JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 91, No. 3, pp. 695–728, DECEMBER 1996

Differential Riccati Equation for the Active Control of a Problem in Structural Acoustics[1]

G. AVALOS[1] AND I. LASIECKA[2]

Communicated by F. E. Udwadia

Abstract. In this paper, we provide results concerning the optimal feedback control of a system of partial differential equations which arises within the context of modeling a particular fluid/structure interaction seen in structural acoustics, this application being the primary motivation for our work. This system consists of two coupled PDEs exhibiting hyperbolic and parabolic characteristics, respectively, with the control action being modeled by a highly unbounded operator. We rigorously justify an optimal control theory for this class of problems and further characterize the optimal control through a suitable Riccati equation. This is achieved in part by exploiting recent techniques in the area of optimization of analytic systems with unbounded inputs, along with a local microanalysis of the hyperbolic part of the dynamics, an analysis which considers the propagation of singularities and optimal trace behavior of the solutions.

Key Words. Structural acoustics, unbounded control input, coupled wave and beam equations, hyperbolic and parabolic-like dynamics, trace regularity, optimization, Riccati equations.

1. Introduction

1.1. Motivation. Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary Γ, Γ_0 a Lipschitz segment of Γ with endpoints a and b, and $[s, T]$ some interval with $0 \leq s < T$. We consider the problem (see Refs. 1–2) of finding functions $z(t, x)$ and $v(t, x)$, corresponding to a fixed $u(t) \in U$, $U \equiv \mathbb{R}^k$,

[1]Research partially supported by National Science Foundation Grant DMS # 9504822 and Army Research Office Grant # 35170–MA.

[2]Postdoctoral Member, Institute for Mathematics and Its Applications, University of Minnesota, Minneapolis, Minnesota.

[3]Professor, Department of Applied Mathematics, Thornton Hall, University of Virginia, Charlottesville, Virginia.

which solve the following system consisting of a coupled wave equation and elastic beamlike equation:

$$z_{tt} = \Delta z, \qquad \text{on } \Omega \times (s, T), \tag{1}$$

$$\partial z/\partial \nu = \begin{cases} v_t & \text{on } \Gamma_0 \times (s, T) \\ 0 & \text{on } \Gamma \setminus \Gamma_0 \times (s, T), \end{cases} \tag{2}$$

$$v_{tt} = -\Delta^2 v - \Delta^2 v_t - z_t + Bu, \qquad \text{on } \Gamma_0 \times (s, T), \tag{3}$$

$$v(a, t) = v(b, t) = \frac{\partial v(a, t)}{\partial x} = \frac{\partial v(b, t)}{\partial x} = 0, \qquad \forall t \in (s, T), \tag{4}$$

$$z(s, x) = z_0, \qquad v(s, x) = v_0, \tag{5a}$$

$$z_t(s, x) = z_1, \qquad v_t(s, x) = v; \tag{5b}$$

here, $B \in \mathscr{L}(U, H^{-\alpha}(\Gamma_0))$, where α is specified to be

$$\alpha \leq 7/4, \qquad \text{when } \Omega \text{ is rectangular,} \tag{6}$$

$$\alpha \leq 5/3, \qquad \text{for } \Omega \text{ an arbitrary smooth domain.} \tag{7}$$

We are looking for a triplet $[z^*, v^*, u^*]^T$ which solves (1)–(5) and which minimizes a given performance index. Our main interest, however, is obtaining a feedback realization of the resulting optimal control via a solution of an appropriate Riccati equation.

In the special case where Ω is rectangular and the operator B is of the form

$$B = \sum_{i=1}^{r} \alpha_i \delta'(x_i),$$

where $\delta'(x_i)$ are derivatives of delta functions evaluated at x_i, the model (1)–(5) was considered in Ref. 1. The physical interpretation for this particular structure of the control operator is that its control action is realized by the strategic placement of piezoelectric ceramic patches on the flexible boundary Γ_0; a voltage is subsequently applied through these patches, and the resulting bending moments can be interpreted as second derivatives of Heaviside functions. Note that the control operator is highly unbounded and is defined only through distribution theory. This is in fact the main difficulty of the problem, which was fully recognized in Ref. 3, wherein the analysis culminated in the existence and uniqueness of the solution $[z, v]^T$ of (1)–(5), for fixed u, defined only in the sense of distributions. Our main goal here is to show the well-posedness of (1)–(5) in this fully unbounded case within a given state space; i.e., $[z, v]^T$ may be taken as continuous functions; moreover, we wish to provide a rigorous theory of feedback control characterized by a solution P to the differential Riccati equation (DRE), and in particular to reveal smoothing properties of the resulting gain operator B^*P, despite the inherent unboundedness of B^*.

To accomplish our goal, we consider a more general version of this problem formulated within an abstract differential equation; the result will then be derived for a broader class of problems, whereby problem (1)–(5) will be deduced as a special case.

It should be noted that the key elements of our analysis rely on:

(i) sharp new regularity properties of the traces of the hyperbolic part of the dynamics [Eq. (1)];

(ii) the theory of analytic semigroups and associated singular integrals, which takes advantage of certain smoothing effects associated with the analytic part of dynamics [Eq. (3)];

(iii) recent results on the characterization of domains of fractional powers of the so-called elastic operators, which in turn allows for a crucial interplay between functional analytical and partial differential equation (PDE) results.

1.2. Abstract Formulation. We wish to recast (1)–(5) into an appropriate functional analytical form for which we need the following facts and definitions:

(i) We set the operator $A: L^2(\Omega) \supset D(A) \to L^2(\Omega)$ to be $A = -\Delta + I$, with

$$D(A) = \{z \in H^2(\Omega) \ni \partial u/\partial v = 0, \text{ on } \Gamma\}. \tag{8}$$

It is well known that A is self-adjoint and positive definite, so fractional powers are well defined.

(ii) With A as above, we also have from Ref. 4 that

$$D(A^\alpha) = H^{2\alpha}(\Omega), \qquad 0 \le \alpha \le 3/4. \tag{9}$$

(iii) We shall consequently use the identification of $D(A^{1/2})$, with $H^1(\Omega)$ throughout. In the sequel, we shall also consider A as a continuous mapping of $D(A^{1/2})$ into its topological dual $[D(A^{1/2})]'$, and we denote this new realization by the same symbol.

(iv) Using (iii), we can moreover define the Neumann map N on $H^s(\Gamma)$ by setting $Ng \equiv z$, $\forall g \in H^s(\Gamma)$ and $s \ge -1/2$, where z is the unique solution of the equation

$$\langle Az, v\rangle_{[D(A^{1/2})]' \times D(A^{1/2})} = \int_\Omega \nabla z \cdot \nabla v + \int_\Omega zv$$
$$= \langle g, v\rangle_{H^s(\Gamma) \times [H^s(\Gamma)]'}, \tag{10}$$

$\forall v \in D(A^{1/2})$. By elliptic regularity (see Ref. 5), we will then have that

$$N \in \mathscr{L}(H^s(\Gamma), H^{s+3/2}(\Omega)). \tag{11}$$

(v) $\gamma_0: H^1(\Omega) \to H^{1/2}(\Gamma)$ will denote the usual Sobolev trace map, and $\gamma: H^1(\Omega) \to H^{1/2}(\Gamma_0)$ will be defined as the restriction of $H^{1/2}(\Gamma)$, the range of γ_0, to Γ_0; i.e.,

$$\gamma(v) = v|_{\Gamma_0}, \qquad \forall v \in H^1(\Omega). \tag{12}$$

Thus, we will have from (10) that $\forall g \in L^2(\Gamma_0)$ and $v \in D(A^{1/2})$,

$$\langle ANg, v \rangle_{[D(A^{1/2})]' \times D(A^{1/2})} = (g, v|_{\Gamma_0})_{L^2(\Gamma_0)}, \tag{13}$$

i.e.,

$$\gamma = N^* A \text{ as elements in } \mathscr{L}(D(A^{1/2}), L^2(\Gamma_0)). \tag{14}$$

(vi) We define the operator $\mathring{A}: L^2(\Gamma_0) \supset D(\mathring{A}) \to L^2(\Gamma_0)$ as

$$\mathring{A} = \Delta^2, \qquad D(\mathring{A}) = H^4(\Gamma_0) \cap H_0^2(\Gamma_0); \tag{15}$$

$\mathring{A}$ is also self-adjoint and positive definite, so fractional powers are well defined. By Ref. 4, we have

$$D(\mathring{A}^a) = H_0^{4a}(\Gamma_0), \qquad 0 \le a < 5/8. \tag{16}$$

(vii) Let

$$A_1 \equiv \begin{bmatrix} 0 & I \\ -A+I & 0 \end{bmatrix}: D(A_1) \to H_1, \tag{17}$$

$$D(A_1) = \{[z_1, z_2]^T \in D(A) \times D(A^{1/2})\}, \tag{18}$$

$$H_1 \equiv D(A^{1/2}) \times L_2(\Omega) = H^1(\Omega) \times L_2(\Omega). \tag{19}$$

(viii) Let

$$A_0 \equiv \begin{bmatrix} 0 & I \\ -\mathring{A} & -\mathring{A} \end{bmatrix}: D(A_0) \to H_0, \tag{20}$$

$$D(A_0) = \{[v_1, v_2]^T \in [D(\mathring{A}^{1/2})]^2 \ni v_1 + v_2 \in D(\mathring{A})\}, \tag{21}$$

$$H_0 \equiv D(\mathring{A}^{1/2}) \times L_2(\Gamma_0) = H_0^2(\Gamma_0) \times L_2(\Gamma_0). \tag{22}$$

(ix) Let

$$C = \begin{bmatrix} 0 & 0 \\ 0 & \gamma^* \end{bmatrix} \in \mathscr{L}(H_0, [D(A^{1/2})]' \times [D(A^{1/2})]'). \tag{23}$$

Note that, from (14), $\gamma^* = AN$. By duality with respect to $H_1 \times H_0$, we then have

$$C^* = \begin{bmatrix} 0 & 0 \\ 0 & N^*A \end{bmatrix} \in \mathscr{L}(D(A^{1/2}) \times D(A^{1/2}), H_0^*). \tag{24}$$

(x) Let

$$\mathscr{A} \equiv \begin{bmatrix} A_1 & C \\ -C^* & A_0 \end{bmatrix}, \qquad \mathscr{A}: H_1 \times H_0 \supset D(\mathscr{A}) \to H_1 \times H_0, \tag{25}$$

$$D(\mathscr{A}) = \{[z_1, z_2, v_1, v_2]^T \in [D(A^{1/2})]^2 \times [D(\mathring{A}^{1/2})]^2, \\ \text{s.t. } -z_1 + Nv_2 \in D(A) \text{ and } v_1 + v_2 \in D(\mathring{A})\}. \tag{26}$$

(xi) Let

$$\mathscr{B} = [0, 0, 0, B]^T \in \mathscr{L}(U, H_1 \times D(\mathring{A}^{1/2}) \times H^{-\alpha}(\Gamma_0)). \tag{27}$$

From (6)–(7) and (25)–(27), we can proceed to show a fortiori that, for $\lambda \in \rho(\mathscr{A})$, $(\lambda - \mathscr{A})^{-1}\mathscr{B} \in \mathscr{L}(U, H_1 \times H_0)$, and consequently,

$$\mathscr{B} \in \mathscr{L}(U, [D(\mathscr{A}^*)]'). \tag{28}$$

So setting

$$Y \equiv [z, z_t, v, v_t]^T, \qquad Y_0 \equiv [z_0, z_1, v_0, v_1]^T,$$

we can then use (28) and the operator definitions above to rewrite (1)–(5) as the abstract dynamical system

$$dY/dt = \mathscr{A}Y + \mathscr{B}u \qquad \text{in } [D(\mathscr{A}^*)]', \tag{29}$$

$$Y(s) = Y_0 \in H_1 \times H_0. \tag{30}$$

Henceforth, our attention is drawn toward solving (29)–(30).

