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Geometry of Optimal Value Functions with Applications to Redundancy in Linear Programming[1]

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Abstract. In 1967, Wets and Witzgall (Ref. 1) made, in passing, a connection between frames of polyhedral cones and redundancy in linear programming. The present work elaborates and formalizes the theoretical details needed to establish this relation. We study the properties of optimal value functions in order to derive the correspondence between problems in redundancy and the frame of a polyhedral cone. The insights obtained lead to schemes to improve the efficiency of procedures to detect redundancy in the areas of linear programming, stochastic programming, and computational geometry.

Key Words. Linear programming, optimal value functions, redundancy in linear programming, convex hull problem, data envelopment analysis.

1. Introduction

Redundancy is a condition indicating excess information. Redundancy in this broad sense appears in different areas. In optimization theory, the concept of redundancy plays a role in classical linear programming and in stochastic programming. In linear programming, a constraint is redundant if it can be omitted from the model without affecting the feasible set. Redundant constraints in linear programming are a common occurrence. These constraints may make problems larger than necessary; reducing model size improves the efficiency of the solution. Also, redundant constraints may be manifestations of modeling inefficiencies and inconsistencies. An additional reason to detect redundancy is that it may lead to

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numerical instability in the simplex method (Ref. 2). Finally, redundancy and degeneracy are dually linked. In stochastic programming, redundancy in the decision variables is an indication of the role of uncertainty in the problem.

Redundancy is also a topic in the study of computational geometry. The convex hull of a finite collection of points reduces to the convex hull of only those points which result in extreme points and all the other points are superfluous. Identifying these extreme points of the convex hull from a finite list makes it more efficient to solve certain problems in computational geometry such as finding the facial decomposition of a polytope and Voronoi diagrams (Ref. 3).

The problems of detecting redundancy in linear programming, stochastic programming, and computational geometry, as presented above, are all equivalent. They are connected by duality relations and can be reduced to two problems: locating unnecessary constraints in a system of linear inequalities and identifying the extreme points of the convex hull of a finite collection of points. Our focus is on redundancy in classical linear programming although our discussion includes results in stochastic programming and computational geometry.

In Ref. 1, it is observed that removing redundant constraints in the system

$$A_j^T \pi \leq c_j, \qquad j = 1, \ldots, n,$$

amounts to finding a frame for the cone spanned by vectors

$$\hat{A}_j = \begin{bmatrix} A_j \\ c_j \end{bmatrix}, j = 1, \ldots, n, \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

One objective of our paper is to elaborate on the theoretical details which lead to this observation. Another objective is to apply the insights gained from this analysis to design efficient schemes to identify some elements of the frame. We believe that this is the first work that formally unifies the correspondence between the problem of redundancy in optimization theory (and other areas) and the problem of identifying the frame of a polyhedral optimal value function.

2. Notation and Assumptions

The purpose of this section is to introduce the relevant forms for the problem we propose to study and to present and discuss the assumptions.

Let A be a matrix with m rows and n columns, with $n > m$, let c be a nonzero vector in $\Re^n$, and let b, ξ be vectors in $\Re^m$. The vector $A_{.j}$

corresponds to the jth column of the matrix A; $A_{\cdot j}$ and $\begin{bmatrix} A_{\cdot j} \\ \alpha \end{bmatrix}$ are points in $\Re^m$ and $\Re^{m+1}$, respectively. This last vector will sometimes appear as $[A_{\cdot j}^T, \alpha]$.

Consider the following linear program:

$$\text{(P)} \quad \min \sum_{j=1}^{n} c_j x_j,$$

$$\text{s.t.} \quad \sum_{j=1}^{n} A_{\cdot j} x_j = b, \qquad x_j \geq 0.$$

The dual to this linear program is defined as follows:

$$\text{(D)} \quad \max \sum_{i=1}^{m} \pi_i b_i,$$

$$\text{s.t.} \quad \pi^T A_{\cdot j} \leq c_j, \qquad j = 1, \ldots, n.$$

The feasible region of (P) is denoted by P and that of (D) by D. Therefore,

$$P = \{x \in \Re^n \mid Ax = b, x \geq 0\}, \tag{1}$$

$$D = \{\pi \in \Re^m \mid \pi^T A \leq c\}. \tag{2}$$

Consider the function $\psi: \Re^m \to \Re$ defined as

$$\text{(F*)} \quad \psi(\xi) = \min \sum_{j=1}^{n} c_j x_j,$$

$$\text{s.t.} \quad \sum_{j=1}^{n} A_{\cdot j} x_j = \xi, \qquad x_j \geq 0.$$

The function ψ is an optimal value (or extremal value, or perturbation, or marginal) function, since for any point ξ in $\Re^m$, we obtain the function value by solving a linear program where the right-hand side is the argument ξ. When ξ is defined on a probability space (Ξ, β, λ), the optimal value function (F*) can be seen as a general statement of the second-stage problem of a stochastic program with fixed recourse (Ref. 4).

