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Some functional equations in the space of uniform distribution functions

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Abstract. In this paper various functional equations which arise in the study of binary operations on the set of uniform probability distribution functions are considered and solved.

1. Introduction.

Under certain rather weak restrictions, any binary operation T on the unit interval $[0, 1]$ induces one or more corresponding binary operations on the space of probability distribution functions. Several such operations and families of operations have been studied in recent years (see [5]). Foremost among these are the τ_T-semigroups which arise naturally in the study of triangle inequalities for probabilistic metric spaces [4, 5, 6, 7, 8]. In addition there are the τ_{T^*}-semigroups, which are in a sense the duals of the τ_T-semigroups and are relevant in the study of betweenness in probabilistic metric spaces, and the operations ρ_C which play a role in the probabilistic extension of the generalized theory of information of Kampé de Fériet and Forte [3, 4].

The aim of this paper is to solve various functional equations which arise when one studies the behavior of the operations τ_T, τ_{T^*}, and ρ_C on the subspace of uniform probability distribution functions, e.g., to determine the functions T for which the τ_T-product of two given uniform distributions is a given uniform distribution.

2. Preliminaries.

Let Δ be the set of one-dimensional probability distribution function, i.e., non-decreasing functions F from $[-\infty, +\infty]$ into $[0, 1]$, which are left continuous on $R=(-\infty, +\infty)$ and such that $F(-\infty)=0, F(+\infty)=1$. The set Δ is naturally

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ordered via $F \leqslant G$ if and only if $F(x) \leqslant G(x)$ for all x in R. Let I denote the closed unit interval $[0, 1]$ and I° the open interval $(0, 1)$, and let $I^2 = I \times I$.

DEFINITION 2.1. A *t-norm* is a two-place function T from $I \times I$ into I such that,

(a) $T(a, 1) = a$, $T(a, 0) = 0$,
(b) $T(a, b) \geqslant T(c, d)$ for $a \geqslant c$, $b \geqslant d$,
(c) $T(a, b) = T(b, a)$,
(d) $T(a, T(b, c)) = T(T(a, b), c)$.

For example, the functions $\mathrm{Min}(a, b)$, $\mathrm{Prod}(a, b) = a \cdot b$, $T_m(a, b) = \mathrm{Max}(a + b - 1, 0)$ and

$$T_w(a, b) = \begin{cases} a, & \text{if} \quad b = 1, \\ b, & \text{if} \quad a = 1, \\ 0, & \text{otherwise,} \end{cases}$$

are t-norms. Moreover, under the usual pointwise ordering of functions, we have $\mathrm{Min} \geqslant \mathrm{Prod} \geqslant T_m \geqslant T_w$ and $\mathrm{Min} \geqslant T \geqslant T_w$, for any t-norm T.

It has been shown [5, 8] that, if T is a left-continuous t-norm, i.e., if $\lim_{x \to a-,\, y \to b-} T(x, y) = T(a, b)$ for all (a, b) different from $(0, 0)$ in $I \times I$, then the function τ_T defined by

$$\tau_T(F, G)(x) = \sup_{u+v=x} T(F(u), G(v)), \tag{2.1}$$

for any F, G in Δ and $-\infty \leqslant x \leqslant \infty$, is an order-preserving binary operation on Δ and that (Δ, τ_T) is a commutative semigroup with unit element ε_0, the step function defined by

$$\varepsilon_0(x) = \begin{cases} 0, & \text{if} \quad x \leqslant 0, \\ 1, & \text{if} \quad x > 0. \end{cases}$$

We remark that the operation τ_T is also well-defined when T is neither associative nor commutative.

DEFINITION 2.2. A (two-dimensional) *copula* is a two-place function C from $I \times I$ into I satisfying the conditions

(a) $C(a, 0) = C(0, a) = 0$, $C(a, 1) = C(1, a) = a$,
(b) $C(a, c) - C(a, d) - C(b, c) + C(b, d) \geqslant 0$, for $a \leqslant b$, $c \leqslant d$.

It is easy to show that Min, Prod and T_m are copulas, that copulas are continuous and non-decreasing in each place, and that $T_m \leq C \leq \text{Min}$, for any copula C [7].

