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# Truss Topology Optimization Including Unilateral Contact[1]

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**Abstract.** This work extends the ground structure approach of truss topology optimization to include unilateral contact conditions. The traditional design objective of finding the stiffest truss among those of equal volume is combined with a second objective of achieving a uniform contact force distribution. Design variables are the volume of bars and the gaps between potential contact nodes and rigid obstacles. The problem can be viewed as that of finding a saddle point of the equilibrium potential energy function (a convex problem) or as that of minimizing the external work among all trusses that exhibit a uniform contact force distribution (a nonconvex problem). These two formulations are related, although not completely equivalent: they give the same design, but concerning the associated displacement states, the solutions of the first formulation are included among those of the second but the opposite does not necessarily hold.

In the classical noncontact single-load case problem, it is known that an optimal truss can be found by solving a linear programming (LP) limit design problem, where compatibility conditions are not taken into account. This result is extended to include unilateral contact and the second objective of obtaining a uniform contact force distribution. The LP formulation is our vehicle for proving existence of an optimal design: by standard LP theory, we need only to show primal and dual feasibility; the primal one is obvious, and the dual one is shown by the Farkas lemma to be equivalent to a condition on the direction of the external load. This method of proof extends results in the classical noncontact case to structures that have a singular stiffness matrix for all designs, including a case with no prescribed nodal displacements.

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Numerical solutions are also obtained by using the LP formulation. It is applied to two bridge-type structures, and trusses that are optimal in the above sense are obtained.

**Key Words.** Unilateral contact, structural optimization, truss topology design, maximum stiffness, constant contact force distribution.

## 1. Introduction

A general structural optimization problem may consist in the finding of the mechanical structure which transmits given forces to given supports in a specified optimal way. The structures allowed to compete in the optimization process can be of different types. The present work deals with truss structures, i.e., structures consisting of one-dimensional bars that are connected, without transmitting moments at nodal points. To achieve a large number of competing trusses the ground structure approach is used, in which a large number of nodal points (up to 231 in this work) and a large number of potential bars (up to 12,800 in this work) are given at the outset, i.e., a ground structure or a structural universe is specified. The optimization process then chooses the magnitude of bar cross-sectional areas and allows for zero bar areas, so that a topology optimization is accomplished.

The ground structure approach was first used by Dorn, Gomory, and Greenberg (Ref. 1) in 1964. They treated essentially the so-called limit design problem, where the weight of the structure is minimized subject to equilibrium constraints. Since compatibility conditions are not taken into account, this is an optimization of a plastic structure and its meaning relies on the theorems of limit analysis; see, for instance, Haftka, Gürdal, and Kamat (Ref. 2). However, Dorn *et al.* also showed that at least one of the optimal structures is statically determinate, and even if such an optimum structure is assumed to be elastic, it satisfies the stress constraints. The mathematical problem of Ref. 1 is a linear programming (LP) problem and therefore numerically appealing.

There has been recently a revived interest in the ground structure approach to truss topology optimization. Rozvany and coworkers [see, for instance, Zhou and Rozvany (Ref. 3)] have used so-called continuum-based optimality criteria methods to solve large truss topology optimization problems. A mathematically oriented approach has been taken in Ben-Tal and Bendsøe (Ref. 4), Achtziger, Bendsøe, Ben-Tal, and Zowe (Ref. 5), Ben-Tal, Kočvara, and Zowe (Ref. 6), Achtziger (Ref. 7), and Ben-Tal and Nemirovskii (Ref. 8). These works treat the minimum compliance problem, which is a nonconvex problem. However, a main result of Ben-Tal and Bendsøe (Ref. 4) is that the minimum compliance problem can be solved by solving another problem that is convex, although nondifferentiable. It is a byproduct of the main development of the present work that the physical meaning of this second problem is somewhat explained. A result of Achtziger, Bendsøe, Ben-Tal, and Zowe (Ref. 5) is that, in the special case of a single-load case, the convex nondifferentiable problem can be solved by solving an LP problem; in fact, it is seen that this problem has the same mathematical structure as the limit design problem of Dorn, Gomory, and Greenberg (Ref. 1). Consequently, by solving the limit design LP problem, the solution of the nonconvex minimum compliance problem is obtained by a simple rescaling. This result was given also in Hemp (Ref. 9) and it is there attributed to Cox. It is also shown in a slightly different setting in Ref. 3 by Zhou and Rozvany.

The present work extends the ground structure approach to include unilateral contact conditions. Recently, shape optimization for contact problems, where the equilibrium potential energy is minimized, has been studied; see Haslinger and Neittaanmäki (Ref. 10), Klarbring (Ref. 11), and Klarbring and Haslinger (Ref. 12). In Ref. 11, a discrete problem, which includes the present truss problem, was considered. It was shown that minimizing the equilibrium potential energy by varying the gaps between the potential contact nodes and the rigid obstacles results in a uniform distribution of contact forces, which is favorable from the point of view of wear reduction and stress concentration. Here, we like to extend the ground structure approach to include not only the traditional objective of stiffness but also the goal of achieving a uniform contact force distribution. Two formulations that reflect these physical objectives are given. The first one, denoted by ($\mathscr{C}$), is a direct extension of the minimum compliance problem of Ben-Tal and Bendsøe (Ref. 4): the work of the external forces is minimized over all admissible equilibrium configurations. As a second problem formulation of the same physical objectives, we give a saddle point problem, denoted by ($S\Psi$), where the potential energy function is maximized with respect to $t$ and minimized with respect to $u$ and $g$. These two formulations are equivalent in the sense that they generate the same optimal bar volumes, but a displacement solution of the former need not be a solution of the latter.

We pose also an extension of the limit design problem to the case of unilateral contact. As with previous problems, a uniform contact force distribution is imposed. This turns out to be an LP problem, denoted by $(\mathrm{LP})_\mathrm{L}$.

Some theoretical results in connection with the above-mentioned problems are:

(i) provided the external forces are not applied at potential contact nodes only (this condition is precisely stated), optimal solutions can be constructed from the set of solutions of a dual pair of LP problems;

(ii) under a quite intuitive condition on the direction of external loads, it is shown that there exist solutions of the dual pair of LP problems; consequently, problems $(S\Psi)$ and $(\mathscr{C})$ also have solutions;

(iii) one of the problems of the dual pair of LP problems is shown to be completely equivalent to problem $(\mathrm{LP})_{\mathrm{L}}$, extending the result of Achtziger, Bendsøe, Ben-Tal, and Zowe (Ref. 5) to contact problems.

## 2. Equations of State Problems

A truss structure which may come into contact with rigid obstacles is considered. Let there be $N$ nodal points. Between each such two points, there may be a bar. The number of bars is $m \leq (1/2)N(N-1)$. In the ground structure approach, we assume a large number of bars and the optimization process then removes bars to produce an optimum set of bars. The mechanical behavior of the bars is expressed through a vector of bar forces $s = \{s_i\} \in \mathbb{R}^m$ (generalized stresses) and a vector of bar elongations $e = \{e_i\} \in \mathbb{R}^m$ (generalized strains). The deformed configuration of the truss structure is represented by a displacement vector $u \in \mathbb{R}^n$. Here $n = \mathrm{d}\, N - p$, where d is 2 for planar trusses and 3 for spatial ones and $p$ is the number of prescribed zero displacement directions. For simplicity, nonzero prescribed displacements are not considered. The forces, work-conjugate to $u$, are similarly given by a force vector $F \in \mathbb{R}^n$. The basic static and kinematic relations of a truss in the case of small displacements are then

$$F = \sum_{i=1}^{m} \gamma_i s_i \Leftrightarrow F = B^T s, \tag{1}$$

$$e_i = \gamma_i^T u, \; i = 1, \dots, m \Leftrightarrow e = Bu, \tag{2}$$

where $\gamma_i$ is a vector of direction cosines, $B$ is a kinematic transformation matrix, the rows of which are $\gamma_i^T$, and the upper index $T$ means the transpose of a vector or matrix.

Further, the bars are assumed to be made of the same material which in the stiffness maximization problems is a linear elastic one with Young modulus $E > 0$, i.e.,

$$s_i = (A_i / l_i) E e_i, \qquad i = 1, \dots, m \Leftrightarrow s = De, \tag{3}$$

where $D = \mathrm{diag}\{A_i E / l_i\}$, and $A_i \geq 0$, $l_i > 0$, $i = 1, \dots, m$, are cross-sectional areas and lengths of bars, respectively. From (1)–(3), we obtain the structural equation

$$F=K(t)u, \qquad K(t):=B^TDB=\sum_{i=1}^{m} t_iK_i, \qquad K_i:=(E/l_i^2)\gamma_i\gamma_i^T, \tag{4}$$

where $t_i=A_il_i$ is the volume of a bar and $t=\{t_i\}\in\mathbb{R}^m$. It is easy to see that $K(t)$ is symmetric and positive semidefinite, since $D$ is a diagonal matrix with nonnegative entries.

