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Duality in Nonlinear Multiobjective Programming Using Augmented Lagrangian Functions[1]

C. Singh,[2] D. Bhatia,[3] and N. Rueda[4]

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Abstract. A vector-valued generalized Lagrangian is constructed for a nonlinear multiobjective programming problem. Using the Lagrangian, a multiobjective dual is considered. Without assuming differentiability, weak and strong duality theorems are established using Pareto efficiency.

Key Words. Augmented Lagrangians, Pareto efficiency, weak duality, strong duality, concavity.

1. Introduction

Many researchers have contributed to the development of duality in multiobjective programming (Refs. 1-10). In Refs. 1 and 5, weak and strong duality results are given under generalized conditions. In Refs. 2, 4, and 9, duality results are developed for classes of nondifferentiable multiobjective problems. Several authors have also studied augmented Lagrangians in nonlinear optimization. Rockafellar proved duality results considering a quadratic augmented Lagrangian (Ref. 11). Bertsekas (Ref. 12) combined primal-dual and penalty methods for equality constrained minimization. Dolecki and Kurcyusz (Ref. 13) studied ϕ-convexity with applications to augmented Lagrangians, and Gould (Ref. 14) considered generalized Lagrangian functions using multiplier functions. Gonen and Avriel (Ref. 15) introduced a primal and dual pair of programming problems as inf sup and sup inf of the same generalized augmented Lagrangian. They derived certain sufficient

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[2]Professor, Department of Mathematics, St. Lawrence University, Canton, New York.
[3]Professor, Department of Mathematics, S.G.T.B. Khalsa College, University of Delhi, Delhi, India.
[4]Assistant Professor, Department of Mathematics and Computer Science, Merrimack College, North Andover, Massachusetts.

conditions for constructing the augmented function such that the extremal values of primal and dual problems were equal.

In this paper, by considering a vector-valued augmented Lagrangian function, we have constructed corresponding multiobjective programming problems and, under similar conditions, we obtain a properly efficient solution of the primal and the dual.

2. Preliminaries and Definitions

Let $f_1, f_2, \ldots, f_k$; $g_1, g_2, \ldots, g_m$ be real-valued functions defined on $S \subseteq R^n$. Consider the following multiobjective nonlinear programming problem:

$$(\text{P}) \quad V_P = \min_{x \in S} \quad f(x) = [f_1(x), f_2(x), \ldots, f_k(x)],$$

$$\text{s.t.} \quad g_j(x) = 0, \qquad j \in \underline{m} = \{1, 2, \ldots, m\},$$

where an optimal point is defined below in terms of efficiency.

Let $g(x) = (g_1(x), \ldots, g_m(x))$, and let the vector-valued generalized augmented Lagrangian function associated with (P) be

$$L(x, y, r) = [L_1(x, y, r), L_2(x, y, r), \ldots, L_k(x, y, r)],$$

$$L_i(x, y, r) = f_i(x) + \phi(g(x), y, r), \qquad i \in \underline{k} = \{1, 2, \ldots, k\},$$

where ϕ is a real-valued function, called augmented multiplier function, defined on some subset of R^{2m+1} and $(y, r) \in T = R^{m+1}$.

Let us define

$$H_i(y, r) = \inf_{x \in S} L_i(x, y, r), \qquad i \in \underline{k}.$$

The multiobjective programming problem that, under certain conditions, may become a dual of (P) is given by

$$(\text{D}) \quad V_D = \max_{(y,r) \in T} \{H_1(y, r), H_2(y, r), \ldots, H_k(y, r)\}.$$

To obtain duality results between problems (P) and (D), we first relate the vector-valued Lagrangian function L to properly efficient solutions of (P) by making certain assumptions on ϕ without making any assumptions on (P) itself.

Definition 2.1. A point $x^* \in S$ is said to be an efficient solution of (P) if there exists no $x \in S$ such that

$$f_i(x) \leqq f_i(x^*), \qquad i \in \underline{k}, \qquad i \neq j,$$

and

$$f_j(x) < f_j(x^*).$$

Definition 2.2. A point x^* is said to be properly efficient for (P) if it is efficient for (P) and if there exists a scalar $M > 0$ such that, for each i, there exists a j with

$$[f_i(x^*) - f_i(x)]/[f_i(x) - f_j(x^*)] \leqq M,$$

$f_j(x) > f_j(x^*)$, whenever x is feasible for (P) and $f_i(x) < f_i(x^*)$.

