta_1)) \cdot (t - T(x_0, \delta_1)) \\ &= x_1(0) \cdot (t - T(x_0, \delta_1)) \\ &= x^* + z_1(T(x_0, d_1)) \cdot (t - T(x_0, d_1)) \\ &= x^*. \end{aligned}$$

Hence, x^* is a finite-time attractor. □

Theorem 3.4 states that the local stability of the PDS depends in a sense on the combination of the regularity at the equilibrium point and the local stability of its MFF. One of the extreme cases, occurring when $S(x^*) = \{0\}$, has been covered by Theorem 3.2, where we see that the local stability of the PDS is implied by the regularity at the equilibrium. When the dimension of $S(x^*)$ increases, the emphasis is expected to shift from the regularity to the local stability of the minimal face flow. Particularly, the extreme case at the other end is when $S(x^*) = R^n$, i.e., when x is in the interior of K. Then, it is clear that

$$z_1(x) = x - x^*, \qquad z_2(x) = 0, \qquad g(z_1) = F(z).$$

Hence, locally, the MFF is just a translation of the PDS from x^* to the origin, and so they both enjoy the same stability.

4. Local and Global Properties under Monotonicity

The previous section aimed to explore the local stability of the projected dynamical system by use of the stability theory of standard dynamical systems. In contrast, we will devote this section to studying local and global stability directly under various monotonicity conditions.

Recall (cf. Ref. 2) the following definitions:

$F(x)$ is locally monotone at x^* if there is a neighborhood $N(x^*)$ of x^*, such that

$$\langle F(x) - F(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in N(x^*); \tag{87}$$

$F(x)$ is locally strictly monotone at x^* if there exists a neighborhood $N(x^*)$ of x^*, such that

$$\langle F(x) - F(x^*), x - x^* \rangle > 0, \qquad \forall x \in N(x^*), x \neq x^*; \tag{88}$$

$F(x)$ is locally strongly monotone at x^* if there is a neighborhood $N(x^*)$ of x^* and $\eta > 0$, such that

$$\langle F(x) - F(x^*), x - x^* \rangle \geq \eta \|x - x^*\|^2, \qquad \forall x \in N(x^*); \tag{89}$$

$F(x)$ is monotone, strictly monotone, strongly monotone at x^* if, respectively, (87), (88), (89) holds true for any $x \in K$.

The above definitions of monotonicity at x^* are easily seen as listed in an order from weak to strong. In what follows, we will establish their correspondence to the stability at x^* in the same order. Namely, monotonicity implies a monotone attractor at x^*; strict monotonicity implies a strictly monotone attractor at x^*; and strong monotonicity implies that x^* is exponentially stable. We begin with the following theorem.

Theorem 4.1. Suppose that x^* solves the $VI(F, K)$. If $F(x)$ is locally monotone at x^*, then x^* is a monotone attractor for the projected dynamical system; if $F(x)$ is monotone, then x^* is a global monotone attractor.

Proof. By Lemma 2.1, for any $x \in K$,

$$\pi(x, -F(x)) = -F(x) + \beta(x)n^*(x), \tag{90}$$

where $\beta(x) \geq 0$ and

$$\langle n^*(x), x-y \rangle \leq 0, \qquad \forall y \in K. \tag{91}$$

Let $N(x^*)$ be a neighborhood of x^* such that (87) holds for $x \in N(x^*)$. Let $x_0 \in N(x^*)$ and $x_0(t)$ solve the initial value problem (10). Define

$$D(t) := \|x_0(t) - x^*\|^2/2. \tag{92}$$

Then,

$$\begin{aligned} \dot{D}(t) &= \langle x_0(t) - x^*, \pi(x_0(t), -F(x_0(t))) \rangle \\ &= \langle x_0(t) - x^*, -F(x_0(t)) \rangle \\ &\quad + \langle x_0(t) - x^*, \beta(x_0(t))n(x_0(t)) \rangle. \end{aligned} \tag{93}$$

Taking $y = x^*$ in (91), we have in (93)

$$\begin{aligned} &\langle x_0(t) - x^*, \beta(x_0(t))n^*(x_0(t)) \rangle \\ &= \beta(x_0(t)) \langle x_0(t) - x^*, n^*(x_0(t)) \rangle \leq 0, \qquad \forall t \geq 0. \end{aligned} \tag{94}$$

