JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 86, No. 3, pp. 669-683, SEPTEMBER 1995

# Natural Frequencies of Structures with Uncertain but Nonrandom Parameters[1]

Z. P. Qiu,[2] S. H. Chen,[3] and I. Elishakoff[4]

Communicated by G. Leitmann

**Abstract.** In this paper, we present a method for computing upper and lower bounds of the natural frequencies of a structure with parameters which are unknown, except for the fact that they belong to given intervals. These parameters are uncertain, yet they are not treated as being random, since no information is available on their probabilistic characteristics. The set of possible states of the system is described by interval matrices. By solving the generalized interval eigenvalue problem, the bounds on the natural frequencies of the structure with interval parameters are evaluated. Numerical results show that the proposed method is extremely effective.

**Key Words.** Eigenvalue problems, elastic structures, uncertainty modeling, interval mathematics.

## 1. Introduction

Uncertainties in the system parameters are usually analyzed through identifying the uncertain parameters with random variables or random fields. A comprehensive review of the studies performed for the analysis of the systems with random parameters was given by Ibrahim (Ref. 1). Conditional probability concepts [Kozin (Ref. 2) and Elishakoff and Spencer (Ref. 3)]

[1]The research reported in this paper has been supported by the PRC National Natural Science Foundation and by the USA National Science Foundation Grant MSM-9215698 (Program Director Dr. K. P. Chong).

[2]PhD Student, Center for Computational Mechanics, Jilin University of Technology, Changchun, People's Republic of China.

[3]Professor, Center for Computational Mechanics, Jilin University of Technology, Changchun, People's Republic of China.

[4]Professor, Department of Mechanical Engineering, Florida Atlantic University, Boca Raton, Florida.

or perturbation techniques [Thomson (Ref. 4)] are usually employed to tackle the problem. The present state of the art on the behavior of systems with stochastic parameters is summarized in the recent monograph by Chen (Ref. 5).

Recently, a stochastic finite-element method was developed extensively to deal with uncertain stochastic structures (Refs. 6–7). Such an analysis requires the knowledge of the probabilistic characteristics of elastic moduli, masses, and other uncertain parameters. However, as Shinozuka (Ref. 8) mentions: "... it is rather difficult to estimate experimentally the autocorrelation function, or equivalently the spectral density function of the stochastic variation of the material properties. In view of this, upper bound results are particularly important, since the bounds derived ... do not require knowledge of the autocorrelation function."

In this study, the mathematical theory of interval analysis (Refs. 9–11) is utilized to deal with the analysis of structures with uncertain parameters. These are not treated as random variables or fields, but rather as interval variables. The idea of the compatibility of interval analysis to treating uncertainty in structures was expressed by Elishakoff (Ref. 12). In this study, we present a method for computing upper and lower bounds for the natural frequencies of a structure with interval parameters.

## 2. Problem Formulation

Consider the eigenvalue problem described by the equation

$$Ku = \lambda M u, \tag{1}$$

subject to constraints representing the uncertainties in the stiffness and mass matrices,

$$\underline{K} \leq K \leq \bar{K}, \tag{2}$$

$$\underline{M} \leq M \leq \bar{M}, \tag{3}$$

where

$$\bar{K} = (\bar{k}_{ij}), \qquad \underline{K} = (\underline{k}_{ij})$$

are upper and lower bounds on the stiffness matrix $K = (k_{ij})$, which is uncertain but nonrandom (or unknown but bounded),

$$\bar{M} = (\bar{m}_{ij}), \qquad \underline{M} = (\underline{m}_{ij})$$

are upper and lower bounds on the mass matrix $M=(m_{ij})$, which is uncertain but nonrandom, $\lambda=\omega^2$ is a squared frequency of the uncertain but nonrandom matrix pair $K$ and $M$, denoted $\langle K, M\rangle$ and $u$ is the associated eigenvector.

We shall study a method for computing the eigenvalues in Eq. (1), in which the elements $k_{ij}$ and $m_{ij}$, with $i, j=1, 2, \ldots, n$, of the matrices $K$ and $M$ are not known precisely. The incomplete information about the elements of the matrices $K$ and $M$ is the result of measurement errors, changes in operating conditions, aging, maintenance-induced errors, and manufacturing errors, etc. In such cases, we do not know precisely the elements of matrices $K$ and $M$. In most if not all cases, we know only the ends of intervals in which the elements of the matrices are confined.

By means of interval matrix notation (Refs. 9–11), Inequalities (2) and (3) can be written as

$$K\in K^I, \qquad M\in M^I, \tag{4}$$

in which $K^I=[\underline{K}, \bar{K}]$ is a positive-semidefinite interval matrix (Ref. 13) and $M^I=[\underline{M}, \bar{M}]$ is a positive-definite interval matrix (Refs. 13–14).

