JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 84, No. 1, pp. 171–180, JANUARY 1995

On Vector Variational Inequalities

A. H. SIDDIQI,[1] Q. H. ANSARI,[2] AND A. KHALIQ[3]

Communicated by F. Giannessi

Abstract. In this paper, we introduce a general form of a vector variational inequality and prove the existence of its solutions with and without convexity assumptions.

Key Words. General vector variational inequalities, existence theorems, convexity, KKM theorem.

1. Introduction

Recently, the vector variational inequality (in short, VVI) has been introduced in a finite-dimensional Euclidean space, and some applications have been shown (see Ref. 1). Later, in a general setting, Chen and Yang (Refs. 2–4) have derived an equivalence between the VVI and the vector extremum problem, an equivalence between the VVI and the vector complementarity problem, and proved the existence of the solution of the VVI. Isac (Ref. 5) and Noor (Ref. 6) have introduced and studied separately a more general form of variational inequality, called general variational inequality (in short, GVI) by Noor (Ref. 6).

Inspired and motivated by the applications of the VVI, we introduce in this paper a more general form of the VVI corresponding to a general variational inequality, which includes the VVI studied by Chen (Ref. 4) as a special case. Moreover, we prove the existence of the solution of this VVI, which may be seen as an extension of the Isac theorems (see Ref. 5) on the existence of solutions for a general (special) variational inequality.

Let X be a Hausdorff topological vector space, and let Y be an ordered Hausdorff topological vector space. Let K be a nonempty, closed, and convex subset of X; let $T: K \to L(X, Y)$ be a mapping, $L(X, Y)$ being the space of all linear continuous operators from X into Y; and let

[1]Professor, Department of Mathematics, Aligarh Muslim University, Aligarh, India.
[2]Lecturer, Department of Mathematics, Aligarh Muslim University, Aligarh, India.
[3]Lecturer, Department of Mathematics and Statistics, Kashmir University, Srinagar, J&K, India.

0022-3239/95/0100-0171$07.50/0

$\{C(x): x \in K\}$ be a family of closed, pointed, and convex cones in Y such that int $C(x) \neq \emptyset$, $\forall x \in K$, where int denotes interior of a set.

Consider the general vector variational inequality (in short, GVVI) problem:

(GVVI) find $x_0 \in K$ such that

$$\langle T(x_0), x - g(x_0) \rangle \notin -\text{int } C(x_0), \qquad \forall x \in K, \tag{1}$$

where $g: K \to K$ is a mapping and $\langle T(x), y \rangle$ denotes the evaluation of the linear operator $T(x)$ at y. Hence, $\langle T(x), \cdot \rangle \in Y$.

Special Cases.

(i) Note that, for $g(x) = x \in K$, (1) is equivalent to finding $x_0 \in K$, such that

$$\text{(VVI)} \quad \langle T(x_0), x - x_0 \rangle \notin -\text{int } C(x_0), \qquad \forall x \in K, \tag{2}$$

which is known as the VVI problem; see Ref. 4.

(ii) If $Y = \mathbb{R}$, $L(X, Y) = X^*$, $C(x) = \mathbb{R}_+$, $\forall x \in K$, then (1) reduces to finding $x_0 \in K$, such that

$$\text{(GVI)} \quad \langle T(x_0), x - g(x_0) \rangle \geq 0, \qquad \forall x \in K. \tag{3}$$

The inequality (3) is known as the GVI problem; see Ref. 6. Such a type of variational inequality has been introduced and studied separately by Isac (Ref. 5) and Noor (Ref. 6).

(iii) If $Y = \mathbb{R}$, $X = \mathbb{R}^n$, $L(X, Y) = X^*$, $C(x) = \mathbb{R}_+$, $\forall x \in K$, and g is the identity mapping, then (1) becomes the usual variatioinal inequality, considered and studied by Hartman and Stampacchia; see Ref. 7.

The special cases (i)–(iii) show that (1) is a general and unifying setting, whose analysis is one of the main motivations of this paper.

2. Existence of Solution for GVVI

In this section, we introduce some existence results for (1). To this end, the following known result (Refs. 8 and 9) will be used.

Lemma 2.1. Let E be a nonempty compact convex set of a Hausdorff topological vector space. Let A be a subset of $E \times E$ having the following properties:

(i) $(x, x) \in A$, $\forall x \in E$;

(ii) $\forall x \in E$, $A_x := \{y \in E: (x, y) \in A\}$ is closed in E;

(iii) $\forall y \in E$, the set $A_y := \{x \in E: (x, y) \notin A\}$ is convex.