In limit design problems, the relevant material property is the constraint that the absolute magnitude of the stress in each bar cannot exceed a limit $\sigma_0>0$, i.e.,

$$-A_i\sigma_0\leq s_i\leq A_i\sigma_0, \qquad i=1,\ldots,m. \tag{5}$$

A node which may come into frictionless unilateral contact with a rigid obstacle is depicted in Fig. 1. Let $r<n$ be the number of such directions of unilateral contact. The kinematic conditions that nodes cannot penetrate rigid obstacles are expressed as

$$\nu_i^Tu\leq g_i, \qquad i=1,\ldots,r \Leftrightarrow Cu\leq g, \tag{6}$$

where $\nu_i$ is a vector of direction cosines of normals of the obstacle surfaces, $C$ is the kinematic transformation matrix formed from these vectors, $g=$

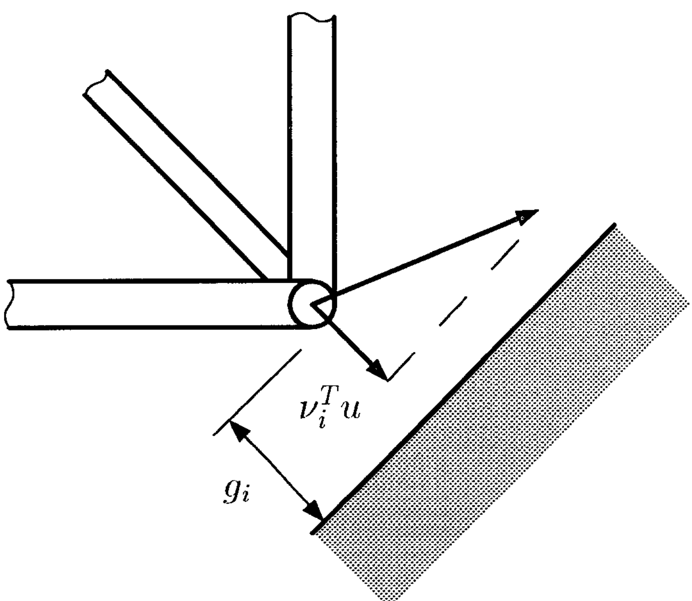


Fig. 1. Node that may come into frictionless unilateral contact.

$\{g_i\} \in \mathbb{R}^r$ is a vector of initial distances between nodes and rigid obstacles. Note that several directions of unilateral contact can be prescribed at each node. The total force $F$ is assumed to be made up from the prescribed forces $f$ and forces due to contact $F_c$, i.e., $F = f + F_c$. The frictionless condition that the forces due to contact are directed normally to the obstacles is expressed as

$$F_c = F - f = \sum_{i=1}^{r} v_i p_i \Leftrightarrow F_c = F - f = C^T p, \tag{7}$$

where $p = \{p_i\} \in \mathbb{R}^r$ is a vector of contact forces, work-conjugate to the vector $Cu$ of contact displacements. Further, adhesionless contact requires that

$$p_i \leq 0, \; i = 1, \ldots, r \Leftrightarrow p \leq 0; \tag{8}$$

finally, ruling out action at a distance gives

$$p_i(v_i^T u - g_i) = 0, \; i = 1, \ldots, r \Leftrightarrow p^T(Cu - g) = 0. \tag{9}$$

## 3. Maximizing Stiffness with Constant Force Distribution

Assume linear elastic bars with no stress constraints. The state problem of finding a displacement $u \in \mathbb{R}^n$ and a contact force $p \in \mathbb{R}^r$ for given cross-sectional areas and contact distances is obtained from (4) and (6)–(9),

$$K(t)u = f + C^T p, \tag{10}$$

$$Cu \leq g, \qquad p \leq 0, \qquad p^T(Cu - g) = 0. \tag{11}$$

It can be seen (Ref. 13) that these conditions are the KKT conditions of the quadratic programming problem of minimizing the potential energy

$$\Pi_t(u) = (1/2)u^T K(t)u - f^T u \tag{12}$$

over the set

$$\mathcal{U}_g = \{u \in \mathbb{R}^n | \, Cu \leq g\}$$

of kinematically admissible displacements.

We note also an extension of the Clapeyron theorem to problems involving unilateral contact,

$$(1/2)u^T K(t)u = (1/2)f^T u + (1/2)p^T g, \tag{13}$$

which follows directly from (10) and (11). Clearly, the familiar statement that "the external work equals the strain energy" does not hold in general unless $g = 0$.

As design variables we take the volume of bars $t=\{t_i\}\in\mathbb{R}^m$ and the contact distances $g=\{g_i\}\in\mathbb{R}^r$. Design constraints are

$$t\in\mathscr{T}=\left\{t\in\mathbb{R}^m \,\middle|\, \sum_{i=1}^{m} t_i\equiv 1_m^T t=V,\ 0\leq t\right\}, \tag{14}$$

$$g\in\mathscr{G}=\left\{g\in\mathbb{R}^r \,\middle|\, \sum_{i=1}^{r} g_i\equiv 1_r^T g=0\right\}. \tag{15}$$

Here, $1_a=(1,\ldots,1)^T$ is a vector of length $a$ and $V>0$ is the given total volume of the bars. It may be noted that (14) defines a compact subset of $\mathbb{R}^m$, while the subset of $\mathbb{R}^r$ defined by (15) is unbounded. The constraint on $g$ represents a constant volume of the gap, which is in fact taken to be zero.

Our goal in this section is to formulate the problem of finding $t\in\mathscr{T}$ and $g\in\mathscr{G}$ such that the structure represented by (10) and (11), or by the equivalent minimization problem, is as stiff as possible among all structures that have a constant contact force distribution. An elementary basic lemma is then given below.

**Lemma 3.1.** The following sets contain exactly the same elements:

$$A=\{u\in\mathbb{R}^n \mid \exists g\in\mathbb{R}^r \text{ such that } Cu\leq g,\ 1_r^T g=0\},$$

$$\mathscr{U}=\{u\in\mathbb{R}^n \mid 1_r^T Cu\leq 0\}.$$

**Proof.**

(i) $A\subset\mathscr{U}$. This follows directly from $1_r^T Cu\leq 1_r^T g=0$.

(ii) $\mathscr{U}\subset A$. For arbitrary $u\in\mathscr{U}$, we will find some $g$ such that $Cu\leq g$ and $1_r^T g=0$. If $1_r^T Cu=0$, we can simply take $g:=Cu$. Suppose that

$$1_r^T Cu=\sum_{i=1}^{r}(Cu)_i<0.$$

Define the two disjoint index sets

$$J=\{i\in\{1,\ldots,r\} \mid (Cu)_i<0\}, \tag{16}$$

$$K=\{i\in\{1,\ldots,r\} \mid (Cu)_i\geq 0\}. \tag{17}$$

Clearly, $J\neq\varnothing$. Denote the number of elements of $J$ by $M$, $M>0$, and define

$$\kappa:=-M^{-1}1_r^T Cu>0. \tag{18}$$

Now, $g \in \mathbb{R}^r$ is constructed as

$$g_i := \begin{cases} (Cu)_i + \kappa, & \text{if } i \in J, \\ (Cu)_i, & \text{if } i \in K. \end{cases}$$

It is easy to see that $g \in \mathscr{G}$ and $Cu \leq g$. □

An element $g \in \mathscr{G}$ corresponding to $u \in \mathscr{U}$ as in the lemma will be denoted by $g_u$.

**Remark 3.1.** For future use, we note that the sets $\mathscr{U}$, $\mathscr{U}_g$, $\mathscr{T}$ all satisfy some constraint qualification (for the KKT conditions), since the functions in the inequalities defining the sets are all affine.

It turns out that there are several different formulations of the intuitive goal of maximizing stiffness while having a constant force distribution. Below we present two such formulations.

**3.1. Minimizing External Work.** We give a theorem that characterizes all instances of (10) and (11) that correspond to a constant contact force distribution.

**Theorem 3.1.** For fixed $t \in \mathscr{T}$, let $(u, p, g, \Lambda) \in \mathbb{R}^n \times \mathbb{R}^r \times \mathscr{G} \times \mathbb{R}$ satisfy

$$K(t)u = f + C^T p, \tag{19}$$

$$Cu \leq g, \qquad p \leq 0, \qquad p^T(Cu - g) = 0, \tag{20}$$

$$p = -\Lambda 1_r, \qquad \Lambda \geq 0; \tag{21}$$

i.e., the structure is in an equilibrium state with constant contact force distribution. Then, $(u, \Lambda) \in \mathbb{R}^n \times \mathbb{R}$ satisfy

$$K(t)u = f - W\Lambda, \qquad W := C^T 1_r, \tag{22}$$

$$W^T u \leq 0, \qquad \Lambda \geq 0, \qquad \Lambda W^T u = 0. \tag{23}$$

Conversely, let $(u, \Lambda) \in \mathbb{R}^n \times \mathbb{R}$ satisfy (22) and (23). Then, there exist $p \in \mathbb{R}^r$ and $g \in \mathscr{G}$ such that $(u, p, g, \Lambda) \in \mathbb{R}^n \times \mathbb{R}^r \times \mathscr{G} \times \mathbb{R}$ satisfy (19)–(21).

**Proof.** The first claim follows since $g \in \mathscr{G}$ satisfies $1_r^T g = 0$. The converse follows from the definition $p := -\Lambda 1_r$ and Lemma 3.1. □

We note that (22) and (23) are sufficient and necessary KKT conditions for $u$ to minimize the potential energy in (12) over the set $\mathscr{U}$.

If we take the external work as a measure of flexibility, the following problem of minimum compliance can be stated:

$$(\mathscr{C}) \quad \min_{(u,\Lambda,t)\in\mathscr{S}} (1/2)f^T u,$$

where

$$\mathscr{S}=\{(u, \Lambda, t)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^m | K(t)u+\Lambda W=f, \Lambda\geq 0,$$
$$\Lambda W^T u=0, W^T u\leq 0, 1_m^T t=V, t\geq 0\}.$$

The interpretation of problem ($\mathscr{C}$) is that, among the whole class of admissible designs with constant contact forces, we pick one that yields the stiffest structure; a posteriori, one can then pick an admissible gap $g$ from $u$ as described above in congruence with the optimal triplet $(u, \Lambda, t)$.