We present our two working assumptions followed by a discussion:

Assumption 2.1. The vectors $A_{\cdot 1}, \ldots, A_{\cdot n}$ positively span $\Re^m$, i.e., any point in $\Re^m$ can be expressed as a nonnegative linear combination of these vectors.

Assumption 2.2. The dual linear program (D) is feasible.

An obvious consequence of the first assumption is that the rank of the $m \times n$ matrix A is m. This assumption also implies that the linear program

(P) is feasible for any right-hand side in $\Re^m$. If the condition in this assumption holds, the set P is unbounded, while the set D is bounded or infeasible (apply Farkas' lemma). We may anticipate that this condition may be satisfied if the linear program (P) has many more columns than rows or in the case where a stochastic linear program has fixed and complete recourse (Ref. 4). In any event, this assumption is not as restrictive as it may appear at first. In Section 4, we discuss how to modify the problem and make the matrix A satisfy this condition without affecting the task of identifying redundancies.

Assumption 2.2 implies that the bounded dual feasible region D is not empty. Therefore, by weak duality, the primal problem (P) is never $-\infty$ for any right-hand side. This translates to the condition that the function ψ is proper; that is, $\psi(\xi)$ is always bounded below for any ξ in $\Re^m$ (see Ref. 5, p. 24). It is not always possible to tell in advance when D is not empty. However, this is the case if all objective cost coefficients are positive, or if there are relatively few negative costs.

3. Polyhedral Cones and Optimal Value Functions

In this section, we introduce the definitions and the important results regarding the geometry of the optimal value function ψ.

A finite or polyhedral cone $\mathscr{C}$ is defined as the positive hull of a finite set of vectors $v^1, \ldots, v^N \in \Re^m$; i.e.,

$$\mathscr{C} = \left\{ t \in \Re^m \mid t = \sum_{j=1}^{N} v^j \lambda_j,\ \lambda_j \geq 0 \right\} := \text{pos}(v^1, \ldots, v^N). \tag{3}$$

The vectors v^j, $j = 1, \ldots, N$, are the generators of $\mathscr{C}$. The frame of a finite cone $\mathscr{C}$ is composed of a subset of its generators such that their positive hull is the cone $\mathscr{C}$ itself, but excluding any element in the frame will result in a finite cone $\hat{\mathscr{C}}$ such that $\mathscr{C} \neq \hat{\mathscr{C}}$. The epigraph of a function in $\Re^m$ is the set of points in $\Re^{m+1}$ situated on or above the graph of the function. The epigraph of the function ψ, denoted by epi ψ, is the set

$$\{(\xi, \alpha) \in \Re^{m+1} \mid \alpha \geq \psi(\xi)\}.$$

Obviously, the point $(\xi, \psi(\xi))$ is always on the boundary of epi ψ. The function ψ is convex and positively homogeneous [see Proposition 2.1 in Birge and Wets (Ref. 6)] and hence it is called sublinear. The function with an epigraph composed of the directions of recession of epi ψ is called the recession function and it is denoted by ψ^∞.

A level set of the function ψ for a given value γ, denoted by $\operatorname{lev}_\gamma \psi$, is defined as follows:

$$\operatorname{lev}_\gamma \psi = \{\xi \in \Re^m \mid \psi(\xi) \leq \gamma\}. \tag{4}$$

Since the epigraph of ψ is polyhedral, its level sets are polyhedra in $\Re^m$ (Ref. 5, p. 174).

The following are standard definitions and results from convex analysis (see, e.g., Ref. 5).

Definition 3.1. A cone is pointed if its lineality space is zero. This is equivalent to saying that it contains no linear subspaces except the origin. Therefore, a pointed cone contains no lines.

Definition 3.2. A half-line v belonging to the cone C is an extreme ray of C if it cannot be expressed as a positive linear combination of two distinct rays in C.

We now proceed to state results regarding the optimal value function ψ and its epigraph. The following results will add insights into the geometry and properties of the level sets and the data in the linear program which define the optimal value function ψ.

Theorem 3.1. The polyhedron $\operatorname{lev}_\gamma \psi$ in $\Re^m$ is bounded if and only if $c_j > 0$, $\forall j$.

Proof. Consider the following result due to Wets (Ref. 7): Suppose that $\psi: \xi \in \Re^m \rightarrow (-\infty, \infty]$ is convex and lower semicontinuous. Then ψ has compact level sets (i.e., ψ is inf-compact) if and only if its recession function, $\psi^\infty > 0$, for all $\xi \neq 0$.

Observe that $\psi^\infty = \psi$ (see Corollary 8.5.2, Ref. 5) and recall that ψ is actually continuous. Now, since $c_j > 0$ if and only if $\psi(\xi) > 0$, $\xi \neq 0$, the result follows directly. □

Due to Theorem 3.1, we know that the level sets of ψ are, in fact, convex polytopes when all the costs in (F*) are strictly positive and that they are unbounded polyhedral convex sets if at least one of the costs is zero or negative.