Therefore, since x^* solves the $VI(F, K)$,

$$\begin{aligned} \dot{D}(t) &\leq \langle x_0(t) - x^*, -F(x_0(t)) \rangle \\ &\leq \langle x_0(t) - x^*, -F(x_0(t)) \rangle + \langle x_0(t) - x^*, F(x^*) \rangle \\ &= -\langle x_0(t) - x^*, F(x_0(t)) - F(x^*) \rangle \leq 0, \qquad \forall t \geq 0, \end{aligned} \tag{95}$$

where the last inequality follows from the local monotonicity of F. Hence, for $x_0 \in N(x^*)$, $\|x_0(t) - x^*\|$ is a nonincreasing function on $[0, +\infty)$. By Definition 2.6, x^* is a monotone attractor.

If F is monotone, then (95) holds for all $x_0 \in K$, so x^* is a global monotone attractor. □

Next, we have the following theorem.

Theorem 4.2. Suppose that x^* solves the $VI(F, K)$. If $F(x)$ is locally strictly monotone at x^*, then x^* is a strictly monotone attractor; if $F(x)$ is strictly monotone at x^*, then x^* is a strictly global monotone attractor.

Proof. Since strict monotonicity implies monotonicity, (90)–(95) in the proof of Theorem 4.1 still hold true here. Moreover, Inequality (95) now has a strict sign, due to local strict monotonicity of $F(x)$ at x^*, that is,

$$\dot{D}(t) \leq -\langle x_0(t) - x^*, F(x_0(t)) - F(x^*)\rangle < 0, \tag{96}$$

when $x_0(t) \neq x^*$. Therefore, $D(t)$ is monotonically decreasing but nonnegative. Let

$$D_- = \lim_{t\to\infty} D(t). \tag{97}$$

If $D_- > 0$, we claim that there exists a sequence $\{t_n\}$, $t_n \to \infty$, as $n \to \infty$, such that

$$\dot{D}(t_n) \to 0, \qquad \text{as } n \to \infty. \tag{98}$$

Suppose that the claim is false. Then, there is a $b > 0$ and a $T > 0$, such that

$$\dot{D}(t) < -b, \qquad \forall t > T, \tag{99}$$

which contradicts that $D_- > 0$, so the claim is true.

Since the sequence $x(t_n)$ is bounded, it has a convergent subsequence $\{t_{n_j}\}_j$ with

$$x_0(t_{n_j}) \to \bar{x}. \tag{100}$$

From (97), we have

$$\lim_{j\to\infty} \|x_0(t_{n_j}) - x^*\|^2/2 = \|\bar{x} - x^*\|^2/2 = D_- > 0, \tag{101}$$

and hence $\bar{x} \neq x^*$. But substituting $\{t_{n_j}\}$ into (96) yields

$$\dot{D}(t_{n_j}) = -\langle x_0(t_{n_j}) - x^*, F(x_0(t_{n_j})) - F(x^*)\rangle < 0. \tag{102}$$

Because of (98), left-hand side of (102) converges to zero as $j \to \infty$. Therefore, by (100),

$$\begin{aligned} &\lim_{j\to\infty} -\langle x_0(t_{n_j}) - x^*, F(x_0(t_{n_j})) - F(x^*)\rangle \\ &= -\langle \bar{x} - x^*, F(\bar{x}) - F(x^*)\rangle = 0. \end{aligned} \tag{103}$$

By strict monotonicity, (103) is a contradiction to the earlier result that $\bar{x} \neq x^*$ from (101). The contradiction shows that $D_- = 0$. Therefore, for any $x_0 \in N(x^*)$,

$$\|x_0(t) - x^*\|^2 \downarrow 0, \qquad \text{as } t \to \infty. \tag{104}$$

By Definition 2.6, x^* is a strictly monotone attractor.

It is clear from (104) that x^* is a strictly global monotone attractor when $F(x)$ is strictly monotone. □

Finally, we have the strongest result under the strongest condition.