Along with the abstract system (29)–(30), we associate a quadratic functional,

$$J_s(Y, u) = (1/2) \int_s^T [\|RY(t)\|_Z^2 + \|u(t)\|_U^2]\, dt, \tag{31}$$

where Z is some Hilbert space and $R \in \mathscr{L}(H_1 \times H_0, Z)$; without loss of generality, we can take Z to be self-dual. The optimal control problem associated to (29)–(30) is then defined as follows: Find

$$[\vec{z}^*(\cdot, s; Y_0), \vec{v}^*(\cdot, s; Y_0), u^*(\cdot, s; Y_0)]^T \\ = [Y^*(\cdot, s; Y_0), u^*(\cdot, s; Y_0)]^T \in L^2(s, T; H_1 \times H_0) \times L^2(s, T; U),$$

which minimize (31) over all $[\vec{z}, \vec{v}, u]^T = [Y, u]^T$ which solve (29)–(30), where the control operator $\mathscr{B}$ is a continuous linear mapping: $U \to H_1 \times$

$D(\mathring{A}^{1/2}) \times [D(\mathring{A}^{\alpha/4})]'$, this boundedness following from the characterization (16).

1.3. Literature. Problems related to feedback control and Riccati equations for the case of unbounded control actions have received considerable attention in the past. In fact, control problems described by analytic semigroups with fully unbounded control operators B have a most comprehensive treatment in the literature (see Refs. 6–7). Also, in the case of hyperbolic problems with a trace-type assumption imposed upon the control operator B, it has been shown that the algebraic Riccati equation (ARE) is solvable with the gain operator usually unbounded, albeit densely defined (see Refs. 8–10). The hyperbolicity of the PDE is critical in the analysis of the gains, whose boundedness can be achieved under additional smoothing-type hypotheses imposed upon the observation R as in Refs. 5, 8 and references therein. Needless to say, our problem here does not fall into any of these categories; more importantly, the techniques developed in these cited works are not readily adaptable to the present situation. The reasons for which we cannot appeal to the earlier theory are threefold:

(i) Our problem consists of coupled hyperbolic/parabolic-like equations with an unbounded control operator; thus, techniques developed specifically for parabolic or hyperbolic cases are not applicable here.
(ii) The coupling between the two dynamics is represented by an unbounded, trace-type operator. This is a source of major difficulties in the treatment of the problem.
(iii) The observation R is not assumed here to be smoothing; thus, there is no benefit of an additional regularity resulting from optimization, unlike the other treatments noted above.

In view of the above, new techniques dealing with the novel facets of the problem need to be developed. We note that sharp regularity of hyperbolic traces (which was necessarily studied by the authors) plays an absolutely major role. In fact, our final result, asserting the solvability of the Riccati equation with the implementation of a bounded gain operator, clearly indicates an interplay between the degree of unboundedness of B which can be allowed in the problem and the geometry of the domain which dictates via the parameter α the regularity of the traces; see Theorem 1.2(ii).

1.4. Statement of Main Results. We consider the control system described by:

(i) the abstract evolution equation (29)–(30), with $\mathscr{A}$ and $\mathscr{B}$ as given by (25)–(27);
(ii) the associated performance index (31).

The following results hold true with α as specified in (6)–(7).

Theorem 1.1. For all prescribed initial data $Y_0 \in H_1 \times H_0$, there exists a unique optimal control and trajectory

$$u^*(\cdot, s; Y_0) \in L^2(s, T; U), \tag{32}$$

$$Y^*(\cdot, s; Y_0) \in C([s, T]; H_1 \times H_0). \tag{33}$$

Moreover, the following additional regularity holds true:

$$u^*(\cdot, s; Y_0) \in C([s, T]; U), \tag{34}$$

$$Y^*(\cdot, s; Y_0) \in C([s, T]; H_1 \times H_0) \cap L^2(s, T; H_1 \times [D(\mathring{A}^{1/2})]^2), \tag{35}$$

with continuous dependence on the data Y_0 (see Remark 2.6).

Theorem 1.2.

(i) The operator $P(t)$, defined by having for all $Y_0 \in H_1 \times H_0$,

$$P(t) Y_0 = \int_t^T e^{\mathscr{A}^*(\tau - t)} R^* R Y^*(\tau, t; Y_0)\, d\tau, \tag{36}$$

is an element of $\mathscr{L}(H_1 \times H_0, C([0, T]; H_1^* \times H_0^*))$, and is moreover self-adjoint, positive semidefinite; see Propositions 3.2 and 3.4(ii).

(ii) $\forall t, 0 \le t \le T, \mathscr{B}^* P(t) \in \mathscr{L}(H_1 \times H_0, U)$, with norm estimate

$$\|\mathscr{B}^* P(t)\|_{\mathscr{L}(H_1^* \times H_0^*, U)} \le C_T (T - t)^{1-\theta}, \tag{37}$$

where $\theta = \alpha/4$; see Proposition 3.3(i).

(iii) The minimum of the functional J_s defined in (31), corresponding to the minimizer $[Y^*(\cdot, s; Y_0), u^*(\cdot, s; Y_0)]^T$, is

$$J_s(Y^*(\cdot, s; Y_0), u^*(\cdot, s; Y_0)) = \langle P(t) Y_0, Y_0 \rangle, \tag{38}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing with respect to the $H_1 \times H_0$ topology; see Proposition 3.4(iii).

(iv) For each $Y_0 \in H_1 \times H_0$, the optimal control $u^*(\cdot, s; Y_0)$ is given in feedback form by

$$u^*(t; s; Y_0) = -\mathscr{B}^* P(t) Y^*(t; s; Y_0), \qquad 0 \le s \le t \le T; \tag{39}$$

see Proposition 3.3(ii).

(v) For $0 < t < T$, $P(t)$ satisfies the following differential Riccati equation (DRE), $\forall Y_0, Y_1 \in D(\mathscr{A})$,

$$\langle \dot{P}(t) Y_0, Y_1 \rangle = -\langle R^* R Y_0, Y_1 \rangle - \langle P(t) \mathscr{A} Y_0, Y_1 \rangle - \langle P(t) Y_0, \mathscr{A} Y_1 \rangle + (\mathscr{B}^* P(t) Y_0, \mathscr{B}^* P(t) Y_1)_U; \quad (40)$$

see Lemma 3.2(i).

(vi) The solution is unique within the class of self-adjoint operators $\hat{P}(t) \in \mathscr{L}(H_1 \times H_0, H_1^* \times H_0^*)$ such that, $\forall Y_0 \in H_1 \times H_0$,

$$\mathscr{B}^* \hat{P}(\cdot) Y_0 \in C([0, T]; U); \quad (41)$$

see Lemma 3.2(ii).

Remark 1.1.

(i) Notice that time regularity of the optimal control given by (34) is not intrinsic to PDE optimization problems with an unbounded control operator $\mathscr{B}$; it is an independent regularity result. In fact, typically one does not have continuity of the optimal controls. In our particular case, this enhanced regularity is a result of a smoothing effect of the analytic part of dynamics, which is propagated onto the entire system in some sense to be made clear presently.

(ii) The key result behind the derivation of the Riccati equation, in the general case of an unbounded operator $\mathscr{B}$, is the regularity of the gain operator $\mathscr{B}^* P(t)$, which appears in the quadratic term of the equation. In our present situation, it is shown [Theorem 1.2(ii)] that the gain operator $\mathscr{B}^* P(t)$ is bounded, despite the unboundedness of $\mathscr{B}$, and consequently $\mathscr{B}^*$.

(iii) It is well known that in a purely hyperbolic case with unbounded $\mathscr{B}$, the gain operators are intrinsically unbounded. What explains the regularity result [Theorem 1.2(ii)] in our particular case is again some propagation of regularity from the analytic part of the dynamics. This is probably the most technical part of the proof, which requires an in-depth microlocal analysis of traces to hyperbolic operators.

Remark 1.2. We notice that the problem considered in Ref. 1 is a special case of our more general setup where Ω is a rectangle,

$$Bu = \sum_{i=1}^{k} \delta'(x_i) u_i, \qquad x_i \in \Gamma_0,\ u_i \in \mathbb{R},\ \forall i.$$

By the Sobolev imbedding theorem,

$$\delta'(x_i) \in H^{-3/2-\epsilon}(\Gamma_0),$$

so the value

$$\alpha = 3/2 + \epsilon < 5/3.$$

Hence, this particular unbounded input, fits within the framework of our problem.

2. Analysis of Open-Loop Dynamics

In this section, we shall prove several properties related to the well-posedness of the open loop control system. These will be critical for the study of feedback control.

By appealing to Ref. 11, we have the following result regarding the dynamics generated by $\mathscr{A}$, and the consequent well-posedness of (1)–(5) in a weak sense.

Theorem 2.1. The operator $\mathscr{A}$ given by (25)–(27) with A_0, A_1, C described by (17)–(23) generates a strongly continuous semigroup on the Hilbert space $H_1 \times H_0$.

2.1. Control → State Map. The main goal of this section is to establish the regularity of the control → state map. That is to say, we consider the map $u \to Y$ defined via the dynamical system (29)–(30) as

$$Y(t) = e^{\mathscr{A}t} Y_0 + \int_0^t e^{\mathscr{A}(t-\tau)} \mathscr{B} u(\tau)\, d\tau; \tag{42}$$

without loss of generality, we take herein the initial time $s=0$. Since $\mathscr{A}$ generates a C_0-semigroup on $H_1 \times H_0$ and $\mathscr{B}: \to [\mathscr{D}(\mathscr{A}^*)]'$, the map L defined by

$$Lu(t) = \int_0^t e^{\mathscr{A}(t-\tau)} \mathscr{B} u(\tau)\, d\tau \in \mathscr{L}(L^2(0, T; U), C([0, T]; [D(\mathscr{A}^*)]'). \tag{43}$$

However, the above a priori regularity is not sufficient for the subsequent analysis. We need more information concerning the smoothing properties of L. In fact, the main goal of this section, is to prove the following lemma.