There are several reasons for making this choice of objective functional. The resulting optimal structures will exhibit features such as uniform stress and contact pressure distributions, small displacements, and constant strain energy density. Furthermore, it will be possible to perform meaningful qualitative analysis; e.g., necessary and sufficient optimality criteria and existence proofs can be obtained.

**3.2. Saddle-Point Formulation.** The physical objectives represented by problem $\mathscr{C}$, i.e., maximum stiffness and uniform contact force distribution, are given an alternative formulation in this subsection.

Consider the following extended potential energy function:

$$(u, g, t)\mapsto \Psi(u, g, t)=\Pi_t(u)+I_{\mathscr{G}}(g)+I_{\mathscr{K}}(u, g)-I_{\mathscr{T}}(t),$$

where for a general set $C$,

$$I_C(x)=\begin{cases}0, & \text{if } x\in C,\\ +\infty, & \text{if } x\notin C,\end{cases}$$

is the indicator function of $C$ and

$$\mathscr{K}=\{(u, g)\in\mathbb{R}^n\times\mathbb{R}^r | Cu\leq g\}.$$

It can be realized, with the aid of, e.g., Corollary 34.2.4 in Ref. 15, that $\Psi$ is a closed property convex–concave function. It is of no concern whether we set $+\infty-\infty$ equal to $+\infty$ or $-\infty$.

As was shown in Ref. 11, minimizing the equilibrium potential energy over $\mathscr{G}$ results in a uniform distribution of contact forces. As indicated previously, the equilibrium potential energy is found by minimizing $\Pi_t(u)$ over $u\in\mathscr{U}_g$. Performing these two minimizations simultaneously gives the following problem for fixed $t \in \mathscr{T}$:

$$\text{Find } (\tilde{u}, \tilde{g}) \in \mathbb{R}^n \times \mathbb{R}^r \text{ such that}$$

$$\Psi(\tilde{u}, \tilde{g}, t) \leq \Psi(u, g, t), \qquad \forall (u, g) \in \mathbb{R}^n \times \mathbb{R}^r.$$

Indeed,

$$\inf_{u,g} \Psi(u, g, t) = \inf_{u,g} \{\Pi_t(u) | (u, g) \in \mathscr{K}, g \in \mathscr{G}\} = \inf_{u} \{\Pi_t(u) | u \in \mathscr{U}\},$$

with the aid of Lemma 3.1. The minimization problem given by the last infimum has (22) and (23) as KKT conditions. Thus, this problem characterizes completely a structure with uniform contact force distribution.

Traditionally, maximum stiffness problems are formulated as maximization of equilibrium potential energy. The motivation for this is the Clapeyron theorem, which gives the equivalence between the potential energy and the negative of external work. As shown previously in Eq. (13), this equivalence generally does not hold for contact problems, due to the contact nonlinearity. Nevertheless, the next lemma shows that it actually holds for the subclass of structures characterized by (22) and (23).

**Lemma 3.2.** For any $(u, \Lambda, t) \in \mathscr{S}$, it holds that

$$\Pi_t(u) = \inf_{v \in \mathscr{U}} \Pi_t(v) = -(1/2) f^T u.$$

**Proof.** The fact that $(u, \Lambda, t) \in \mathscr{S}$ means that the sufficient Kuhn–Tucker conditions for

$$\Pi_t(u) = \min_{v \in \mathscr{U}} \Pi_t(v)$$

are satisfied. Moreover,

$$\Pi_t(u) = (1/2) u^T K(t) u - f^T u = (1/2) u^T (f - \Lambda W) - f^T u$$
$$= -(1/2) f^T u - (1/2) \Lambda W^T u = -(1/2) f^T u,$$

from the definition of $\mathscr{S}$. □

Thus, besides selecting $g$ to get a structure of uniform force distribution, we want to maximize the potential energy to get a stiff structure. This objective results in the following saddle-point problem:

$(S\Psi)$ Find $(\tilde{u}, \tilde{g}, \tilde{t}) \in \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^m$ such that

$$\Psi(\tilde{u}, \tilde{g}, t) \leq \Psi(\tilde{u}, \tilde{g}, \tilde{t}) \leq \Psi(u, g, \tilde{t}), \qquad \forall (u, g, t) \in \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^m.$$

Similarly to the fact that $g$ does not appear explicitly in problem ($\mathscr{C}$), it is possible to reduce problem $(S\Psi)$ to a problem not containing $g$.

**Theorem 3.2.** Let $(\tilde{u}, \tilde{g}, \tilde{t}) \in \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^m$ be a solution of problem $(S\Psi)$. Then, $(\tilde{u}, \tilde{t}) \in \mathscr{U} \times \mathscr{T}$ solves the following problem:

(*S*Π) Find $(\tilde{u}, \tilde{t}) \in \mathscr{U} \times \mathscr{T}$ such that

$$\Pi_t(\tilde{u}) \leq \Pi_{\tilde{t}}(\tilde{u}) \leq \Pi_{\tilde{t}}(u), \qquad \forall (u, t) \in \mathscr{U} \times \mathscr{T}.$$

Conversely, if $(\tilde{u}, \tilde{t}) \in \mathscr{U} \times \mathscr{T}$ is a solution of problem (*S*Π), then there exists $\tilde{g} \in \mathscr{G}$ such that $(\tilde{u}, \tilde{g}, \tilde{t}) \in \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^m$ solves problem $(S\Psi)$.

**Proof.** Let $(\tilde{u}, \tilde{g}, \tilde{t})$ solve problem $(S\Psi)$. Such a solution belongs, by, e.g., Corollary 36.3.1 in Ref. 15, to the effective domain of $\Psi$; by Lemma 3.1, $\tilde{u} \in \mathscr{U}$. Therefore,

$$\Psi(\tilde{u}, \tilde{g}, t) = \Pi_t(\tilde{u}), \qquad \forall t \in \mathscr{T}.$$

Also,

$$\Psi(u, g_u, \tilde{t}) = \Pi_{\tilde{t}}(u), \qquad \forall u \in \mathscr{U};$$

so, by choosing $g = g_u$ in problem $(S\Psi)$, the first claim follows.

If $(\tilde{u}, \tilde{t})$ solves problem (*S*Π), it can be seen that $(\tilde{u}, \tilde{g}, \tilde{t})$ solves problem $(S\Psi)$ by taking $\tilde{g} := g_{\tilde{u}}$ and observing that

$$\Pi_{\tilde{t}}(u) = \Psi(u, g_u, \tilde{t}) \leq \Psi(u, g, \tilde{t}), \qquad \forall (u, g) \in \mathscr{U} \times \mathbb{R}^r. \qquad \square$$

For any pair $(\tilde{u}, \tilde{t})$ solving problem (*S*Π), it is understood that $\tilde{u}$ is the equilibrium state for $\tilde{t}$; furthermore, for any $t$ with equilibrium displacement $u_t$, we have

$$\Pi_t(u_t) \leq \Pi_t(\tilde{u}) \leq \Pi_{\tilde{t}}(\tilde{u}).$$

Hence, the equilibrium potential energy is maximized, and this saddle point represents the maximum stiffness also.

**Remark 3.2.** Associated with any saddle-point problem, there are two dual optimization problems; see Rockafellar (Ref. 15). In the case of problem (*S*Π), the objective functions of these dual problems are

$$\mathscr{T} \ni t \mapsto \varphi(t) = \inf_{u \in \mathscr{U}} \Pi_t(u) \in \mathbb{R} \cup \{-\infty\}, \tag{24}$$

$$\mathscr{U} \ni u \mapsto \psi(u) = \sup_{t \in \mathscr{T}} \Pi_t(u) \in \mathbb{R}. \tag{25}$$

The first one [see (24)] will be utilized in Section 8. Concerning the second one [see (25)], note that the sup is actually attained (hence, no $+\infty$ to the right), and it can be shown that

$$\psi(u)=V \max_{i=1,\dots,m} \{(1/2)u^T K_i u\} - f^T u, \tag{26}$$

which turns out to be the convex nondifferentiable problem defined and used extensively by Ben-Tal and Bendsøe (Ref. 5).

**3.3. Assumptions.** Subsequently, we will show some existence and equivalence results. To that end, the following assumptions will be needed:

(A1) for every $\Lambda \geq 0$, it holds that $C^T 1_r \Lambda \neq f$;
(A2) for all $u \in \mathcal{U}$ such that $Bu=0$, it holds that $f^T u \leq 0$.

Loosely speaking, Assumption (A2) says that any kinematically permissible rigid body displacement $u$ and the applied force $f$ form an obtuse angle (see Fig. 2); Assumption (A1) says that $f$ is not entirely applied at potential contact nodes. Whenever any of these assumptions is used, it will be mentioned explicitly.

## 4. Limit Design Problem

For given cross-sectional areas and contact distances, the statically admissible vectors of bar forces and contact forces satisfy

$$-A_i\sigma_0 \leq s_i \leq A_i\sigma_0, \qquad i=1,\dots,m, \tag{27}$$

$$B^T s = f + C^T p, \qquad p \leq 0. \tag{28}$$

The lower-bound theorem of limit analysis can be extended to unilateral contact [see Telega (Ref. 14)]; thus, any load $f \in \mathbb{R}^m$ for which an element $(s,p) \in \mathbb{R}^m \times \mathbb{R}^r$ that satisfies (27) and (28) exists is a safe load.