Then, $\exists y_0 \in E$ such that $E \times \{y_0\} \subset A$.

Let $D \subset K$ be a nonempty compact convex set. The bilinear form $\langle \cdot , \cdot \rangle$ is supposed to be continuous.

Theorem 2.1. If $T: K \to L(X, Y)$ and $g: K \to K$ are continuous, if the multivalued map $W(x) = Y \backslash \{-\text{int } C(x)\}$ is upper semicontinuous on K, and if $\langle T(x), x - g(x) \rangle \notin -\text{int } C(x)$, $\forall x \in D$, then $\exists x_0 \in D \subset K$ such that

$$\langle T(x_0), x - g(x_0) \rangle \notin -\text{int } C(x_0), \qquad \forall x \in D \subset K.$$

Proof. Let

$$A = \{(x, y) \in D \times D: \langle T(y), x - g(y) \rangle \notin -\text{int } C(y)\}.$$

The thesis is proved if we show that (i)–(iii) of Lemma 2.1 are satisfied. From the definition of A, we deduce that

$$\forall x \in D, \qquad (x, x) \in A \Leftrightarrow \langle T(x), x - g(x) \rangle \notin -\text{int } C(x).$$

Now, we will show that

$$A_x := \{y \in D: (x, y) \in A\}, \qquad x \in D,$$

is closed. Let $\{y_n\}$ be a sequence in A_x such that $y_n \to y$. Since $y_n \in A_x$, we have

$$\langle T(y_n), x - g(y_n) \rangle \in W(y_n) := Y \backslash [-\text{int } C(y_n)].$$

Since T, g and $\langle \cdot , \cdot \rangle$ are continuous, we have

$$\langle T(y_n), x - g(y_n) \rangle \to \langle T(y), x - g(y) \rangle.$$

The upper semicontinuity of the multifunction $W(y)$ implies that

$$\langle T(y), x - g(y) \rangle \in W(y) = Y \backslash [-\text{int } C(y)],$$

and thus $y \in A_x$. Hence, A_x is closed. It remains to show that, $\forall y \in D$,

$$A_y := \{x \in D: (x, y) \notin A\},$$

is convex. Indeed, if $x_1, x_2 \in A_y$ and $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha + \beta = 1$, then since $C(y)$ is cone, we have

$$\langle T(y), \alpha x_1 - \alpha g(y) \rangle \in -\text{int } C(y),$$

$$\langle T(y), \beta x_2 - \beta g(y) \rangle \in -\text{int } C(y),$$

which imply that

$$\langle T(y), \alpha x_1 + \beta x_2 - (\alpha + \beta) g(y) \rangle \in -\text{int } C(y)$$

$$\Rightarrow \langle T(y), \alpha x_1 + \beta x_2 - g(y) \rangle \in -\text{int } C(y)$$

$$\Rightarrow \alpha x_1 + \beta x_2 \in A_y.$$

Hence, A_y is convex. Now, Lemma 2.1 gives the existence of $x_0 \in D$ such that $D \times \{x_0\} \subset A$. This implies that

$$x_0 \in D: \langle T(x_0), x - g(x_0) \rangle \notin -\text{int } C(x_0), \qquad \forall x \in D. \qquad \square$$

Remark 2.1. If $Y = \mathbb{R}$, $L(X, Y) = X^*$, and $C(x) = \mathbb{R}_+, \forall x \in K$, then Theorem 2.1 collapses to Proposition 2 of Ref. 5.

A multivalued map $F: D \subset X \to X$ is called a KKM map if

$$\text{conv}\{x_1, x_2, \ldots,\} \subset \bigcup_{i=1}^{n} F(x_i),$$

for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of D; conv denotes the convex hull.

We need the following results for the proof of the next theorem.

Lemma 2.2. See Ref. 8. Let D be an arbitrary nonempty set in a Hausdorff topological vector space X. Let $F: D \to X$ be a KKM map. If all the sets $F(x)$ are closed in X and if one is compact, then

$$\bigcap_{x \in D} F(x) \neq \emptyset.$$

Lemma 2.3. See Ref. 4. Let (X, P) be an ordered topological vector space equipped with a closed, pointed, and convex cone P such that int $P \neq \emptyset$. Then, $\forall y, z \in X$, we have:

(i) $y - z \in \text{int } P$ and $y \notin \text{int } P \Rightarrow z \notin \text{int } P$;
(ii) $y - z \in P$ and $y \notin \text{int } P \Rightarrow z \notin \text{int } P$;
(iii) $y - z \in -\text{int } P$ and $y \notin -\text{int } P \Rightarrow z \notin -\text{int } P$;
(iv) $y - z \in -P$ and $y \notin -\text{int } P \Rightarrow z \notin -\text{int } P$.