The following lemma is of independent interest and will be important to our development.

Lemma 3.1. The system $\pi^T A \leq -c$ has at most one solution. It has exactly one solution if and only if the system $\pi^T A \leq c$ also has exactly one solution. These two solutions solve the systems $\pi A = -c$ and $\pi A = c$, respectively.

Proof. By our assumptions, the rank of A is m and D is nonempty. Suppose that the set $\{\pi \mid \pi A \leq -c\}$ is nonempty containing the point $\tilde{\pi}$. Select the point $\hat{\pi} \in D$. Then,

$$\hat{\pi} A \leq c, \qquad \tilde{\pi} A \leq -c. \tag{5}$$

This means that $\hat{\pi} + \tilde{\pi}$ is a homogeneous solution to the system $\pi A \leq c$. Our assumptions on the matrix A imply that D is bounded, which in turn implies that only the trivial homogeneous solution exists; i.e., $\tilde{\pi} = -\hat{\pi}$. This establishes that $\hat{\pi} A \geq c$, implying that $\hat{\pi} A = c$, which is unique when the rank of A is m. The converse follows immediately. □

Theorem 3.2. The lineality of $\psi: \Re^m \to \Re$ is the difference between m and the dimension of D.

Proof. Let $\mathscr{A}$ be the smallest affine set containing D; then $\dim \mathscr{A} = \dim D$. Consider the case where $\dim \mathscr{A} < m$. In this event, there exists a vector, $\hat{\xi}$, belonging to the $(m - \dim \mathscr{A})$-dimensional orthogonal complement of $\mathscr{A}$, such that

$$\max_D \hat{\xi} = -\max_D \pi(-\hat{\xi}).$$

This is true since $\hat{\xi}$ is perpendicular to all feasible directions anywhere in D. Therefore, by strong duality,

$$-\psi(\hat{\xi}) = \psi(-\hat{\xi}).$$

Since ψ is sublinear, this implies that ψ is linear in all directions $\hat{\xi}$ belonging to the orthogonal complement of $\mathscr{A}$. In the case where $\dim \mathscr{A} = m$, it is clear that

$$-\psi(\hat{\xi}) = \psi(-\hat{\xi})$$

is never possible. □

As a corollary to Lemma 3.1 and Theorem 3.2 above, the cone epi ψ is pointed only when D has dimension m. Notice that, if $c_j > 0$ for all j, then

$$\psi(\xi) + \psi(-\xi) > 0, \qquad \text{for any } \xi,$$

and epi ψ is necessarily pointed. The presence of a linearlity space is a particularly serious complication when it is of full dimension; that is, when the lineality of ψ is m, or equivalently when ψ is linear. Another consequence of Lemma 3.1 and Theorem 3.2 above is that the function ψ is linear only when the vector $c \in \Re^n$ belongs to the row space of A. We may verify this by checking if the least squares projection of the vector c onto the row space of A is c itself. This is an unusual occurrence, since it requires that the dual feasible region be a single degenerate extreme point. We will refer to sets of this sort as degenerate.

Theorem 3.3. Assume that the set D is not degenerate. Then,

$$\text{epi}\,\psi = \text{pos}\left(\begin{bmatrix} A_{\cdot j} \\ c_j \end{bmatrix}, j = 1, \ldots, n\right).$$

Proof. First we demonstrate that

$$\text{epi}\,\psi \subseteq \text{pos}\left(\begin{bmatrix} A_{\cdot j} \\ c_j \end{bmatrix}, j = 1, \ldots, n\right).$$

Suppose that it is not true. Then, for an arbitrary point $(\hat{\xi}, \hat{\alpha}) \in \text{epi}\,\psi$, the system

$$\begin{bmatrix} \hat{\xi} \\ \hat{\alpha} \end{bmatrix} = \begin{bmatrix} A_{\cdot 1} \\ c_1 \end{bmatrix} \lambda_1 + \cdots + \begin{bmatrix} A_{\cdot n} \\ c_n \end{bmatrix} \lambda_n, \qquad \lambda_1 \geq 0, \ldots, \lambda_n \geq 0, \tag{6}$$

has no solution. This implies by Farkas' lemma that there exists a vector $\pi \in \Re^m$ and a scalar σ such that the two relations

$$[\pi^T, \sigma] \begin{bmatrix} A \\ c^T \end{bmatrix} \leq 0, \qquad [\pi^T, \sigma] \begin{bmatrix} \hat{\xi} \\ \hat{\alpha} \end{bmatrix} > 0 \tag{7}$$