Lemma 2.1. $L \in \mathscr{L}(L^2(0, T; U), C([0, T]; H_1 \times H_0))$.

Remark 2.1. The result of Lemma 2.1 should be contrasted with that of Ref. 3, which shows that the operator L is defined as a mapping into a space of distributions.

The proof of Lemma 2.1 follows through a sequence of propositions. A critical role is played by the following results.

Theorem 2.2. See Refs. 12–13. Let A_0 be defined as in (20)–(21). Then:

(i) A_0 generates a C_0-semigroup of contractions $\{e^{A_0(\cdot)}\}$, which is also analytic on H and

$\rho(A_0) \subseteq \{\lambda \ni \mathrm{Re}(\lambda) \geq 0\}$.

(ii) Moreover, for $0 \leq \eta \leq 1/2$, we have the following characterization of the fractional powers of $(-A_0)^\eta$:

$$D((-A_0)^\eta) = D(\mathring{A}^{1/2}) \times D(\mathring{A}^\eta). \tag{44}$$

(iii) The map

$$e^{A_0(\cdot)} \in \mathscr{L}(H_0, L^2(0, T; [D(\mathring{A}^{1/2})]^2)). \tag{45}$$

Remark 2.2. Note that $\{e^{A_0 t}\}_{t \geq 0}$ has a greater smoothing effect on initial data than the standard results given for analytic semigroups; see Ref. 14, p. 295.

The next theorem pertains to the regularity of solutions of the Neumann problem with given Dirichlet boundary data (the so-called Neumann–Dirichlet map); this result will be used in the work ahead.

Theorem 2.3. Let z be a weak solution to the following Neumann boundary-value problem (BVP):

$$z_{tt} = \Delta z, \qquad \text{in } \Omega \times (0, T),$$

$$\partial z / \partial \nu = g \in L^2(0, T; H^{4\beta}(\Gamma)), \qquad \text{with } \zeta \leq 4\beta < 1/2,$$

$$z(0) = z_t(0) = 0,$$

where $\zeta = 1/4$ if Ω is rectangular and $\zeta = 1/3$ in the general case of a smooth boundary Γ. Then, we have that

$$[z, z_t, z_{t|\Gamma}]^T \in C([0, T]; H^1(\Omega)) \times C([0, T]; L^2(\Omega)) \times L^2(0, T; H^{-4\beta}(\Gamma)),$$

with continuous dependence on the datum g; viz., $\forall t \in [0, T]$, there exists C independent of t, such that

$$\|z(t)\|^2_{H^1(\Omega)} + \|z_t(t)\|^2_{L^2(\Omega)} + \int_0^T \|z_t(s)\|^2_{H^{-4\beta}(\Gamma)}\, ds \leq C \|g\|^2_{L^2(0,T;\, H^{4\beta}(\Gamma))}.$$

Remark 2.3. In the case of rectangular domains, Theorem 2.3 was proved in Ref. 15. In the case of smooth boundaries, the result of Theorem 2.3 follows from techniques used in Ref. 16.

Remark 2.4. Theorem 2.3 states sharp regularity results for the Neumann–Dirichlet map which do not follow from standard PDE theory. In fact, the standard estimates (see Ref. 17) require that $g \in H^{1/2,1/2}((0, T) \times \Gamma)$ in order to obtain only that $z \in C([0, T]; H^1(\Omega))$. Thus, our result improves the regularity by a $1/2 - 4\beta$ space derivative and a $1/2$ time derivative. More recent and refined PDE estimates, such as those in Ref. 18 with initial datum $g \in L^2(0, T; H^{1/2}(\Gamma))$, produce $z \in C([0, T]; H^1(\Omega))$ and $z_t \in C([0, T]; L^2(\Omega))$. Our result in Ref. 15 still betters that of Miyatake by a $1/2 - 4\beta$ space derivative to obtain the same state regularity and, more importantly, by allowing the trace $z_{t|\Gamma}$ to be defined in an appropriate negative Sobolev space. These improvements for the Neumann–Dirichlet map are indispensable for the following analysis.

As $\mathscr{B} \in \mathscr{L}(U, [D(A^*)]')$, then $e^{\mathscr{A}t}\mathscr{B}$ is well defined as an element of $[D(\mathscr{A}^*)]'$. Our main task now is to show that, for $t > 0$, $e^{\mathscr{A}t}\mathscr{B}$ may actually be taken as an element of $\mathscr{L}(U, H_1 \times H_0)$.

Proposition 2.1.

(i) Let η satisfy $\beta \leq \eta \leq 1/2$, where β is in the range $\zeta \leq \beta < 1/8$, with $\zeta = 1/16$ if Ω is rectangular and $\zeta = 1/12$ if Γ is a smooth arbitrary boundary, and define the map K_0 by having for every $\vec{v} \in L^2(0, T; D((-A_0)^\eta)$,

$$K_0\vec{v}(t) = \int_0^t e^{A_0(t-s)}C^* \int_0^s e^{A_1(s-\tau)}C\vec{v}(\tau)\, d\tau\, ds. \tag{46}$$

Then, $K_0 \in \mathscr{L}(L^2(0, T; D((-A_0)^\eta))$.

(ii) $I - K_0$ is boundedly invertible on $L^2(0, T; D((-A_0)^\eta))$.

Proof of (i). From Theorem 2.2(ii) above, we have that

$$D((-A_0)^\beta) = D(\mathring{A}^{1/2}) \times D(\mathring{A}^\beta), \tag{47}$$

and from (16),

$$D(\mathring{A}^\eta) \subseteq D(\mathring{A}^\beta) = H^{4\beta}(\Gamma_0). \tag{48}$$

We then note that, for arbitrary $\tilde{v}=[v, v_t]^T \in L^2(0, T; D((-A_0)^\beta))$, the term

$$\begin{bmatrix} z(\cdot) \\ z_t(\cdot) \end{bmatrix} \equiv \int_0^{(\cdot)} e^{A_1(\cdot - \tau)} \begin{bmatrix} 0 \\ ANv_t(\tau) \end{bmatrix} d\tau = \int_0^{(\cdot)} e^{A_1(\cdot - \tau)} C\tilde{v}(\tau)\, d\tau \tag{49}$$

is a fortiori (see Ref. 16) the unique weak solution of the following BVP:

$$z_{tt} = \Delta z, \tag{50a}$$

$$\partial z/\partial \nu = \begin{cases} v_t & \text{on } \Gamma_0 \times (0, T) \\ 0 & \text{on } \Gamma \backslash \Gamma_0 \times (0, T), \end{cases} \tag{50b}$$

$$z(0) = z_t(0) = 0. \tag{50c}$$

From the given regularity of v_t, we deduce from Theorem 2.2(ii), Theorem 2.3, and the characterization (48) that the map

$$\tilde{z}(\cdot) \to C^* \int_0^{(\cdot)} e^{A_1(\cdot - \tau)} C\tilde{v}(\tau)\, d\tau = \begin{bmatrix} 0 \\ z_{t|\Gamma_0} \end{bmatrix}$$

is an element of

$$\mathscr{L}(L^2(0, T; D((-A_0)^\eta), L^2(0, T; (D(\mathring{A}^{1/2}) \times H^{-4\beta}(\Gamma_0)). \tag{51}$$

Now considering the inner integral of (46), if we define j_0 as the inclusion map from $H^{-4\beta}(\Gamma_0)$ into $H^{-2}(\Gamma_0) = [D(\mathring{A}^{1/2})]'$, and set

$$\mathscr{J}_{1/2} = \begin{bmatrix} 0 & 0 \\ 0 & j_0 \end{bmatrix}, \tag{52}$$

then $\forall \tilde{v} = [v, v_t]^T \in L^2(0, T; (D(\mathring{A}^{1/2})) \times H^{-4\beta}(\Gamma_0))$, we will have by the explicit representation of $(-A_0)^{1/2}$, given in Ref. 19, p. 62, that

$$(-A_0)^{-1/2} \mathscr{J}_{1/2} \tilde{v} = \begin{bmatrix} \mathring{A}^{-3/4}(2I + \mathring{A}^{1/2})^{-1/2} j_0 v_t \\ \mathring{A}^{-1/4}(2I + \mathring{A}^{1/2})^{-1/2} j_0 v_t \end{bmatrix}. \tag{53}$$

As $\mathring{A}$ is self-adjoint, we deduce that

$$(-A_0)^{-1/2} \mathscr{J}_{1/2} \in \mathscr{L}((D(\mathring{A}^{1/2}) \times H^{-4\beta}, H_0). \tag{54}$$

As A_0 is an analytic generator [Theorem 2.2(i)], we can consequently make use of the well-developed theory for analytic semigroups (see Ref. 7), after agreeing to identify j_0 as simply the identity, to establish that the map

$$\vec{v}(\cdot)\to\int_0^{(\cdot)} e^{A_0(\cdot-\tau)}\vec{v}(\tau)\,d\tau\in\mathscr{L}(L^2(0,T;\{0\}\times H^{-4\beta}(\Gamma_0)), L^2(0,T;D((-A_0)^{1/2}))). \tag{55}$$

As $D((-A_0)^{1/2})\subseteq D((-A_0)^{\eta})$, upon combining (51) and (55) we have that

$$K_0\in\mathscr{L}(L^2(0,T;D((-A_0)^{\eta}))), \tag{56}$$

and (i) is proved.

Proof of (ii). We have further that the operator norm of K_0 is bounded by a constant dependent upon time. To wit, $\forall T_0$, $0<T_0\leq T$, and $\vec{v}\in L^2(0,T_0;D((-A_0)^{\eta}))$,

$$\|K_0\vec{v}\|_{L^2(0,T_0;D((-A_0)^{\eta}))}\leq C_0\|K_0\vec{v}\|_{L^2(0,T_0;D((-A_0)^{1/2}))} \tag{57a}$$

$$\leq C_{T_0}\left\|C^*\int_0^{(\cdot)} e^{A_1(\cdot-\tau)}C\vec{v}(\tau)\,d\tau\right\|_{L^2(0,T_0;H^{-4\beta}(\Gamma_0))} \tag{57b}$$

$$\leq C_{T_0}\|\vec{v}\|_{L^2(0,T_0;D((-A_0)^{\eta}))}. \tag{57c}$$

Note that (57a) yields (57b) by the norm estimate given in Ref. 7, p. 455 and Theorem 2.3; also note that (57b) yields (57c) because of (51), where $C_{T_0}<1$ for T_0 small enough. Then, for integer N, we can define the maps $K_{0,J}$ for $1\leq J\leq N$ by having, $\forall\,\vec{v}\in L^2(((J-1)/N)T, (J/N)T; D((-A_0)^{\eta}))$,

$$K_{0,J}\vec{v}(\cdot)=\int_{(J-1)T/N}^{(\cdot)} e^{A_0(\cdot-s)}C^*\int_{(J-1)T/N}^{s} e^{A^1(s-\tau)}C\vec{v}(\tau)\,d\tau\,ds. \tag{58}$$

For N large enough, we will have from (57) that $\|K_{0,J}\|<1$, $\forall\,J$, and we can use repeated iteration with the maps $K_{0,J}$ to show the bijectivity of $I-K_0$ on $L^2(0,T;D((-A_0)^{\eta}))$, and the bounded invertibility of $I-K_0$ will follow from the open mapping theorem. □

We now use Proposition 2.1 to establish the crucial norm estimate which will ultimately validate the input operator L given in (43).

