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Minimax Theorems and Cone Saddle Points of Uniformly Same-Order Vector-Valued Functions[1]

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Abstract. This paper is concerned with minimax theorems in vector-valued optimization. A class of vector-valued functions which includes separated functions $f(x, y) = u(x) + v(y)$ as its proper subset is introduced. Minimax theorems and cone saddle-point theorems for this class of functions are investigated.

Key Words. Uniformly same-order functions, vector-valued functions, sequentially compact sets, cone saddle points, minimax theorems.

1. Introduction

Minimax problems for real-valued functions $f: X_0 \times Y_0 \to R$ have been investigated extensively. It is well known that the equality

$$\inf_{Y_0}\sup_{X_0} f(x, y) = \sup_{X_0}\inf_{Y_0} f(x, y)$$

holds under suitable conditions (Refs. 1-2). In recent years, some authors have studied minimax theorems for vector-valued functions (Refs. 3-8). In Ref. 3, Nieuwenhuis first proved that

$$\min_{Y_0}\max_{X_0}{}_W f(x, y) \subset \max_{X_0}\min_{Y_0}{}_W f(x, y) - K,$$

$$\max_{X_0}\min_{Y_0}{}_W f(x, y) \subset \min_{Y_0}\max_{X_0}{}_W f(x, y) + K,$$

where the vector-valued function $f(x, y)$ is limited to be of form

$$f(x, y) = x + y.$$

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Then, in Ref. 4, Tanaka proved the above relationships for general separated vector-valued functions

$$f(x, y) = u(x) + v(y).$$

In this paper, we give a class of more general vector-valued functions, which includes that of separated functions as its proper subset, and establish relationships similar to the above. Also, some results on cone saddle points and values are established without hypotheses of convexity type; therefore, the results of Ref. 4 are improved.

2. Uniformly Same-Order Functions

Throughout this paper, X, Y, Z denote real normed linear spaces; K denotes the pointed, closed convex cone in Z. We always assume that K^0 (interior of K) $\neq \emptyset$.

Let Z_0 be a nonempty subset of Z, $\hat{z} \in Z_0$. If

$$(Z_0 - \hat{z}) \cap K = \{\theta\}, \tag{1}$$

then $\hat{z}$ is said to be a K-maximal point of Z_0. The set of all K-maximal points of Z_0 is denoted by $\max Z_0$. If

$$(\hat{z} - Z_0) \cap K = \{\theta\}, \tag{2}$$

then $\hat{z}$ is said to be a K-minimal point of Z_0. The set of all K-minimal points of Z_0 is denoted by $\min Z_0$. If

$$(Z_0 - \hat{z}) \cap K^0 = \emptyset, \tag{3}$$

then $\hat{z}$ is said to be a weak K-maximal point of Z_0. The set of all weak K-maximal points of Z_0 is denoted by $\max_W Z_0$. If

$$(\hat{z} - Z_0) \cap K^0 = \emptyset, \tag{4}$$

then $\hat{z}$ is said to be a weak K-minimal point of Z_0. The set of all weak K-minimal points of Z_0 is denoted by $\min_W Z_0$ (Ref. 6).

Lemma 2.1. If Z_0 is a nonempty compact set, then $\max Z_0 \neq \emptyset$ and $\min Z_0 \neq \emptyset$.

Proof. The proof can be found in Ref. 6. □

Lemma 2.2. If Z_0 is a nonempty compact set, then

$$Z_0 \subset \max Z_0 - K, \qquad Z_0 \subset \min Z_0 + K.$$

Proof. We only prove the first inclusion relationship; the second can be proved similarly.