In the case where $g \in \mathbb{R}^r$ is a design variable, we may expect intuitively that this vector can be adjusted so that the contact force vector becomes a constant vector, i.e.,

$$p=-1_r \Lambda, \qquad 0 \leq \Lambda \in \mathbb{R}. \tag{29}$$

Equation (28) is then replaced by

$$B^T s = f - W\Lambda, \qquad 0 \leq \Lambda. \tag{30}$$

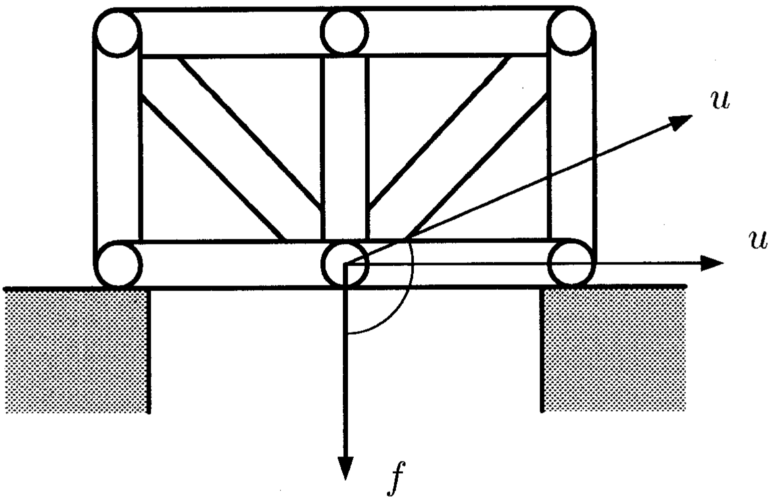


Fig. 2. Directions of kinematically admissible displacements and applied force.

A limit design problem where the objective function is the volume of the structure can now be stated:

$$(\mathrm{LP})_{\mathrm{L}} \quad \min_{(t,s,\Lambda)} \sum_{i=1}^{m} t_i,$$

$$\text{s.t.} \quad (t, s, \Lambda) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^+,$$

$$-t_i\sigma_0 \le s_i l_i \le t_i\sigma_0, \qquad i = 1, \ldots, m,$$

$$B^T s = f - W\Lambda.$$

In the next section, we will see how problems ($\mathscr{C}$) and ($\mathscr{S}\Pi$) relate to problem $(\mathrm{LP})_{\mathrm{L}}$. Note that the nonnegativity constraints for the $t_i$ are superfluous, because of the inequality constraints in $(\mathrm{LP})_{\mathrm{L}}$; note that we have supposed $\sigma_0 > 0$.

## 5. Optimality Conditions

In this section, we study the optimality conditions for problems ($\mathscr{C}$) and ($\mathscr{S}\Pi$). We start with the second problem.

**Theorem 5.1.** Problem ($\mathscr{S}\Pi$) has a solution $(\tilde{u}, \tilde{t}) \in \mathscr{U} \times \mathscr{T}$ if and only if there exists $\tilde{\Lambda} \in \mathbb{R}$ such that:

(j) $(\tilde{u}, \tilde{\Lambda}, \tilde{t}) \in \mathscr{S}$;
(jj) $\tilde{t}_i = 0$, whenever $(1/2)\tilde{u}^T K_i \tilde{u} < \max_{i=1,\ldots,m} (1/2)\tilde{u}^T K_i \tilde{u}$.

**Proof.** The function

$$(u, t) \mapsto \Phi(u, t) = \Pi_t(u) + I_{\mathscr{U}}(u) - I_{\mathscr{T}}(t)$$

has a saddle point at $(\tilde{u}, \tilde{t})$ if and only if the convex function $\Phi(\cdot, \tilde{t})$ achieves its minimum at $\tilde{u}$ and the concave function $\Phi(\tilde{u}, \cdot)$ achieves its maximum at $\tilde{t}$. Such a saddle point necessarily belongs to the effective domain, and the saddle value is finite; see, e.g., Corollary 36.3.1 in Ref. 15. Therefore, $(\tilde{u}, \tilde{t})$ is a solution of problem $(S\Pi)$ if and only if

$$0 \in \partial_1 \Phi(\tilde{u}, \tilde{t}), \tag{31}$$

$$0 \in \partial_2 \Phi(\tilde{u}, \tilde{t}), \tag{32}$$

where $\partial_1 \Phi$ is the convex subgradient of $\Phi$ with respect to the first argument and $\partial_2 \Phi$ is the concave subgradient of $\Phi$ with respect to the second argument. Since $\Pi_t(u)$ is strictly differentiable, (31) is equivalent to

$$0 \in \{K\tilde{u} - f\} + N_{\mathscr{U}}(\tilde{u}), \tag{33}$$

where $N_{\mathscr{U}}$ is the normal cone of $\mathscr{U}$. It necessarily holds that $\tilde{t} \in \mathscr{T}$; therefore (33) is exactly condition (j). Analogously, (32) means that, for some $\lambda \in \mathbb{R}$ and $\xi_i \in \mathbb{R}$, $i = 1, \ldots, m$, it holds that

$$\tilde{u}^T K_i \tilde{u} = \lambda - \xi_i, \qquad i = 1, \ldots, m,$$

$$\xi_i \tilde{t}_i = 0,\ \xi_i \geq 0,\ \tilde{t}_i \geq 0, \qquad i = 1, \ldots, m,$$

$$1_m^T \tilde{t} = V;$$

by taking

$$\lambda = \max_{i=1,\ldots,m} (\tilde{u}^T K_i \tilde{u}),$$

this is easily seen to be equivalent to condition (jj) and $\tilde{t} \in \mathscr{T}$. □

**Theorem 5.2.** If $(\tilde{u}, \tilde{\Lambda}, \tilde{t})$ satisfy conditions (j) and (jj) in Theorem 5.1, then $(\tilde{u}, \tilde{\Lambda}, \tilde{t})$ is a solution to problem $(\mathscr{C})$.

From the above theorem, in order for a triplet $(u, \Lambda, t)$ to solve the maximum stiffness problem $(\mathscr{C})$, it suffices to check that: (j) $(u, \Lambda, t)$ is feasible in $(\mathscr{C})$; and (jj) a bar is removed (its volume is zero), as soon as its strain energy density is not on the maximum level.

**Proof.** Let $(u, \Lambda, t) \in \mathscr{S}$. Then from Lemma 3.2, it holds that

$$\Pi_t(\tilde{u}) \geq \Pi_t(u).$$

Moreover, we can write

$$\Pi_t(\tilde{u}) = \Pi_{\tilde{t}}(\tilde{u}) + \sum_{i=1}^{m} (t_i - \tilde{t}_i)(1/2)\tilde{u}^T K_i \tilde{u}.$$

It will be shown that the last term is nonpositive; then, the theorem follows from Lemma 3.2. Let $\bar{R}$ be the subset of $\{1, \ldots, m\}$ such that $\tilde{t}_i \neq 0$; let $\bar{R}$ be its complement; and let

$$\lambda := \max_{i=1,\ldots,m} ((1/2)\tilde{u}^T K_i \tilde{u}).$$

Then, from $1_m^T t = V$ and condition (jj), one finds that

$$\sum_{i=1}^{m} (t_i - \tilde{t}_i)(1/2)\tilde{u}^T K_i \tilde{u} = -V\lambda + \sum_{i \in R} t_i \lambda + \sum_{i \in \bar{R}} t_i (1/2)\tilde{u}^T K_i \tilde{u}$$

$$= \sum_{i \in \bar{R}} t_i ((1/2)\tilde{u}^T K_i \tilde{u} - \lambda),$$

and the last term is clearly nonpositive. □

That the converse [i.e., that $(\tilde{u}, \tilde{\Lambda}, \tilde{t})$ solves problem $(\mathscr{C})$ implies that $(\tilde{u}, \tilde{t})$ solves $(S\Pi)$] does not hold can be understood from the following example.

Let $t_0 := (V/m)1_m$ represent a sufficiently large ground structure, and suppose that $(\tilde{u}, \tilde{\Lambda}, \tilde{t})$ solves problem $(\mathscr{C})$. It is more than likely that $\tilde{t}$ contains many zero components, so that

$$\dim \mathscr{N}(K(\tilde{t})) > \dim \mathscr{N}(K(t_0));$$

here, $\dim \mathscr{N}(A)$ means the dimension of the null space of $A$. Let $u_s$ be such that $u_s \in \mathscr{N}(K(\tilde{t}))$, but $u_s \perp \mathscr{N}(K(t_0))$. Then, $u_s$ can be a displacement at a node that is such that any bar connected to the node has zero stiffness; see Fig. 3. The node is neither a potential contact node, nor is such that it is on the boundary where forces are prescribed. Now,

$$u^* := \tilde{u} + u_s,$$

together with $\tilde{\Lambda}$ and $\tilde{t}$, will still solve problem $\mathscr{C}$, since

$$(u^*, \tilde{\Lambda}, \tilde{t}) \in \mathscr{S} \quad \text{and} \quad \Pi_{\tilde{t}}(u^*) = \Pi_{\tilde{t}}(\tilde{u}).$$

If $(u^*, \tilde{t})$ solves problem $(S\Pi)$, then

$$\Pi_t(u^*) \leq \Pi_{\tilde{t}}(u^*), \qquad \forall t \in \mathscr{T}. \tag{34}$$

Picking $t = t_0$, realizing that $\Pi_{t_0}(\cdot)$ is coercive for our choices of $u^*$, and letting $|u_s| \to +\infty$, one obtains a contradiction to (34). Note that this $u^*$ also fails to satisfy condition (jj) of Theorem 5.1.