We associate (P) to the following scalar-valued problem

$$(\mathrm{P}_\lambda) \quad V_{P_\lambda} = \inf_{x \in S} \sum_{i=1}^{k} \lambda_i f_i(x),$$

$$\text{s.t.} \quad g_j(x) = 0, \qquad j \in \underline{m},$$

$$\lambda_i > 0, \qquad i \in \underline{k}, \qquad \sum_{i=1}^{k} \lambda_i = 1.$$

Similarly, we associate (D) to the following problem.

$$(\mathrm{D}_\lambda) \quad V_{D_\lambda} = \sup_{(y,t) \in T} \sum_{i=1}^{k} \lambda_i H_i(y, r),$$

$$\text{s.t.} \quad H_i(y, r) = \inf_{x \in S} L_i(x, y, r), \qquad i \in \underline{k},$$

$$\lambda_i > 0, \qquad i \in \underline{k}, \qquad \sum_{i=1}^{k} \lambda_i = 1.$$

The generalized augmented Lagrangian (Ref. 15) corresponding to problem (P_λ) is defined as

$$L_\lambda(x, y, r) = \sum_{i=1}^{k} \lambda_i f_i(x) + \phi(g(x), y, r).$$

We let

$$H_\lambda(y, r) = \sum_{i=1}^{k} \lambda_i H_i(y, r).$$

We can derive duality results between multiobjective programming problems (P) and (D) by finding relationships between L, L_λ, V_{P_λ}, and the set of properly efficient solutions of (P), and by using certain assumptions on ϕ stated in Ref. 15, but without making convexity assumptions on (P) itself.

Assumption A1. See Ref. 15. For every $(y, r) \in T$, $\phi(0, y, r) = 0$.

Assumption A2. See Ref. 15. For every $u \in R^m$ $u \neq 0$, and every $c \in R^1_+$, there exists a $(y, r) \in T$ such that $\phi(u, y, r) > c$.

Let E_P denote the set of all properly efficient solutions of (P); similarly, let E_D denote the set of all properly efficient solutions of (D). In terms of vector-valued Lagrangian function L, we consider the following problems:

$$(\mathrm{P_L}) \quad \underset{x \in S}{\text{V-min}} \left[\sup_{(y,r) \in T} L_1(x, y, r), \sup_{(y,r) \in T} L_2(x, y, r), \ldots, \sup_{(y,r) \in T} L_k(x, y, r) \right],$$

$$(\mathrm{D_L}) \quad \underset{(y,r) \in T}{\text{V-max}} \left[\inf_{x \in S} L_1(x, y, r), \inf_{x \in S} L_2(x, y, r), \ldots, \inf_{x \in S} L_k(x, y, r) \right],$$

where $(\mathrm{D_L}) = (\mathrm{D})$ the solutions to these problems are given in terms of efficiency.

It is clear that

$$L_\lambda(x, y, r) = \sum_{i=1}^{k} \lambda_i L_i(x, y, r),$$

and from Assumptions A1 and A2 it follows that

$$\sup_{(y,r) \in T} L_i(x, y, r) = \begin{cases} f_i(x), & g(x) = 0, \\ +\infty, & g(x) \neq 0. \end{cases}$$

Hence,

$$(\mathrm{P_L}) \quad \underset{x \in S}{\text{V-min}} \left[\sup_{(y,r) \in T} L_1(x, y, r), \sup_{(y,r) \in T} L_2(x, y, r), \ldots, \sup_{(y,r) \in T} L_k(x, y, r) \right]$$

is the same as

$$(\mathrm{P}) \quad \underset{x \in S}{\text{V-min}} \quad [f_1(x), f_2(x), \ldots, f_k(x)],$$

$$\text{s.t.} \quad g(x) = 0.$$

3. Duality Theory

Theorem 3.1. Weak Duality Theorem. If ϕ satisfies Assumptions A1 and A2, then

$$\text{V-min } P \not\leq \text{V-max } D,$$

where V-min and V-max are taken with respect to the sets E_P and E_D.