Theorem 2.2. Assume that:

(a) the mappings $g: K \to K$ and $T: K \to L(X, Y)$ are continuous;
(b) $C: K \to Y$ is a multivalued mapping such that, $\forall x \in K$, $C(x)$ is a closed, pointed, and convex cone with int $C(x) \neq \emptyset$;
(c) $W: K \to Y$ is an upper semicontinuous multivalued mapping defined by $W(x) := Y \backslash \{-\text{int } C(x)\}$;
(d) there exists a function $h: K \times K \to Y$ such that:
    (i) $h(x, y) - \langle T(x), y - g(x) \rangle \in -\text{int } C(x)$;
    (ii) the set $\{y \in K: h(x, y) \in -\text{int } C(x)\}$ is convex, $\forall x \in K$;
    (iii) $h(x, x) \notin -\text{int } C(x)$, $\forall x \in K$;
    (iv) there exists a nonempty, compact and convex subset $D \subset K$, such that, $\forall x \in K \backslash D$, $\exists y \in D$ such that $\langle T(x), y - g(x) \rangle \in -\text{int } C(x)$.

Then, $\exists x_0 \in D$ such that

$$\langle T(x_0), y - g(x_0) \rangle \notin -\text{int } C(x_0), \qquad \forall y \in K.$$

Proof. $\forall y \in K$,

$$D(y) := \{x \in D: \langle T(x), y - g(x) \rangle \notin -\text{int } C(x)\}.$$

From assumptions (a) and (c), we have that $D(y)$ is closed in D. Since every element

$$x_0 \in \bigcap_{y \in K} D(y),$$

is a solution of (1), we have to prove that

$$\bigcap_{y \in K} D(y) \notin \varnothing.$$

Because of the compactness of D, it is sufficient to show that the family $\{D(y)\}_{y \in K}$ has the finite intersection property. Indeed, let $y_1, y_2, \ldots, y_m \in K$ be given; we see that

$$A := \text{conv}(D \cup \{y_1, y_2, \ldots, y_m\})$$

is a compact and convex subset of K. We consider the following multivalued mappings:

$$F_1(y) = \{x \in A: \langle T(x), y - g(x) \rangle \notin -\text{int } C(x)\},$$

$$F_2(y) = \{x \in A: h(x, y) \notin -\text{int } C(x)\}, \qquad \forall y \in K.$$

Because of the continuity of the bilinear form $\langle \cdot, \cdot \rangle$ and because of assumptions (a) and (c), we have that $F_1(y)$ is compact, since it is a closed subset of the compact (convex) set A. From assumption (d)(i) and (d)(iii), we have

$$h(x, x) - \langle T(x), x - g(x) \rangle \in -\text{int } C(x),$$

$$h(x, x) \notin -\text{int } C(x).$$

Then, by Lemma 2.3 we have

$$\langle T(x), x - g(x) \rangle \notin -\text{int } C(x).$$

Hence, $F_1(y) \neq \varnothing$. Now, we prove that F_2 is a KKM map. Indeed, if we suppose that $\exists v_1, v_2, \ldots, v_n \in A$ and $\exists \alpha_i \geq 0, i = 1, 2, \ldots, n$, with $\sum_{i=1}^n \alpha_i = 1$, such that

$$\sum_{i=1}^{n} \alpha_i v_i \notin \bigcup_{j=1}^{n} F_2(v_j),$$

then we have that

$$h\left(\sum_{i=1}^{n} \alpha_i v_i, v_j\right) \in -\text{int } C\left(\sum_{i=1}^{n} \alpha_i v_i\right).$$

By assumption (d)(ii), we have

$$h\left(\sum_{i=1}^{n} \alpha_i v_i, \sum_{i=1}^{n} \alpha_i v_i\right) \in -\text{int } C\left(\sum_{i=1}^{n} \alpha_i v_i\right),$$

which contradicts assumption (d)(iii). Therefore, F_2 is a KKM map. Since from assumption (d)(i) we have $F_2(y) \subset F_1(y)$, $\forall y \in K$, we obtain that also $F_1(y)$ is a KKM map. Applying Lemma 2.2 to F_1, we get

$$\bigcap_{y \in A} F_1(y) \neq \varnothing,$$

or the existence of a point $x_0 \in A$, such that

$$\langle T(x_0), x - g(x_0) \rangle \notin -\text{int } C(x_0), \qquad \forall x \in A.$$

By assumption (d)(iv), we have that $x_0 \in D$; moreover, $x_0 \in D(y_i)$, for every $1 \leq i \leq m$. Hence, $\{D(y)\}_{y \in K}$ has the finite intersection property. This completes the proof. □

Remark 2.2. If $Y = \mathbb{R}$, $L(X, Y) = X^*$, and $C(x) = \mathbb{R}_+$, $\forall x \in K$, then Theorem 2.2 becomes Theorem 8 of Ref. 5.