In terms of Eq. (4), the relations (1)–(3) can be simply written as

$$K^I u=\lambda M^I u. \tag{5}$$

Equation (5) is called a generalized interval eigenvalue problem. Because $K^I$ and $M^I$ are defined as interval matrices, the associated eigenvalues of $K^I$ and $M^I$ similarly constitute the interval variables,

$$\lambda^I=[\underline{\lambda}, \bar{\lambda}]=(\lambda_i^I).$$

It should be stressed that the set-theoretic representation of uncertainty or unknown but bounded models in parametric space is motivated by the lack of detailed probabilistic information on the possible distributions of the parameters. Nonprobabilistic, set-theoretical representations of uncertainty have been employed in a wide range of engineering applications (Refs. 15–21). Several topics in the eigenvalue problem pertaining to an interval matrix were reviewed in Ref. 10. Based on the invariance properties of the characteristic vector entries, Deif (Ref. 11) presented a method of computing interval eigenvalues for the standard interval eigenvalue problem. Reference 25 extended the Deif results to the generalized interval eigenvalue problem. Because there exists no efficient criterion for judging the invariance properties of the signs of the components of the eigenvector under interval matrix operations before computing the interval eigenvalues, application of the Deif results is restricted. In this paper, we present an alternative method for computing the interval eigenvalues of the generalized interval eigenvalue problem.

## 3. Analysis

In this section, we present a solution of the eigenvalue problem (5), which serves a wide range of applications. The basic problem to be solved herein is as follows: Given the central matrices

$$K^c = (\bar{K} + \underline{K})/2, \qquad M^c = (\bar{M} + \underline{M})/2$$

of $K^I$ and $M^I$, and given the deviation amplitude matrices

$$\Delta K = (\bar{K} - \underline{K})/2, \qquad \Delta M = (\bar{M} - \underline{M})/2$$

of $K^I$ and $M^I$, find a multidimensional rectangle containing all the eigenvalues,

$$\Gamma = \{\lambda \colon \lambda \in R,\ Ku = \lambda Mu,\ u \neq 0,\ K \in K^I,\ M \in M^I\}, \tag{6}$$

for the interval matrices

$$K^I = [\underline{K}, \bar{K}] = \{K \colon |K - K^c| \leq \Delta K\},$$

$$M^I = [\underline{M}, \bar{M}] = \{M \colon |M - M^c| \leq \Delta M\}.$$

In other words, we seek the upper and lower bounds, or interval eigenvalues, on the set (6); i.e.,

$$\lambda^I = [\underline{\lambda}, \bar{\lambda}] = (\lambda_i^I), \qquad \lambda_i^I = [\underline{\lambda}_i, \bar{\lambda}_i], \qquad i = 1, 2, \ldots, n, \tag{7}$$

where

$$\underline{\lambda}_i = \min_{K \in K^I, M \in M^I} \lambda_i(\langle K, M \rangle), \tag{8}$$

$$\bar{\lambda}_i = \max_{K \in K^I, M \in M^I} \lambda_i(\langle K, M \rangle), \tag{9}$$

with

$$\lambda_i(\langle K, M \rangle) = \min_{\phi_i \in R^n} \max_{\substack{u \in \phi_i \\ u \neq 0}} (u^T K u / u^T M u). \tag{10}$$

Under the constraint conditions (2) and (3), let us consider the Rayleigh quotient for the structural vibration,

$$\lambda_i = \min_{\phi_i \in R^n} \max_{\substack{u \in \phi_i \\ u \neq 0}} (u^T K u / u^T M u), \qquad i = 1, 2, \ldots, n. \tag{11}$$

Clearly, the eigenvalue $\lambda_i$ is considered a function of the elements $k_{ij}$ and $m_{ij}$. Then, by means of the natural interval extension (Refs. 9–11), from Eq. (11) we obtain

$$\lambda_i^I = \min_{\phi_i \in R^n} \max_{\substack{u \in \phi_i \\ u \neq 0}} (u^T K^I u / u^T M^I u), \qquad i = 1, 2, \ldots, n, \tag{12}$$

where

$$K^I = [\underline{K}, \bar{K}] = [K^c - \Delta K, K^c + \Delta K],$$

$$M^I = [\underline{M}, \bar{M}] = [M^c - \Delta M, M^c + \Delta M].$$

To find the upper and lower bounds of each eigenvalue $\lambda_i$, we shall assume that the deviation amplitude matrices

$$\Delta K = (\bar{K} - \underline{K})/2, \qquad \Delta M = (\bar{M} - \underline{M})/2$$