Theorem 4.3. Suppose that x^* solves the $VI(F, K)$. If $F(x)$ is locally strongly monotone at x^*, then x^* is exponentially stable; if $F(x)$ is strongly monotone at x^*, then x^* is globally exponentially stable.

Proof. Since strong monotonicity implies strict monotonicity, it follows from (96) that

$$\dot{D}(t) \leq -\langle x_0(t) - x^*, F(x_0(t)) - F(x^*)\rangle \leq -\eta \|x_0(t) - x^*\|^2. \quad (105)$$

Letting $d(x_0, t) = \|x_0(t) - x^*\|$, we have

$$\dot{d}(x_0, t) < -\eta d(x_0, t). \quad (106)$$

If there is some $t_0 \geq 0$ with $d(x_0, t_0) = 0$, because $d(x_0, \cdot)$ is monotone nonincreasing, we have

$$\|x_0 \cdot t - x^*\| = 0, \qquad \forall t \geq t_0. \quad (107)$$

Let $B = \exp(\eta t_0)$; then,

$$\|x_0 \cdot t - x^*\| \leq \|x_0 - x^*\| \leq B\|x_0 - x^*\| \exp(-\eta t). \quad (108)$$

Combining (107) and (108), it follows that

$$\|x_0 \cdot t - x^*\| \leq B\|x_0 - x^*\| \exp(-\eta t), \quad (109)$$

so x^* is exponentially stable.

Now, suppose that $d(x_0, t) \neq 0$, $\forall t \geq 0$. Dividing (106) by $d(x_0, t)$ and taking the integral, we have

$$\log d(x_0, t) \leq \log d(x_0, 0) - \eta t,$$

or

$$\|x_0(t) - x^*\| \leq \|x_0 - x^*\| \exp(-\eta t). \quad (110)$$

Hence, x^* is exponentially stable.

When $F(x)$ is monotone at x^*, then (106) has no restriction for x_0. The same arguments above will apply and give either (108) or (110), with no restriction for the initial value x_0. Therefore, x^* is globally exponentially stable. □

5. Summary and Conditions

This paper initiates the study of the stability of a class of projected dynamical systems (PDS), whose stationary points are solutions to the associated variational inequality problems (VIP). Since the ordinary differential equations that define such PDS have discontinuous right-hand sides, corresponding to the feasibility constraints in the VIP, the stability of PDS around their stationary points can be quite distinct from that of standard dynamical systems, as illustrated in the examples presented in Section 2.

However, despite the distinctness, it is appealing to discover the connection between the stability of PDS and the stability of classical dynamical systems, and to explore a way of studying the former via the latter, which is a more developed discipline. Toward this end, in Section 3 we brought in the definition of a regular solution to a VIP and introduced the concept of a minimal face flow (MFF), induced by the PDS at its stationary point. We managed to show that, at regular solutions to the associated VIP, the PDS enjoys stability similar to its MFF. Since a MFF is a standard dynamical system in a subspace of R^n, we can, in this case, access the classical stability theory of dynamical systems for analyzing the stability at the stationary points of PDS.

In Section 4, we conducted the study following another line. Namely, we developed directly a series of stability results of PDS at their stationary points, under various monotonicity conditions. Interestingly, but not surprisingly, the stronger the monotonicity condition that is imposed, the better the stability enjoyed by the stationary points.

The stability study undertaken here provides a mathematical approach for the analysis of the stability of equilibria in many social and economic competitive systems with natural constraints. In fact, some well-known equilibrium problems, such as oligopolistic market equilibrium problems, traffic network equilibrium problems, general economic equilibrium problems, and spatial price equilibrium problems, have already been formulated as PDS for their dynamical model counterparts (cf. Refs. 1 and 3). Hence, it would be of much interest to launch the stability study for these models at their equilibrium points. Moreover, for those problems with multiple equilibria, the research interest to date has focused on the calculation of these equilibrium points, using typically among others variational inequality methods. However, it is clearly necessary to associate stability analysis with such calculations, since from an application point of view the VIP solutions with poor dynamical stability are not regarded as true equilibria in practice. It is expected that potential research as such will directly benefit from the theoretical results established in this paper.

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