Proposition 2.2. Let α be as specified in (6)–(7). Then:

(i) $\forall\,u\in U$, the function $e^{A(\cdot)}\mathscr{B}u$ is an element in

$$C([0,T];H_1)\times L^2(0,T;D((-A_0)^{1/2-\alpha/4}))\cap_{\alpha/4}C([0,T];H_0), \tag{59}$$

where, $\forall \eta \in \mathbb{R}$, ${}_\eta C([0, T]; H_0)$ is the Banach space

$$\left\{\vec{v} \in C((0, T]; H_0) \ni \|\vec{v}\|_{{}_\eta C((0,T];H_0)} = \sup_{0<t\le T} t^\eta \|\vec{v}(t)\|_{H_0} < \infty\right\}.$$

(ii) $\forall 0 < t \le T$, $e^{\mathscr{A}t}\mathscr{B} \in \mathscr{L}(U, H_1 \times H_0)$, and there exists a constant θ, $0 < \theta < 1/2$, such that the following norm estimate is satisfied:

$$\|e^{\mathscr{A}t}\mathscr{B}u\|_{\mathscr{L}(U, H_1 \times H_0)} \le C_0/t^\theta, \tag{60}$$

where C_0 is some positive constant, and in fact $\theta \equiv \alpha/4$.

Proof of (i). If we consider, for fixed u, the quantity

$$e^{\mathscr{A}t}\mathscr{B}u \equiv \begin{bmatrix} z(t) \\ z_t(t) \\ v(t) \\ v_t(t) \end{bmatrix} = \begin{bmatrix} \vec{z}(t) \\ \vec{v}(t) \end{bmatrix} \in [D(\mathscr{A}^*)]', \tag{61}$$

then a formal differentiation yields that $\vec{z}(\cdot)$ and $\vec{v}(\cdot)$ satisfy the coupled system

$$z_{tt} = \Delta z, \tag{62a}$$

$$\partial z/\partial \nu = \begin{cases} v_t & \text{on } \Gamma_0 \times (s, T), \\ 0 & \text{on } \Gamma\backslash\Gamma_0 \times (s, T), \end{cases} \tag{62b}$$

$$z(0) = z_t(0) = 0, \tag{62c}$$

$$v_{tt} + \mathring{A}v + \mathring{A}v_t = -z_t, \tag{62d}$$

$$v(0) = 0, \tag{62e}$$

$$v_t(0) = Bu. \tag{62f}$$

By the uniqueness theory for ordinary differential equations, finding the solution of the above coupled system will be tantamount to solving the following pair of coupled integral equations in $L^2(0, T; H_1 \times H_0)$:

$$\vec{z}(\cdot) = \int_0^{(\cdot)} e^{A_1(\cdot - \tau)} C\vec{v}(\tau)\, d\tau, \tag{63}$$

$$\vec{v}(\cdot) = e^{A_0(\cdot)} \begin{bmatrix} 0 \\ Bu \end{bmatrix} - \int_0^{(\cdot)} e^{A_0(\cdot - \tau)} C^* \vec{z}(\tau)\, d\tau. \tag{64}$$

Formally then, solving the coupled system (63)–(64) is equivalent to finding $\vec{v}$ which satisfies

$$\vec{v} + K_0 \vec{v}(\cdot) = e^{A_0(\cdot)} \begin{bmatrix} 0 \\ Bu \end{bmatrix}, \tag{65}$$

where K_0 is as defined in (46).

Now, considering the right-hand side of (65), we have by the definition of B that

$$[0, B]^T \in \mathscr{L}(U, D(\mathring{A}^{1/2}) \times H^{-\alpha}(\Gamma_0)), \tag{66}$$

and from (16), $H_0^\alpha(\Gamma_0) = D(\mathring{A}^{\alpha/4})$. So we can deduce from Theorem 2.2(ii)–(iii) that

$$e^{A_0(\cdot)} \begin{bmatrix} 0 \\ B \end{bmatrix} = (-A_0)^{\alpha/4} e^{A_0(\cdot)} (-A_0)^{-\alpha/4} \begin{bmatrix} 0 \\ B \end{bmatrix}$$
$$\in \mathscr{L}(U, L^2(0, T; D((-A_0)^{1/2-\alpha/4}))). \tag{67}$$

As $1/2 - \alpha/4 \geq \zeta$, with $\zeta = 1/16$ if Ω is a rectangle and $\zeta = 1/12$ if boundary Γ is smooth, we can apply to (65) the inverse $(I + K_0)^{-1}$, whose existence is ensured by Proposition 2.1(ii), to obtain

$$\begin{bmatrix} v(\cdot) \\ v_t(\cdot) \end{bmatrix} \equiv \vec{v}(\cdot)$$
$$= (I + K_0)^{-1} e^{A_0(\cdot)} \begin{bmatrix} 0 \\ Bu \end{bmatrix} \in L^2(0, T; D((-A_0)^{1/2-\alpha/4})). \tag{68}$$

Going back to the variable $\vec{z}$, defining (with the newly found $\vec{v}(\cdot)$)

$$\begin{bmatrix} z(\cdot) \\ z_t(\cdot) \end{bmatrix} = \vec{z}(\cdot) \equiv \int_0^{(\cdot)} e^{A_1(\cdot - \tau)} C \vec{v}(\tau)\, d\tau \tag{69}$$

will yield a fortiori via Theorem 2.3, which again is directly applicable here, as $1/3 \leq 2 - \alpha < 1/2$ in the case of a general smooth domain and $1/4 \leq 2 - \alpha < 1/2$ if Ω is a rectangle, the unique (weak) solution of (63). Consequently, we have that

$$\vec{z} \in C([0, T]; H_1). \tag{70}$$

In addition, using the properties of A_0, as spelled out in Theorem 2.2, we can show, in a manner akin to that which was used in establishing Proposition 2.2(i), that

$$K_0 \in \mathscr{L}(L^2(0, T; D((-A_0)^{1/2-\alpha/4})), C([0, T]; H_0)); \tag{71}$$

hence, we deduce from the integral representation of $\vec{v}(\cdot)$ in (64) that

$$\vec{v}(\cdot) \in_{\alpha/4} C([0, T]; H_0), \tag{72}$$

where the singularity at 0 is of course due to the presence of the unbounded operator B. So, defining

$$e^{\mathscr{A}(\cdot)}\mathscr{B}u \equiv \begin{bmatrix} \vec{z}(\cdot) \\ \vec{v}(\cdot) \end{bmatrix}, \tag{73}$$

where $[\vec{z}, \vec{v}]$ solves (63)–(64), will yield the result in (i) via the regularity properties (68), (70), and (72).

Proof of (ii). With $[\vec{z}, \vec{v}]$ as defined by (73), we use the a posteriori estimates given from both the regularity result of Theorem 2.3 and the inversion of the operator $I + K_0$ to majorize $\|\vec{z}\|_{C([0,T];H_1)}$, where $\vec{z}$ is as given in (69); namely, we have

$$\|\vec{z}\|_{C([0,T];H_1)} \leq C_1 \|v_t\|_{L^2(0,T;H^{2-\alpha}(\Gamma_0))} \tag{74a}$$

$$\leq C_2 \|(I+K_0)^{-1}\|_{\mathscr{L}(L^2(0,T;D((-A_0)^{1/2-\alpha/4})))} \cdot \left\| e^{A_0(\cdot)} \begin{bmatrix} 0 \\ Bu \end{bmatrix} \right\|_{L^2(0,T;D((-A_0)^{1/2-\alpha/4}))} \tag{74b}$$

$$\leq C_3 \left\| (-A_0)^{-\alpha/4} \begin{bmatrix} 0 \\ Bu \end{bmatrix} \right\|_{H_0} \tag{74c}$$

$$\leq C_4 \|u\|_U. \tag{74d}$$

Note that (74c) arises from (74b) in light of Theorem 2.2(iii). Moreover, using the representation for $\vec{v}(\cdot)$ in (65) and the continuity of K_0 as spelled out in (71), upon taking norms we obtain the pointwise estimate, valid on $(0, T]$ because of the regularity given in (72),

$$\|\vec{v}(t)\|_{H_0} \leq \left\| e^{A_0 t} \begin{bmatrix} 0 \\ Bu \end{bmatrix} \right\|_{H_0} + C\|\vec{v}\|_{L^2(0,T;D((-A_0)^{1/2-\alpha/4}))}. \tag{75}$$

After considering the boundedness of $I + K_0$ given in Proposition 2.1(ii), along with that of

$$e^{A_0(\cdot)} \begin{bmatrix} 0 \\ Bu \end{bmatrix},$$

posted in (67), we have

$$(75) \leq \left\| e^{A_0 t} \begin{bmatrix} 0 \\ Bu \end{bmatrix} \right\|_{H_0} + C \|u\|_U. \tag{76}$$

In addition, from the analyticity of A_0 we have again

$$\left\| e^{A_0 t} \begin{bmatrix} 0 \\ Bu \end{bmatrix} \right\|_{H_0} = \left\| (-A_0)^{\alpha/4} e^{A_0 t} (-A_0)^{-\alpha/4} \begin{bmatrix} 0 \\ Bu \end{bmatrix} \right\|_{H_0}$$

$$\leq \left(\frac{C_T}{t^{\alpha/4}} \right) \|u\|_U. \tag{77}$$

Coupling (74) and (75)–(77), we indeed obtain the desired estimate,

$$\|e^{\mathscr{A} t} \mathscr{B} u\|_{H_1 \times H_0} \leq (C/t^\theta) \|u\|_U, \tag{78}$$

where $\theta \equiv \alpha/4$, and Proposition 2.2(ii) is proved. □

Proof of Lemma 2.1. This is a simple consequence of Proposition 2.2. Indeed from (43), $\forall u \in L^2(0, T; U)$, we have

$$\|Lu(t)\|_{H_1 \times H_0} \leq \int_0^t \|e^{\mathscr{A}(t-s)} \mathscr{B} u(s)\|_{H_1 \times H_0} \, ds$$

$$\leq \int_0^t \left[\frac{1}{(t-s)^{\alpha/4}} \right] \|u(s)\|_U \, ds$$

$$\leq C_T \|u\|_{L_2(0,T;\, U)}, \tag{79}$$

after using the estimate (60) of Proposition 2.2(ii), which implies that

$$\|Lu(t)\|_{L^\infty(0,T;H_1 \times H_0)} \leq C_T \|u\|_{L^2(0,T;U)}.$$

The usual density argument, together with the completeness of $C([0, T]; U)$ proves continuity in time as asserted. □

In the sequel, we shall need the properties of the map L_s: $L^2(s, T; U) \to L^2(s, T; H_1 \times H_0)$, defined as

$$(L_s u)(t) \equiv \int_s^t e^{\mathscr{A}(t-\tau)} \mathscr{B} u(\tau) \, d\tau, \tag{80}$$

where $s < t < T$.