are positive semidefinite. Thus, for $u \in \phi_i$ and $\bar{K} - \underline{K} = 2\Delta K$, we have

$$u^T(\bar{K} - \underline{K})u = 2u^T \Delta K u \geq 0, \tag{13}$$

which implies

$$u^T \bar{K} u \geq u^T \underline{K} u. \tag{14}$$

Analogously, we have

$$u^T \bar{M} u \geq u^T \underline{M} u. \tag{15}$$

In terms of the Inequalities (14) and (15) and interval operations (Refs. 9–11), Eq. (12) can be written as follows:

$$\lambda_i^I = \min_{\phi_i \in R^n} \max_{\substack{u \in \phi_i \\ u \neq 0}} [u^T \underline{K} u, u^T \bar{K} u] / [u^T \underline{M} u, u^T \bar{M} u]. \tag{16}$$

By the interval division (Refs. 9–11), we obtain

$$\lambda_i = \min_{\phi_i \in R^n} \max_{\substack{u \in \phi_i \\ u \neq 0}} [u^T \underline{K} u / u^T \bar{M} u, u^T \bar{K} u / u^T \underline{M} u]. \tag{17}$$

Further, bearing in mind that

$$\underline{K} = \bar{K} - 2\Delta K, \qquad \underline{M} = \bar{M} - 2\Delta M, \tag{18}$$

and that

$$u^T \Delta K u \geq 0, \qquad u^T \Delta M u \geq 0, \tag{19}$$

we arrive at

$$
\begin{aligned}
&\min_{\phi_i \in R^n} \max_{\substack{u \in \phi_i \\ u \neq 0}} (u^T \underline{K} u / u^T \bar{M} u) \\
&= \min_{\phi_i \in R^n} \max_{\substack{u \in \phi_i \\ u \neq 0}} [(u^T \bar{K} u - 2u^T \Delta K u)/(u^T \underline{M} u + 2u^T \Delta M u)] \\
&\leq \min_{\phi_i \in R^n} \max_{\substack{u \in \phi_i \\ u \neq 0}} (u^T \bar{K} u / u^T \underline{M} u).
\end{aligned} \tag{20}
$$

Thus, from Eq. (17), we have

$$
\lambda_i^T = \left[ \min_{\phi_i \in R^n} \max_{\substack{u \in \phi_i \\ u \neq 0}} (u^T \underline{K} u / u^T \bar{M} u), \min_{\phi_i \in R^n} \max_{\substack{u \in \phi_i \\ u \neq 0}} (u^T \bar{K} u / u^T \underline{M} u) \right]. \tag{21}
$$

According to the necessary and sufficient conditions of equality of interval variables (Refs. 9–11), we obtain

$$
\underline{\lambda}_i = \min_{\phi_i \in R^n} \max_{\substack{u \in \phi_i \\ u \neq 0}} (u^T \underline{K} u / u^T \bar{M} u), \tag{22}
$$

$$
\bar{\lambda}_i = \min_{\phi_i \in R^n} \max_{\substack{u \in \phi_i \\ u \neq 0}} (u^T \bar{K} u / u^T \underline{M} u). \tag{23}
$$

The stationarity condition of the Rayleigh quotient is equivalent to the algebraic eigenvalue problem (Refs. 26–28). Thus, the eigenvalue problem corresponding to the lower bound of Eq. (22) is

$$
\underline{K} \underline{u}_i = \underline{\lambda}_i \bar{M} \underline{u}_i, \tag{24}
$$

where $\underline{u}_i$ is the eigenvector associated with $\underline{\lambda}_i$. Similarly, the eigenvalue problem corresponding to the upper bound of Eq. (23) is

$$
\bar{K} \bar{u}_i = \bar{\lambda}_i \underline{M} \bar{u}_i, \tag{25}
$$

where $\bar{u}_i$ is the eigenvector associated with $\bar{\lambda}_i$.

Thus, we arrive at the following theorem.

**Theorem 3.1.** If $K^I = [\underline{K}, \bar{K}] = [K^c - \Delta K, K^c + \Delta K]$ is a positive-semidefinite interval matrix and if $M^I = [\underline{M}, \bar{M}] = [M^c - \Delta M, M^c + \Delta M]$ is a positive-definite interval matrix, $\Delta K$ and $\Delta M$ are also positive-semidefinite real matrices, then the eigenvalues $\lambda_i$, $i=1,2,\ldots,n$, of $K\in K^I$ and $M\in M^I$ range over the interval, i.e.,

$$\lambda_i^I=[\underline{\lambda}_i,\bar{\lambda}_i],\qquad i=1,2,\ldots,n, \tag{26}$$

where the lower bounds $\underline{\lambda}_i$ satisfy

$$\underline{K}\underline{u}_i=\underline{\lambda}_i\bar{M}\underline{u}_i,\qquad i=1,2,\ldots,n, \tag{27}$$

and the upper bounds $\bar{\lambda}_i$ satisfy

$$\bar{K}\bar{u}_i=\bar{\lambda}_i\underline{M}\bar{u}_i,\qquad i=1,2,\ldots,n. \tag{28}$$