In the sequel, $\mathcal{L}$ will denote the set of all two-place functions S from $I \times I$ into I which are non-decreasing in each place and such that $T_w \leq S \leq \text{Min}$. The uniform distribution function on the interval $[x, y]$, $x, y \in R$, will be denoted by U_{xy} (or $U_{x,y}$), so that, $U_{xx} = \varepsilon_x$ is the step function given by $\varepsilon_x(t) = \varepsilon_0(t-x)$, and for $x < y$,

$$U_{xy}(t) = \begin{cases} 0, & \text{if} \quad t \leq x, \\ \dfrac{t-x}{y-x}, & \text{if} \quad x \leq t \leq y, \\ 1, & \text{if} \quad t \geq y. \end{cases}$$

3. τ_T operations and uniform distributions.

Our chief concern in this section is the functional equation $\tau_T(U_{ab}, U_{cd}) = U_{ef}$, where a, b, c, d are given and T is to be found.

LEMMA 3.1. *If $S \in \mathcal{L}$ then we have*

(i) $\tau_S(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b}$, *for any a, b in R;*
(ii) $\tau_S(\varepsilon_a, U_{cd}) = U_{a+c,a+d}$, *for any a, c, d in R with $c < d$.*

It is well-known (see, e.g., [2]) that τ_{Min} admits the representation

$$\tau_{\text{Min}}(F, G) = (F^\wedge + G^\wedge)^\wedge,$$

where for any H in Δ, $H^\wedge$ is the quasi-inverse of H, given by $H^\wedge(0) = -\infty$ and $H^\wedge(t) = \sup\{x \mid H(x) < t\}$ for $t \in (0, 1]$. In particular, $U_{ab}^\wedge(x) = (b-a)x + a$ for $x \in (0, 1]$ whence we have:

LEMMA 3.2. *If $a < b$ and $c < d$ then $\tau_{\text{Min}}(U_{ab}, U_{cd}) = U_{a+c,b+d}$.*

THEOREM 3.1. *Let $S \in \mathcal{L}$ and let $a < b$ and $c < d$. If $\tau_S(U_{ab}, U_{cd}) = U_{ef}$, for some $e \leq f$, then $f = b + d$ and $a + c \leq e \leq \text{Min}(a+d, b+c)$, consequently, $e < f$.*

Proof. Consider the inequalities,

$$\varepsilon_b \leq U_{ab} \leq \varepsilon_a, \; \varepsilon_d \leq U_{cd} \leq \varepsilon_c.$$

Since τ_S is non-decreasing, using Lemma 3.1 we have

$$\varepsilon_{b+d} = \tau_S(\varepsilon_b, \varepsilon_d) \leqslant \tau_S(U_{ab}, U_{cd}) = U_{ef} \leqslant \tau_S(\varepsilon_a, \varepsilon_c) = \varepsilon_{a+c},$$

whence $a + c \leqslant e \leqslant f \leqslant b + d$. Moreover, from $\tau_S \leqslant \tau_{\mathrm{Min}}$ and Lemma 3.2 we obtain $U_{ef} \leqslant U_{a+c,b+d}$, so that $f \geqslant b+d$ and we conclude $f = b+d$. Using $T_w \leqslant S$, $\tau_S(U_{ab}, U_{cd})\ (e) = 0$, and $\tau_{T_w}(U_{ab}, U_{cd})\ (x) > 0$ whenever $x > \mathrm{Min}\,(a+d, b+c)$, it further follows that $e \leqslant \mathrm{Min}\,(a+d, b+c)$.

Theorem 3.1 suggests the study of the family of functional equations

(FE I) $\tau_S(U_{ab}, U_{cd}) = U_{a+c,b+d}$,
(*FE* II) $\tau_S(U_{ab}, U_{cd}) = U_{\mathrm{Min}(a+d,b+c),b+d}$,
(FE III) $\tau_S(U_{ab}, U_{cd}) = U_{a+c+\theta(\mathrm{Min}(a+d,b+c)-a-c),b+d}$,
for some given θ in I°.

Note that (FE I) and FE II) are the limiting cases $\theta = 0$ and $\theta = 1$ of (FE III), respectively. The rest of this section is concerned with these equations.