It is now a straightforward result, via Lemma 2.1, that the continuity of the map L_s: $L^2(s, T; U) \to C([s, T]; H_1 \times H_0)$ is preserved uniformly in the parameter s. In fact, we have the following corollary.

Corollary 2.1.

(i) The operator L_s defined on U, $\forall\ 0<s<T$, given by

$$L_s u(\cdot) \equiv \int_s^{(\cdot)} e^{\mathscr{A}(\cdot - \tau)} \mathscr{B} u(\tau)\, d\tau, \tag{81}$$

is an element of $\mathscr{L}(L^2(s, T; U), C([s, T]; H_1 \times H_0))$ with a norm estimate uniform with respect to s.

(ii) The unique solution to (29)–(30), for fixed $u \in L^2(s, T; U)$, is given by

$$\begin{bmatrix} \bar{z}(\cdot) \\ \bar{v}(\cdot) \end{bmatrix} \equiv Y(\cdot) = e^{\mathscr{A}(\cdot - s)} Y_0 + L_s u(\cdot) \in C([s, T]; H_1 \times H_0). \tag{82}$$

(iii) The adjoint operator

$$L_s^* \in \mathscr{L}(L^2(s, T; H_1^* \times H_0^*), L^2(s, T; U))$$

is given by

$$\begin{aligned} L_s^* \hat{Y}(\cdot) &= \int_{(\cdot)}^T \mathscr{B}^* e^{\mathscr{A}^*(\tau - \cdot)} \hat{Y}(\tau)\, d\tau \\ &= \mathscr{B}^* \int_{(\cdot)}^T e^{\mathscr{A}^*(\tau - \cdot)} \hat{Y}(\tau)\, d\tau. \end{aligned} \tag{83}$$

Proof. Given the norm estimate (60), the proofs of (i)–(iii) are straightforward and may be omitted. Moreover, it can readily be shown that, $\forall \hat{Y} \in L^2(s, T; D(\mathscr{A}^*))$,

$$L_s^* \hat{Y}(\cdot) = \mathscr{B}^* \int_{(\cdot)}^T e^{\mathscr{A}^*(\tau - \cdot)} \hat{Y}(\tau)\, d\tau, \tag{84}$$

so L_s^*, as given by the first equality of (83), can be considered as the unique extension of the map

$$\hat{Y}(\cdot) \to \mathscr{B}^* \int_{(\cdot)}^T e^{\mathscr{A}^*(\tau - \cdot)} \hat{Y}(\tau)\, d\tau, \qquad \forall \hat{Y}(\cdot) \in L^2(s, T; D(\mathscr{A}^*)); \tag{85}$$

hence, the second equality in (83) is entirely justified. □

2.2. Associated Optimal Control Problem: Proof of Theorem 1.1

Given the existence and uniqueness established in Section 2.1 for the solution $[\vec{z}, \vec{v}]^T = Y$ of (29)–(30), we can proceed to consider the following optimal control problem for $0 \leq s < T$:

$$\min J_s(Y, u) = (1/2) \int_s^T [\|RY(t)\|_Z^2 + \|u(t)\|_U^2]\, dt, \tag{86}$$

where J_s is as defined in (31), over all $[Y, u]^T \in L^2(s, T; H_1 \times H_0) \times L^2(s, T; U)$ which satisfy

$$Y(\cdot) = e^{\mathscr{A}(\cdot - s)} Y_0 + L_s u(\cdot),$$

where Y_0 is denoted to be the initial data $[z_0, z_1, v_0, v_1]^T \in H_1 \times H_0$.

Lemma 2.2. There exists a unique optimal pair

$$\begin{aligned} &[\vec{z}^*(\cdot, s; Y_0), \vec{v}^*(\cdot, s; Y_0), u^*(\cdot, s; Y_0)]^T \\ &= [Y^*(\cdot, s; Y_0), u^*(\cdot, s; Y_0)]^T \\ &\in C([s, T]; H_1 \times H_0) \times L^2(s, T; U), \end{aligned} \tag{87}$$

which solves (86) and which is given explicitly by

$$u^*(\cdot, s; Y_0) = -(I_s + L_s^* R^* R L_s)^{-1} L_s^* R^* R\, e^{\mathscr{A}(\cdot - s)} Y_0, \tag{88}$$

$$\begin{bmatrix} \vec{z}^*(\cdot, s; Y_0) \\ \vec{v}^*(\cdot, s; Y_0) \end{bmatrix} \equiv Y^*(\cdot, s; Y_0) = e^{\mathscr{A}(\cdot - s)} Y_0 + L_s u^*(\cdot, s; Y_0), \tag{89}$$

where $(I_s + L_s^* R^* R L_s)^{-1} \in \mathscr{L}(L^2(0, T; U))$; here, I_s is the identity map on $L^2(0, T; U)$.

Proof. Let us define $V \in \mathscr{L}(L^2(s, T; H_1 \times H_0) \times L^2(s, T; U), L^2(s, T; H_1 \times H_0))$ by having $\forall\, [Y, u]^T \in L^2(s, T; H_1 \times H_0) \times L^2(s, T; U)$,

$$V(Y, u) = Y(\cdot) - e^{\mathscr{A}(\cdot - s)} Y_0 - \int_0^{(\cdot)} e^{\mathscr{A}(\cdot - \tau)} \mathscr{B} u(\tau)\, d\tau \tag{90}$$

(see Lemma 2.1 and Corollary 2.1). Then, we have that (86) is equal to the following optimization problem:

$$\min J_s(Y, u) = (1/2) \int_s^T [\|RY(t)\|_Z^2 + \|u(t)\|_U^2]\, dt,$$

over $[Y, u]^T \in L^2(s, T; H_1 \times H_0) \times L^2(s, T; U)$, s.t. $V(Y, u) = 0$. (91)

Subsequently, by the standard theory of convex optimization (see Ref. 20, p. 35), there exists a unique pair

$$\begin{aligned} &[Y^*(\cdot, s; Y_0), u^*(\cdot, s; Y_0)]^T \\ &= [\vec{z}^*(\cdot, s; Y_0), \vec{v}^*(\cdot, s; Y_0), u^*(\cdot, s; Y_0)] \\ &\in L^2(s, T; H_1 \times H_0) \times L^2(s, T; U), \end{aligned} \tag{92}$$

which solves (91), and from Corollary 2.1(ii), $Y^*(\cdot, s; Y_0) \in C([s, T]; H_1 \times H_0)$. Moreover, the Frechét derivative of V can easily be computed at every $[Y, u]^T \in L^2(s, T; H_1 \times H_0) \times L^2(s, T; U)$ as the constant V', where V' is defined by having, for every $[Y, u]^T$,

$$V' \begin{bmatrix} Y \\ u \end{bmatrix} = \begin{bmatrix} I_s & 0 \\ 0 & -L_s \end{bmatrix} \begin{bmatrix} Y \\ u \end{bmatrix}; \tag{93}$$

here, I_s denotes the identity on $L^2(s, T; H_1 \times H_0)$. V' is evidently surjective; to wit, $\forall Y \in L^2(s, T; H_1 \times H_0)$,

$$V' \begin{bmatrix} Y \\ 0 \end{bmatrix} = Y,$$

so by the Liusternik Lagrange multiplier theorem (see Ref. 21, p. 243), there exists a $\lambda^* \in L^2(s, T; H_1^* \times H_0^*)$ such that

$$J_s'(Y^*(\cdot, s; Y_0), u^*(\cdot, s; Y_0)) + \left\langle \lambda^*, V' \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \right\rangle = 0, \tag{94}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $L^2(s, T; H_1 \times H_0)$ and its dual; i.e., $\forall [Y, u]^T \in L^2(s, T; H_1 \times H_0) \times L^2(s, T; U)$,

$$\begin{aligned} 0 &= \langle R^* R Y^*(\cdot, s; Y_0), Y \rangle \\ &\quad + \langle \lambda^*, Y - L_s u \rangle + (u^*(\cdot, s; Y_0), u)_{L^2(s,T;U)}. \end{aligned} \tag{95}$$

We thus deduce from (95) that

$$R^* R Y^*(\cdot, s; Y_0) = -\lambda^*, \tag{96}$$

$$u^*(\cdot, s; Y_0) = L_s^* \lambda^* = -L_s^* R^* R Y^*(\cdot, s; Y_0), \tag{97}$$

$$Y^*(\cdot, s; Y_0) = e^{\mathcal{A}(\cdot - s)} Y_0 + L_s u^*(\cdot, s; Y_0). \tag{98}$$

Thus, (96)–(98) yield

$$u^*(\cdot, s; Y_0) = -L_s^* R^* R\, e^{\mathcal{A}(\cdot - s)} Y_0 - L_s^* R^* R L_s u^*(\cdot, s; Y_0), \tag{99}$$

or

$$u^*(\cdot, s; Y_0) = -(I_s + L_s^* R^* R L_s)^{-1} L_s^* R^* R\, e^{\mathscr{A}(\cdot - s)} Y_0, \tag{100}$$

after using the fact that $I_s + L_s^* R^* R L_s$ is invertible on $L^2(s, T; U)$ by the Lax–Milgram theorem, together with the regularity results of Corollary 2.1; of course, this time I_s is the identity on $L^2(s, T; U)$. □

2.3. Additional Regularity Properties of the Optimal Solution. In this section, we show that the optimal control $u^*(\cdot, s; Y_0)$, as given in Lemma 2.2, is actually continuous in time, and that $\vec{v}^*(\cdot, s; Y_0)$ of (89) has greater regularity in the spatial variable than previously stated. These properties are critical for derivation of the Riccati equation.

Proposition 2.3. Let $\vec{v}(\cdot)$ be as in (82). Then, $\vec{v} \in L^2(s, T; [D(\mathring{A}^{1/2})]^2)$.