Let $z \in Z_0$. If $z \in \max Z_0$, then $z \in \max Z_0 - K$. If $z \notin \max Z_0$, let

$$E_z := \{z' \in Z_0 \mid z' - z \in K\}.$$

It is clear that $E_z \neq \varnothing$ and E_z is a closed set; therefore, E_z is a nonempty compact subset of Z_0. Let $z^0 \in \max E_z$; then, $z \in z^0 - K$. We claim that $z^0 \in \max Z_0$. Indeed, if $z^0 \notin \max Z_0$, then there exists $z' \in Z_0$ such that $z' - z^0 \in K/\{\theta\}$, and so $z' - z \in K$, that is, $z' \in E_z$. This contradicts $z^0 \in \max E_z$. Therefore, $z \in \max Z_0 - K$. □

Let $X_0 \subset X$, $Y_0 \subset Y$, and let $f: X_0 \times Y_0 \to Z$ be a vector-valued function. Now, we introduce a class of vector-valued functions.

Definition 2.1. A vector-valued function $f(x, y)$ is said to be $K(K^0)$-uniformly same-order on X_0 with respect to $(y', y'') \in Y_0 \times Y_0$, if

$$f(x, y') - f(x, y'') \in K/\{\theta\} \ (K^0)$$

for all $x \in X_0$ when there exists $x_0 \in X_0$ such that

$$f(x_0, y') - f(x_0, y'') \in K/\{\theta\} \ (K^0).$$

Moreover, if f is $K(K^0)$-uniformly same-order on X_0 with respect to any $(y', y'') \in Y_0 \times Y_0$, then f is said to be $K(K^0)$-uniformly same-order on X_0.

The definition that $f(x, y)$ is said to be $K(K^0)$-uniformly same-order on Y_0 is similar. If $f(x, y)$ is both $K(K^0)$-uniformly same-order on X_0 and Y_0, then $f(x, y)$ is said to be $K(K^0)$-uniformly same-order on $X_0 \times Y_0$.

It is easy to see that the separated vector-valued function $f(x, y) = u(x) + v(y)$ must be $K(K^0)$-uniformly same-order on $X_0 \times Y_0$. The following example illustrates that the set of $K(K^0)$-uniformly same-order vector-valued functions includes some unseparated vector-valued functions.

Example 2.1. Let

$$X = Y = Z = R^2,$$

$$X_0 = \{(x_1, x_2) \mid 1 \leq x_i \leq 2 \ (i = 1, 2)\},$$

$$Y_0 = \{(y_1, y_2) \mid 1 \leq y_i \leq 2 \ (i = 1, 2)\},$$

$$K = \{(z_1, z_2) \mid z_1 \geq 0, z_2 \geq 0\},$$

$$f(x, y) = (x_1 y_1, x_2 y_2).$$

It is easy to show that the $f(x, y)$ is $K(K^0)$-uniformly same-order on $X_0 \times Y_0$; however, it is an unseparated vector-valued function.

Lemma 2.3. Let $f: X_0 \times Y_0 \to Z$ be a vector-valued function. Then:

(i) if $f(x, y)$ is K-uniformly same-order on Y_0, $f(\hat{x}, \hat{y}) \in \max f(X_0, \hat{y})$ $[\min f(X_0, \hat{y})]$ implies that $f(\hat{x}, y) \in \max f(X_0, y)$ $[\min f(X_0, y)]$, for all $y \in Y_0$;

(ii) if $f(x, y)$ is K^0-uniformly same-order on Y_0, $f(\hat{x}, \hat{y}) \in \max_W f(X_0, \hat{y})$ $[\min_W f(X_0, \hat{y})]$ implies that $f(\hat{x}, y) \in \max_W f(X_0, y)$ $[\min_W f(X_0, y)]$, for all $y \in Y_0$.

Proof.

(i) Let $f(\hat{x}, \hat{y}) \in \max f(X_0, \hat{y})$. If there exists $y' \in Y_0$, such that $f(\hat{x}, y') \notin \max f(X_0, y')$, then by (1), there exists $x^0 \in X_0$ such that

$$f(x^0, y') - f(\hat{x}, y') \in K/\{\theta\}.$$

Since $f(x, y)$ is K-uniformly same-order on Y_0, we have

$$f(x^0, \hat{y}) - f(\hat{x}, \hat{y}) \in K/\{\theta\},$$

which contradicts $f(\hat{x}, \hat{y}) \in \max f(X_0, \hat{y})$.