have a solution. If this is true, then the first part implies that $\pi^T A \leq -\sigma c$ is feasible. But by Lemma 3.1, since the region D is not degenerate, necessarily $\sigma < 0$. From the second part we know that $\pi^T \hat{\xi} > -\sigma \hat{\alpha}$ or that $\pi'^T \hat{\xi} > \hat{\alpha}$ if we let $\pi' = -(1/\sigma)\pi$. Now recall that $\psi(\hat{\xi}) = \hat{\pi}^T \hat{\xi}$, where $\hat{\pi}$ solves the linear program $\max \pi^T \hat{\xi}$ and $\psi(\hat{\xi}) \leq \hat{\alpha}$, since $(\hat{\xi}, \hat{\alpha}) \in \text{epi}\,\psi$. Positioning these relations, we obtain that

$$\hat{\pi}^T \hat{\xi} = \psi(\hat{\xi}) \leq \hat{\alpha} < \pi'^T \hat{\xi},$$

which is the contradiction we seek, since $\hat{\pi}^T \hat{\xi} < \pi'^T \hat{\xi}$ is impossible when $\hat{\pi}$ solves the dual $\max_D \pi^T \hat{\xi}$.

Next, we demonstrate the reverse inclusion. $\sum_{j=1}^n \begin{bmatrix} A_{\cdot j} \\ c_j \end{bmatrix} \lambda_j$ belongs to epi ψ for $\lambda_j \geq 0$, $j = 1, \ldots, n$, if $c_j \geq \psi(A_{\cdot j})$. This inequality follows since a feasible solution to the linear program (P) when $b = A_{\cdot j}$ is $x_j = 1$. At this solution, the objective function value is precisely c_j, which may or may not be optimal. □

As we have seen, if the set D is degenerate, then the function ψ is linear. In this event,

$$\text{pos}\left(\begin{bmatrix} A_{\cdot j} \\ c_j \end{bmatrix}, j = 1, \ldots, n\right) \subset \text{epi}\,\psi, \tag{8}$$

with the converse not being true at all. Essentially, the convex hull of the graph of ψ does not contain the epigraph. However, if the vector $[0^T, 1] \in \Re^{m+1}$ is added to the list, then the inclusion goes the other way and the epigraph will be the convex hull of the graph.

Theorem 3.4. The vector $[\xi^*, \psi(\xi^*)] \in \Re^{m+1}$ is an extreme ray of epi ψ if and only if the point $\xi^* \in \Re^m$ is an extreme point of some level set $\text{lev}_{\psi(\xi^*)}\,\psi$ provided that $\psi(\xi^*) \neq 0$.

Proof. Suppose that $[\xi^*, \psi(\xi^*)]$ is not an extreme ray of epi ψ. Since $\psi(\xi^*) \neq 0$, we can select two distinct vectors,

$$v^1 = [\xi^1, \gamma_1] \in \text{epi}\,\psi, \qquad v^2 = [\xi^2, \gamma_2] \in \text{epi}\,\psi,$$

such that the last components $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$ have the same sign as $\psi(\xi^*)$ and

$$[\xi^*, \psi(\xi^*)] = \text{pos}[v^1, v^2].$$

Multiply v^1 and v^2 by $\alpha_1 > 0$ and $\alpha_1 > 0$ respectively, where

$$\alpha_1 = \psi(\xi^*)/\gamma_1, \qquad \alpha_2 = \psi(\xi^*)/\gamma_2.$$

This way,

$$[\xi^*, \psi(\xi^*)] = \text{pos}[\hat{v}^1, \hat{v}^2],$$

where

$$\hat{v}^1 = \alpha_1 v^1, \qquad \hat{v}^2 = \alpha_2 v^2.$$

However, since the $(m + 1)$th components of the three vectors are all equal, the positive multipliers necessary to express $[\xi^*, \psi(\xi^*)]$ as the positive hull of $\hat{v}^1$ and $\hat{v}^2$ must also add to one. This establishes that a convex combination of the points $\alpha_1 \xi^1$ and $\alpha_2 \xi^2$ in $\text{lev}_{\psi(\xi^*)}\,\psi$ contain ξ^*, arriving at the desired result. To prove the converse, suppose that ξ^* is not extreme. Recall that $\psi(\xi^*) \neq 0$. Then, there exist two points ξ^1, ξ^2 both belonging to $\text{lev}_{\psi(\xi^*)}\,\psi$ such that

$$\alpha\xi^1 + (1 - \alpha)\xi^2 = \xi^*, \qquad \text{for some } \alpha \in (0, 1).$$