Proof. Suppose that, contrary to the conclusion,

$$\text{V-min } P \leq \text{V-max } D,$$

i.e.,

$$f_i(x) \leqq H_i(y, r), \quad i \in \underline{k},\ i \neq j, \qquad x \in E_P,$$

$$f_j(x) < H_j(y, r),$$

i.e.,

$$\sup_{(y,r)\in T} L_i(x, y, r) \leqq \inf_{s\in S} L_i(s, y, r),$$

$$\sup_{(y,r)\in T} L_j(x, y, r) < \inf_{s\in S} L_j(s, y, r),$$

which imply that

$$\sum_{i=1}^{k} \lambda_i \sup_{(y,r)\in T} L_i(x, y, r) < \sum_{i=1}^{k} \lambda_i \inf_{s\in S} L_i(s, y, r),$$

or

$$\sup_{(y,r)\in T} \sum_{i=1}^{k} \lambda_i L_i(x, y, r) \leqq \sum_{i=1}^{k} \lambda_i \sup_{(y,r)\in T} L_i(x, y, r)$$

$$< \sum_{i=1}^{k} \lambda_i \inf_{s\in S} L_i(s, y, r) \leqq \inf_{s\in S} \sum_{s\in S}^{k} \sum_{i=1}^{k} \lambda_i L_i(s, y, r).$$

Since

$$\sup(\mathrm{a}+\mathrm{b}) \leqq \sup \mathrm{a} + \sup \mathrm{b},$$

$$\inf(\mathrm{a}+\mathrm{b}) \geqq \inf \mathrm{a} + \inf \mathrm{b},$$

i.e.,

$$\sup_{(y,r)\in T} L_\lambda(x, y, r) < \inf_{s\in S} L_\lambda(s, y, r),$$

or

$$\inf_{x\in S} \sup_{(y,r)\in T} L_\lambda(x, y, r) < \sup_{(y,r)\in T} \inf_{s\in S} L_\lambda(s, y, r),$$

contradicting the weak duality between the scalar-valued problems (P_λ) and (D_λ). This completes the proof. □

For every $x \in S$ and $u \in R^m$, we define

$$F_i(x, u) = \begin{cases} f_i(x), & \text{if } \beta(g(x)) \leqq \beta(u), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\beta: R^m \to R^m_+$ is an operator such that

$$\beta(0)=0 \quad \text{and} \quad \|u\| \leqq \|v\| \Rightarrow \beta(u) \leqq \beta(v).$$

We define the perturbation function associated with (P) as

$$W_\lambda(u) = \inf_{x \in S} \sum_{i=1}^{k} \lambda_i F_i(x, u).$$

Lemma 3.1. See Ref. 15. If ϕ is concave in $(y, r) \in T$ for every $u \in R^m$, then $\sum_{i=1}^k \lambda_i L_i$ and $\sum_{i=1}^k \lambda_i H_i$ are concave in $(y, r) \in T$ for $\lambda_i > 0$, $i \in \underline{k}$. Further, if ϕ is upper semicontinuous in $(y, r) \in T$ for every $u \in R^m$, then $\sum_{i=1}^k \lambda_i L_i$ and $\sum_{i=1}^k \lambda_i H_i$ for $\lambda_i > 0$, $i \in \underline{k}$ are also upper semicontinuous in $(y, r) \in T$ for every $u \in R^m$.

Assumption A3. Isotonicity Assumption. See Ref. 15. For every $u^1 \in R^m$, $u^2 \in R^m$, and $(y, r) \in T$,

$$\beta(u^1) \geqq \beta(u^2) \Rightarrow \phi(u^1, y, r) \geqq \phi(u^2, y, r).$$

Lemma 3.2. See Ref. 15. If ϕ satisfies Assumption A3, then

(i) $L_\lambda(x, y, r) = \sum_{i=1}^k \lambda_i L_i(x, y, r) = \inf_{u \in R^m} [\sum_{i=1}^k \lambda_i F_i(x, u) + \phi(u, y, r)]$, for every $x \in S$ and $(y, r) \in T$, $\sum_{i=1}^k \lambda_i = 1$;

(ii) $H_\lambda(y, r) = \inf_{u \in R^m} [W_\lambda(u) + \phi(u, y, r)]$, for every $(y, r) \in T$.

Proof.