Proof. Without loss of generality, we can take $s \equiv 0$. Finding the solution

$$Y(\cdot) = \begin{bmatrix} \vec{z}(\cdot) \\ \vec{v}(\cdot) \end{bmatrix}$$

of (29)–(30) is formally tantamount to solving the coupled system of integral equations

$$\vec{z}(t) = e^{A_1 t} z_0 + \int_0^t e^{A_1(t-\tau)} C \vec{v}(\tau)\, d\tau, \tag{101}$$

$$\vec{v}(t) = e^{A_0 t} v_0 - \int_0^t e^{A_0(t-\tau)} C^* \vec{z}(\tau)\, d\tau + \int_0^t e^{A_0(t-\tau)} \begin{bmatrix} 0 \\ Bu(\tau) \end{bmatrix} d\tau, \tag{102}$$

which in turn is formally equivalent to solving

$$\vec{v}(t) + K_0 \vec{v}(t) = e^{A_0 t} v_0 - \int_0^t e^{A_1(t-\tau)} C^*\, e^{A_1 \tau} z_0\, d\tau + \int_0^t e^{A_0(t-\tau)} \begin{bmatrix} 0 \\ Bu(\tau) \end{bmatrix} d\tau, \tag{103}$$

where K_0 is as defined in (46). Now by Ref. 18,

$$C^* e^{A_1(\cdot)} z_0 \in L^2(0, T; D(\mathring{A}^{1/2}) \times H^{-1/2}(\Gamma_0)), \tag{104}$$

so we can use this fact along with the same argument used in establishing (56) of Proposition 2.1(i) to have that

$$\text{RHS of (103)} \in L^2(0, T; D((-A_0)^{1/2})), \tag{105}$$

where RHS stands for right-hand side. Consequently by Proposition 2.1(ii), we can apply the inverse $(I+K_0)^{-1}$ to both sides of (103) to obtain

$$\begin{aligned}\begin{bmatrix} v \\ v_t \end{bmatrix} = \vec{v}(\cdot) = (I+K_0)^{-1} \Bigg[e^{A_0(\cdot)} v_0 - \int_0^{(\cdot)} e^{A_0(\cdot - \tau)} C^* e^{A_1 \tau} z_0 \, d\tau \\ + \int_0^{(\cdot)} e^{A_0(\cdot - \tau)} \begin{bmatrix} 0 \\ Bu(\tau) \end{bmatrix} d\tau \Bigg] \\ \in L^2(0, T; [D(\mathring{A}^{1/2})]^2),\end{aligned} \tag{106}$$

after using Theorem 2.2(ii). We can subsequently derive the solution $\vec{z}(\cdot)$ of (101) by setting

$$\begin{aligned}\begin{bmatrix} z \\ z_t \end{bmatrix} = \vec{z}(\cdot) \equiv e^{A_1(\cdot)} z_0 + \int_0^{(\cdot)} e^{A_1(\cdot - \tau)} C[(I+K_0)^{-1} e^{A_0(\cdot)} v_0]_\tau \, d\tau \\ - \int_0^{(\cdot)} e^{A_1(\cdot - \tau)} C \left[(I+K_0)^{-1} \int_0^{(\cdot)} e^{A_0(\cdot - \sigma)} C^* e^{A_1 \sigma} z_0 \, d\sigma \right]_\tau d\tau \\ + \int_0^{(\cdot)} e^{A_1(\cdot - \tau)} C \left[(I+K_0)^{-1} \int_0^{(\cdot)} e^{A_0(\cdot - \sigma)} \begin{bmatrix} 0 \\ Bu(\sigma) \end{bmatrix} d\sigma \right]_\tau d\tau.\end{aligned} \tag{107}$$

Note that $\vec{z}$ is in $C([0, T]; H_1)$ by Theorem 2.3, which is directly applicable, as we can readily compute that the solution of (101), which we have obtained here, satisfies

$$z_{tt} = \Delta z \tag{108a}$$

$$\partial z / \partial \nu = \begin{cases} v_t & \text{on } \Gamma_0 \times (0, T) \\ 0 & \text{on } \Gamma \backslash \Gamma_0 \times (0, T), \end{cases} \tag{108b}$$

$$[z(0), z_t(0)]^T = [z_0, z_1]^T \in H^1(\Omega) \times L^2(\Omega). \tag{108c}$$

A fortiori then, the solution $[\vec{z}, \vec{v}]^T$ defined in (106)–(107) solves the ODE (29)–(30), and a uniqueness argument concludes the proof. □

Remark 2.5. As we have just shown in Proposition 2.3 that every $\vec{v}(\cdot)$ of (82), corresponding to some fixed $u \in L^2(0, T; U)$, is in $L^2(s, T; [D(\mathring{A}^{1/2})]^2)$, then in particular the optimal $\vec{v}^*(\cdot, s; Y_0)$ given by (89) has the stated regularity.

Proposition 2.4. Let L_s^* be as given in (83). Then, $L_s^* \in \mathscr{L}(C([s, T]; H_1^* \times H_0^*), C([s, T]; U))$, and $\forall \hat{Y} \in C([s, T]; H_1^* \times H_0^*)$, we have the estimate

$$\|L_s^* \hat{Y}\|_{C([s,T];U)} \leq C_T \|\hat{Y}\|_{C([s,T]; H_1^* \times H_0^*)}, \tag{109}$$

where C_T is independent of s, $0 \leq s < T$.

Proof. $L_s^* \hat{Y}$ is easily seen to be in $C([s, T]; U)$, $\forall \hat{Y}$ in $C([s, T]; H_1^* \times H_0^*)$; hence, we need only show that there is no singularity at the endpoint T. So $\forall\ t$, $s \leq t < T$, we have by (60),

$$\begin{aligned} \|L_s^* \hat{Y}(t)\|_U &\leq \int_t^T \|B^* e^{\mathscr{A}^*(\tau - t)} \hat{Y}(\tau)\|_U \, d\tau \\ &= \int_t^T [C_0/(\tau - t)^\theta] \|\hat{Y}(\tau)\|_{H_1^* \times H_0^*} \, d\tau \\ &\leq [C_0(T-s)^{1-\theta}/(1-\theta)] \|\hat{Y}\|_{C([s,T]; H_1^* \times H_0^*)}, \end{aligned} \tag{110}$$

where θ is as in (60), and we have the result. □

Proposition 2.5. $L_s \in \mathscr{L}(C([s, T]; U), C([s, T]; H_1 \times H_0))$ and $\forall u \in C([s, T]; U)$ we have the estimate

$$\|L_s u\|_{C([s,T]; H_1 \times H_0)} \leq C_T \|u\|_{C([s,T];U)}, \tag{111}$$

where C_T is independent of s, $0 \leq s < T$.

Proof. This is similar to that of Proposition 2.4. □

Lemma 2.3. The operator $I_s + L_s^* R^* R L_s$ is boundedly invertible on $C([s, T]; U)$ with the following norm estimate:

$$\|(I_s + L_s^* R^* R L_s)^{-1}\|_{\mathscr{L}(C([s,T];U))} \leq C_T, \tag{112}$$

where C_T is independent of s, $0 \leq s < T$.

Proof. We make use here of a bootstrap argument. For an arbitrary $g \in C([s, T]; U)$, we wish to show that there exists a unique $f \in C([s, T]; U)$, such that

$$f + L_s^* R^* R L_s f = g, \qquad \text{in } C([s, T]; U). \tag{113}$$

As $I_s + L_s^* R^* R L_s$ is invertible in $L^2(s, T; U)$ by the Lax–Milgram theorem, then certainly there exists a unique $h \in L^2(s, T; U)$ such that

$$h + L_s^* R^* R L_s h = L_s^* R^* R L_s g \qquad \text{in } L^2(s, T; U). \tag{114}$$

Thus, by Propositions 2.4–2.5,

$$h = L_s^* R^* R L_s g - L_s^* R^* R L_s h \qquad \text{in } C([s, T]; U), \tag{115}$$

and it is easy to see that

$$f \equiv g - h \in C([s, T]; U)$$

will be the unique solution to (113). The result then follows upon application of the open mapping theorem. The norm estimate (112) is a consequence of the estimates given in (109) and (111). □

Corollary 2.2. $u^*(\cdot, s; Y_0)$ given by (88) is in $C([s, T]; U)$.

Proof. By Proposition 2.4, $L_s^* R^* R\, e^{\mathscr{A}(\cdot)} Y_0 \in C([s, T]; U)$, and the result follows from Lemma 2.3. □

Corollary 2.3. We have the following estimates for the optimal pair:

$$\|\vec{v}^*(\cdot, s; Y_0)\|_{L^2(s,T;[D(\mathring{A}^{1/2})]^2)} \le C_T \|Y_0\|_{H_1 \times H_0}, \tag{116}$$

$$\|u^*(\cdot, s; Y_0)\|_{C([s,T];U)} \le C_T \|Y_0\|_{H_1 \times H_0}, \tag{117}$$

$$\|Y^*(\cdot, s; Y_0)\|_{C([s,T];H_1 \times H_0)} \le C_T \|Y_0\|_{H_1 \times H_0}. \tag{118}$$

Proof. Estimates (117), (118) follow from (97)–(98), (100), and the results of Propositions 2.4–2.5 and Lemma 2.3. For the component $\vec{v}^*(\cdot, s; Y_0)$, we have from (106) that

$$\vec{v}^*(\cdot, s; Y_0) = (I + K_0)^{-1} \left[e^{A_0(\cdot - s)} v_0 - \int_s^{(\cdot)} e^{A_0(\cdot - \tau)} C^* e^{A_1(\tau - s)} z_0 \, d\tau \right.$$
$$\left. + \int_s^{(\cdot)} e^{A_0(\cdot - \tau)} \begin{bmatrix} 0 \\ B u^*(\tau, s; Y_0) \end{bmatrix} d\tau \right], \tag{119}$$

where equality can be taken in $L^2(s, T; [D(\mathring{A}^{1/2})]^2)$. Now, for the right-hand side of (119).

$$\|(I+K_0)^{-1} e^{A_0(\cdot - s)} v_0\|_{L^2(s,T;[D(\mathring{A}^{1/2})]^2)} \leq C_T \|e^{A_0(\cdot - s)} v_0\|_{L^2(s,T;(D(-A_0)^{1/2}))}$$
$$\leq C_T \|v_0\|_{H_0}, \tag{120}$$

after making use of the continuity of the map $(I+K_0)^{-1}$ and Theorem 2.2(ii)–(iii).