## 4. Numerical Example

The undamped free vibrations of a multidegree-of-freedom linear system is governed by the mass matrix $M$ and the stiffness matrix $K$, where $M$ is positive definite and $K$ is positive semidefinite. If the central (nominal) stiffness matrix $K^c=(k_{ij}^c)$ is given and its deviation amplitude matrix $\Delta K=(\Delta K_{ij})$ is obtained, the stiffness interval matrix

$$K^I=[\underline{K},\bar{K}]=[K^c-\Delta K, K^c+\Delta K]$$

can be formulated. In the same way, the interval mass matrix

$$M^I=[\underline{M},\bar{M}]=[M^c-\Delta M, M^c+\Delta M]$$

can also be obtained. In reality, the systems are continuous and their parameters are distributed. However, in many cases, it is possible to simplify the analysis by replacing the distributed characteristics of the system by discrete ones. This is accomplished by a suitable lumping of the continuous system. These systems have a special property, i.e., the uncertainties of the interval stiffness matrix and the interval mass matrix are positive semidefinite. To illustrate this, let us consider the example of a frame shown in Fig. 1. The deviation matrix associated with the interval stiffness matrix is

$$\Delta K=\begin{bmatrix}\Delta k_1+\Delta k_2 & \Delta k_2 & 0 & 0 & 0\\ \Delta k_2 & \Delta k_2+\Delta k_3 & \Delta k_3 & 0 & 0\\ 0 & \Delta k_3 & \Delta k_3+\Delta k_4 & \Delta k_4 & 0\\ 0 & 0 & \Delta k_4 & \Delta k_4+\Delta k_5 & \Delta k_5\\ 0 & 0 & 0 & \Delta k_5 & \Delta k_5\end{bmatrix}. \tag{29}$$

Obviously, $\Delta K$ is diagonally dominant. According to the Gershgorin disk theorem (Ref. 28), a diagonally dominant matrix is positive semidefinite.

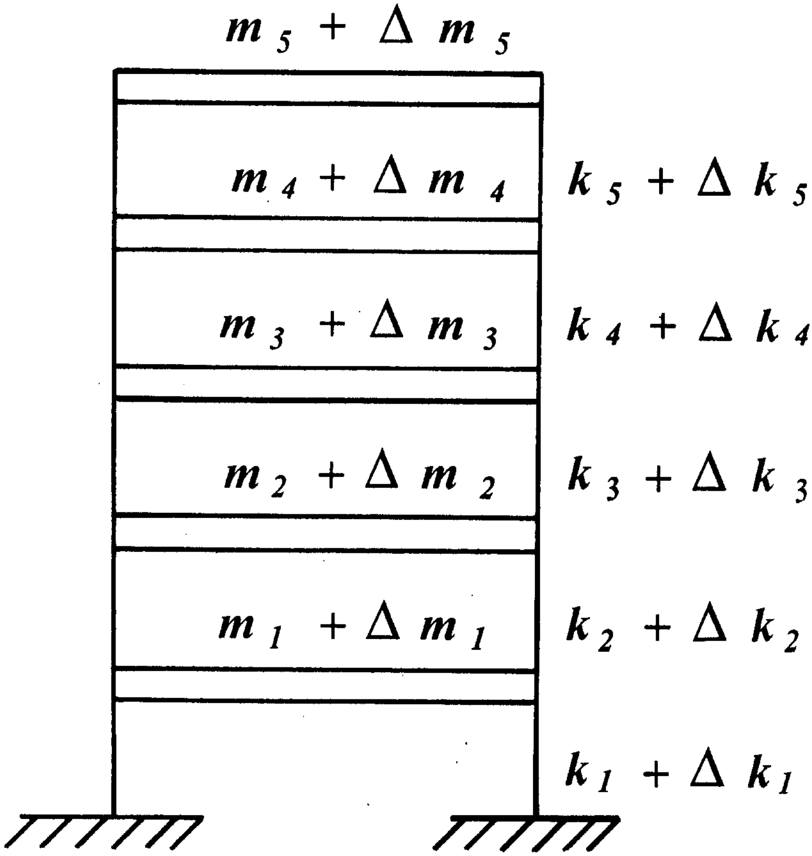


Fig. 1. Frame of a multistory structure.