LEMMA 3.3. *If* $S \in \mathscr{L}$ then $\tau_S(U_{ab}, U_{cd}) = U_{e,b+d}$ *is equivalent to the statement that, for any* x *in* I,

$$\sup_{\Omega(x)} S(u, v) = x, \tag{3.1}$$

where $\Omega(x) = \{(u, v) \in I \times I | (b-a)u + (d-c)v = (b+d-e)x + e - a - c\}$.

Proof. For any $x \in I$, the point $W_x = (b+d-e)x + e$ is in $[e, b+d]$. Consequently, using (2.1), we have

$$x = U_{e,b+d}(W_x) = \sup\,\{S(U_{ab}(z), U_{cd}(t)) | z + t = W_x\}$$
$$= \sup\left\{S\left(U_{01}\left(\frac{z-a}{b-a}\right), U_{01}\left(\frac{t-c}{d-c}\right)\right) \Big| z + t = W_x\right\}$$

which is equivalent to (3.1) by the change of variables $u = (z-a)/(b-a)$, $v = (t-c)/(d-c)$.

THEOREM 3.2. (FE I) *Let* $a < b$, $c < d$ *and let* S *be in* $\mathscr{L}$. *Then* $\tau_S(U_{ab}, U_{cd}) = U_{a+c,b+d}$ *if and only if* $S = \mathrm{Min}$.

Proof. Sufficiency follows from Lemma 3.2. To prove necessity, assume that $\tau_S(U_{ab}, U_{cd}) = U_{a+c,b+d}$ or equivalently, in view of Lemma 3.3, that for any $x \in I^\circ$, (3.1) holds with $\Omega(x)=\{(u, v)\in I\times I|Au+Bv=(A+B)x\}$, where $A=b-a$ and $B=d-c$. For any fixed $x\in I°$, choose $\varepsilon>0$ so that $x-\varepsilon/2>0$. Then

$$\sup_{\Omega(x-\varepsilon/2)} S(u, v)=x-\varepsilon/2.$$

Consequently for $\delta=\varepsilon/4\,\mathrm{Min}\,(A/B, B/A)$ there exist u_0, v_0 such that $(u_0, v_0)\in\Omega(x-\varepsilon/2)$ and

$$x-(\varepsilon/2)-S(u_0, v_0)<\delta. \tag{3.2}$$

Since $\delta<\varepsilon/2$, we have

$$x-\varepsilon<x-(\varepsilon/2)-\delta<S(u_0, v_0). \tag{3.3}$$

Suppose $x<u_0$. Then, since $(u_0, v_0)\in\Omega(x-\varepsilon/2)$, we have $v_0<x-(\varepsilon/2)-(A/B)(\varepsilon/2)<u_0$, which together with (3.3) yields

$$x-(\varepsilon/2)-\delta<S(u_0, v_0)\leq\mathrm{Min}\,(u_0, v_0)=v_0<x-(\varepsilon/2)-(A/B)(\varepsilon/2),$$

i.e., $\delta>(A/B)(\varepsilon/2)$, which is a contradiction. Analogously the assumption $x<v_0$ yields the contradiction $\delta>(B/A)(\varepsilon/2)$. Thus $u_0\leq x$, $v_0\leq x$, whence from (3.3) and the fact that $S\leq\mathrm{Min}$, we have

$$x-\varepsilon<S(u_0, v_0)\leq S(x, x)\leq x.$$

It follows that $S(x, x)=x$ for every x in $I°$, which in turn implies $S=\mathrm{Min}$.

We now turn to the study of (FE II). To this end, for any $\lambda\in I$, let T_λ be the function from $I\times I$ into I defined by

$$T_\lambda(u, v)=\mathrm{Max}\,[\mathrm{Min}\,(u, v)+\lambda\,\mathrm{Max}\,(u, v)-\lambda, 0]. \tag{3.4}$$

Clearly $T_\lambda\in\mathscr{L}$, is continuous, commutative, and $\mathrm{Min}=T_0\geq T_\lambda\geq T_1=T_m$. (See Figure 2. The graph of the surface corresponding to T_λ is the special case in which $P_2=(0, 1, 0)$, $P_4=(1, 0, 0)$ and $P_1=(\lambda/1+\lambda, \lambda/1+\lambda, 0)$.) Moreover, since $\lambda\leq\lambda'$ implies $T_\lambda\geq T_{\lambda'}$, the collection $\{T_\lambda|\lambda\in I\}$ is a decreasing family of functions on $I\times I$, with maximum $T_0=\mathrm{Min}$ and minimum $T_1=T_m$.