(ii) This can be proved similarly. □

3. Minimax Theorems

Let $X_0 \subset X$ and $Y_0 \subset Y$ be nonempty sets, and let the vector-valued function $f(x, y)$ be continuous on $X_0 \times Y_0$. It can be easily proved that $f(X_0, y)$ and $f(x, Y_0)$ are both compact subsets of Z for any $(x, y) \in X_0 \times Y_0$. So, by Lemma 2.1, the sets

$$h(y) := \max_W f(X_0, y), \tag{5}$$

$$g(x) := \min_W f(x, Y_0) \tag{6}$$

are both nonempty for any $(x, y) \in X_0 \times Y_0$. Thus, h and g form two set-valued maps from Y_0 to Z and from X_0 to Z, respectively.

Now, we introduce a notion concerning set-valued maps and give several propositions.

Definition 3.1. A set-valued map h is said to be sequentially compact at $\hat{y} \in Y_0$ if, for any sequence $\{y_n\} \subset Y_0$ with $y_n \to \hat{y}$ and the sequence $\{z_n\}$ with $z_n \in h(y_n)$, there exists a subsequence $\{z_j\}$ of $\{z_n\}$ such that $z_j \to \hat{z}$ and $\hat{z} \in h(\hat{y})$. For any $y \in Y_0$, if h is sequentially compact at y, then h is said to be sequentially compact on Y_0.

Proposition 3.1. The graph of a set-valued map h on Y_0, which is denoted by

$$\mathrm{graph}_{Y_0} h := \{(y, z) \mid z \in h(y), y \in Y_0\},$$

is a compact set in the space $Y \times Z$ if and only if the set Y_0 is compact and the set-valued map h is sequentially compact on Y_0.

Proof. This follows directly from Definition 3.1 and the definition of $\mathrm{graph}_{Y_0} h$. □

Proposition 3.2. If Y_0 is a compact set and h is a sequentially compact set-valued map on Y_0, then

$$h(Y_0) := \bigcup_{y \in Y_0} h(y)$$

is a compact set.

Proof. Let the sequence $\{z_n\} \subset h(Y_0)$; then, there exists a sequence $\{y_n\} \subset Y_0$ such that $z_n \in h(y_n)$. Since Y_0 is compact, we may assume, without loss of generality, that $y_n \to \hat{y} \in Y_0$. By the sequential compactness of h on Y_0, there exists a subsequence $\{z_j\}$ of $\{z_n\}$ such that $z_j \to \hat{z} \in h(\hat{y}) \subset h(Y_0)$. That is, $h(Y_0)$ is compact. □

Now, we reconsider the set-valued maps in (5) and (6).

Lemma 3.1. If $X_0 \subset X$ and $Y_0 \subset y$ are nonempty compact sets, and if $f: X_0 \times Y_0 \to Z$ is a continuous vector-valued function, then:

(i) the set-valued maps $h(y)$ and $g(x)$, which are defined by (5) and (6), are sequentially compact on Y_0 and X_0, respectively;
(ii) $h(Y_0)$ and $g(X_0)$ are compact sets.

Proof.

(i) Let $\hat{x} \in X_0$; then, $g(\hat{x}) \neq \emptyset$. Let $(x_n\} \subset X_0$ be any sequence with $x_n \to \hat{x}$, and let $z_n = f(x_n, y_n) \in g(x_n)$, with $y_n \in Y_0$ for all $n \geq 1$. Since Y_0 is compact, there exists a subsequence $\{y_j\}$ of $\{y_n\}$ such that $y_j \to \hat{y} \in Y_0$. From the continuity of f, we have $z_j = f(x_j, y_j) \to \hat{z} = f(\hat{x}, \hat{y})$. It can be shown that $\hat{z} \in g(\hat{x})$. In fact, if $\hat{z} \notin g(\hat{x})$, then, by (6), (4), and $\hat{z} = f(\hat{x}, \hat{y}) \in f(\hat{x}, Y_0)$, there exists $y_0 \in Y_0$ such that

$$f(\hat{x}, \hat{y}) - f(\hat{x}, y_0) = k^0 \in K^0.$$

Hence, from

$$f(x_j, y_j) - f(x_j, y_0) \to f(\hat{x}, \hat{y}) - f(\hat{x}, y_0),$$

we have

$$f(x_j, y_j) - f(x_j, y_0) \in K^0, \qquad \text{for } j \text{ large enough.}$$