For comparison purposes, let us study the example considered in Ref. 25. The five-story frame is shown in Fig. 1. The interval stiffness matrix of the frame with interval parameters reads as follows:

$$K^I = \begin{bmatrix} [3800, 3870] & -[1800, 1850] & 0 & 0 & 0 \\ -[1800, 1850] & [3400, 3480] & -[1600, 1630] & 0 & 0 \\ 0 & -[1600, 1630] & [3000, 3050] & -[1400, 1420] & \\ 0 & 0 & -[1400, 1420] & [2600, 2630] & -[1200, 1210] \\ 0 & 0 & 0 & -[1200, 1210] & [1200, 1210] \end{bmatrix}. \tag{30}$$

The interval mass matrix, which is diagonal, is given by

$$M^I = \begin{bmatrix} [29, 30] & 0 & 0 & 0 & 0 \\ 0 & [26, 28] & 0 & 0 & 0 \\ 0 & 0 & [26, 28] & 0 & 0 \\ 0 & 0 & 0 & [24, 26] & 0 \\ 0 & 0 & 0 & 0 & [17, 19] \end{bmatrix}. \tag{31}$$

The stiffness deviation amplitude matrix reads

$$\Delta K = \begin{bmatrix} 35 & 25 & 0 & 0 & 0 \\ 25 & 40 & 15 & 0 & 0 \\ 0 & 15 & 25 & 10 & 0 \\ 0 & 0 & 10 & 15 & 5 \\ 0 & 0 & 0 & 5 & 5 \end{bmatrix}. \tag{32}$$

The mass deviation amplitude matrix reads

$$\Delta M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{33}$$

Obviously, the matrices $\Delta K$ and $\Delta M$ are positive semidefinite; $\Delta K$ and $\Delta M$ satisfy the conditions of the present method.

The interval eigenvalues of the frame are summarized in Table 1. To facilitate comparison, the interval eigenvalues and the basic quantities which

Table 1. Interval eigenvalues obtained by the present method.

| $\lambda_i^I$ | $\underline{\lambda}_i$ | $\bar{\lambda}_i$ | $\lambda_{ci}$ | $\Delta\lambda_i$ | $\gamma_i = \Delta\lambda_i/\lambda_{ci}$ |
|---|---|---|---|---|---|
| $\lambda_1^I$ | 4.6215 | 7.8303 | 6.2259 | 1.6044 | 0.2577 |
| $\lambda_2^I$ | 41.0091 | 47.7025 | 44.3558 | 3.3467 | 0.0755 |
| $\lambda_3^I$ | 99.5956 | 108.9858 | 104.2907 | 4.6951 | 0.0450 |
| $\lambda_4^I$ | 106.1150 | 172.8947 | 166.5048 | 6.3898 | 0.0384 |
| $\lambda_5^I$ | 211.5044 | 227.9488 | 219.7266 | 8.2222 | 0.0374 |

Table 2. Interval eigenvalues obtained by the Deif method (Ref. 22).

| $\lambda_i^I$ | $\underline{\lambda}_i$ | $\bar{\lambda}_i$ | $\lambda_{ci}$ | $\Delta\lambda_i$ | $\underline{\lambda}_i=\Delta\lambda_i/\lambda_{ci}$ |
|---|---|---|---|---|---|
| $\lambda_1^I$ | 4.6166 | 7.8303 | 6.2234 | 1.6069 | 0.2582 |
| $\lambda_2^I$ | 40.6428 | 47.8200 | 44.2314 | 3.5886 | 0.0811 |
| $\lambda_3^I$ | 98.1800 | 109.3993 | 103.7897 | 5.6096 | 0.0540 |
| $\lambda_4^I$ | 157.8433 | 174.0029 | 165.9231 | 8.0798 | 0.0487 |
| $\lambda_5^I$ | 209.5148 | 230.0845 | 219.7997 | 10.2849 | 0.0468 |

are calculated by the Deif method (Ref. 25) are listed in Table 2. It is seen that the present method yields tighter bounds; namely, the lower bounds within the present method are larger than those predicted by the Deif method. Likewise, the upper bounds furnished by the present technique are smaller than those yielded by the Deif approach. The tables also list the ratio $\gamma_i=\Delta\lambda_i/\lambda_{ci}$, which characterizes the magnitude of the variation. As is seen, as predicted by the present theory, this coefficient is smaller for each natural frequency than that predicted by the Deif method. This feature demonstrates clearly that the present method is advantageous over that of Deif.