Then the ray in $\Re^{m+1}$

$$\begin{bmatrix} \xi^* \\ \psi(\xi^*) \end{bmatrix} = \alpha \begin{bmatrix} \xi^1 \\ \psi(\xi^*) \end{bmatrix} + (1-\alpha) \begin{bmatrix} \xi^2 \\ \psi(\xi^*) \end{bmatrix}, \tag{9}$$

and $[\xi^*, \psi(\xi^*)]$ is not extreme. □

Theorem 3.4 can be extended to extreme points of the convex polyhedra which result from intersecting epi ψ with a hyperplane in $\Re^{m+1}$. That is, extreme points of any such polyhedra correspond to extreme rays of epi ψ, and extreme rays of epi ψ that intersect the hyperplane at a single point are extreme points of the intersection set. Note that, if epi ψ is pointed, there exists a hyperplane that intersects all the extreme rays, creating an intersection set which is a bounded polyhedron. The approach for identifying these extreme rays of epi ψ for the case where it is pointed is as follows. Consider a hyperplane that supports the cone epi ψ at the origin only. This hyperplane exists when the cone is pointed. The defining normal for this hyperplane belongs to the interior of the polar cone. That means that we must find a solution to the system of inequalities that describe the polar,

$$\pi^T A_{\cdot j} + \sigma c_j \leq 0, \qquad j = 1, \ldots, n, \tag{10}$$

or

$$\pi^T A_{\cdot j} \leq -\sigma c_j, \qquad j = 1, \ldots, n. \tag{11}$$

Recall though that, from the proof of Theorem 3.3, necessarily $\sigma < 0$, implying that the problem reduces to finding a point in the interior of the set D. This is immediate if ψ is inf-compact (see proof of Theorem 3.1), since $\tilde{\pi} = 0$ is strictly interior to D; otherwise, we may apply efficient iterative techniques such as the one proposed in Ref. 8. So, if $(\tilde{\pi}^T, -1) \in \Re^{m+1}$ belongs to the interior of the polar of epi ψ whenever $\tilde{\pi}$ is in the interior of D, then the hyperplane defined as the solutions to the equation

$$\sum_{i=1}^{m} \tilde{\pi}_i \xi_i - \xi_{m+1} = a$$

intersects with epi ψ whenever $a < 0$. The multipliers, $\alpha_1, \ldots, \alpha_n$, that scale each of the vectors $(A_{\cdot j}^T, c_j)$, $j = 1, \ldots, n$, to the point of intersection with the hyperplane at a given value, say $a = -1$, are given by

$$\alpha_j = -1/(\tilde{\pi}^T A_{\cdot j} - c_j), \qquad j = 1, \ldots, n. \tag{12}$$

Observe that, since $\tilde{\pi}$ is strictly interior to D, the denominator never vanishes and the values of all α_j are always positive. Now, the convex hull of the points in $\mathfrak{R}^{m+1}$

$$\alpha_j \begin{bmatrix} A_{\cdot j} \\ c_j \end{bmatrix}, \qquad j = 1, \ldots, n, \tag{13}$$

define a bounded polytope. Its extreme points correspond to the extreme rays of epi ψ. In Section 4, we apply ideas from this scheme to identify nonredundant constraints.

The scheme proposed in the previous paragraph will identify the extreme rays of epi ψ when it is pointed, and this gives us the frame of the cone. If epi ψ is not pointed, we face other complications. Lineality spaces imply the possibility that the frame is not unique. It also means that the frame may contain nonextreme rays of epi ψ, and that there may be a choice of frames with different cardinalities. In this situation, we may proceed as follows. Begin by finding a point in the relative interior of D; denote this by $\tilde{\pi}$. All columns in A such that $\tilde{\pi}^T A_j = c_j$ belong to the lineality space. Collect these columns into a matrix called A^*; then, the lineality space is given by pos A^*. Suppose that the rank of A^* is r; then, the dimension of pos A^* is r. A frame for the subspace pos A^* may be composed of as few as $r+1$ columns and as many as $2r$ columns of A^*. A frame for epi ψ is composed of the union of the columns in the frame of the subspace pos A^* along with the extreme rays of epi ψ.

4. Applications

We will use the detection of redundancy in linear programming as an example of the applications of the results, ideas, and developments presented in Section 3. We start by defining the terms used in this topic along with some background. Note that the discussion in this section assumes that ψ is not a linear function.

Definition 4.1. A constraint in the linear program (D) is redundant if it may be removed without affecting the feasible set.

There are two types of redundant constraints: those which can never be active or necessary, since the intersection of the defining hyperplane and the feasible region is empty; and those which could be numerically perceived as active although unnecessary in the definition of the feasible set. In this case, the intersection between the hyperplane defining the constraint and the feasible region is not empty. This is, of course, a cause for degeneracy. We refer to the first type of redundancy as strong and the second as weak. Note that a weakly redundant constraint may be removed, but in so doing it may happen that another weakly redundant constraint may become necessary to the description of the feasible region. For example, consider $D \subset \Re^2$ as follows:

$$g_1\colon x_1 \geq 0, \quad g_2\colon x_2 \leq 1, \quad g_3\colon x_1 + x_2 \leq 1, \quad g_4\colon x_1 + x_2 \geq 1, \quad g_5\colon x_1 - x_2 \leq 1.$$

Individually, constraints g_1 and g_2 act as weakly redundant in the sense that either may be removed. However, once one is removed, the other becomes necessary. Constraints g_1 and g_2 are therefore relatively weakly redundant.