This implies that

$$f(x_j, y_j) \notin g(x_j), \qquad \text{for } j \text{ large enough.}$$

This leads to a contradiction. Thus, we have proved that g is sequentially compact at $\hat{x}$. Therefore, g is sequentially compact on X_0.

Similarly, we can prove that $h(y)$ is sequentially compact on Y_0.

(ii) This follows directly from (i) and Proposition 3.2. □

By Lemma 3.1 and Lemma 2.1, we get immediately the following theorem.

Theorem 3.1. Let X_0 and Y_0 be nonempty compact sets, and let $f(x, y)$ be a continuous vector-valued function on $X_0 \times Y_0$. Then,

$$\min_{Y_0}\max_{X_0}{}_W f(x, y) := \min h(Y_0) \neq \emptyset,$$

$$\max_{X_0}\min_{Y_0}{}_W f(x, y) := \max g(X_0) \neq \emptyset.$$

Lemma 3.2. Let X_0 and Y_0 be nonempty compact sets, and let $f(x, y)$ be a continuous vector-valued function on $X_0 \times Y_0$. Then:

(i) if $f(x, y)$ is K^0-uniformly same-order on Y_0,

$$\min_{Y_0}{}_W \max_{X_0}{}_W f(x, y) \subset g(X_0) \cap h(Y_0);$$

(ii) if $f(x, y)$ is K^0-uniformly same-order on X_0,

$$\max_{X_0}{}_W \min_{Y_0}{}_W f(x, y) \subset g(X_0) \cap h(Y_0).$$

Proof.

(i) We first have

$$\min_{Y_0}{}_W \max_{X_0}{}_W f(x, y) \neq \emptyset,$$

from Theorem 3.1. Let

$$\hat{z} \in \min_{Y_0}{}_W \max_{X_0}{}_W f(x, y).$$

Then, there exists $\hat{y}, \hat{x}$ such that $\hat{z} \in \max_W f(X_0, \hat{y})$ and $\hat{z} = f(\hat{x}, \hat{y})$. Thus, $f(\hat{x}, \hat{y}) \in h(\hat{y})$ by (5). We further claim that $f(\hat{x}, \hat{y}) \in \min_W f(\hat{x}, Y_0) = g(\hat{x})$. Indeed, if $f(\hat{x}, \hat{y}) \notin g(\hat{x})$, there exists $f(\hat{x}, y_0)$ such that

$$f(\hat{x}, \hat{y}) - f(\hat{x}, y_0) \in K^0. \tag{7}$$

Since $f(\hat{x}, \hat{y}) \in \max_W f(X_0, \hat{y})$, $f(\hat{x}, y_0) \in \max_W f(X_0, y_0)$ by Lemma 2.3(i). This implies that

$$f(\hat{x}, \hat{y}) \notin \min_{Y_0} {}_W \max_{X_0} {}_W f(x, y),$$

which leads to a contradiction.

(ii) This can be proved similarly. □

We now show one of the main results of this paper.

Theorem 3.2. Let X_0 and Y_0 be nonempty compact sets, and let $f(x, y)$ be a continuous vector-valued function on $X_0 \times Y_0$. Then:

(i) if $f(x, y)$ is K^0-uniformly same-order on Y_0,

$$\min_{Y_0} {}_W \max_{X_0} {}_W f(x, y) \subset \max_{X_0} {}_W \min_{Y_0} {}_W f(x, y) - K;$$

(ii) if $f(x, y)$ is K^0-uniformly same-order on X_0,

$$\max_{X_0} {}_W \min_{Y_0} {}_W f(x, y) \subset \min_{Y_0} {}_W \max_{X_0} {}_W f(x, y) + K.$$

Proof.