## 5. Conclusions

It is often desirable in a variety of dynamic structural problems to obtain frequencies of the dynamic system $Ku=\lambda Mu$ in which both $K$ and $M$ are affected by uncertainties. One becomes therefore concerned with determining the tolerances in the eigenvalues $\lambda_i$, knowing the tolerances inherent in the elements $k_{ij}$ and/or $m_{ij}$. Such a problem pertains usually to a mathematical model whose data are gathered from field or experimental observations which are too limited to justify a probabilistic analysis. In this paper, a method was proposed for computing interval eigenvalues of structures with interval parameters.

## 6. Appendix: Mathematical Background

In order to treat the eigenvalue problem of structures with interval parameters, we need to introduce the basics of interval analysis (Refs. 9–14). In interval mathematics, a subset of real numbers $R$ of the form

$$[a_1, a_2]=\{t: a_1\le t\le a_2, a_1, a_2\in R\} \tag{34}$$

is called a closed real interval or an interval. Here, we denote the closed real interval as $X^I = [\underline{x}, \bar{x}]$, where $\underline{x}$ and $\bar{x}$ are the lower and upper bounds. The set of all closed real intervals is denoted by $I(R)$.

The center and deviation amplitude of an interval $X^I = [\underline{x}, \bar{x}]$ are defined as

$$X^c = (\bar{x} + \underline{x})/2, \tag{35}$$

$$\Delta X = (\bar{x} - \underline{x})/2. \tag{36}$$

Two intervals $X_1^I = [\underline{x}_1, \bar{x}_1]$ and $X_2^I = [\underline{x}_2, \bar{x}_2]$ are called equal, if their corresponding endpoints are equal. Thus, $X_1^I = X_2^I$, if $\underline{x}_1 = \underline{x}_2$ and $\bar{x}_1 = \bar{x}_2$.

By an $n$-dimensional interval vector, we mean an ordered $n$-tuple of intervals

$$X^I = (X_1^I, X_2^I, \ldots, X_n^I)^T. \tag{37}$$

The set of all interval vectors is denoted by $I(R^n)$. We define the midvector and deviation amplitude vector of an interval vector,

$$X^c = (X_1^c, X_2^c, \ldots, X_n^c)^T, \tag{38}$$

$$\Delta X = (\Delta X_1, \Delta X_2, \ldots, \Delta X_n)^T, \tag{39}$$

where $X_i^c$ and $\Delta X_i$ are given by (2) and (3).

A matrix whose elements are intervals is called an interval matrix and is denoted by $A^I = [\underline{A}, \bar{A}]$. The set of all interval matrices is denoted by $I(R^{m \times n})$.

Similarly, we can define the central and deviation amplitude matrices associated with an interval matrix $A^I$ as

$$A^c = (\bar{A} + \underline{A})/2 \quad \text{or} \quad a_{ij}^c = (\bar{a}_{ij} + \underline{a}_{ij})/2, \tag{40}$$

$$\Delta A = (\bar{A} - \underline{A})/2 \quad \text{or} \quad \Delta a_{ij} = (\bar{a}_{ij} - \underline{a}_{ij})/2, \tag{41}$$

where

$$A^c = (a_{ij}^c) \quad \text{and} \quad \Delta A = (\Delta a_{ij}).$$

An arbitrary interval $X^I \in I(R)$ can be written as the sum of a real number $X^c$ and an interval $\Delta X^I = [-\Delta X, \Delta X]$; i.e.,

$$X^I = X^c + \Delta X^I. \tag{42}$$

Similar expressions exist for an interval vector and interval matrix. For the matrix $A^I \in I(R^{m \times n})$, we have

$$A^I = A^c + \Delta A^I, \tag{43}$$

where

$$\Delta A^I = [-\Delta A, \Delta A].$$

$A^I$ is called a symmetric interval matrix (Refs. 10, 13), if $A$ is symmetric for every real matrix $A \in A^I$; $A^I$ is called a positive-semidefinite interval matrix (Refs. 10, 13), if $A$ is positive-semidefinite for every real matrix $A \in A^I$. A similar definition holds for a positive-definite interval matrix. $A^I$ is called a nonsingular interval matrix, if $A$ is nonsingular for every real matrix $A \in A^I$.

Let $f$ be a real-valued function of $n$ real variables $x_1, x_2, \ldots, x_n$. By an extension of $f$, we mean an interval-valued function $F$ of $n$ interval variables $X_1^I, X_2^I, \ldots, X_n^I$ for all $x_i \in X_i^I$, $i = 1, 2, \ldots, n$, with the property that

$$F([\underline{x}_1, \bar{x}_1], [\underline{x}_2, \bar{x}_2], \ldots, [\underline{x}_n, \bar{x}_n]) = f(x_1, x_2, \ldots, x_n). \tag{44}$$

Thus, an interval extension of $f$ is an interval-valued function which has real values when the arguments are all real and coincides with $f$ in this case.