(i) If $x \in S$, $u \in R^m$ such that $\beta(g(x)) \leqq \beta(u)$, then by Assumption A3,

$$\phi(g(x), y, r) \leqq \phi(u, y, r), \text{ for every } (y, r) \in T.$$

If $\beta(g(x)) \not\leqq \beta(u)$, then

$$F_i(x, u) = +\infty, \qquad i \in \underline{k}.$$

Hence,

$$\begin{aligned} L_i(x, y, r) &= f_i(x) + \phi(g(x), y, r) \\ &\leqq F_i(x, u) + \phi(u, y, r). \end{aligned}$$

Therefore,

$$L_i(x, y, r) \leqq \inf_{u \in R^m} [F_i(x, u) + \phi(u, y, r)].$$

Hence,

$$L_\lambda(x, y, r) = \sum_{i=1}^{k} \lambda_i L_i(x, y, r) \leqq \sum_{i=1}^{k} \lambda_i \inf_{u \in R^m} [F_i(x, u) + \phi(u, y, r)].$$

Also, since

$$\sum_{i=1}^{k} \lambda_i = 1,$$

we have

$$\begin{aligned} L_\lambda(x, y, r) &= \sum_{i=1}^{k} \lambda_i [F_i(x, g(x)) + \phi(g(x), y, r)] \\ &= \sum_{i=1}^{k} \lambda_i F_i(x, g(x)) + \sum_{i=1}^{k} \lambda_i \phi(g(x), y, r) \\ &= \sum_{i=1}^{k} \lambda_i F_i(x, g(x)) + \phi(g(x), y, r) \\ &\geqq \inf_{u \in R^m} [\textstyle\sum \lambda_i F_i(x, u) + \phi(u, y, r)] \\ &\geqq \sum_{i=1}^{k} \lambda_i \{\inf [F_i(x, u) + \phi(u, y, r)]\}. \end{aligned}$$

Hence,

$$L_\lambda(x, y, r) = \sum_{i=1}^{k} \lambda_i \left[\inf_{u \in R^m} [F_i(x, u) + \phi(u, y, r)] \right].$$

(ii) It follows from Lemma 4 of Ref. 15. □

We now introduce additional assumptions on ϕ to obtain weak duality results under conditions different from those given in Theorem 3.1.

Assumption A4. Behavior of ϕ for $y = 0$. See Ref. 15. Assume that $r \geq 0$. Then:

(i) for $u \neq 0$, $\phi(u, 0, r)$ is nondecreasing in r;
(ii) for $u \neq 0$, $\lim_{r \to \infty} \phi(u, 0, r) = +\infty$;
(iii) let $N(0) \subset R^n$ be any spherical neighborhood of the origin in R^m, and let $M \subset R^m$ be the complement of $N(0)$; then the function θ_s, defined by

$$\theta_s(u, r) = \phi(u, 0, r) - \phi(u, 0, s),$$

is uniformly unbounded on M for every $s < r$;

(iv) for every r, there exists an $N(0) \subset R^m$ where $\phi(u, 0, r)$ is nonnegative.

Assumption A4(iii) can be rewritten in a simpler way if $\phi(u, 0, r)$ is differentiable with respect to r.

Lemma 3.3. See Ref. 15. Let $N(0)$ and M be as in A4(iii), and assume that $\phi(u, 0, r)$ is differentiable with respect to r for all $u \in M$. If there exists $r_0 \in R_+$ and $\epsilon > 0$ such that $\partial\phi/\partial r(u, 0, r) > \epsilon$ for all $u \in M$ and $r > r_0$, then ϕ satisfies A4(iii).

Assumption A5. See Ref. 15. Let $N(0)$ and M be as in A4(iii). There exists a real nonnegative function ψ, defined on $R^m \times (0, +\infty)$, satisfying the following properties:

(i) for every $u \in R^m$, $(y, r) \in T$, $(z, s) \in T$, $r > s$,

$$\phi(u, y, r) - \phi(u, z, s) \geqq -\psi(y - z, r - s);$$

(ii) for every $a \in R^m$,

$$\lim_{b \to +\infty} \psi(a, b) = 0.$$

Lemma 3.4. If ϕ satisfies Assumptions A1 and A4(i), then $\sum_{i=1}^k \lambda_i L_i(x, 0, r)$ and $\sum_{i=1}^k \lambda_i H_i(0, r)$ are nondecreasing in r.