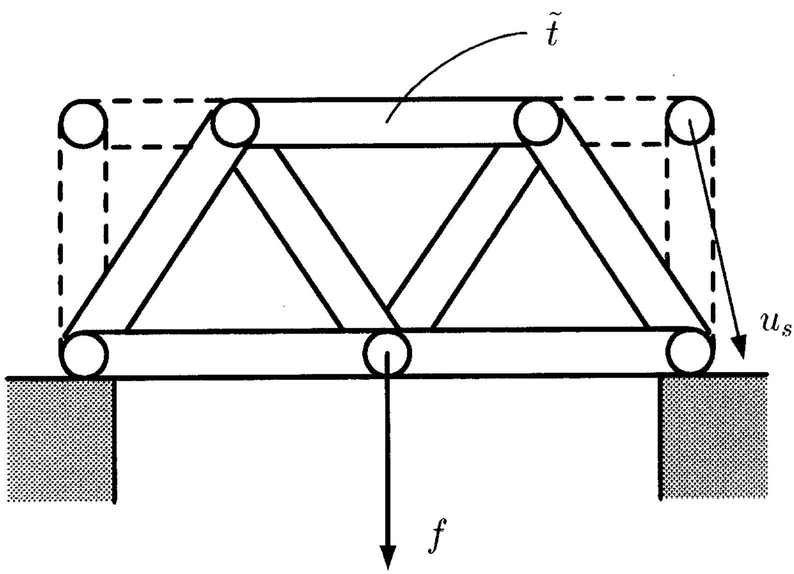


Fig. 3. Illustration of example showing that the converse of Theorem 5.2 does not hold.

In the above sense, the solutions of problem ($S\Pi$) are finer than those of problem ($\mathscr{C}$).

## 6. Linear Programming Problems

The last section made clear that problems ($\mathscr{C}$) and ($S\Pi$) are both solved if we can find a triplet $(\tilde{u}, \tilde{\Lambda}, \tilde{t})$ that satisfies conditions (j) and (jj). In this section, we will see that this quest can actually be achieved by solving a linear programming (LP) problem.

For fixed matrix $B$, vectors $W$ and $v = k(l_1, \ldots, l_m)^T$, and scalar $k = \sqrt{2/VE}$, we state the following conditions for $\bar{u} \in \mathbb{R}^n$, $\rho^+ = \{\rho_i^+\} \in \mathbb{R}^m$, $\rho^- = \{\rho_i^-\} \in \mathbb{R}^m$, and $\Lambda \in \mathbb{R}$:

$$W\Lambda + B^T(\rho^+ - \rho^-) = f, \tag{35}$$

$$W^T\bar{u} \leq 0, \tag{36}$$

$$B\bar{u} \leq v, \qquad -B\bar{u} \leq v, \tag{37}$$

$$\Lambda \geq 0, \qquad \rho^+ \geq 0, \qquad \rho^- \geq 0, \tag{38}$$

$$W^T\bar{u}\Lambda = 0, \tag{39}$$

$$(B\bar{u} - v)^T\rho^+ = 0, \tag{40}$$

$$(B\bar{u} + v)^T\rho^- = 0. \tag{41}$$

Standard LP theory shows that these conditions are the necessary and sufficient optimality conditions of the following LP problem and its dual:

$$\begin{aligned}
(\mathrm{LP}) \quad & \max_{\bar{u}\in\mathbb{R}^n} \; f^T\bar{u}, \\
& \text{s.t.} \quad W^T\bar{u} \le 0, \\
& \qquad\;\; B\bar{u} \le v, \\
& \qquad\;\; -B\bar{u} \le v;
\end{aligned}$$

$$\begin{aligned}
(\mathrm{LP})_{\mathrm{d}} \quad & \min_{(\rho^+,\rho^-,\Lambda)} \; v^T(\rho^+ + \rho^-), \\
& \text{s.t.} \quad (\rho^+, \rho^-, \Lambda) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}, \\
& \qquad\;\; W\Lambda + B^T(\rho^+ - \rho^-) = f, \\
& \qquad\;\; \Lambda \ge 0, \; \rho^+ \ge 0, \; \rho^- \ge 0.
\end{aligned}$$

The next two theorems show that these LP problems are somewhat equivalent to conditions (j) and (jj); as a consequence of Theorem 5.1, they are also equivalent to problem ($\mathcal{S}\Pi$).

**Theorem 6.1.** Suppose that $\bar{u}\in\mathbb{R}^n$ solves problem (LP) and that $(\rho^+, \rho^-, \tilde{\Lambda})\in\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ solves problem $(\mathrm{LP})_{\mathrm{d}}$, or equivalently that they satisfy (35) through (41). Define $\tilde{u}\in\mathbb{R}^n$ and $\tilde{t}\in\mathbb{R}^m$ as

$$\tilde{u} := \mu\bar{u}, \qquad \tilde{t} := \mu^{-1}(\omega^+ + \omega^-),$$

where

$$\omega^+ = \{\omega_i^+\}, \qquad \omega^- = \{\omega_i^-\},$$

and

$$\omega_i^+ = \rho_i^+\sqrt{V/2El_i}, \quad \omega_i^- = \rho_i^-\sqrt{V/2El_i}, \qquad i = 1, \ldots, m,$$

$$\mu := V^{-1}1_m^T(\omega^+ + \omega^-).$$

Then, provided Assumption (A1) holds, $\mu > 0$; so, $\tilde{t}$ is well defined and the triplet $(\tilde{u}, \tilde{\Lambda}, \tilde{t})$ satisfies conditions (j) and (jj).

**Proof.** First, we show that $\mu > 0$. Clearly, $\mu \ge 0$ since all of the following quantities are nonnegative: $V$, $l_i$, $\rho_i^+$, $\rho_i^-$. Suppose that $\mu = 0$. Then, (35) gives

$$\Lambda W = f, \tag{42}$$

since necessarily

$$\rho_i^+ = \rho_i^- = 0, \qquad \text{if } \mu = 0;$$

note that $l_i > 0$, $V > 0$. Now, (42) is impossible because of Assumption (A1).

(Claim that $K(\tilde{t})\tilde{u} + \tilde{\Lambda} W = f$). From $(\mathrm{LP})_d$, we know that

$$\tilde{\Lambda} W + B^T(\rho^+ - \rho^-) = f,$$

so it is sufficient to show that

$$B^T(\rho^+ - \rho^-) = K(\tilde{t})\tilde{u}. \tag{43}$$

From (40) and (41), we have

$$(\gamma_i^T \bar{u} - kl_i)\rho_i^+ = 0, \qquad \forall i, \tag{44}$$

$$(\gamma_i^T \bar{u} + kl_i)\rho_i^- = 0, \qquad \forall i. \tag{45}$$

Now from (4),

$$K(\tilde{t})\tilde{u} = \sum_{i=1}^{m} \tilde{t}_i K_i \tilde{u} = \sum_{i=1}^{m} (\omega_i^+ + \omega_i^-) E / l_i^2 \gamma_i \gamma_i^T \bar{u}$$

$$= \sum_{i=1}^{m} (\rho_i^+ + \rho_i^-)(1/kl_i)\gamma_i \gamma_i^T \bar{u}. \tag{46}$$

Taking (44) and (45) into account, (46) yields (43). In fact,

$$K(\tilde{t})\tilde{u} = \sum_{i=1}^{m} (\gamma_i / kl_i)(kl_i \rho_i^- - kl_i \rho_i^-)$$

$$= \sum_{i=1}^{m} (\rho_i^+ - \rho_i^-)\gamma_i = B^T(\rho^+ - \rho^-). \tag{47}$$

(Claim that $1_m^T \tilde{t} = V$, $\tilde{t} \geq 0$). For any $i$, clearly $\omega_i^+ \geq 0$, $\omega_i^- \geq 0$, and $\mu > 0$, so necessarily $\tilde{t} \geq 0$. Moreover,

$$1_m^T \tilde{t} = \mu^{-1} 1_m^T(\omega^+ + \omega^-) = V,$$

by the definition of $\mu$.

(Claim that $\tilde{\Lambda} \geq 0$, $\tilde{\Lambda} W^T \tilde{u} = 0$, $W^T \tilde{u} \leq 0$). From (36), we get

$$W^T \tilde{u} = \mu W^T \bar{u} \leq 0,$$

and from $(\mathrm{LP})_d$,

$$\Lambda \geq 0, \qquad \tilde{\Lambda} W^T \tilde{u} = 0$$

follow from (38) and (39). Now, condition (j) is established and condition (jj) remains. Suppose that

$$(1/2)\tilde{u}^T K_i \tilde{u} < \max_{i=1,\dots,m} ((1/2)\tilde{u}^T K_i \tilde{u}),$$

for some particular $i$. We have to show that $\tilde{t}_i = 0$. Note first that, for any $i$,

$$-v \leq B\bar{u} \leq v \Rightarrow (\gamma_i^T \bar{u})^2 \leq k^2 l_i^2 = 2l_i^2/VE. \tag{48}$$

Since $\mu > 0$, what we have supposed implies that

$$\bar{u}^T K_i \bar{u} < \max_{i=1,\dots,m} (\bar{u}^T K_i \bar{u}).$$

By (4), this means that

$$(E/l_i^2)\bar{u}^T \gamma_i \gamma_i^T \bar{u} = (E/l_i^2)(\gamma_i^T \bar{u})^2 < \max_{i=1,\dots,m} \{(E/l_i^2)(\gamma_i^T \bar{u})^2\}, \tag{49}$$

and by (48),

$$(E/l_i^2)(\gamma_i^T \bar{u})^2 < 2/V. \tag{50}$$

Now, if either $\rho_i^+ \neq 0$ or $\rho_i^- \neq 0$, it follows from (44) and (45) that

$$(\gamma_i^T \bar{u})^2 = k^2 l_i^2 \Leftrightarrow (E/l_i^2)(\gamma_i^T \bar{u})^2 = 2/V.$$

This contradicts (50), and hence $\rho_i^+ = \rho_i^- = 0$, and necessarily $\tilde{t}_i = 0$. □

