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On a functional inequality arising in the construction of the product of several metric spaces

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Suppose that (X_1, d_1) and (X_2, d_2) are two metric spaces. Then it is well known that there are several ways one can induce a metric on the product space which induces a topology equivalent to the Tychonoff topology, the usual product topology. In particular, one can require that such a metric be a function of the two metrics d_1 and d_2, i.e., that there be a function H from $\mathbf{R}^+ \times \mathbf{R}^+$, the first quadrant, into $\mathbf{R}^+$, the non-negative reals, such that for every x_1, y_1 in X_1 and every x_2, y_2 in X_2, we have:

$$d((x_1, x_2), (y_1, y_2)) = H(d_1(x_1, y_1), d_2(x_2, y_2)). \tag{1}$$

It is then natural to impose the following conditions on H:

$$H(a, 0) = H(0, a) = a, \quad \text{for every} \quad a \varepsilon \mathbf{R}^+; \tag{2}$$

$$H(a+b, c+d) \leqslant H(a, c) + H(b, d) \quad \text{for all} \quad a, b, c, d \text{ in } \mathbf{R}^+; \tag{3}$$

$$H(a, b) \leqslant H(c, d) \quad \text{whenever} \quad a \leqslant c \quad \text{and} \quad b \leqslant d; \tag{4}$$

$$H \text{ is continuous}; \tag{5}$$

$$H(H(a, b), c) = H(a, H(b, c)) \quad \text{for all } a, b, c \text{ in } \mathbf{R}^+. \tag{6}$$

Conditions (2) and (3) then guarantee that the function defined by (1) actually is a metric on $X_1 \times X_2$; conditions (2)–(5), that this metric induces a topology on the product space which is equivalent to the Tychonoff topology; and the addition of condition (6) allows us to extend the definition of the product to three or more

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metric spaces without ambiguity. It should be emphasized that conditions (2)–(6) are sufficient, but not necessary, for these results; in particular, to obtain the Tychonoff topology on $X_1 \times X_2$, one only needs the continuity of H at $(0, 0)$. Now if (4) is replaced by the stronger condition

$$H(a, b) < H(c, b), H(a, b) < H(a, d) \quad \text{whenever} \quad a < c, b < d; \tag{7}$$

then, as is well-known (cf. [1], p. 256), H admits the representation

$$H(a, b) = h^{-1}(h(a) + h(b)) \tag{8}$$

where h: $\mathbf{R}^+ \to \mathbf{R}^+$ is a continuous, strictly increasing function with inverse h^{-1} such that $h(0) = 0$. Furthermore, from (3) it follows that h must satisfy the inequality:

$$h^{-1}(h(a+b) + h(c+d)) \leq h^{-1}(h(a) + h(c)) + h^{-1}(h(b) + h(d)) \tag{9}$$

for all a, b, c, d in $\mathbf{R}^+$. Conversely, if h: $\mathbf{R}^+ \to \mathbf{R}^+$ is a continuous, strictly increasing function with $h(0) = 0$ satisfying (9), and if H is then defined by (8), then it follows without difficulty that H satisfies (2)–(7), and in addition, the symmetry condition

$$H(a, b) = H(b, a) \quad \text{for all} \quad a, b \text{ in } \mathbf{R}^+. \tag{10}$$

Condition (10), when applied in (1), insures that the metric induced on $X_1 \times X_2$ is isometric to that induced on $X_2 \times X_1$.

Inequality (9) is the subject of this paper. Henceforth we shall assume that h: $\mathbf{R}^+ \to \mathbf{R}^+$ is a continuous strictly increasing function with $h(0) = 0$. It is easily shown that such functions h exist; for example, $h(x) = x^p$, for any $p \geq 1$. In this case (9) is the Minkowski inequality. Another family of functions is exhibited in Example 2.

Such functions h were considered in [4] by Th. Motzkin. There he stated, without proof, that if $h(0) = 0$, $h' \geq 0$, $h'' \geq 0$ and $h'h''' - 2(h'')^2 \geq 0$, then h satisfies (9). It will be shown in Theorem 1 that the only functions h which satisfy all of Motzkin's conditions simultaneously must be of the form $h(x) = \alpha x$ for some $\alpha \geq 0$. Thus, Motzkin's assertion is correct, but far too restrictive, and in particular, cannot be used to obtain the usual Minkowski inequality. In Theorem 2 a different set of sufficient conditions will be presented in order for a function h to satisfy (9). These conditions will be strong enough to allow a proof of the Minkowski inequality.