Proof. From Assumption A1, we have $\phi(g(x), 0, r) = 0$ for every x such that $g(x) = 0$ and for every r. If $g(x) \neq 0$, then by Assumption A4(i) we know that $\phi(g(x), 0, r)$ is nondecreasing in r. Hence, each $L_i(x, 0, r)$, $i \in \underline{k}$, is nondecreasing in r, and it follows that

$$L_\lambda(x, 0, r) = \sum_{i=1}^k \lambda_i L_i(x, 0, r)$$

is nondecreasing in r. Similarly, we can show that $H_\lambda(0, r)$ is also nondecreasing in r. □

Lemma 3.5. If ϕ satisfies Assumptions A3 and A5(i), then for every $r > 0$,

$$H_\lambda(y, r) \geqq \sup_{\substack{(x,s) \in T \\ r > s > 0}} [H_\lambda(z, s) - \psi(y - z, r - s)].$$

The proof follows as $H_\lambda(y, r)$ is a scalar-valued function.

Corollary 3.1. If ϕ satisfies Assumptions A3 and A5(i), and if there exists a $(z, s) \in T$ such that $H_i(z, s)$ is finite for $i \in \underline{k}$, then $H_i(y, r) \neq -\infty$, $i \in \underline{k}$, for every $(y, r) \in T$ satisfying $r > s$.

Proof. It follows from Lemma 7 of Ref. 15 that

$$H_i(y, r) \geqq H_i(z, s) - \psi(y - z, r - s), \qquad i \in \underline{k}.$$

Since ψ is real valued, $H_i(y, r)$, $i \in \underline{k}$, is bounded below for every (y, r) such that $r > s$. □

Corollary 3.2. If ϕ satisfies Assumptions A3, A5(i), A5(ii), then

$$V_{D_\lambda} = \sup_{(z,s) \in T} \sum_{i=1}^{k} \lambda_i H_i(z, s) = \lim_{r \to \infty} H_\lambda(y, r), \qquad \text{for every } y \in R_+^m.$$

Proof. It follows from Corollary 2 of Ref. 15. □

Definition 3.1. Program (P) is said to satisfy the boundedness condition if there exists an $r \in R_+^1$ such that each $L_i(x, 0, r)$, $i \in \underline{k}$, is bounded below for every $x \in S$.

The boundedness condition certainly holds if ϕ is bounded below on $R^m \times T$ and $\sum_{i=1}^k \lambda_i f_i$ is bounded below on S, or certainly if S is compact and $\sum_{i=1}^k \lambda_i f_i$, $\lambda_i > 0$, $i \in \underline{k}$ is lower semicontinuous.

Lemma 3.6. Suppose that ϕ satisfies Assumptions A1, A3, A4(i), A5(i), A5(ii). Then,

$$V_{D_i} \neq -\infty, \text{ where } V_{D_i} \text{ is the } i\text{th component of } V_D,$$

if and only if (P) satisfies the boundedness condition.

Proof. This result can be established along the lines of Lemma 8 of Ref. 15. □

Theorem 3.2. If ϕ satisfies Assumptions A1, A3, A4(ii), A4(iii), A4(iv), A5(i), A5(ii), and if (P) satisfies the boundedness condition, then

$$\text{V-min } P \nleq \text{V-max } D,$$

where V-min and V-max are taken with respect to the sets E_P and E_D.

Proof. Suppose that, contrary to the conclusion of the theorem,

$$\text{V-min } P \leq \text{V-max } D,$$

i.e.,

$$f_i(x) \leqq H_i(y, r), \qquad i \in \underline{k},\ i \neq j,$$

$$f_j(x) < H_j(y, r).$$

Therefore,

$$\sum_{i=1}^{k} \lambda_i f_i(x) < \sum_{i=1}^{k} \lambda_i H_i(y, r). \tag{1}$$

Now, from the definition of $W_\lambda(0)$,

$$W_\lambda(0) = \inf_{x \in S} \sum_{i=1}^{k} \lambda_i F_i(x, 0)$$

and

$$F_i(x, 0) = \begin{cases} f_i(x), & \text{if } \beta(g(x)) \leqq \beta(0) = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Therefore,

$$W_\lambda(0) = \inf_{x \in S} \sum_{i=1}^{k} \lambda_i f_i(x) = V_{P_\lambda}.$$

Moreover, from Lemma 3.2 and using A1,

$$H_\lambda(0, r) = \inf_{u \in R^m} [W_\lambda(u) + \phi(u, 0, r)]$$

$$\leqq \inf_{u \in R^m} [W_\lambda(u) + \phi(0, 0, r)] \leqq \liminf_{u \to 0} W_\lambda(u),$$

$$V_{D_\lambda} = \lim_{r \to +\infty} H_\lambda(0, r) \leqq \lim_{u \to 0} \{\inf W_\lambda(u)\}.$$