Again from Ref. 18,

$$C^* e^{A_1(\cdot - s)} \in \mathscr{L}(H_1, L^2(s, T; D(\mathring{A}^{1/2}) \times H^{-1/2}(\Gamma_0))),$$

so the boundedness of this map and that of $(I+K_0)^{-1}$, in conjunction with the unbounded control theory derived for analytic semigroups (detailed in Ref. 7), yield

$$\left\| (I+K_0)^{-1} \left[\int_s^{(\cdot)} e^{A_0(\cdot - \tau)} C^* e^{A_1(\tau - s)} z_0 \, d\tau \right] \right\|_{L^2(s,T;[D(\mathring{A}^{1/2})]^2)}$$
$$\leq C_T \left\| \int_s^{(\cdot)} e^{A_0(\cdot - \tau)} C^* e^{A_1(\tau - s)} z_0 \, d\tau \right\|_{L^2(s,T;[D(\mathring{A}^{1/2})]^2)}$$
$$\leq \|C^* e^{A_1(\cdot - s)} z_0\|_{L^2(s,T;D(\mathring{A}^{1/2}) \times H^{-1/2}(\Gamma_0))}$$
$$\leq C_T \|z_0\|_{H_1}. \tag{121}$$

Likewise, by the boundedness of $(I+K_0)^{-1}$ and the aforementioned theory for analytic semigroups, we obtain the estimate

$$\left\| (I+K_0)^{-1} \left[\int_s^{(\cdot)} e^{A_0(\cdot - \tau)} \begin{bmatrix} 0 \\ Bu^*(\tau, s; Y_0) \end{bmatrix} d\tau \right] \right\|_{L^2(s,T;[D(\mathring{A}^{1/2})]^2)}$$
$$\leq C_T \|u^*(\cdot, s; Y_0)\|_{L^2(s,T;U)}. \tag{122}$$

Using the characterization of $u^*(\cdot, s; Y_0)$ in (100), we consequently have from Proposition 2.5 and (122) that

$$\left\| (I+K_0)^{-1} \left[\int_s^{(\cdot)} e^{A_0(\cdot - \tau)} \begin{bmatrix} 0 \\ Bu^*(\tau, s; Y_0) \end{bmatrix} d\tau \right] \right\|_{L^2(s,T;[D(\mathring{A}^{1/2})]^2)}$$
$$\leq C_T \|Y_0\|_{H_1 \times H_0}. \tag{123}$$

Thus, combining (119)–(121) and (123), we arrive at

$$\|\vec{v}^*(\cdot, s; Y_0)\|_{L^2(s,T;[D(\mathring{A}^{1/2})]^2)} \leq C_T \|Y_0\|_{H_1 \times H_0}, \tag{124}$$

and the estimate (116) is hence established. □

Remark 2.6. The proof of Theorem 1.1 follows from Lemma 2.2, Proposition 2.3, and Corollaries 2.2–2.3.

3. Riccati Operator

3.1. Definition of the Operator $\Phi(\cdot,\cdot)$. At this point, we introduce the evolution operator $\Phi(t,s)$ defined by

$$\Phi(t,s)Y_0 = Y^*(t,s;Y_0), \qquad Y_0 \in H_1 \times H_0, \qquad 0 \le s \le t \le T, \tag{125}$$

and one can readily establish that

$$\Phi(t,t)Y_0 = Y_0, \tag{126}$$

$$\Phi(t,\tau)\Phi(\tau,s) = \Phi(t,s), \qquad 0 \le s \le \tau \le t \le T. \tag{127}$$

Furthermore, one has the following proposition.

Proposition 3.1. With $\Phi(\cdot,\cdot)$ as defined in (125), the following properties hold:

(i) for fixed $s, 0 \le s \le T$, the map $t \to \Phi(t,s)Y_0$ is continuous in $H_1 \times H_0$, $s \le t \le T$;
(ii) for fixed $t, s \le t \le T$ and $Y_0 \in H_1 \times H_0$, the map $s \to \Phi(t,s)Y_0$ is continuous in $H_1 \times H_0$.

Proof. Having established the regularity properties of Corollary 2.3, the proof is the same as in Ref. 9, and hence is omitted. □

With $\Phi(\cdot,\cdot)$ defined as in (125), we subsequently define

$$P(t)Y_0 = \int_t^T e^{\mathscr{A}^*(\tau-t)}R^*R\Phi(\tau,t)Y_0\,d\tau, \tag{128}$$

and a fortiori

$$P(t) \in \mathscr{L}(H_1 \times H_0, H_1^* \times H_0^*), \qquad \forall t, 0 \le t \le T,$$

and

$$P(\cdot) \in \mathscr{L}(H_1 \times H_0, L^\infty(0,T;H_1^* \times H_0^*)). \tag{129}$$

Indeed, in a standard way, we can show that the following proposition holds.

Proposition 3.2. $P(\cdot)\in\mathscr{L}(H_1\times H_0, C([0,T]; H_1^*\times H_0^*))$.

The key result in this section is the following regularity property of the gain $\mathscr{B}^*P$.

Proposition 3.3.

(i) $\forall t, 0\le t<T, \mathscr{B}^*P(t)\in\mathscr{L}(H_1\times H_0, U)$ with the pointwise norm estimate

$$\|\mathscr{B}^*P(t)\|_{\mathscr{L}(H_1^*\times H_0^*, U)}\le C_T(T-t)^{1-\theta}, \tag{130}$$

where θ is as given in (60).

(ii) For each $Y_0\in H_1\times H_0$, the optimal control $u^*(\cdot, s; Y_0)$ is given in feedback form by

$$u^*(t,s;Y_0)=-\mathscr{B}^*P(t)Y^*(t,s;Y_0),\qquad 0\le s\le t<T. \tag{131}$$

Proof of (i). $\forall Y_0\in H_1\times H_0$, we have $\forall t, 0\le t<T$,

$$\|\mathscr{B}^*P(t)Y_0\|_{H_1^*\times H_0^*}\le\int_t^T\|\mathscr{B}^*e^{\mathscr{A}^*(\tau-t)}R^*R\Phi(\tau,t)Y_0\|_{H_1^*\times H_0^*}\,d\tau \tag{132a}$$

$$\le\int_t^T[C_0/(\tau-t)^\theta]\|Y_0\|_{H_1\times H_0}\,d\tau, \tag{132b}$$

$$=[C_0(T-t)^{1-\theta}/(1-\theta)]\|Y_0\|_{H_1\times H_0}. \tag{132c}$$

Note that (132c) ensues from (132b) in light of Proposition 2.2(ii), duality, and (118).

Proof of (ii). Using the characterization of $u^*(\cdot,s;Y_0)$ given by (97), we have

$$u^*(t,s;Y_0)=-\int_t^T\mathscr{B}^*e^{\mathscr{A}^*(\tau-t)}R^*R[e^{\mathscr{A}(\tau-s)}Y_0-L_su^*(\cdot,s;Y_0)(\tau)]\,d\tau \tag{133a}$$

$$=-\int_t^T\mathscr{B}^*e^{\mathscr{A}^*(\tau-t)}R^*R\Phi(\tau,s)Y_0\,d\tau \tag{133b}$$

$$=-\mathscr{B}^*\int_t^T e^{\mathscr{A}^*(\tau-t)}R^*R\Phi(\tau,t)\Phi(t,s)Y_0\,d\tau \tag{133c}$$

$$=-\mathscr{B}^*P(t)Y^*(t,s;Y_0). \tag{133d}$$

Note that (133d) ensues from (133c) after using property (127). □

Proposition 3.4.

(i) For $0 \leq t \leq T$ and $\forall Y_0, Y_1 \in H_1 \times H_0$, we have

$$\langle P(t)Y_0, Y_1\rangle = \int_t^T (R\Phi(\tau, t)Y_0, R\Phi(\tau, t)Y_1)_z \, d\tau + \int_t^T (\mathscr{B}^* P(\tau)\Phi(\tau, t)Y_0, \mathscr{B}^* P(\tau)\Phi(\tau, t))_U \, d\tau. \quad (134)$$

(ii) $P(t) \in \mathscr{L}(H_1 \times H_0, H_1^* \times H_0^*)$ is self-adjoint and positive semidefinite.

(iii) With J_t as defined in (31), we have that the constrained minimum of J_t, corresponding to the minimizer $[Y^*(\cdot, t; Y_0), u^*(\cdot, t; Y_0)]^T$, is

$$J_t(Y^*(\cdot, Y_0), u^*(\cdot, t; Y_0)) = < P(t)Y_0, Y_0\rangle, \quad (135)$$

where the duality product is taken between $H_1 \times H_0$ and $H_1^* \times H_0^*$.

Proof of (i). Using the definition of $P(t)$ in (128), we have for Y_0, $Y_1 \in H_1 \times H_0$,

$$\langle P(t)Y_0, Y_1\rangle = \int_t^T (R\Phi(\tau, t)Y_0, R\, e^{\mathscr{A}(\tau-t)}Y_1)_Z \, d\tau. \quad (136)$$

Now, using (125) and (82), we have that $\forall Y \in H_1 \times H_0$

$$R\Phi(\tau, t)Y = R\, e^{\mathscr{A}(\tau-t)}Y + RL_t u^*(\cdot, t; Y)(\tau). \quad (137)$$

Applying (137) to (136) then yields

$$\begin{aligned}\langle P(t)Y_0, Y_1\rangle &= \int_t^T (R\Phi(\tau, t)Y_0, R\Phi(\tau, t)Y_1)_Z \, d\tau \\ &\quad - \int_t^T (R\Phi(\tau, t)Y_0, RL_t u^*(\cdot, t; Y_1)(\tau))_Z \, d\tau \\ &= \int_t^T (R\Phi(\tau, t)Y_0, R\Phi(\tau, t)Y_1) \, d\tau \\ &\quad + \int_t^T (u^*(\tau, t; Y_0), u^*(\tau, t; Y_1))_U \, d\tau, \end{aligned} \quad (138)$$

after taking the adjoints of R and L_t and using the characterization of $u^*(\cdot, t; Y_0)$ in (97). The result follows upon applying Proposition 3.3(ii).