LEMMA 1. *Let a be a fixed real number, and g be a non-decreasing, convex function defined on $[a, \infty)$. Then either g is unbounded from above, or g is constant.*

Proof. For fixed positive α, consider the ray $P_\alpha = \{(t, g(a)+\alpha(t-a))/t > a\}$ issuing from $(a, g(a))$. If there is an $\alpha > 0$ such that the graph of g intersects P_α, then by the convexity of g, the graph of g cannot fall below P_α in the half-plane to the right of the point of intersection, whence g is unbounded from above. If, on the other hand, the graph of g fails to intersect P_α for any $\alpha > 0$, then for any $t > a$ we have $g(t) \leq g(a)$, whence g, being non-decreasing, is constant.

THEOREM 1. *Suppose h: $\mathbf{R}^+ \to \mathbf{R}^+$ is such that $h(0) = 0$, $h' \geq 0$, $h'' \geq 0$, and $h'h''' - 2(h'')^2 \geq 0$. Then $h(x) = \alpha x$ for some $\alpha \geq 0$.*

Proof. Either $h'(t) = 0$ for all $t \geq 0$, in which case $h(t) = 0$ for all $t \geq 0$, or there is an $a \geq 0$ such that $h'(a) > 0$. In the latter case, we have $h'(t) > 0$ for all $t \geq a$ (since $h'' > 0$, h' is non-decreasing), thus g, defined by

$$g(t) = -\frac{1}{h'(t)}, \quad t \geq a,$$

is a well-defined function on $[a, \infty)$. Direct calculation yields $g' \geq 0$, $g'' \geq 0$, so that g is non-decreasing and convex, and Lemma 1 applies. Since g is bounded above (by 0), g is constant, and there is a number $\alpha > 0$ such that $h'(t) = \alpha$ for all $t \geq a$. We now show that we can take $a = 0$. For if not, then there is a number $b \geq 0$ such that $h'(t) = 0$ for $0 \leq t \leq b$, $h'(t) > 0$ for $t > b$. But the argument above shows that there is a fixed $\alpha > 0$ such that $h'(t) = \alpha$ for $t > b$, whence h' is discontinuous at b. Since this is impossible, it follows that we can take $a = 0$, and the theorem is proved.

We now turn our attention to providing necessary conditions for the function h to satisfy (9). We begin with:

LEMMA 2. *If* (9) *holds, then h is convex and superadditive.*

Proof. Let x and y be non-negative real numbers, and assume without loss of generality, that $x \leq y$. Let $a = h^{-1}(x)$, $b = h^{-1}((x+y)/2) - h^{-1}(x)$, $c = h^{-1}((y-x)/2)$, and $d = 0$. Substituting these values into (9) yields:

$$\frac{h^{-1}(x) + h^{-1}(y)}{2} \leq h^{-1}\left(\frac{x+y}{2}\right);$$

whence h^{-1} is concave and thus h is convex. Since $h(0)=0$, it follows (cf. [3] p. 239) that h is superadditive.

If h satisfies (9), then by Lemma 2, h^{-1} is subadditive, and by assumption $h^{-1}(0)=0$; therefore h^{-1} is a gauge [2, p. 63].

The converse of Lemma 2 is false; i.e. a continuous increasing convex function h with $h(0)=0$ need not satisfy (9).

EXAMPLE 1. Let $h(x)=\begin{cases} x^2, & \text{for } 0\leq x\leq 1; \\ 2x-1 & \text{for } x\geq 1. \end{cases}$

It is clear that h is a continuous, strictly increasing convex function with $h(0)=0$. But if $a=c=d=1$ and $b=\frac{1}{2}$, then (9) fails.

Although h being convex does not imply (9), it does imply the special case of (9) with $a=0$.

LEMMA 3. *If h is convex, then* $h^{-1}(h(b)+h(c+d))\leq c+h^{-1}(h(b)+h(d))$.

Proof. Fix b and d and consider the function

$$F(x)=h^{-1}(h(b)+h(x+d))-x-h^{-1}(h(b)+h(d)).$$

If $0\leq x<y$, then

$$F(x)-F(y)=h^{-1}(h(b)+h(x+d))-h^{-1}(h(b)+h(y+d))+y-x$$

Since h is increasing and h^{-1} is concave, it follows from [5; Lemma 15, p. 108] that $F(x)-F(y)\geq 0$. Thus F is decreasing. Since $F(0)=0$, the result now follows.