3. Existence Result without Convexity

In this section, by using the technique of Chen (Ref. 4), we prove an existence theorem for a special case of (1) by replacing the convexity assumption with merely topological properties. The following definitions can be found in Ref. 10.

Definition 3.1. Let X be a topological space, and let $\{\Gamma_A\}$ be a given family of nonempty contractible subsets of X, indexed by finite subsets of X.

(i) A pair $(X, \{\Gamma_A\})$ is said to be a H-space, if $A \subset B$ implies $\Gamma_A \subset \Gamma_B$. A subset $D \subset X$ is called H-convex, if for every finite subset $A \subset D$, it follows that $\Gamma_A \subset D$.

(ii) A subset $D \subset X$ is called weakly H-convex, if $\Gamma_A \cap D$ is nonempty and contractible for every finite subset $A \subset D$. This is equivalent to saying that the pair $(D, \{\Gamma_A \cap D\})$ is an H-space.

(iii) A subset $K \subset X$ is called H-compact, if there exists a compact and weakly H-convex set $D \subset X$, such that $K \cup A \subset D$ for every finite subset $A \subset X$.

(iv) A multivalued mapping $F: X \to X$ is called H-KKM, if $\Gamma_A \subset \bigcup_{x \in A} F(x)$, for every finite subset $A \subset X$.

Lemma 3.1. See Ref. 10. Let $(X, \{\Gamma_A\})$ be an H-space, and let $F: X \to X$ be an H-KKM multivalued mapping, such that:

(a) $\forall x \in X$, $F(x)$ is compactly closed, that is, $B \cap F(x)$ is closed in B for every compact set $B \subset X$;

(b) there exist a compact set $L \subset X$ and an H-compact set $K \subset X$ such that, for each weakly H-convex set D with $K \subset D \subset X$, we have $\bigcap_{x \in D} (F(x) \cap D) \subset L$.

Then,

$$\bigcap_{x \in X} F(x) \neq \varnothing.$$

We now consider a special case of (1), but in a more general context,

$$x_0 \in X, \langle T(x_0), x - g(x_0) \rangle \notin -\text{int}\, P, \qquad \forall x \in X, \tag{4}$$

where (Y, P) is an ordered Banach space with $\text{int}\, P \neq \varnothing$ and $T: X \to L(X, Y)$, $g: X \to X$ are mappings.

Theorem 3.1. Let $(X, \{\Gamma_A\})$ be an H-space, and let (Y, P) be an ordered topological vector space equipped with a closed, pointed, and convex cone P such that $\text{int}\, P \neq \varnothing$. Assume that:

(a) the mappings $T: X \to L(X, Y)$ and $g: X \to X$ are continuous;

(b) $\langle T(y), y - g(y) \rangle \notin -\text{int}\, P$, $\forall y \in X$;

(c) $\forall y \in X$, $B_y := \{x \in X: \langle T(y), x - g(y) \rangle \in \text{int}\, P\}$ is either H-convex or empty;

(d) there exist a compact set $L \subset X$ and an H-compact set $K \subset X$ such that, for every weakly H-compact set D with $K \subset D \subset X$, we have

$$\{y \in D: \langle T(y), x - g(y) \rangle \notin -\text{int}\, P, \forall x \in D\} \subset L.$$

Then, (4) is solvable.

Proof. Let

$$F(x) := \{y \in X: \langle T(y), x - g(y) \rangle \notin -\text{int}\, P\}, \qquad x \in X.$$

We will prove the inequality

$$\bigcap_{x\in X} F(x) \neq \varnothing.$$

Then, the thesis will be a consequence of the fact that every element

$$x_0 \in \bigcap_{x\in X} F(x)$$

is a solution of (4). The inequality follows from Lemma 3.1, if we prove that F is an H-KKM mapping and the conditions (a) and (b) of Lemma 3.1 hold. Suppose that F is not an H-KKM mapping. Then, there exists a finite subset $A \subset X$, such that

$$\Gamma_A \not\subset \bigcup_{x\in X} F(x).$$

Thus, $\exists z \in \Gamma_A$ such that

$$z \notin F(x), \qquad \forall x \in A \Rightarrow \langle T(z), x - g(z)\rangle \in -\text{int}\, P, \qquad \forall x \in A.$$

By assumption (c) and since B_z is H-convex, we have

$$\Gamma_A \subset \Gamma_B, \qquad \text{for } A \subset B_z.$$

Therefore,

$$z \in B_z \Rightarrow \langle T(z), z - g(z)\rangle \in -\text{int}\, P,$$

which contradicts assumption (b). Thus,

$$\Gamma_A \subset \bigcup_{x\in X} F(x),$$

for every finite subset $A \subset X$, so that F is an H-KKM mapping.