THEOREM 3.3. *T_λ is associative if and only if $\lambda=0$ or $\lambda=1$.*

Proof. $T_0=\mathrm{Min}$ and $T_1=T_m$ are associative. If T_λ were associative for some

$\lambda \in I^\circ$, then, with $a = b = (4\lambda + 3)/(4\lambda + 4)$ and $c = \lambda/(\lambda + 1)$, we would have

$$(3\lambda - \lambda^2)/(4\lambda + 4) = T_\lambda(T_\lambda(a, b), c) = T_\lambda(a, T_\lambda(b, c)) = 2\lambda/(4\lambda + 4),$$

i.e., $\lambda(\lambda - 1) = 0$, which is a contradiction.

A straighforward computation shows that all the T_λ are copulas. For $\lambda \in I^\circ$, the mass of T_λ is concentrated on the three line segments which join the point $(\lambda/(1+\lambda), \lambda/(1+\lambda))$ to the points $(0, 1)$, $(1, 0)$ and $(1, 1)$, respectively. Thus we have a family of copulas which are not t-norms.

THEOREM 3.4. (FE II) *Let a, b, c, d be fixed and such that $a < b$, $c < d$, and let $S \in \mathcal{L}$ be commutative. Then $\tau_s(U_{ab}, U_{cd}) = U_{\mathrm{Min}(a+d,\, b+c),\, b+d}$ if and only if $S \leqslant T_\mu$, where $\mu = \mathrm{Min}\,(b - a, d - c)/\mathrm{Max}\,(b - a, d - c)$.*

Proof. We will assume, without loss of generality, that $A = b - a \leqslant d - c = B$. Then $\mu = A/B$ and $\mathrm{Min}\,(a + d, b + c) = b + c$.

Suppose that $\tau_S(U_{ab}, U_{cd}) = U_{b+c,b+d}$. Then by Lemma 3.3 we have,

$$\sup\{S(u, v) | \mu u + v - \mu = x\} = x, \tag{3.5}$$

for all $x \in I$. Since S is commutative, for any $x \in I$ we also have

$$\sup\{S(u, v) | u + \mu v - \mu = x\} = x. \tag{3.6}$$

Letting $x = 0$ in (3.5) and (3.6) yields that $S(u, v) = 0 = T_\mu(u, v)$, whenever $\mu u + v - \mu \leqslant 0$ or $u + \mu v - \mu \leqslant 0$. Next, if $u_0, v_0 \in I^\circ$ are such that $\mu u_0 + v_0 - \mu$ and $u_0 + \mu v_0 - \mu$ are both in I°, then from (3.5) and (3.6) we have

$$\sup\{S(u, v) | \mu u + v - \mu = \mu u_0 + v_0 - \mu\} = \mu u_0 + v_0 - \mu \geqslant S(u_0, v_0),$$

$$\sup\{S(u, v) | u + \mu v - \mu = u_0 + \mu v_0 - \mu\} = u_0 + \mu v_0 - \mu \geqslant S(u_0, v_0),$$

whence $S(u_0, v_0) \leqslant \mathrm{Min}\,(u_0, v_0) + \mu\,\mathrm{Max}\,(u_0, v_0) - \mu = T_\mu(u_0, v_0)$. Thus $S \leqslant T_\mu$.

Conversely, suppose that $S \leqslant T_\mu$ and let $x \in I$. Some calculation yields that for all $(u, v) \in I \times I$ which lie on the line $Au + Bv = Bx + A$, we have $T_\mu(u, v) \leqslant x$, so that

$$S(u, v) \leqslant T_\mu(u, v) \leqslant x = S(1, x),$$

Consequently

$$\sup\{S(u, v) | \mu u + v - \mu = x\} = x,$$

whence, by Lemma 3.3, $\tau_S(U_{ab}, U_{cd}) = U_{b+c,b+d}$.