Definition 4.2. A variable x_j in the linear program (P) is extraneous if it is not necessary as a basic variable in the definition of an optimal solution, for any right-hand side.

If a variable in (P) is extraneous, the corresponding dual constraint is redundant. Since an extraneous variable is never in the optimal basis of the linear program (P), independently of the right-hand side elements, the corresponding column may be removed without affecting the solution to the problem. There are variables that may appear as basic at optimality for some right-hand sides but for which there is always an alternate optimal basis where this variable is nonbasic. This is the case of weakly extraneous variables, and it corresponds to a weakly redundant constraint in the dual. The status of a weakly extraneous variable may depend on the presence of other weakly extraneous variables, and the removal of a weakly extraneous variable may change the status of other weakly extraneous variables. This category of variables will be termed relatively weakly extraneous.

We will now apply the results on the properties of polyhedral optimal value functions to the detection of nonextraneous variables and necessary constraints. But first we present the following results.

Theorem 4.1. The vector $[A_{\cdot r}^T, \psi(A_{\cdot r})]$ is an extreme ray of epi ψ if and only if $\psi(A_{\cdot r}) = c_r$ and $x_r = 1$ is the unique solution to the corresponding linear program.

Proof. An extreme ray of a polyhedral cone is a one-dimensional edge. A point on an extreme ray can only be expressed as the positive multiple of that specific generating element. If the point $(A_{\cdot r}^T, \psi(A_{\cdot r}))$ is on an extreme ray, then the unique solution to the linear program (P), with $A_{\cdot r}$ as the right-hand side element, is $x_r = 1$ with all other x_j zero, and the optimal value of the linear program is c_r.

To show the converse, consider the linear program (P) with right-hand side $A_{\cdot r}$, and let $x_r = 1$ be an optimal solution. Then, the point $(A_{\cdot r}, c_r) \in \Re^{m+1}$ is on the boundary of epi ψ. If in addition this solution is unique, the point $(A_{\cdot r}, c_r)$ is necessarily on an extreme ray, since otherwise there would exist an alternate representation with other variables having nonzero multipliers. □

An important corollary to this result is that, if the vector $[A_{\cdot r}^T, \psi(A_{\cdot r})]$ is an extreme ray of epi ψ, then the variable x_r in the linear program (P) is nonextraneous. These results permit us to make the following statements about the column corresponding to the variable x_r in (P):

(i) If $c_r = \psi(A_{\cdot r})$ and $x_r = 1$ is the unique solution to the corresponding linear program, then the variable x_r is nonextraneous and the rth dual constraint in (D) is necessary.

(ii) If $c_r = \psi(A_{\cdot r})$, but $x_r = 1$ is not the unique solution to the corresponding linear program, then the variable x_r is weakly extraneous and the rth dual constraint in (D) is weakly redundant. If the vector $[A_{\cdot r}^T, c_r]$ belongs to a lineality space of epi ψ, then it is relatively weakly redundant.

(iii) If $c_r > \psi(A_{\cdot r})$, then the variable x_r is strongly extraneous and the rth dual constraint in (D) is redundant.

A frame of epi ψ is composed of the columns of A. When the cone epi ψ is pointed, the frame is unique and it is composed of the columns of A which correspond to extreme rays of epi ψ. When the cone epi ψ is not pointed, it may be possible to have more than one frame and frames may contain vectors of the cone which are not extreme rays. The elements of the frame correspond, by definition, to necessary variables of the linear program (P), which correspond to necessary constraints in the dual linear program (D). All other variables of the linear program can be removed without affecting the problem. The multiplicity of frames for the cone epi ψ is a result of lineality spaces in the cone. It is the presence of columns in the matrix A corresponding to relatively weakly extraneous variables in the lineality space that account for the multiplicity of frames. With this, we approach the problem of redundancy in linear program as one of identifying a frame or, if not possible, some elements of a frame. This means that our task is to identify extreme rays of epi ψ.

There are several works in the area of redundancy in linear programming. Most notable among these is the monograph by Karwan, Lofti, Telgen, and Zionts (Ref. 9). In this work, several procedures are analyzed for identifying redundant constraints in linear programs such as (D). The most general of these and the one most commonly known requires the partial or complete solutions to several linear programs, possibly as many as there are constraints in the program. Each linear program is formulated to measure the distance (via the slack) between the constraint being tested and the feasible region. It will be established that the constraint is not redundant in the course of the solution using the simplex method before it achieves optimality. If the simplex terminates in an optimal solution and the distance is greater than (or equal) to zero, then this establishes that the constraint is strongly (weakly) redundant.

The dually equivalent approach for the methods described by Karwan *et al.* is to identify extraneous variables in (P) by finding the columns in A where $\psi(A_{.j}) = c_j$, for $j = 1$ to n, such that $x_j = 1$ is a solution. Any gain from knowing *a priori* what the optimal solution should be is offset by the fact that this optimal solution is highly degenerate. However, the geometry of the optimal value function can be exploited to reveal, in advance, several of the extreme rays of its epigraph, identifying thus some nonexraneous variables and avoiding having to solve linear programs for some of the columns. This way, a linear program of the form (P) may be preprocessed, and consequently the task of searching for extraneous variables may be reduced.