LEMMA 4. *Let f be a function from $\mathbf{R}^+$ into $\mathbf{R}^+$ that is positive for positive arguments. Let g be the function defined on the entire real line by*

$$g(x)=\log(f(e^x)) \tag{11}$$

Then the following statements are equivalent:

(a) The function g is convex; i.e. $f(e^x)$ is log convex;

(b) For all $x>0$, y, $z\geq 0$ we have:

$$\frac{f(x+y)\cdot f(x+z)}{f(x)}\leq f\left(\frac{(x+y)(x+z)}{x}\right). \tag{12}$$

Moreover, (assuming h continuous), equality holds in (12) for all $x>0$, y, $z\geq 0$ if *and only if g is linear, i.e., if and only if* $f(x) = cx^p$ *for some positive c and fixed real* p.

(N.B. The requirement that g be convex does not imply the convexity of f (take $f(x) = \sqrt{x}$). Hence this requirement is weaker than the requirement that f itself be log-convex.)

Proof. Letting $a = \log(x+y)$, $b = \log(x)$, and $c = \log(x+z)$ yields that

$$\frac{f(x+y)\cdot f(x+z)}{f(x)} \leq f\left(\frac{(x+y)(x+z)}{x}\right) \quad \text{if and only if}$$

$$g(a) + g(c) \leq g(a+c-b) + g(b) \quad \text{for} \quad -\infty < b \leq \mathrm{Min}\,(a, c). \tag{13}$$

First, assume that (13) holds. Then let $u, v \in \mathbf{R}$. Let $a = c = ((u+v)/2)$, and let $b = \mathrm{Min}\,(u, v)$. Note that $b \leq \mathrm{Min}\,(a, c)$. Substituting these values for a, b, and c into (13) yields:

$$2g\left(\frac{u+v}{2}\right) \leq g(v) + g(u). \tag{14}$$

Thus g is convex.

In the other direction, suppose that g is convex. Consider the function

$$F(b) = g(a) + g(c) - g(a+c-b) - g(b), \quad \text{for} \quad -\infty < b \leq \mathrm{Min}\,(a, c).$$

Since $F(\mathrm{Min}\,(a, c)) = 0$, it suffices to show that F is non-decreasing on $(-\infty, \mathrm{Min}\,(a, c))$. Let $b_1 < b_2 < \mathrm{Min}\,(a, c)$. Then $b_1 < b_2 < ((a+c)/2)$, whence there is an $\alpha \in (0, 1)$ such that $b_2 = (1-\alpha)b_1 + \alpha((a+c)/2)$. From this it follows that $a + c - b_2 = (1-\alpha)(a+c-b_1) + \alpha((a+c)/2)$. Since g is convex,

$$g(b_2) \leq \alpha g\left(\frac{a+c}{2}\right) + (1-\alpha)g(b_1);$$

$$g(a+c-b_2) \leq \alpha g\left(\frac{a+c}{2}\right) + (1-\alpha)g(a+c-b_1); \quad \text{and}$$

$$g\left(\frac{a+c}{2}\right) \leq \frac{g(b_1) + g(a+c-b_1)}{2}.$$

Adding the first two inequalities and then using the third yields

$$g(b_2) + g(a+c-b_2) \leq g(a+c-b_1) + g(b_1).$$

from which it follows that $F(b_1) \leq F(b_2)$, for $b_1 \leq b_2$.

Finally, if $f(x) = cx^p$ for some fixed positive c and fixed real p, then it is clear that equality holds in (12). On the other hand, if equality holds in (12) then (14) is the Jensen Equation, and the continuous solutions of (14) are of the form $\alpha x + \beta$ [1, p. 43].

THEOREM 2. *Let* h: $\mathbf{R}^+ \to \mathbf{R}^+$ *be a continuous, strictly increasing, everywhere-differentiable, convex function with* $h(0) = 0$. *Suppose further that if* $f(x) = h'(x)$, *then the function* g *defined by* (11) *is convex. Then* (9) *holds for all* a, b, c, d, *in* $\mathbf{R}^+$ *if it holds asymptotically, i.e., if for all fixed* c, d *in* $\mathbf{R}^+$, *we have*

$$\limsup_{a+b\to\infty} \frac{h^{-1}(h(a+b)+h(c+d))}{h^{-1}(h(a)+h(c))+h^{-1}(h(b)+h(d))} \leq 1. \tag{15}$$

Proof. Let $A = h^{-1}(h(a)+h(c))$, $B = h^{-1}(h(b)+h(d))$, and $C = h^{-1}(h(a+b)+h(c+d))$. Then if either c or d is zero, the conclusion of the theorem follows from Lemma 3. Therefore, we may assume that both c and d are not zero. If this is the case, then both A and B are not zero, and thus, for any fixed c, $d > 0$, the two place function

$$G(a, b) = C/(A+B)$$

is continuous and well defined on $\mathbf{R}^+ \times \mathbf{R}^+$. It is clear that the conclusion of the Theorem will follow if we can show that $G(a, b) \leq 1$ for all pairs $(a, b) \in \mathbf{R}^+ \times \mathbf{R}^+$.