(i) By Lemma 3.2, we have

$$\min_{Y_0} {}_W \max_{X_0} {}_W f(x, y) \subset g(X_0).$$

Since $g(X_0)$ is a compact set by Lemma 3.1(ii), we have

$$\min_{Y_0} {}_W \max_{X_0} {}_W f(x, y) \subset \max_{X_0} {}_W \min_{Y_0} {}_W f(x, y) - K,$$

by Lemma 2.2.

(ii) This can be proved similarly. □

The following example shows that the assumption of $f(x, y)$ as uniformly same-order is important in Theorem 3.2.

Example 3.1. Let

$$X = Y = R^1, \qquad Z = R^2,$$

$$X_0 = \{x \mid 0 \le x \le 2\}, \qquad Y_0 = \{y \mid -1 \le y \le 1\},$$

$$K = \{(z_1, z_2) \mid |z_1| \leq z_2/2\},$$

$$f(x, y) = (y, y^2 x).$$

It is easy to show that $f(x, y)$ is K^0-uniformly same-order on Y_0, but it is not K^0-uniformly same-order on X_0. We observe with a simple geometric analysis that

$$\min_{Y_0}{}_W \max_{X_0}{}_W f(x, y) = \{(y, 2y^2) \mid -1/2 \leq y \leq 1/2\},$$

$$\max_{X_0}{}_W \min_{Y_0}{}_W f(x, y) = \{(y, 2y^2) \mid -1/2 \leq y \leq 1/2\} \cup \{(y, y) \mid 1/2 < y \leq 1\} \cup \{(y, -y) \mid -1 \leq y \leq -1/2\}.$$

Hence,

$$\min_{Y_0}{}_W \max_{X_0}{}_W f(x, y) \subset \max_{X_0}{}_W \min_{Y_0}{}_W f(x, y),$$

but

$$\max_{X_0}{}_W \min_{Y_0}{}_W f(x, y) \not\subset \min_{Y_0}{}_W \max_{X_0}{}_W f(x, y) + K.$$

4. Cone Saddle Points

In this section, we establish the existence theorem for cone saddle points. The following definition of cone saddle point is equivalent to that in Ref. 9.

Definition 4.1. A point $(x_0, y_0) \in X_0 \times Y_0$ is said to be a K-saddle point of the vector-valued function $f(x, y)$ with respect to $X_0 \times Y_0$ if

$$f(x_0, y_0) \in \max f(X_0, y_0) \cap \min f(x_0, Y_0).$$

The set of all K-saddle points of $f(x, y)$ with respect to $X_0 \times Y_0$ is denoted by S.

The following definition of weak K-saddle point is from Ref. 4.

Definition 4.2. A point $(x_0, y_0) \in X_0 \times Y_0$ is said to be a weak K-saddle point of the vector-valued function $f(x, y)$ with respect to $X_0 \times Y_0$ if

$$f(x_0, y_0) \in \max{}_W f(X_0, y_0) \cap \min{}_W f(x_0, Y_0).$$

The set of all weak K-saddle points of $f(x, y)$ with resepct to $X_0 \times Y_0$ is denoted by S^W.

It is obvious that $S \subset S^W$.

Remark 4.1. Note that an existence theorem for a weak K-saddle point has actually been given in the proof of Lemma 3.2.

We further establish the existence theorem for K-saddle points. In Ref. 3, Nieuwenhuis proved that, if X_0 and Y_0 are nonempty convex compact sets, and if $f(x, y)$ is continuous on $X_0 \times Y_0$, is convex in x for every $y \in Y_0$, and is concave in y for every $x \in X_0$, then $f(x, y)$ has at least one saddle point on $X_0 \times Y_0$.