Now, we will prove that, $\forall x \in X$, $F(x)$ is closed. Indeed, suppose that $\{y_n\} \subset F(x)$, $x \in X$, such that $y_n \to y$. As T, g and $\langle \cdot , \cdot \rangle$ are continuous, we have

$$\langle T(y_n), x - g(y_n)\rangle \to \langle T(y), x - g(y)\rangle,$$

since

$$\langle T(y_n), x - g(y_n)\rangle \notin -\text{int}\, P, \qquad \forall n,$$

i.e.,

$$\langle T(y_n), x - g(y_n)\rangle \in W = Y \backslash \{-\text{int}\, P\}.$$

But $W = Y\backslash(-\text{int}\, P)$ is closed. Thus, we have

$$\langle T(y), x - g(y)\rangle \in W,$$

i.e.,

$$\langle T(y), x - g(y) \rangle \notin -\text{int } P.$$

Hence, $y \in F(x)$. Therefore, $F(x)$ is closed, $\forall x \in X$; i.e., the condition (a) of Lemma 3.1 holds. It is easy to see that the present assumption (d) is the same as assumption (b) of Lemma 3.1. Thus, by Lemma 3.1,

$$\bigcap_{x \in X} F(x) \neq \varnothing;$$

i.e., $\exists x_0 \in X$ such that

$$\langle T(x_0), x - g(x_0) \rangle \notin -\text{int } P, \qquad \forall x \in X. \qquad \square$$

Remark 3.1. If g is an identity mapping, then Theorem 3.1 collapses to Theorem 3 of Ref. 4.

Acknowledgment

The authors would like to thank Prof. F. Giannessi, University of Pisa, for his valuable suggestions in the preparation of this paper.

References

1. GIANNESSI, F., *Theorems of the Alternative, Quadratic Programs, and Complementary Problems*, Variational Inequalities and Complementarity Problems, Edited by R. W. Cottle, F. Giannessi, and J. L. Lions, John Wiley and Sons, New York, New York, pp. 151–186, 1980.
2. CHEN, Y., and CHENG, G. M., *Vector Variational Inequalities and Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, Springer Verlag, Berlin, Germany, Vol. 285, pp. 408–416, 1987.
3. CHEN, G. Y., and YANG, X. Q., *Vector Complementarity Problem and Its Equivalence with a Weak Minimal Element in Ordered Space*, Journal of Mathematical Analysis and Applications, Vol. 153, pp. 136–158, 1990.
4. CHEN, G. Y., *Existence of Solutions for a Vector Variational Inequality: An Extension of the Hartmann–Stampacchia Theorem*, Journal of Optimization Theory and Applications, Vol. 74, pp. 445–456, 1992.
5. ISAC, G., *A Special Variational Inequality and the Implicit Complementarity Problem*, Journal of the Faculty of Sciences of the University of Tokyo, Vol. 37, pp. 109–127, 1990.
6. NOOR, M. A., *General Variational Inequality*, Applied Mathematical Letters, Vol. 1, pp. 119–122, 1988.
7. HARTMAN, P., and STAMPACCHIA, G., *On Some Nonlinear Elliptic Differential Functional Equations*, Acta Mathematica, Vol. 115, pp. 271–310, 1966.

8. FAN, K., *A Generalization of Tychonoff's Fixed-Point Theorem*, Mathematische Annalen, Vol. 142, pp. 305–310, 1961.
9. FAN, K., *A Minimax Inequality and Applications*, Inequalities III, Edited by O. Shisha, Academic Press, New York, New York, pp. 103–113, 1972.
10. BARDARO, C., and CEPPITELLI, R., *Some Further Generalizations of the Knaster–Kuratowski–Mazurkiewicz Theorem and Minimax Inequalities*, Journal of Mathematical Analysis and Applications, Vol. 132, pp. 484–490, 1988.
11. JAMESON, G., *Ordered Linear Spaces*, Springer Verlag, Heidelberg, Germany, 1970.