COROLLARY 3.1. *Under the hypotheses of Theorem* 3.4, $\tau_S(U_{ab}, U_{cd}) = U_{\mathrm{Min}(a+d,b+c),b+d}$, *for all* $a<b$, $c<d$, *if and only if* $S \leq T_m$.

COROLLARY 3.2. *If* C *is a copula then* $\tau_C(U_{ab}, U_{cd}) = U_{\mathrm{Min}(a+d,b+c),b+d}$, *for all* $a<b$, $c<d$ *if, and only if,* $C = T_m$.

Note that, for any fixed $a<b$ and $c<d$, (FE I) has the unique solution $S = \mathrm{Min}$, while (FE II) admits an infinity of solutions (dependent on the parameters a, b, c, d), of which T_μ is the strongest.

To analyse (FE III) we need to introduce some additional machinery. Given two parameters $\alpha \in (0, 1]$ and $\theta \in I$, let (see Fig. 1),

$$L_\alpha^\theta = \{(u, v) \in I^2; u + \alpha v \leq \alpha\theta \quad \text{or} \quad \alpha u + v \leq \alpha\theta\},$$

$$M^\theta = \{(u, v) \in I^2; (1-\theta)u + \theta \leq v \quad \text{or} \quad (1-\theta)v + \theta \leq u\}.$$

Let W_α^θ be defined on $I \times I$ by

$$W_\alpha^\theta(u, v) = \begin{cases} \mathrm{Min}\,(u, v), & \text{if} \quad (u, v) \in M^\theta, \\ \mathrm{Max}\left[\dfrac{\alpha\,\mathrm{Max}\,(u, v) + \mathrm{Min}\,(u, v) - \alpha\theta}{1 + \alpha - \alpha\theta}, 0\right], & \text{if} \quad (u, v) \notin M^\theta. \end{cases}$$

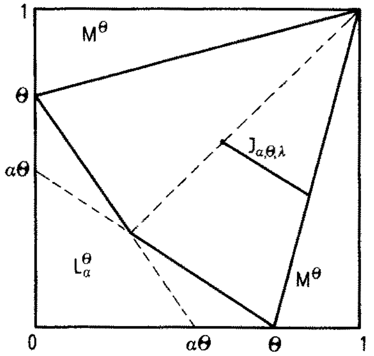

Figure 1

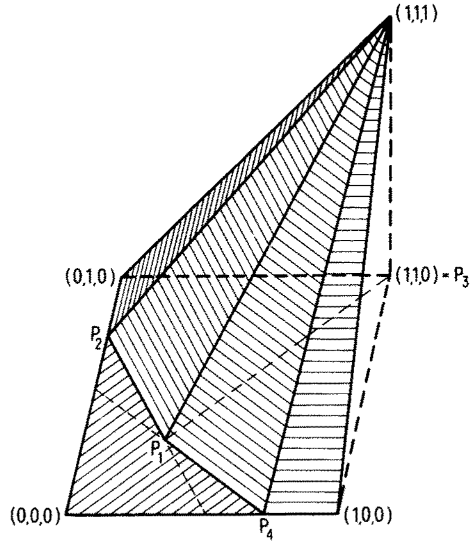

Figure 2

The graph of the surface corresponding to W_α^θ is shown in Figure 2, where $P_1 = (\alpha\theta/(1+\alpha), \alpha \cdot \theta/(1+\alpha), 0)$, $P_2 = (0, \theta, 0)$ and $P_4 = (\theta, 0, 0)$.

It is not hard to show that W_α^θ is a commutative copula and that

$$W_1^1 = T_m \leqslant W_\alpha^1 = T_\alpha \leqslant W_\alpha^\theta \leqslant \mathrm{Min} = W_\alpha^0,$$

where T_α is defined by (3.4). (For $\alpha, \theta \in I^\circ$, the mass of W_α^θ is concentrated on the line segments P_1P_2, P_2P_3, P_3P_4, P_4P_1 and P_1P_3, where P_1, P_2, P_4 are as above and $P_3 = (1, 1, 0)$.