We prove that the conditions in Theorem 3.2 are sufficient to ensure the existence of the K-saddle points. To this end, we introduce the following symbols:

$$A := \{x \in X_0 \,|\, f(x, y) \in \max f(X_0, y), \text{ for all } y \in Y_0\},$$

$$B := \{y \in Y_0 \,|\, f(x, y) \in \min f(x, Y_0), \text{ for all } x \in X_0\},$$

$$A^W := \{x \in X_0 \,|\, f(x, y) \in \max_W f(X_0, y), \text{ for all } y \in Y_0\},$$

$$B^W := \{y \in Y_0 \,|\, f(x, y) \in \min_W f(x, Y_0), \text{ for all } x \in X_0\}.$$

Theorem 4.1. Let X_0 and Y_0 be nonempty compact sets, and let $f(x, y)$ be a continuous vector-valued function on $X_0 \times Y_0$. Then:

(i) if f is K-uniformly same-order on $X_0 \times Y_0$,

$$A \neq \varnothing, \qquad B \neq \varnothing, \qquad S = A \times B;$$

(ii) if f is K^0-uniformly same-order on $X_0 \times Y_0$,

A^W and B^W are both nonempty compact sets, $S^W = A^W \times B^W$.

Proof.

(i) For any $y_0 \in Y_0$, there exists $x_0 \in X_0$ such that $f(x_0, y_0) \in \max f(X_0, y_0)$. By Lemma 2.3(i), we have

$$f(x_0, y) \in \max f(X_0, y), \qquad \text{for all } y \in Y_0.$$

Therefore, $x_0 \in A$; that is, $A \neq \varnothing$. The proof of $B \neq \varnothing$ is analogous.

Now, we turn to the proof of $S = A \times B$. Let $(x_0, y_0) \in S$. Then,

$$f(x_0, y_0) \in \max f(X_0, y_0) \cap \min f(x_0, Y_0),$$

from Definition 4.1. It follows that

$$f(x_0, y) \in \max f(X_0, y), \qquad \text{for all } y \in Y_0,$$

by Lemma 2.3(i). Thus, $x_0 \in A$. Similarly, $y_0 \in B$. Therefore, $S \subset A \times B$. The converse inclusion relationship is clear. Hence, we have $S = A \times B$.

(ii) We can prove that $A^W \neq \emptyset$, $B^W \neq \emptyset$, and $S^W = A^W \times B^W$. It remains to show that A^W and B^W are compact.

Let $\{x_n\} \subset A^W$ with $x_n \to \hat{x}$. We take an arbitrary sequence $\{y_n\} \subset Y_0$ with $y_n \to \hat{y} \in Y_0$. From the definition of A^W and the continuity of $f(x, y)$, we have

$$z_n = f(x_n, y_n) \in h(y_n)$$

and

$$z_n \to \hat{z} = f(\hat{x}, \hat{y}).$$

Since the set-valued map h is sequentially compact at y by Lemma 3.1(i), $f(\hat{x}, \hat{y}) \in h(\hat{y})$. That is,

$$f(\hat{x}, \hat{y}) \in \max_W f(X_0, \hat{y}).$$

Thus, for any $y \in Y_0$, one has

$$f(\hat{x}, y) \in \max_W f(X_0, y)$$

by Lemma 2.3(ii); hence, $\hat{x} \in A^W$. Thus, A^W is a closed subset of the compact set X_0, and hence A^W is compact.

Similarly, we can prove that B^W is compact. □

Remark 4.2. Note that Theorem 4.1 also depicts the structures of the sets S and S^W.

The value $f(x_0, y_0)$, for which (x_0, y_0) is a (weak) K-saddle point of $f(x, y)$ on $X_0 \times Y_0$, is called the (weak) K-saddle value of $f(x, y)$ at (x_0, y_0).