By Lemma 3, $G(0, b) \leq 1$ and $G(a, 0) \leq 1$. Since $\limsup_{a+b\to\infty} G(a, b) \leq 1$, it follows that if there is a point $(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+$ such that $G(x, y) > 1$, then $\sup_{a,b} G(a, b)$ exists and is attained at an interior point, (x_0, y_0), of $\mathbf{R}^+ \times \mathbf{R}^+$, and $G(x_0, y_o) > 1$.

Since both A and B are not 0, and since h is increasing on $(0, \infty)$, both $h'(A)$ and $h'(B)$ are not 0. Therefore at (x_0, y_0),

$$0 = \frac{\partial G}{\partial a} = \frac{1}{(A+B)^2}\left((A+B)\cdot\frac{h'(x_0+y_0)}{h'(C)} - \frac{C \cdot h'(x_0)}{h'(A)}\right);$$

and

$$0 = \frac{\partial G}{\partial b} = \frac{1}{(A+B)^2}\left((A+B)\cdot\frac{h'(x_0+y_0)}{h'(C)} - \frac{C \cdot h'(y_0)}{h'(B)}\right).$$

Since the function g defined by (11) is convex, it follows from Lemma 4 that

$$\frac{C}{A+B}=\frac{h'(A)\cdot h'(x_0+y_0)}{h'(C)\cdot h'(x_0)}\leqslant\frac{1}{h'(C)}\cdot h'\left(\frac{(x_0+y_0)A}{x_0}\right) \tag{16}$$

$$\frac{C}{A+B}=\frac{h'(B)\cdot h'(x_0+y_0)}{h'(C)\cdot h'(y_0)}\leqslant\frac{1}{h'(C)}\cdot h'\left(\frac{(x_0+y_0)B}{y_0}\right) \tag{17}$$

If either of the quantities on the right hand side of (16) or (17) is less than or equal to one, then $G(x_0, y_0)=(C/(A+B))\leqslant 1$, and this is a contradiction. Thus

$$h'\left(\frac{(x_0+y_0)A}{x_0}\right)>h'(C); \quad \text{and} \quad h'\left(\frac{(x_0+y_0)B}{y_0}\right)>h'(C).$$

Since h is convex, h' is increasing. Therefore

$$\frac{(x_0+y_0)A}{x_0}>C>A+B; \quad \text{and} \quad \frac{(x_0+y_0)B}{y_0}>C>A+B. \tag{18}$$

But these inequalities imply $Ay_0>Bx_0$ and $Bx_0>Ay_0$ which is again a contradiction. Therefore $G(a, b)\leqslant 1$ for all a and b.

COROLLARY 1. (*Minkowski Inequality*) *If* $h(x)=x^p$, $p\geqslant 1$, *then* h *satisfies the hypotheses of Theorem* 2.

Proof. It is clear that if $p\geqslant 1$ then $h(x)$ and $g(x)=\log(h'(e^x))=\log p+(p-1)x$ are both convex. Next (15) follows from the fact that for any fixed c, d;

$$G(a, b)=\frac{((a+b)^p+(c+d)^p)^{1/p}}{(a^p+c^p)^{1/p}+(b^p+d^p)^{1/p}}\leqslant\frac{((a+b)^p+(c+d)^p)^{1/p}}{a+b}$$

EXAMPLE 2. We conclude by exhibiting another family of functions satisfying (9). Let $h(x)=\exp(x^n)-1$, $n\geqslant 1$. Then $h''(x)\geqslant 0$ so that h is convex, and $\log(h'(e^x))=\log n+e^{nx}+(n-1)x$ is also convex. Therefore it remains only to show that $\limsup_{a+b\to\infty} G(a, b)\leqslant 1$. To see this note that $\log(x+1)$ is subadditive.

Therefore,

$$
\begin{aligned}
G(a,b) &= \frac{(\log(\exp((a+b)^n)-1+\exp((c+d)^n)-1+1))^{1/n}}{(\log(\exp(a^n)+\exp(c^n)-1))^{1/n}+(\log(\exp(b^n)+\exp(d^n)-1))^{1/n}} \\
&\leqslant \frac{(\log(\exp((a+b)^n)+\exp(c+d)^n)-1))^{1/n}}{a+b} \\
&\leqslant \left\{\frac{1}{(a+b)^n}[\log\exp((a+b)^n)+\log\exp((c+d^n))]\right\}^{1/n} \\
&= \left(1+\left(\frac{c+d}{a+b}\right)^n\right)^{1/n}
\end{aligned}
$$

from which (15) follows.

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