In order to simplify the notation, if $A \subset I^2$, then $A^\wedge$ will denote the reflection of A in the main diagonal of the unit square, i.e., $A^\wedge = \{(u, v) | (v, u) \in A\}$. If $A = A^\wedge$ we will say that A is *symmetric*.

THEOREM 3.5. (FE I, II, III) *Let a, b, c, d, θ be fixed and such that $a < b$, $c < d$, $\theta \in I$, and let $S \in \mathscr{L}$ be commutative and continuous. Then $\tau_S(U_{ab}, U_{cd}) = U_{a+c+\mathrm{Min}(b-a,\, d-c)\theta,\, b+d}$, if and only if there exists a subset K_α^θ of I^2, with $\alpha =$* $\mathrm{Min}\,(b-a, d-c)/\mathrm{Max}\,(b-a, d-c)$, *such that*

(i) *$S \leqslant W_\alpha^\theta$ on I^2 and $S = W_\alpha^\theta$ on K_α^θ.* (See Fig. 1, 2).
(ii) *K_α^θ is symmetric and $L_\alpha^\theta \subset K_\alpha^\theta$.*

(iii) $K_\alpha^\theta \cap J_{\alpha,\theta,\lambda} \neq \varnothing$, *for each* $\lambda \in I$, *where* $J_{\alpha,\theta,\lambda}$ *is the segment* $\{(u, \alpha\theta+(1+\alpha-\alpha\theta)\lambda-\alpha u) \mid ((1+\alpha-\alpha\theta)\lambda+\alpha\theta)/(1+\alpha) \leq u \leq (1-\theta)\lambda+\theta\}$.

Proof. Again, without loss of generality we will consider the case $A = b-a \geq d-c = B$ so that $\alpha = B/A \in (0, 1]$.

Suppose that S is a solution of (FE III), i.e., that

$$\tau_S(U_{ab}, U_{cd}) = U_{a+c+B\theta,\, b+d}.$$

Then, by Lemma 3.3 and the commutativity of S, for any $x \in I$ we have

$$\sup_{L(x)} S(u, v) = \sup_{L(x)^\wedge} S(u, v) = x, \tag{3.7}$$

where $L(x) = \{(u, v) \in I^2 \mid \alpha u + v - \alpha\theta = (1+\alpha-\alpha\theta)x\}$. Taking $x = 0$ in (3.7) yields $S(u, v) = 0$ whenever $(u, v) \in L(0) \cup L(0)^\wedge (= P_1P_2 \cup P_1P_4)$, whence, since S is non-decreasing, $S(u, v) = 0 = W_\alpha^\theta(u, v)$ on L_α^θ. Obviously $S(u, v) \leq \operatorname{Min}(u, v) = W_\alpha^\theta(u, v)$ whenever $(u, v) \in M^\theta$. Now suppose $(u, v) \in I^2$- $(L_\alpha^\theta \cup M^\theta)$ and $v \leq u$. Then there is a unique x such that $(u, v) \in L(x)$, i.e.,

$$x = \frac{\alpha u + v - \alpha\theta}{1+\alpha-\alpha\theta} = \frac{\alpha \operatorname{Max}(u, v) + \operatorname{Min}(u, v) - \alpha\theta}{1+\alpha-\alpha\theta} = W_\alpha^\theta(u, v).$$

Hence, by (3.7), we have $S(u, v) \leq x = W_\alpha^\theta(u, v)$. Similarly, if $u \leq v$, it follows that $S(u, v) \leq W_\alpha^\theta(u, v)$. Thus $S \leq W_\alpha^\theta$ on all of $I \times I$. Next, since S is continuous, for each $x \in (0, 1]$ there exists a point $(u_x, v_x) \in L(x)$ such that

$$x = \sup_{L(x)} S(u, v) = S(u_x, v_x) \leq W_\alpha^\theta(u_x, v_x) \leq \sup_{L(x)} W_\alpha^\theta(u, v).$$

Looking at the definition of W_α^θ and $L(x)$ it is easily seen that W_α^θ is constant and equal to x on the segment $J_{\alpha\theta x}$ and strictly less than x at any other point of $L(x)$. Thus $(u_x, v_x) \in J_{\alpha\theta x}$ and $S(u_x, v_x) = W_\alpha^\theta(u_x, v_x) = x$.