**Corollary 6.1.** Suppose that $\bar{u}$ solves (LP) and that Assumption (A1) holds. Then, problem ($\mathscr{C}$) has a solution; between the optimal values of problems ($\mathscr{C}$) and (LP), the following relation holds:

$$\mathrm{opt}(\mathscr{C}) = (1/4)[\mathrm{opt}(\mathrm{LP})]^2. \tag{51}$$

**Proof.** Suppose that $\bar{u}$ solves problem (LP); cf. Theorem 5.1. Then,

$$\mathrm{opt}(\mathscr{C}) = (1/2)f^T \tilde{u} = (\mu/2) f^T \bar{u}. \tag{52}$$

By the definition of $\mu$, we have

$$\begin{aligned}\mu &= (1/V)\sum_{i=1}^m \sqrt{V/2E}(\rho_i^+ + \rho_i^-) l_i = (1/k\sqrt{2EV})\sum_{i=1}^m k l_i(\rho_i^+ + \rho_i^-)\\ &= (1/2)v^T(\rho^+ + \rho^-) = (1/2)\,\mathrm{opt}(\mathrm{LP})_{\mathrm{d}}.\end{aligned} \tag{53}$$

It is well known that, if problem $(\mathrm{LP})_{\mathrm{d}}$ has a solution, then so does problem (LP) and the optimum objective values coincide. Hence, (52) and (53) together imply that

$$\mathrm{opt}(\mathscr{C}) = (1/2) f^T \bar{u}(1/2)\, \mathrm{opt}(\mathrm{LP}) = (1/4)[\mathrm{opt}(\mathrm{LP})]^2. \tag{54}$$

□

**Theorem 6.2.** Let the triplet $(\tilde{u}, \tilde{\Lambda}, \tilde{t})$ satisfy conditions (j) and (jj). Then, provided Assumption (A1) holds,

$$\mu^2 := (V/2) \max_{i=1,\dots,m} (\tilde{u}^T K_i \tilde{u}) > 0.$$

Furthermore, define

$$\bar{u} := \mu^{-1} \tilde{u}$$

and, for $i = 1, \dots, m$,

$$\rho_i^+ := \begin{cases} \mu\sqrt{2E/V}(\tilde{t}_i/l_i), & \text{if } \gamma_i^T \tilde{u} = k l_i \mu, \\ 0, & \text{otherwise,} \end{cases}$$

$$\rho_i^- := \begin{cases} \mu\sqrt{2E/V}(\tilde{t}_i/l_i), & \text{if } \gamma_i^T \tilde{u} = -k l_i \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\bar{u}$ solves problem (LP) and $(\rho^+, \rho^-, \tilde{\Lambda})$ solves problem $(\mathrm{LP})_d$, or equivalently they satisfy (35) through (41).

**Proof.** We will first show the strict positiveness of $\mu^2$ and then verify (37), (40), (41), (36), (38), (39), and finally (35).

Condition (jj) can equivalently be written as

(jj) there exists $\mu \geq 0$ such that

$$(V/2)\tilde{u}^T K_i \tilde{u} \leq \mu^2, \qquad i = 1, \dots, m,$$

$$\tilde{t}_i((V/2)\tilde{u}^T K_i \tilde{u} - \mu^2) = 0, \qquad i = 1, \dots, m.$$

Furthermore,

$$(1/2)\tilde{u}^T K(\tilde{t})\tilde{u} = \sum_{i=1}^{m} \tilde{t}_i (1/2)\tilde{u}^T K_i \tilde{u} = \mu^2,$$

so $\mu^2$ equals the strain energy. Assumption (A1) and condition (j) imply that

$$K(\tilde{t})\tilde{u} \neq 0.$$

In order to show that $\mu^2 > 0$, it now suffices to come to the conclusion that

$$(1/2)\tilde{u}^T K(\tilde{t})\tilde{u} = 0 \text{ implies } K(\tilde{t})\tilde{u} = 0.$$

Since

$$K(\tilde{t}) = B^T D B,$$

we have

$$\tilde{u}^T K(\tilde{t})\tilde{u} = 0 \Rightarrow (B\tilde{u})^T D(B\tilde{u}) = 0,$$

and from the definitions of B and D, this means that

$$\sum_{i=1}^{m} (B\tilde{u})_i^2 (E\tilde{t}_i/l_i^2) = 0 \Rightarrow \tilde{t}_i (B\tilde{u})_i = \tilde{t}_i \gamma_i^T \tilde{u} = 0, \qquad \forall i.$$

Hence,

$$\tilde{u}^T K(\tilde{t})\tilde{u} = 0 \text{ implies } K(\tilde{t})\tilde{u} = 0.$$

In fact, by (4),

$$K(\tilde{t})\tilde{u} = \sum_{i=1}^{m} \tilde{t}_i K_i \tilde{u} = \sum_{i=1}^{m} (E/l_i^2)\gamma_i(\gamma_i^T \tilde{t}_i \tilde{u}) = 0.$$

Therefore,

$$K(\tilde{t})\tilde{u} \neq 0 \Rightarrow \mu^2 > 0.$$

We use this in condition (jj) and introduce the vector $\bar{u}$ as in the theorem. Equivalently to condition (jj), we then have the following condition:

$$\text{(jj)} \quad \bar{u}^T K_i \bar{u} \leq 2/V, \qquad i = 1, \ldots, m,$$
$$\tilde{t}_i(\bar{u}^T K_i \bar{u} - 2/V) = 0, \qquad i = 1, \ldots, m.$$

We now take into account the structure of $K_i$, i.e.,

$$K_i = (E/l_i^2)\gamma_i \gamma_i^T.$$

Then, for $i = 1, \ldots, m$,

$$\bar{u}^T K_i \bar{u} \leq 2/V \Leftrightarrow (\gamma_i^T \bar{u})^2 (E/l_i^2) \leq 2/V \Leftrightarrow \begin{cases} \gamma_i^T \bar{u} \leq l_i \sqrt{2/EV}, \\ -\gamma_i^T \bar{u} \leq l_i \sqrt{2/EV}, \end{cases}$$

which are conditions (37). The second condition in (jj) now implies

$$\mu \tilde{t}_i (\gamma_i^T \bar{u} - l_i \sqrt{2/EV})(\gamma_i^T \bar{u} + l_i \sqrt{2/EV}) = 0, \qquad \forall i. \tag{55}$$

Define the vectors $\omega^+ = \{\omega_i^+\}$ and $\omega^- = \{\omega_i^-\}$ according to, for $i = 1, \ldots, m$,

$$\omega_i^+ := \begin{cases} \mu \tilde{t}_i, & \text{if } \gamma_i^T \bar{u} = \sqrt{2/EV} l_i, \\ 0, & \text{otherwise,} \end{cases}$$

$$\omega_i^- := \begin{cases} \mu \tilde{t}_i, & \text{if } \gamma_i^T \bar{u} = -\sqrt{2/EV} l_i, \\ 0, & \text{otherwise.} \end{cases}$$

By (55), it can be verified that these vectors possess the following properties:

$$\omega_i^+(\gamma_i^T\bar{u} - l_i\sqrt{2/EV}) = 0, \tag{56}$$

$$\omega_i^-(\gamma_i^T\bar{u} + l_i\sqrt{2/EV}) = 0, \tag{57}$$

$$\tilde{t} = \mu^{-1}(\omega^- + \omega^+), \tag{58}$$

$$\omega^- \geq 0, \qquad \omega^+ \geq 0.$$

The vectors $\rho^-$ and $\rho^+$ are now related to $\omega^+$ and $\omega^-$ as in Theorem 6.1; therefore, (56) and (57) imply (40) and (41). It is also immediate that condition (j), with a rescaling by $\mu > 0$, implies (36), (38), and (39).

What is now left to show is that the second term in (35) equals $K(\tilde{t})\tilde{u}$. Using (56) through (58) and the definition of $\bar{u}$, one finds that

$$K(\tilde{t})\tilde{u} = \sum_{i=1}^{m} \tilde{t}_i K_i \tilde{u} = \sum_{i=1}^{m} (\omega_i^+ + \omega_i^-)(E/l_i^2)\gamma_i\gamma_i^T\bar{u}$$

$$= \sum_{i=1}^{m} \sqrt{2E/V}(\omega_i^+ - \omega_i^-)(\gamma_i/l_i),$$

from which

$$K(\tilde{t})\tilde{u} = B^T(\rho^+ - \rho^-)$$

follows, by taking the relation between the $\rho$'s and the $\omega$'s into account. □

**Theorem 6.3.** Suppose that $(\tilde{t}, \tilde{s}, \tilde{\Lambda})$ solves problem $(\mathrm{LP})_\mathrm{L}$. Define

$$\tilde{\rho}_i^+ := \max\{0, \tilde{s}_i\}, \qquad \tilde{\rho}_i^- := \max\{0, -\tilde{s}_i\}.$$

Then, the triplet $(\tilde{\rho}^+, \tilde{\rho}^-, \tilde{\Lambda})$ solves problem $(\mathrm{LP})_\mathrm{d}$. Conversely, suppose that the triplet $(\tilde{\rho}^+, \tilde{\rho}^-, \tilde{\Lambda})$ solves problem $(\mathrm{LP})_\mathrm{d}$. Define

$$\tilde{s} := \tilde{\rho}^+ - \tilde{\rho}^-, \qquad \tilde{t}_i := (l_i/\sigma_0)(\tilde{\rho}_i^+ + \tilde{\rho}_i^-).$$

Then, the triplet $(\tilde{t}, \tilde{s}, \tilde{\Lambda})$ solves problem $(\mathrm{LP})_\mathrm{L}$.