We denote by V^W the set of all weak K-saddle values of $f(x, y)$ with respect to $X_0 \times Y_0$. That is,

$$V^W := \{f(x_0, y_0) \mid (x_0, y_0) \in S^W\}.$$

By Theorem 4.1, we have

$$V^W = f(A^W, B^W),$$

and V^W is compact under the conditions in Theorem 4.1.

Theorem 4.2. Let X_0 and Y_0 be nonempty compact sets, and let $f(x, y)$ be a continuous vector-valued function on $X_0 \times Y_0$. Then:

(i) if $f(x, y)$ is K^0-uniformly same-order on Y_0,

$$\min_{Y_0}{}_W \max_{X_0}{}_W f(x, y) = \min_W V^W,$$

$$\min_{Y_0}\max_{X_0}{}_W f(x, y) = \min V^W;$$

(ii) if $f(x, y)$ is K^0-uniformly same-order on X_0,

$$\max_{X_0}{}_W \min_{Y_0}{}_W f(x, y) = \max_W V^W,$$

$$\max_{X_0}\min_{Y_0}{}_W f(x, y) = \max V^W.$$

Proof.

(i) We only prove the first equation. Let $f(\hat{x}, \hat{y}) \in \min_W V^W$; then, $f(\hat{x}, \hat{y}) \in h(Y_0)$. If $f(\hat{x}, \hat{y}) \notin \min_W \max_W f(x, y)$, then there exists $f(x', y') \in h(Y_0)$ such that

$$f(\hat{x}, \hat{y}) - f(x', y') \in K^0. \tag{8}$$

Since $h(Y_0)$ is compact by Lemma 3.1(ii), we have

$$f(x', y') \in \min_{Y_0}{}_W \max_{X_0}{}_W f(x, y) + K,$$

by Lemma 2.2. That is,

$$f(x', y') = f(x_0, y_0) + k', \tag{9}$$

where

$$f(x_0, y_0) \in \min_{Y_0}{}_W \max_{X_0}{}_W f(x, y), \qquad k' \in K.$$

Thus, we have

$$f(\hat{x}, \hat{y}) - f(x_0, y_0) \in K^0,$$

from (8) and (9), and

$$f(x_0, y_0) \in V^W,$$

by Lemma 3.2(i). This contradicts $f(\hat{x}, \hat{y}) \in \min_W V^W$. Therefore,

$$\min_W V^W \subset \min_{Y_0}{}_W \max_{X_0}{}_W f(x, y).$$

Next, we show the converse inclusion relationship. Let

$$f(\hat{x}, \hat{y}) \in \min_{Y_0}{}_W \max_{X_0}{}_W f(x, y).$$

From Lemma 3.2, we have $f(\hat{x}, \hat{y}) \in V^W$. If $f(\hat{x}, \hat{y}) \notin \min_W V^W$, there exists $f(x_0, y_0) \in V^W$ such that

$$f(\hat{x}, \hat{y}) - f(x_0, y_0) \in K^0.$$

However,

$$f(x_0, y_0) \in \max_W f(X_0, y_0) \cap \min_W f(x_0, Y_0).$$

This contradicts

$$f(\hat{x}, \hat{y}) \in \min_{Y_0}{}_W \max_{X_0}{}_W f(x, y).$$

Therefore,

$$\min_{Y_0}{}_W \max_{X_0}{}_W f(x, y) \subset \min_W V^W.$$

(ii) This can be proved similarly. □

Since V^W is compact, the following corollary is a direct consequence of Lemma 2.2 and Theorem 4.2.

Corollary 4.1. If the assumptions in Theorem 4.2 hold, then

$$V^W \subset \max_{X_0}{}_W \min_{Y_0}{}_W f(x, y) - K,$$

$$V^W \subset \min_{Y_0}{}_W \max_{X_0}{}_W f(x, y) + K.$$

That is,

$$V^W \subset \left(\min_{Y_0}{}_W \max_{X_0}{}_W f(x, y) + K\right) \cap \left(\max_{X_0}{}_W \min_{Y_0}{}_W f(x, y) - K\right).$$

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