So let

$$K_\alpha^\theta = \{(u_x, v_x), (v_x, u_x) \mid x \in (0, 1]\} \cup L_\alpha^\theta.$$

By construction S satisfies (i) and K_α^θ satisfies (ii) and (iii).

Conversely, suppose there exists a set K_α^θ satisfying the conditions (i), (ii) and (iii). Since $L_\alpha^\theta \subset K_\alpha^\theta$ and since $S = W_\alpha^\theta = 0$ on L_α^θ, it follows that (3.7) holds for $x = 0$. Next choose any $x \in I^0$. By (ii), $K_\alpha^\theta \cap J_{\alpha,\theta,x} \neq \varnothing$, whence there exists a point

$(u_x, v_x) \in K_\alpha^\theta \cap J_{\alpha,\theta,x}$ such that

$$S(u_x, v_x) = W_\alpha^\theta(u_x, v_x) = \frac{\alpha u_x + v_x - \alpha\theta}{1 + \alpha - \alpha\theta} = x,$$

and consequently (3.7) holds because S and W_α^θ are commutative and K_α^θ is symmetric.

Note that the conclusions of Theorems 3.2 and 3.4 are simple consequences of Theorem 3.5. In the case $\theta = 0$ we have $K_\alpha^0 = \{(x, x) \mid x \in I\}$ and $S = W_\alpha^0 = \mathrm{Min}$; and in the case $\theta = 1$ we have $K_\alpha^1 = L_\alpha^1 \cup \{(x, 1), (1, x) \mid x \in I\}$ and $S \le W_\alpha^1 = T_\alpha$. But note also that, whereas the continuity of S plays a crucial role in the proof of Theorem 3.5, it is not needed in the proofs of Theorem 3.2 and Theorem 3.4.

COROLLARY 3.3 (FE III, in global form). *Under the hypotheses of the above theorem, for a fixed* $\theta \in I^0$, $\tau_S(U_{ab}, U_{cd}) = U_{a+c+\mathrm{Min}(b-a,\, d-c)\theta,\, b+d}$, *for all* $a < b$, $c < d$, *if and only if* $S \le W_1^\theta$ *and* $S = W_1^\theta$ *on* K_1^θ.

The commutativity of S postulated in Theorem 3.5 can be reduced to the assumption that $\tau_S(U_{ab}, U_{cd}) = \tau_s(U_{cd}, U_{ab})$.

4. Dual operations and uniform distributions.

In the study of probabilistic metric spaces and related topics, certain operations τ_{S^*}, which are in a sense the duals of the τ_S, play a vital role. These are defined by

$$\tau_{S^*}(F, G)(x) = \inf_{u+v=x} S^*(F(u), G(v)),\ F, G \in \Delta,\ -\infty \le x \le +\infty,$$

where S^* is a two-place function from $I \times I$ into I satisfying

(a′) $S^*(a, 1) = S^*(1, a) = 1$, $S^*(a, 0) = S^*(0, a) = a$,
(b′) $S^*(a, b) \le S^*(c, d)$ if $a \le c$ and $b \le d$.

If $S \in \mathcal{L}$ then the function

$$S^*(a, b) = 1 - S(1 - a, 1 - b) \tag{4.1}$$

satisfies the conditions above and, dually, for any S^* the function

$$S(a, b) = 1 - S^*(1 - a, 1 - b)$$

is in $\mathscr{L}$. Thus the behavior of the operations τ_{S^*} on uniform distributions will be dual to that of the operations τ_S studied in Section 3 and we have at once:

LEMMA 4.1. *Let $S \in \mathscr{L}$ and let S^* be the dual operation given by* (4.1). *Then* $\tau_{S^*}(U_{ab}, U_{cd}) = U_{ef}$, *if and only if* $\tau_S(U_{ab}, U_{cd}) = U_{a+b+c+d-f, a+b+c+d-e}$.