**Proof.** Suppose that the triplet $(\tilde{t}, \tilde{s}, \tilde{\Lambda})$ solves problem $(\mathrm{LP})_\mathrm{L}$. Set

$$\tilde{\rho}_i^+ := \max\{0, \tilde{s}_i\}, \qquad \tilde{\rho}_i^- := \max\{0, -\tilde{s}_i\}.$$

Now, the triplet $(\tilde{\rho}^+, \tilde{\rho}^-, \tilde{\Lambda})$ is feasible in problem $(\mathrm{LP})_\mathrm{d}$, since $\tilde{s} = \tilde{\rho}^+ - \tilde{\rho}^-$. Take any feasible triplet $(\rho^+, \rho^-, \Lambda)$ in $(\mathrm{LP})_\mathrm{d}$ and

$$t_i := \sigma_0^{-1} l_i(\rho_i^+ + \rho_i^-), \qquad s_i = \rho_i^+ - \rho_i^-.$$

By inspection,

$$B^T s = B^T(\rho^+ - \rho^-) = f - W;$$

since $\rho_i^+$ and $\rho_i^-$ are nonnegative, we have

$$-\sigma_0 t_i \leq -\rho_i^- l_i \leq s_i l_i \leq \rho_i^+ l_i \leq \sigma_0 t_i,$$

so the triplet $(t, s, \Lambda)$ constructed this way is feasible in problem $(\mathrm{LP})_\mathrm{L}$. Therefore,

$$\sum_{i=1}^m \tilde{t}_i \leq \sum_{i=1}^m \sigma_0^{-1} l_i(\rho_i^+ + \rho_i^-). \tag{59}$$

Since the $t_i$'s are minimized in problem $(\mathrm{LP})_\mathrm{L}$, it follows that the upper or lower bounds in the inequality constraints are attained for $\tilde{t}$. Hence,

$$\sigma_0^{-1} l_i |\tilde{s}_i| = \tilde{t}_i, \qquad i = 1, \ldots, m. \tag{60}$$

Since

$$|\tilde{s}_i| = \tilde{\rho}_i^+ + \tilde{\rho}_i^-,$$

(60) in (59) yields that the triplet $(\tilde{\rho}^+, \tilde{\rho}^-, \tilde{\Lambda})$ solves problem $(\mathrm{LP})_\mathrm{d}$.

The converse will be shown in a very similar way: Suppose that the triplet $(\tilde{\rho}^+, \tilde{\rho}^-, \tilde{\Lambda})$ solves problem $(\mathrm{LP})_\mathrm{d}$. As above, it is realized that the triplet $(\tilde{t}, \tilde{s}, \tilde{\Lambda})$ is feasible in problem $(\mathrm{LP})_\mathrm{L}$,

$$\tilde{t}_i := \sigma_0^{-1} l_i(\tilde{\rho}^+ + \tilde{\rho}^-), \qquad \tilde{s} := \tilde{\rho}^+ - \tilde{\rho}^-.$$

For arbitrary $(t, s, \Lambda)$ feasible in problem $(\mathrm{LP})_\mathrm{L}$, the triplet $(\rho^+, \rho^-, \Lambda)$ defined by

$$\rho_i^+ := \max\{0, s_i\}, \qquad \rho_i^- := \max\{0, -s_i\}$$

is feasible in problem $(\mathrm{LP})_\mathrm{d}$, and hence

$$\sum_{i=1}^m k l_i(\tilde{\rho}_i^+ + \tilde{\rho}_i^-) \leq \sum_{i=1}^m k l_i(\rho_i^+ + \rho_i^-)$$

$$\Rightarrow \sum_{i=1}^m k\sigma_0 \tilde{t}_i \leq \sum_{i=1}^m k l_i |s_i| \leq \sum_{i=1}^m k t_i \sigma_0$$

$$\Rightarrow \sum_{i=1}^m \tilde{t}_i \leq \sum_{i=1}^m t_i. \qquad \square$$

**Corollary 6.2.** Between the optimal values of problems (LP) and $(\mathrm{LP})_\mathrm{L}$, the following relation holds:

$$\mathrm{opt(LP)} = k\sigma_0\, \mathrm{opt(LP)}_\mathrm{L}. \tag{61}$$

**Proof.** Let $\tilde{\rho}^+$, $\tilde{\rho}^-$, $\tilde{t}_i$ be solutions as described in Theorem 6.3. Then,

$$\begin{aligned}\text{opt}(\text{LP})_\text{d} &= \sum_{i=1}^{m} kl_i(\tilde{\rho}_i^+ + \tilde{\rho}_i^-) = \sum_{i=1}^{m} k\sigma_0 \tilde{t}_i \\ &= k\sigma_0 \sum_{i=1}^{m} \tilde{t}_i = k\sigma_0 \,\text{opt}(\text{LP})_\text{L}.\end{aligned}$$

Indeed,

$$\text{opt}(\text{LP}) = \text{opt}(\text{LP})_\text{d},$$

and (61) follows. □

Apparently, the solutions $\tilde{t}$ obtained from problem $(\text{LP})_\text{L}$ give the optimal topology of the truss for both the plastic limit design problem and the maximum stiffness problem ($\mathscr{C}$). Identifying the matrices $B$ and $C$ from the structure under study and $f$, it is only a question of using an LP solver, capable of determining dual variables/multipliers, to solve problem $(\text{LP})_\text{d}$, or $(\text{LP})_\text{L}$ if preferable, and by a simple rescaling obtain solutions to problems ($\mathscr{C}$) and $(\text{LP})_\text{L}$. This will be done in Section 9.

## 7. Existence of Solutions

The details about the relations between the major formulations are dealt with at this stage, and we will turn to the question of existence of solutions. Assumption (A2) will ensure the nonemptiness of the feasible sets, and this in turn will be sufficient for the existence.

First, Assumption (A2) is brought into a more practicable form.

**Lemma 7.1.** Assumption (A2) holds if and only if there exists a triplet $(\rho^+, \rho^-, \Lambda) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ such that

$$\Lambda W + B^T(\rho^+ - \rho^-) = f, \tag{62}$$

with $\Lambda \geq 0$, $\rho^+ \geq 0$, $\rho^- \geq 0$.

**Proof.**

(If) Take arbitrarily $u$ such that $W^T u \leq 0$ and $Bu = 0$. Multiplying the transpose of (62) with this $u$ gives

$$\Lambda W^T u + (\rho^+ - \rho^-)^T Bu = f^T u,$$

and hence $f^T u \leq 0$.

(Only If) By the Farkas lemma (cf. Ref. 15), we have that either

$$\begin{bmatrix} W \\ B^T \\ -B^T \end{bmatrix} \begin{bmatrix} \Lambda \\ \rho^+ \\ \rho^- \end{bmatrix} = f, \qquad \begin{matrix} \Lambda \geq 0 \\ \rho^+ \geq 0, \\ \rho^- \geq 0 \end{matrix} \tag{63}$$

has a solution, or

$$\begin{bmatrix} W^T \\ B \\ -B \end{bmatrix} u \leq 0, \qquad f^T u > 0 \tag{64}$$

does. Given Assumption (A2), we can see that (64) cannot hold for any $u \in \mathbb{R}^n$; therefore, (63) necessarily holds for some triplet $(\Lambda, \rho^+, \rho^-)$. Indeed, (63) is (62). □

Now, we are ready to show the existence of solutions of problems (LP) and $(\mathrm{LP})_\mathrm{d}$; after that, we will obtain the existence of solutions to all of the interesting optimization problems stated so far, in a rather immediate corollary.

**Theorem 7.1.** There exist solutions to problems (LP) and $(\mathrm{LP})_\mathrm{d}$ provided Assumption (A2) holds.

**Proof.** By duality theory for LP problems, it is sufficient to show that the feasible sets for problems (LP) and $(\mathrm{LP})_\mathrm{d}$ are both nonempty. Clearly, $0 \in \mathbb{R}^n$ belongs to the feasible set in problem (LP); problem $(\mathrm{LP})_\mathrm{d}$ has a nonempty feasible set due to Assumption (A2) and Lemma 7.1. □

**Corollary 7.1.** Problems $(\mathscr{C})$ and $(S\Pi)$ both have solutions whenever Assumptions (A1) and (A2) hold.

**Proof.** By Theorem 7.1, there are solutions to problems (LP) and $(\mathrm{LP})_\mathrm{d}$, and hence also to conditions (j) and (jj) according to Theorem 6.1. Now, the statement follows from Theorem 5.1 and 5.2. □

**Remark 7.1.** One might suggest to show the existence of solutions to problem $(S\Pi)$ above by the general theory, e.g., from Ekeland and Temam (Ref. 16), but because of the rather weak Assumption (A2), the potential energy function does not in general have the required coercivity in $u$.

## 8. Further Equivalence

Theorems 5.1 and 5.2 show that a vector $\tilde{t}$ of volumes that is part of a solution of problem $(S\Pi)$ is also part of a solution of problem $(\mathscr{C})$. Once we have the existence results of the previous section, it is possible to show the converse of this result. To that end, define the function $\varphi$ as

$$\mathscr{T} \ni t \mapsto \varphi(t) = \inf_{u \in \mathscr{U}} \Pi_t(u) \in \mathbb{R} \cup \{-\infty\}, \tag{65}$$

and consider the following related problem:

$(\varphi)$ Find $\tilde{t} \in \mathscr{T}$ such that

$$\varphi(\tilde{t}) \geq \varphi(t), \qquad \forall t \in \mathscr{T}.$$

From some simple properties of saddle points, we have the following lemma.