For many applications, it is important to perform a computation of an interval extension. We say that an interval-valued function $F$ of the interval variables $X_1^I, X_2^I, \ldots, X_n^I$ is inclusion monotonic, if

$$Y_1^I \subseteq X_1^I, \qquad i = 1, 2, \ldots, n, \tag{45}$$

implies

$$F(Y_1^I, Y_2^I, \ldots, Y_n^I) \subseteq F(X_1^I, X_2^I, \ldots, X_n^I). \tag{46}$$

Real rational functions of $n$ real variables have natural interval extensions. Given a rational expression in real variables, we can replace the real variables by corresponding interval variables and replace the real arithmetic operations by the corresponding interval arithmetic operations to obtain a rational interval function which is a natural extension of the real rational function.

The following theorem holds: If $F$ is an inclusion monotonic interval extension of $f$, then

$$\{f(x) : x_i \in X_i^I, i = 1, 2, \ldots, n\} \subseteq F(X_1^I, X_2^I, \ldots, X_n^I). \tag{47}$$

In other words, an interval value of $F$ contains the range of values of the corresponding real function $f$, when the real arguments of $f$ lie in the intervals shown. Thus, rational interval functions are inclusion monotonic, as are natural interval extension of all the standard functions used in computing; rounded interval arithmetic operations are also inclusion monotonic.

This theorem provides us with a means for the evaluation of upper and lower bounds on the ranges of values of real rational functions over a dimensional space. For example, consider the function

$$f(x_1, x_2, x_3) = [(x_1 + x_2)/(x_1 - x_2)]x_3. \tag{48}$$

Suppose that we wish to calculate the range of values of $f(x_1, x_2, x_3)$ when $x_1, x_2, x_3$ are any numbers in the intervals $[1, 2]$, $[5, 10]$, $[2, 3]$, respectively. A natural interval extension of $f$ is the interval function

$$F(X_1^I, X_2^I, X_3^I) = [(X_1^I + X_2^I)/(X_1^I - X_2^I)]X_3^I. \tag{49}$$

Computing $F([1, 2], [5, 10], [2, 3])$, we obtain

$$\begin{aligned} F([1, 2], [5, 10][2, 3]) &= ([1, 2] + [5, 10])/([1, 2] - [5, 10])[2, 3] \\ &= [-12, -12/9]. \end{aligned} \tag{50}$$

We can also rewrite $f(x_1, x_2, x_3)$ in the following form:

$$f(x_1, x_2, x_3) = x_3[1 + 2/(x_1/x_2 - 1)], \tag{51}$$

which is equivalent in real arithmetic to the original form. A natural interval extension of $f(x_1, x_2, x_3)$, written in this form, reads

$$\begin{aligned} F([1, 2], [5, 10], [2, 3]) &= [2, 3][1 + 2/([1, 2]/[5, 10] - 1)] \\ &= [-12, -22/9]. \end{aligned} \tag{52}$$

The exact range of values for the above function $f(x_1, x_2, x_3)$ for $x_1 \in [1, 2]$, $x_2 \in [5, 10]$, $x_3 \in [2, 3]$ is $[-12, -22/9]$, which is a tighter bound than that given in Eq. (50).

In the example above, for polynomials, the nested form

$$A_c + X^I(A_1 + X^I(A_2 + \cdots + X^I(A_n) \cdots) \tag{53}$$

is never worse (and is usually better) than the sum of powers

$$A_c + A_1 X^I + A_2 X^I \cdot X^I + \cdots + A_n X^I \cdot X^I \cdots X^I, \tag{54}$$

because of subdistributivity.

The following theorem is instrumental for interval computations: Any natural interval extension of a rational function in which each variable occurs only once (if at all) and to the first power only will compute the exact range of value providing that no division by an interval containing zero occurs (Ref. 14).