By Theorem 2 of Ref. 15,

$$V_{P_\lambda} \geqq V_{D_\lambda},$$

i.e.,

$$\sum_{i=1}^{k} \lambda_i f_i(x) \geqq \sum_{i=1}^{k} \lambda_i H_i(0, r),$$

which is a contradiction to (1). □

Definition 3.2. Problem (P) is said to be stable of degree 0 if and only if (P_λ) is stable of degree 0; i.e., (P) is stable of degree 0 if there is a real function θ defined on $N(0)$, an open neighborhood of the origin in R^m, such that θ is continuous and

(i) $W_\lambda(u) \geqq \theta(u)$, for every $u \in N(0)$;
(ii) $W_\lambda(0) = \theta(0)$.

Theorem 3.3. See Ref. 15. Program (P) is stable of degree 0 if and only if

$$W_\lambda(0) = \lim_{u \to 0} \inf[W_\lambda(u)], \text{ whenever } W_\lambda(0) \text{ is finite.}$$

Theorem 3.4. Let ϕ satisfy Assumptions A1, A3, A4, A5. If (P) satisfies the boundedness condition and is stable of degree zero, then

$$\text{V-min } P = \text{V-max } D.$$

Proof. It follows from Theorem 4 of Ref. 15 and Lemma 1 of Ref. 16. □

References

1. EGUDO, R. R., and HANSON, M. A., *Multiobjective Duality with Invexity*, Journal of Mathematical Analysis and Applications, Vol. 126, pp. 469–477, 1987.
2. IACOB, P., *Saddle-Point Duality Theorems for Pareto Optimization*, Analyse Numerique et Theorie de L'Approximation, Vol. 15, pp. 37–40, 1986.
3. EGUDO, R. R., *Efficiency and Generalized Convex Duality for Multiobjective Programs*, Journal of Mathematical Analysis and Applications, Vol. 138, pp. 84–94, 1989.
4. MOND, B., HUSAIN, I., and DURGA PRASAD, M. V., *Duality for a Class of Nondifferentiable Multiple-Objective Programming Problems*, Journal of Information and Optimization Sciences, Vol. 9, pp. 331–341, 1988.
5. PREDA, V., *On Efficiency and Duality for Multiobjective Programs*, Journal of Mathematical Analysis and Applications, Vol. 166, pp. 365–377, 1992.
6. BITRAN, G. R., *Duality for Nonlinear Multiple-Criteria Optimization Problems*, Journal of Optimization Theory and Applications, Vol. 35, pp. 367–401, 1981.
7. SINGH, C., and HANSON, M. A., *Generalized Proper Efficiency in Multiobjective Programming*, Journal of Information and Optimization Sciences, Vol. 12, pp. 139–144, 1991.
8. SINGH, C., and HANSON, M. A., *Erratum*, Journal of Information and Optimization Sciences, Vol. 13, p. 173, 1992.
9. WEIR, T., MOND, B., and CRAVEN, B. D., *Weak Minimization and Duality*, Numerical Functional Analysis and Optimization, Vol. 9, pp. 181–192, 1987.

10. WEIR, T., and MOND, B., *Preinvex Functions in Multiple-Objective Optimization*, Journal of Mathematical Analysis and Applications, Vol. 136, pp. 29–38, 1988.
11. ROCKAFELLAR, R. T., *Lagrange Multipliers and Duality in Nonlinear Optimization*, Quinta Escuela Latino-Americana de Matematica, Mar del Plata, Argentina, 1980.
12. BERTSEKAS, D. P., *Constrained Optimization and Lagrange Multiplier Methods*, Academic Press, London, England, 1982.
13. DOLECKI, S., and KURCYUSZ, S., *On ϕ-Convexity in Extremal Problems*, SIAM Journal on Control and Optimization, Vol. 16, pp. 277–300, 1978.
14. GOULD, F. J., *Extensions of Lagrange Multipliers in Nonlinear Programming*, SIAM Journal on Applied Mathematics, Vol. 17, pp. 1280–1297, 1969.
15. GONEN, A., and AVRIEL, M., *Duality in Nonlinear Programs Using Augmented Lagrangian Functions*, Journal of Mathematical Analysis and Applications, Vol. 121, pp. 39–56, 1987.
16. GEOFFRION, A. M., *Proper Efficiency and Theory of Vector Maximization*, Journal of Mathematical Analysis and Applications, Vol. 22, pp. 618–630, 1968.