**Lemma 8.1.** Suppose that Assumptions (A1) and (A2) are satisfied. Then, any $\tilde{t} \in \mathscr{T}$ that solves problem $(\varphi)$ is part of a saddle point of problem $(S\Pi)$. Conversely, given any saddlepoint $(\tilde{u}, \tilde{t}) \in \mathscr{U} \times \mathscr{T}$, we have that $\tilde{t}$ solves problem $(\varphi)$. Furthermore, the optimum value of problem $(\varphi)$ coincides with the saddle-point value.

**Theorem 8.1.** Suppose that Assumptions (A1) and (A2) hold. If $(\tilde{u}, \tilde{\Lambda}, \tilde{t}) \in \mathscr{S}$ solves problem $(\mathscr{C})$, then $\tilde{t} \in \mathscr{T}$ is part of a solution of problem $(S\Pi)$.

**Proof.** By the first statement of Lemma 8.1, what we have to conclude is that

$$\text{opt}(\varphi) = \varphi(\tilde{t}).$$

From Lemma 3.2, we have

$$\text{opt}(\mathscr{C}) = (1/2) f^T \tilde{u} = -\Pi_{\tilde{t}}(\tilde{u}) = -\varphi(\tilde{t}). \tag{66}$$

Moreover, by Corollary 7.1, there exists $(u_s, t_s)$ that solves problem $(S\Pi)$; recalling the definition of problem $(S\Pi)$, the last assertion of Lemma 8.1 yields

$$\text{opt}(\varphi) = \varphi(t_s) = \Pi_{t_s}(u_s). \tag{67}$$

Theorems 5.1 and 5.2 give the existence of some $\Lambda_s \geq 0$ such that the triplet $(u_s, \Lambda_s, t_s)$ solves problem $(\mathscr{C})$, and hence

$$\text{opt}(\mathscr{C}) = (1/2) f^T u_s. \tag{68}$$

Rewriting (68) according to Lemma 3.2 yields

$$(1/2) f^T u_s = -\Pi_{t_s}(u_s). \tag{69}$$

Comparison between (67) through (69) results in

$$\text{opt}(\varphi) = -\text{opt}(\mathscr{C}),$$

and hence by (66),

$$\text{opt}(\varphi) = \varphi(\tilde{t}).$$

□

## 9. Numerical Results

In this section, we will obtain structures of maximum stiffness and constant contact force distribution by solving linear programming problems, as indicated in Theorem 6.1. It will be shown how the condition of constant contact forces influences the topology.

The implementation was done in FORTRAN 77 and executed on a SUN SPARC-ELC workstation. As solution procedure for the linear programming problem, we have used the SIMPLEX code in the XMP library; see Marsten (Ref. 17). The LP models are typically sparse: the number of nonzero elements in the constraint matrix is limited between 0.43% and 1.90%.

We mention that it is possible to generalize to cases where the truss structure has $z$ disjoint regions of potential contact nodes, in each of which the distribution of contact forces is constant. For details, see Ref. 18. Instead of solving problems (LP) and $(\text{LP})_d$ in the previous form, one solves problem (LP) with $W$ replaced by $W_i = C_i^T 1 n_i$, corresponding to the $i$th contact region, and solves problem $(\text{LP})_d$ with $W\Lambda$ replaced by $\sum_{i=1}^{z} W_i \Lambda_i$. As a special case of the generalization, namely when $r = z$, one can obtain the optimal topology in the case where the initial gap is not a design variable but fixed to zero at all potential contact nodes. This is done in test example B2.

We have chosen two bridge-type plane structures; see Table 1. The structure is unilaterally supported from below at the two ends; between these, there are downward forces acting on the lower part. The definition of the forces is as follows: black arrows are external loads; white arrows are contact forces.

Table 1. Data for the test examples.

| Test example | Number of nodes | Number of bars | Total number of contact nodes | Number of regions of constant contact forces |
|---|---|---|---|---|
| B1 | 231 | 12806 | 14 | 1 |
| B2 | 231 | 12806 | 14 | 14 |

Table 2. Computational results.

| Test example | Computing time (s) | Compliance |
|---|---|---|
| B1 | 6144 | $2.55 \cdot 10^{-6}$ |
| B2 | 8763 | $1.72 \cdot 10^{-6}$ |

The structures do not have any prescribed displacements; in fact, both test examples B1 and B2 have singular stiffness matrices, whereas Assumptions (A1) and (A2) are satisfied.

In Fig. 4, the structure of the test examples are shown. Here, we include only a subset of the number of bars in the ground structure. To include all bars will, due to the great number, just give a black box. In order to reduce the number of bars in the ground structure, longer bars that overlap shorter ones are removed. This does not have any consequence concerning loss in topology information.

In Table 2, we give the computing time and the compliance value of the optimal topology for the test examples. Recall that the value of the

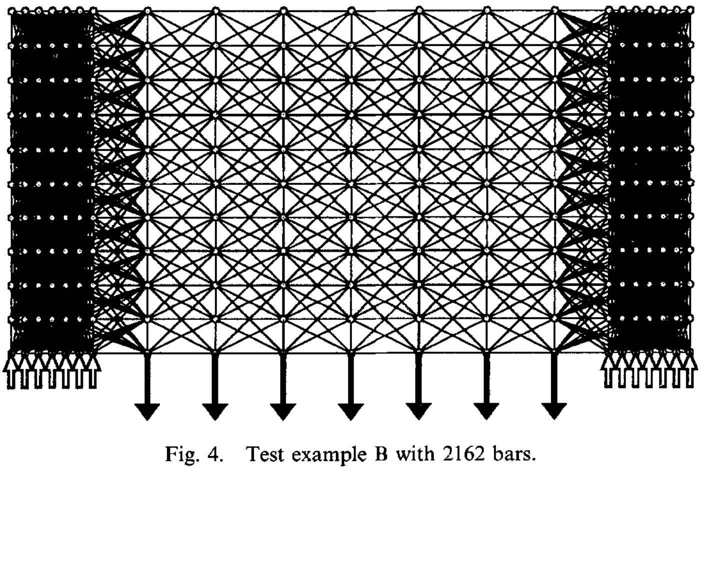

Fig. 4. Test example B with 2162 bars.

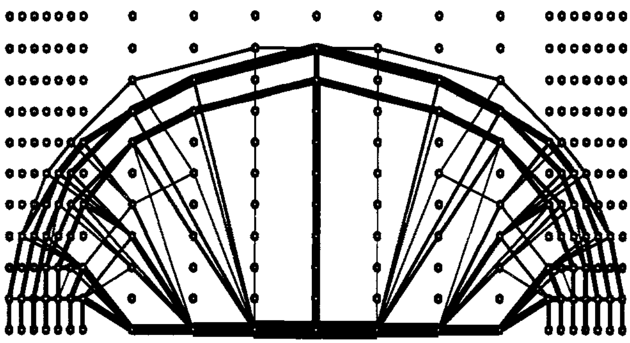

Fig. 5. Optimal topology for test example B1.

compliance is related to the obtained objective value of the LP problem, by the simple expression given in Corollary 6.1.

The optimal solution for test example B1, which is depicted in Fig. 5, has 48% higher compliance than for test example B2, which is depicted in Fig. 6, but the maximum contact force is seven times larger in test example B2.

In test example B2, the contact forces are not zero ($\Lambda \neq 0$); since $\Lambda W^T u = 0$, this means that $W^T u = 0$; hence, we can take $g := Cu$ as pointed out in the proof of Lemma 3.1. In Fig. 7, we give on the $y$-axis the initial gaps and on the $x$-axis the nodes where contact forces act for test example B1. The curve describing the initial gaps is slightly concave.

A special feature of the test examples is that they can be very degenerate i.e., many basic variables are zero. As an example, we mention a particular one (among several additional ones in Ref. 18 not present here) where the number of nonzero variables in problem $(LP)_d$ is 14, which means that the remaining 228 basic variables are zero.

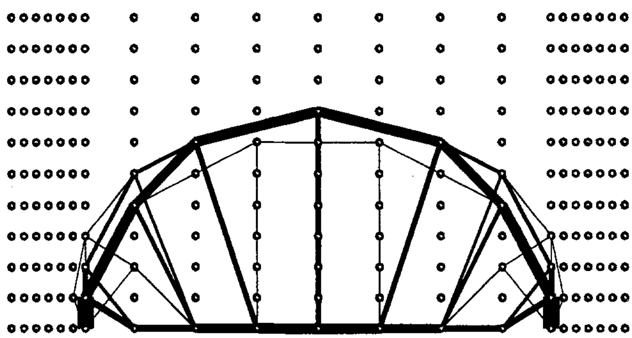

Fig. 6. Optimal topology for test example B2.

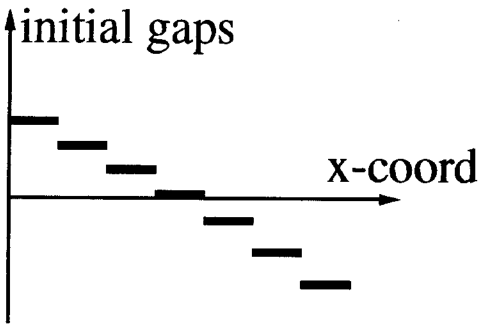


Fig. 7. Initial gaps for the left-end contact nodes for test example B1.

Due to the degeneracy, it is necessary to use high accuracy in the solution procedure. Another feature is that many Phase I iterations (i.e., finding a feasible solution) of the SIMPLEX method are required. This amounts to about one-third of the overall number of iterations.

We tested a larger Michell-type structure with 225 nodal points, which give 450 equalities, and it took much more time to solve the overall LP problem. Therefore, it would be interesting to see how another method for linear programming works, e.g., an interior-point method. This is because such methods are regarded to work more efficiently for degenerate problems. In Ref. 8, an interior-point algorithm was considered, but applied to much smaller problems.

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