THEOREM 4.1. *Let $a<b$, $c<d$ and let S^* be a commutative function from $I \times I$ into I satisfying* (a′) *and* (b′). *Then*

(i) *If* $\tau_{S^*}(U_{ab}, U_{cd}) = U_{ef}$ *then, necessarily,* $e = a+c$ *and* $f \geq \text{Max}(a+d, b+c)$;
(ii) $\tau_{S^*}(U_{ab}, U_{cd}) = U_{a+c, b+d}$ *if and only if* $S^*(u, v) = \text{Max}(u, v)$;
(iii) $\tau_{S^*}(U_{ab}, U_{cd}) = U_{a+c, \text{Max}(a+d, b+c)}$ *if and only if* $S^*(u, v) \geq S^*_\alpha(u, v) = \text{Min}(1, \text{Max}(u, v) + \alpha \,\text{Min}(u, v))$, where $\alpha = \text{Min}(b-a, d-c)/\text{Max}(b-a, d-c)$;
(iv) *If S^* is continuous and $\theta \in I^0$ then*

$$\tau_{S^*}(U_{ab}, U_{cd}) = U_{a+c, \text{Min}(b-a, d-c)\theta + \text{Max}(a+d, b+c)}$$

if and only if there exists a symmetric set K^θ_α, as given in Theorem 3.5, *such that*

$$S^* = (W^\theta_\alpha)^* \text{ on } K^\theta_\alpha \text{ and } S^* \geq (W^\theta_\alpha)^* \text{ on } I^2.$$

In particular, Theorem 4.1 holds for the operations τ_{T^*}, where T is a t-norm and $T^*(x, y) = 1 - T(1-x, 1-y)$ is its associated t-conorm.

A second family of binary operations is given ([4]) by the following:

DEFINITION 4.1. For any copula C, ρ_C is the binary operation on Δ defined by

$$\rho_C(F, G)(x) = \inf_{u+v=x}(F(u) + G(v) - C(F(u), G(v))).$$

For any copula C the function $T_C(x, y) = x + y - 1 + C(1-x, 1-y)$ is continuous and satisfies conditions (a) and (b) of Definition 2.1. It is immediate that $T^*_C(x, y) = x + y - C(x, y)$ and consequently $\rho_C = \tau_{T^*_C}$. Thus the ρ_C operations are particular cases of τ_{S^*} operations and Theorem 4.1 applies whenever C is commutative. Since any copula C is continuous and satisfies $C \geq T_m$, we can state part (iii) of Theorem 4.1 as follows.

COROLLARY 4.1. *Let C be a commutative copula. Then* $\rho_C(U_{ab}, U_{cd}) = U_{a+c, \text{Max}(a+d, b+c)}$ *if and only if* $C \leq C_\alpha$, *where* $\alpha = \text{Min}(b-a, d-c)/$

$\operatorname{Max}(b-a, d-c)$ *and*

$$C_\alpha(x, y) = \operatorname{Max}[(1-\alpha)\operatorname{Min}(x, y), T_m(x, y)].$$

In particular $\rho_C(U_{ab}, U_{c+b-a}) = U_{a+c,b+c}$, if and only if $C = T_m$; and $\rho_C(U_{ab}, U_{cd}) = U_{a+c,\operatorname{Max}(a+d,b+c)}$, for all $a<b$, $c<d$ if and only if $C = T_m$.

Note that for $\alpha \in I$, C_α is a commutative copula with mass concentrated on the line segments Q_1Q_2, Q_2Q_3, Q_2Q_4, where $Q_1 = (0, 0)$, $Q_2 = (1/(1+\alpha), 1/(1+\alpha))$ $Q_3 = (0, 1)$ and $Q_4 = (1, 0)$. It is easy to see that $C_0 = \operatorname{Min} \geq C_{\alpha'} \geq C_\alpha \geq T_m = C_1$, whenever $\alpha \geq \alpha'$. Moreover, if $\alpha \in I^0$ and we let $x = (1-\alpha)/(2+2\alpha)$ and $y = 1/(1+\alpha)$ we have

$$C_\alpha(x, C_\alpha(x, y)) = (1-\alpha)x \neq (1-\alpha)^2 x = C_\alpha(C_\alpha(x, x), y),$$

i.e., C_α is a t-norm if and only if $\alpha = 0$ or $\alpha = 1$.

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