Before we proceed, we will resolve the difficulty resulting from Assumption 2.1. This assumption requires that $\text{pos}\, A = \Re^m$; that is, that the columns of the matrix A positively span the space, or equivalently that the dual feasible region D is never unbounded. This may appear to be a restrictive assumption. However, the matrix A can be modified to satisfy this condition without affecting the task at hand of identifying redundancies as long as it has full rank. First, we must ascertain whether the columns of A positively span the space. To do this, generate a vector $\bar{\xi}$ in the interior of $\text{pos}\, A$, for example

$$\bar{\xi} = (1/n) \sum_{j=1}^{n} A_{.j}.$$

Next, we check if the linear system

$$Ax = -\bar{\xi}, \qquad x \geq 0,$$

has a solution. If affirmative, we may conclude $\text{pos}\, A = \Re^m$; otherwise, obviously, $\text{pos}\, A \neq \Re^m$. This is because both systems

$$Ax = \bar{\xi}, \qquad x \geq 0,$$

and

$$Ax = -\bar{\xi}, \qquad x \geq 0,$$

have a feasible solution only when $\text{pos}\, A = \Re^m$. If the columns of the matrix A do not positively span the space and the region D is not empty, it is unbounded and the condition in Assumption 2.1 is violated. Note though that, by adding a single column to A, we can remedy this. This column should be the negative of any vector in int(pos A), for example $-(1/n)\sum_{j=1}^{n} A_{\cdot j}$. Selecting the corresponding cost coefficient to be large enough will guarantee that the dual feasible region will be bounded and furthermore, it will contain all of its previous extreme points.

Next, we present a list of schemes for detecting extreme rays of epi ψ. Extreme rays correspond to nonextraneous variables in the linear program (P) and these, in turn, are associated with necessary constraints in the dual.

(i) This idea was proposed by Wets and Witzgall (Ref. 1). It consists of observing that, if the ith row of the matrix $\begin{bmatrix} A_{\cdot j} \\ c_j \end{bmatrix}$, $j = 1, \ldots, n$, contains exactly one negative entry, then the corresponding column vector belongs to the frame and the associated variable is nonextraneous. This is so because the column where the unique negative entry occurs cannot be expressed as a positive linear combination of the remaining columns, since these do not have negative values in their ith row. Therefore, this column corresponds to an extreme ray of epi ψ.

(ii) This approach is based on the following result. The list of m basic variables in a unique optimal solution for an arbitrary right-hand side is, necessarily, nonextraneous. If one proceeds to evaluate the linear program using the negative of this right-hand side vector, the solution, if unique, will yield another list of m basic variables revealing the status of as many as $2m$ variables. Changing the right-hand sides using Monte Carlo approaches may lead to interesting hit-and-run algorithms of the type in Berbee *et al.* (Ref. 10).

The schemes that follow are based on identifying the frame of epi ψ by intersecting the cone with a suitable hyperplane defined by a vector in the interior of the dual feasible region. This is the procedure discussed at the end of Section 3. Initially, we assume no lineality spaces in epi ψ.

Generate a list of points $p^1, \ldots, p^n$ in $\Re^{m+1}$ as follows:

$$p^j = \alpha_j \begin{bmatrix} A_{\cdot j} \\ c_j \end{bmatrix}, \qquad j = 1, \ldots, n, \tag{14}$$

where

$$\alpha_j = -1/(\tilde{\pi}^T A_{\cdot j} - c_j)$$

is as in expression (12) and where $\tilde{\pi}$ belongs to the interior of the set D. Since no lineality spaces are present in epi ψ, and since $\tilde{\pi}$ is strictly interior to D, the denominator of α_j is always strictly negative. If all c_j are positive, then the vector $(0^T, -1) \in \Re^{m+1}$ is in the interior of D and the conditions above reduce to

$$p^j = \begin{bmatrix} (1/c_j)A_{\cdot j} \\ 1 \end{bmatrix}, \qquad j = 1, \ldots, n. \tag{15}$$

This means that we need only look at the convex hull of the points $(1/c_j)A_{\cdot j}$, $j = 1, \ldots, n$, in $\Re^m$.

A consequence of Theorems 3.3 and 4.1 is that the extreme points of the polytope defined as the convex hull of $p^j, j = 1, \ldots, n$, denoted as cov p^j, $j = 1, \ldots, n$, correspond directly to the extreme rays of epi ψ when no lineality space is present. The following schemes work on identifying these extreme points.