## References


1. IBRAHIM, R., *Structural Dynamics with Parameter Uncertainties*, Applied Mechanics Reviews, Vol. 40, pp. 309-328, 1987.
2. KOZIN, F., *On the Probability Densities of the Output of Some Random Systems*, Journal of Applied Mechanics, Vol. 28, pp. 161-164, 1961.
3. ELISHAKOFF, I., and SPENCER, B. F., JR., *Reliability of an Uncertain Structure*, Journal of Sound and Vibration, Vol. 114, pp. 399-404, 1987.
4. THOMSON, W. T., *Parameter Uncertainty in Dynamic Systems*, Shock and Vibration Digest, Vol. 7, pp. 3-9, 1975.
5. CHEN, S. H., *Vibration Theory of Structures with Random Parameters*, Jilin Science and Technology Press, Changchun, China, 1992 (in Chinese).
6. ANANTA RAMU, S., and GANESAN, R., *Parametric Instability of Stochastic Columns*, International Journal of Solids and Structures, Vol. 30, pp. 1339-1354, 1993.
7. ANANTA RAMU, S., and GANESAN, R., *A Galerkin Finite-Element Technique for Stochastic Field Problems*, Computer Methods in Applied Mechanics and Engineering, Vol. 105, pp. 315-331, 1993.
8. SHINOZUKA, M., *Structural Response Variability*, Journal of Engineering Mechanics, Vol. 113, pp. 825-842, 1987.
9. MOORE, R., *Methods and Applications of Interval Analysis*, SIAM Publications, Philadelphia, Pennsylvania, 1979.
10. ALEFELD, G., and HERZBERGER, J., *Introduction to Interval Computations*, Academic Press, New York, New York, 1983.
11. DEIF, A., *Advanced Matrix Theory for Scientists and Engineers*, Abacus Press, Tunbridge Wells, England, 1991.
12. ELISHAKOFF, I., *Some Questions in Eigenvalue Problems in Engineering*, Numerical Treatment of Eigenvalue Problems, Edited by J. Albrecht, L. Collatz, P. Hagedorn, and W. Welte, Birkhauser Publishers, Basel, Switzerland, pp. 71-107, 1991.
13. SHI, Z. C., and GAO, G. B., *A Necessary and Sufficient Condition for the Positive Definiteness of Interval Symmetric Matrices*, International Journal of Control, Vol. 41, pp. 325-328, 1986.
14. BIALAS, S., *A Necessary and Sufficient Condition for the Stability of Interval Matrices*, International Journal of Control, Vol. 37, pp. 717-722, 1983.
15. SCHWEPPE, F. C., *Uncertain Dynamical Systems*, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.
16. LEITMANN, G., *On One Approach to the Control of Uncertain Systems*, Journal of Dynamic Systems, Measurement, and Control, Vol. 115, pp. 373-380, 1993.
17. BEN-HAIM, Y., and ELISHAKOFF, I., *Convex Models of Uncertainty in Applied Mechanics*, Elsevier Science Publishers, Amsterdam, The Netherlands, 1990.
18. ELISHAKOFF, I., and BEN-HAIM, Y., *Dynamics of a Thin Cylindrical Shell under Impact with Limited Deterministic Information on Initial Imperfection*, Structural Safety, Vol. 8, pp. 103-112, 1990.
19. ELISHAKOFF, I., *Convex Versus Probabilistic Modeling of Uncertainty in Structural Dynamics*, Structural Dynamics: Recent Advances, Edited by M. Petyt, M.

F. Wolfe, and C. Mei, Elsevier Applied Science Publishers, London, England, pp. 3–21, 1991.

20. ELISHAKOFF, I., *Essay on Uncertainties in Elastic and Viscoelastic Structures: From A. M. Freudenthal's Criticisms to Modern Convex Modeling*, Technical Report, College of Engineering, Florida Atlantic University, 1993.
21. ELISHAKOFF, I., LI, Y. W., and STARNES, J.H., JR., *A Deterministic Method to Predict the Effect of Unknown-but-Bounded Elastic Moduli in the Buckling of Composite Structures*, Computer Methods in Applied Mechanics and Engineering, Vol. 111, pp. 155–167, 1994.
22. ELISHAKOFF, I., ELISSEEFF, P., and GLEGG, S. A. L., *Nonprobabilistic, Convex-Theoretic Modeling of Scatter in Material Properties*, AIAA Journal, Vol. 32, pp. 843–849, 1994.
23. ELISHAKOFF, I., LIN, Y. K., and ZHU, L. P., *Probabilistic and Convex Modeling of Acoustically Excited Structures*, Elsevier Science Publishers, Amsterdam, The Netherlands, 1994.
24. HOLLOT, C., and BARTLETT, A., *On the Eigenvalues of Interval Matrices*, Technical Report, Department of Electrical Engineering and Computer Engineering, University of Massachusetts, 1987.
25. QIU, Z. P., CHEN, S. H., and NA, J. X., *The Rayleigh Quotient Method for Computing Eigenvalue Bounds of Vibrational Systems with Interval Parameters*, Acta Mechanica Sinica, Vol. 6, pp. 309–318, 1993 (English Edition).
26. HU, H. C., *The Theory of Natural Vibration of MDOF Structures*, Science Press of China, Beijing, pp. 1–80, 1987 (in Chinese).
27. CHEN, S. H., *Matrix Perturbation Theory in Structural Dynamics*, International Academic Publishers, Beijing, P.R. China, 1993.
28. LANCASTER, P., *Theory of Matrices*, Academic Press, New York, New York, pp. 289–290 and 350–380, 1985.