(iii) Some of the extreme points of cov $p^j, j = 1, \ldots, n$, can be readily identified. Select one of the points from the list and call it p^k. Calculate the Euclidean distance between each $p^j, j = 1, \ldots, n$, and p^k. The point or points farthest away from p^k are extreme points of cov $p^j, j = 1, \ldots, n$. Note that, if there are ties for the maximum distance, then all points participating in the tie are extreme. This procedure may be executed over all points in the level set, possibly identifying other extreme points.

(iv) Given a direction of optimization v, the point p^j, $j = 1, \ldots, n$, that maximizes or minimizes the inner product $\langle p^j, v \rangle$, if unique, is an extreme point of the set. If exactly two points are involved in a tie, both are extreme. If three or more tie, not all may be extreme. Identifying extreme points using this principle does not require the solution of a linear program, only the evaluation of inner products. One approach is to select the direction of optimization to be a direction along an axis. This corresponds to finding the maximum or minimum component of a row of the matrix composed of the points $p^j, j = 1, \ldots, n$, as columns. Note that this generalizes procedure (i) above. This way, searching for the maximum and minimum component of each row of the matrix may reveal the status of as many as $2m$ points in the polytope, with an effort equivalent to sorting each row.

The presence of lineality spaces complicates these last two procedures. A lineality space will cause an equality in the denominator of α_j in expression (14) for any value of $\tilde{\pi}$ in the relative interior of D. To properly treat this complication requires maneuvers that detract from the simplicity of the scheme. However, we may still gain knowledge about the extreme rays of epi ψ by considering only the points that remain after removing those associated with the columns in A for which $\hat{\pi}^T A_{\cdot j} - c_j$ is zero. The convex hull of the points in (14) that correspond to columns of A such that $\hat{\pi}^T A_{\cdot j} - c_j < 0$ correspond to extreme rays of epi ψ. Note, however, that some extreme rays of this cone will avoid detection if they belong to the linearity space.

We close this section by discussing other related applications of the results above.

The problem of identifying redundancies in linear programming reduces by dual relations to the identification of vertices of the convex hull of a finite set of this points. This is one version of the convex hull problem in computational geometry; see Ref. 11, Section 2. The procedures presented in this paper can be used in this problem directly. Consider a finite set of points in $\Re^m$, $v^1, \ldots, v^n$. One version of the convex hull problem consists of identifying the extreme point of the convex hull of the vectors v^j, $j = 1, \ldots, n$. Constructing the matrix A by making the v^j, $j = 1, \ldots, n$, its columns and setting the cost vector to be all 1's, the extreme points of the level set lev_1 of the resultant (proper) optimal value function are the extreme points of the convex hull of the points p^j, $j = 1, \ldots, n$, as given in expression (15). We may use the schemes presented above as preprocessors to identify some, and possibly many, of these extreme rays.

The convex hull problem described above has important applications. The problem appears directly in computer graphics, design automation, and pattern recognition. It is a subproblem in the solution of other problems such as the construction of Voronoi diagrams (Ref. 3) and the facial decomposition of polytopes. The convex hull problem also appears in applied statistics. Observations from a multivariate statistical sample are points in multidimensional space, and extreme points are, in effect, outliers. The so-called Gastwirth estimators (Ref. 12) can be used to discount the effect of the outliers by discarding the extreme points of the convex hull of the sample. This approach may be implemented recursively removing several layers of extreme points to a specified depth. Identifying each layer is a convex hull problem. Also, the problem of identifying the extreme points of the convex hull is connected directly to the methodology for measuring the comparative efficiency among many economic firms known as data envelopment analysis or DEA. Finally, determining the importance of randomness in the second-stage problem of a stochastic linear program with recourse requires the identification of the generating elements of the frame of an optimal value function of the form of (F*); see Ref. 13.

5. Conclusions

The analysis of polyhedral cones defined as the epigraph of an optimal value function with a linear program as the underlying optimization problem provides insight into the geometry of linear programs. This, in turn, results in ideas for preprocessors that may be applied to the task of identifying extreme points in bounded polytopes and redundant constraints in linear inequalities systems.

The generating elements in the frame of the epigraph of an optimal value function correspond to nonextraneous variables in the underlying linear program. The number of generating elements is at most the number of columns in the coefficient matrix of the linear program, and not in any way combinatorially as large as the number of extreme points in the dual feasible region D. By preprocessing, it is possible to identify efficiently a subset from the complete list of generators of the frame reducing the numerical requirements of standard procedures used for detecting redundancies.

We have seen that detecting redundancies and identifying the frame of an optimal value function reduce to the convex hull problem of finding the extreme points of a finite set of points. Finding the extreme points of the convex hull of a finite set of points is equivalent to identifying the frame of a specific optimal value function. This last problem is not fully explored in multiple dimensions. Results here have direct applications in computational geometry, e.g., pattern recognition and artificial vision and in stochastic programming. In the sequel to this paper (Ref. 14), we present ideas and implementation results on preprocessing and resolving the problem of identifying the extreme points of the convex hull of a finite set of points.

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