Proof of (ii) and (iii). This follows immediately from (i). □

Lemma 3.1. Let $\Phi(\cdot,\cdot)$ be as defined in (125). Then $\forall\tau$, $0\le\tau<T$ and $Y_0\in D(\mathscr{A})$, we have that $(d/d\tau)\Phi(\cdot,\tau)Y_0$ exists as an element of ${}_\theta C([\tau,T];H_1\times H_0)$, where θ is as given in (60); moreover,

$$(d/d\tau)\Phi(\cdot,\tau)Y_0$$
$$=(I_\tau+L_\tau L_\tau^*R^*R)^{-1}[e^{\mathscr{A}(\cdot-\tau)}\mathscr{B}L_\tau^*R^*R\Phi(\cdot,\tau)(\tau)-e^{\mathscr{A}(\cdot-\tau)}\mathscr{A}Y_0]. \tag{139}$$

Proof. Using the definition of $Y^*(\cdot,\tau;Y_0)$ in (89) and the representation of $u^*(\cdot,\tau;Y_0)$ in (97), we have that for $Y_0\in H_1\times H_0$,

$$\Phi(\cdot,\tau)Y_0=(I_\tau+L_\tau L_\tau^*R^*R)^{-1}e^{\mathscr{A}(\cdot-\tau)}Y_0, \tag{140}$$

after using the fact that $I_\tau+L_\tau L_\tau^*R^*R$ is invertible on $L^2(\tau,T;H_1\times H_0)$ by the Lax–Milgram theorem. Thus for fixed t, $\tau<t\le T$, and $Y_0\in D(\mathscr{A})$,

$$(d/d\tau)\Phi(t,\tau)+(d/d\tau)(L_\tau L_\tau^*R^*R\Phi(\cdot,\tau)Y_0(t))$$
$$=-e^{\mathscr{A}(t-\tau)}\mathscr{A}Y_0,\qquad \text{in } H_1\times H_0, \tag{141}$$

where each term on the left-hand side of (141) is well defined initially in the variable τ at least in the sense of distributions, i.e., as elements of $\mathscr{D}'(0,t;H_1\times H_0)$; see Ref. 22, p. 101. Now, for fixed t, the distributional derivative of $L_\tau L_\tau^*R^*R\Phi(\cdot,\tau)(t)$ can be computed straightforwardly as

$$(d/d\tau)(L_\tau L_\tau^*R^*R\Phi(\cdot,\tau)Y_0(t))$$
$$=-e^{\mathscr{A}(t-\tau)}\mathscr{B}(L_\tau L_\tau^*R^*R\Phi(\cdot,\tau)Y_0)(\tau)$$
$$+(L_\tau L_\tau^*R^*R(d/d\tau)\Phi(\cdot,\tau)Y_0)(t) \tag{142}$$

in $\mathscr{D}'(0,t;H_1\times H_0)$. Thus, using (141) and (142) we have that for fixed t, $\tau<t\le T$,

$$[(I_\tau+L_\tau L_\tau^*R^*R)(d/d\tau)\Phi(\cdot,\tau)Y_0](t)$$
$$=e^{\mathscr{A}(t-r)}\mathscr{B}[(L_\tau^*R^*R\Phi(\cdot,\tau)Y_0)(\tau)]-e^{\mathscr{A}(t-\tau)}\mathscr{A}Y_0 \tag{143}$$

in $\mathscr{D}'(0,t;H_1\times H_0)$, given that $Y_0\in D(\mathscr{A})$. But for fixed τ, the function

$$e^{\mathscr{A}(\cdot-\tau)}\mathscr{B}[(L_\tau^*R^*R\Phi(\cdot,\tau)Y_0)(\tau)]-e^{\mathscr{A}(\cdot-\tau)}\mathscr{A}Y_0 \tag{144}$$

is an element of ${}_\theta C([\tau,T];H_1\times H_0)$, and, furthermore, we can use Propositions 2.4 and 2.5 and the same bootstrap argument employed in the proof of Lemma 2.3 to find that $I_\tau+L_\tau L_\tau^*R^*R$ is boundedly invertible on ${}_\theta C([\tau,T];H_1\times H_0)$. We thus deduce that $(d/d\tau)\Phi(\cdot,\tau)Y_0$ can be taken as an element of ${}_\theta C([\tau,T];H_1\times H_0)$ with the representation in (139) being valid. □

Lemma 3.2.

(i) $P(t)$ as defined in (128) satisfies the following differential Riccati equation (DRE): $\forall Y_0, Y_1 \in D(\mathscr{A})$ and $\forall t \in (0, T)$,

$$\begin{aligned}\langle \dot{P}(t) Y_0, Y_1 = &-\langle R^* R Y_0, Y_1\rangle \\ &-\langle P(t)\mathscr{A} Y_0, Y_1\rangle - \langle P(t) Y_0, \mathscr{A} Y_1\rangle \\ &+\langle \mathscr{B}^* P(t) Y_0, \mathscr{B}^* P(t) Y_1\rangle, \end{aligned} \tag{145}$$

where the duality pairing $\langle \cdot, \cdot \rangle$ is taken between $H_1 \times H_0$ and $H_1^* \times H_0^*$.

(ii) The solution $P(t)$ is unique within the class of self-adjoint operators $\hat{P}(t) \in \mathscr{L}(H_1 \times H_0, H_1^* \times H_0^*)$ which satisfy $\forall Y_0 \in H_1 \times H_0$,

$$\mathscr{B}^* \hat{P}(\cdot) Y_0 \in C([0, T]; U). \tag{146}$$

Proof of (i). $\forall Y_0, Y_1 \in D(\mathscr{A})$ and $\forall t$, $0 \le t \le T$, we have by the definition of $P(t)$,

$$\langle P(t) Y_0, Y_1\rangle = \int_t^T \langle R^* R \Phi(\tau, t) Y_0, e^{\mathscr{A}(\tau - t)} Y_1\rangle \, d\tau, \tag{147}$$

and differentiating both sides yields

$$\begin{aligned}(d/dt)\langle P(t) Y_0, Y_1\rangle = &-\langle R^* R Y_0, Y_1\rangle \\ &+ \int_t^T (d/dt)\langle R^* R \Phi(\tau, t) Y_0, e^{\mathscr{A}(\tau - t)} Y_1\rangle \, d\tau. \end{aligned} \tag{148}$$

Now, using Lemma 3.1 to move the differentiation on the right-hand side above inside the brackets yields

$$\begin{aligned}\text{LHS of (148)} = &-\langle R^* R Y_0, Y_1\rangle \\ &- \int_t^T \langle R^* R[(I_t + L_t L_t^* R^* R)^{-1} e^{\mathscr{A}(\cdot - t)} \mathscr{A} Y_0](\tau), e^{\mathscr{A}(\tau - t)} Y_1\rangle \, d\tau \\ &+ \int_t^T \langle R^* R[(I_t + L_t L_t^* R^* R)^{-1} e^{\mathscr{A}(\cdot - t)} \mathscr{B}](\tau), e^{\mathscr{A}(\tau - t)} Y_1\rangle \, d\tau \\ &- \int_t^T \langle R^* R \Phi(\tau, t) Y_0, e^{\mathscr{A}(\tau - t)} \mathscr{A} Y_1\rangle \, d\tau \end{aligned}$$

$$= -\langle R^*RY_0, Y_1\rangle - \langle P(t)\mathscr{A}Y_0, Y_1\rangle$$

$$+ \left\langle \mathscr{B}^*P(t)Y_0, \int_t^T \mathscr{B}^* e^{\mathscr{A}^*(\tau - t)} \right.$$

$$\left. \times [(I_t + L_t L_t^* R^*R)^{-*} R^*R\, e^{\mathscr{A}(\cdot - t)} Y_1](\tau)\, d\tau \right\rangle$$

$$-\langle P(t)Y_0, \mathscr{A}Y_1\rangle, \tag{149}$$

where LHS stands for left-hand side. Now, from Proposition 3.4(ii),

$$P(t) = P^*(t),$$

and from (140),

$$\Phi(\cdot, t)Y_0 = (I_t + L_t L_t^* R^*R)^{-1} e^{\mathscr{A}(\cdot - t)} Y_0. \tag{150}$$

So by directly computing the adjoint $P^*(t)$ to handle the third term on the right-hand side of (149), we have that indeed

$$\text{LHS of (149)} = -\langle R^*RY_0, Y_1\rangle - \langle P(t)\mathscr{A}Y_0, Y_1\rangle$$
$$+ \langle \mathscr{B}^*P(t)Y_0, \mathscr{B}^*P(t)Y_1\rangle - \langle P(t)Y_0, \mathscr{A}Y_1\rangle. \tag{151}$$

Proof of (ii). It suffices to prove the uniqueness of the solution in the given class (146) to the following Riccati integral equation:

$$\langle P(t)Y_0, Y_1\rangle = \int_\tau^T (R\, e^{\mathscr{A}(\tau - t)} Y_0, R\, e^{\mathscr{A}(\tau - t)} Y_1)_z\, dt$$
$$- \int_t^T (\mathscr{B}^*P(\tau)\, e^{\mathscr{A}(\tau - t)} Y_0, \mathscr{B}^*P(\tau)\, e^{\mathscr{A}(\tau - t)} Y_1)_U\, d\tau, \tag{152}$$

$\forall Y_0, Y_1 \in H_1 \times H_0$. To this end, if $P_1(\cdot)$ and $P_2(\cdot)$ both solve the DRE for $0 < t < T$ and are of the class (146), then setting $Q(\cdot) \equiv P_1(\cdot) - P_2(\cdot)$, one has necessarily that $\forall Y_0, Y_1 \in H_1 \times H_0$,

$$\langle Q(t)Y_0, Y_1\rangle = \int_t^T (\mathscr{B}^*P_2(\tau)\, e^{\mathscr{A}(\tau - t)} Y_0, \mathscr{B}^*Q(\tau)\, e^{\mathscr{A}(\tau - t)} Y_1)_U\, d\tau$$
$$- \int_t^T (\mathscr{B}^*Q(\tau)\, e^{\mathscr{A}(\tau - t)} Y_0, \mathscr{B}^*P_1(\tau)\, e^{\mathscr{A}(\tau - t)} Y_1)_U\, d\tau, \tag{153}$$

or

$$Q(t)Y_0=\int_t^T e^{\mathscr{A}^*(\tau-t)}(\mathscr{B}^*Q(\tau))^*\mathscr{B}^*P_2(\tau)\, e^{\mathscr{A}(\tau-t)}Y_0\,d\tau$$
$$-\int_t^T e^{\mathscr{A}^*(\tau-t)}(\mathscr{B}^*P_1(\tau))^*\mathscr{B}^*Q(\tau)\, e^{\mathscr{A}(\tau-t)}Y_0\,d\tau. \tag{154}$$

Then, setting $V(t)=\mathscr{B}^*Q(t)$, we have, after applying $\mathscr{B}^*$ to both sides of (154), the equation

$$V(t)Y_0=\int_t^T \mathscr{B}^*\, e^{\mathscr{A}^*(\tau-t)}V^*(\tau)\mathscr{B}^*P_2(\tau)\, e^{\mathscr{A}(\tau-t)}Y_0\,d\tau \tag{155}$$
$$-\int_t^T \mathscr{B}^*\, e^{\mathscr{A}^*(\tau-t)}(\mathscr{B}^*P_1(\tau))^*V(\tau)\, e^{\mathscr{A}(\tau-t)}Y_0\,d\tau. \tag{156}$$

As $P_1(t)$, $P_2(t)$ are in the specified class (146), then by the Banach–Steinhaus theorem one has the following estimate $\forall Y_0\in H_1\times H_0$, $0\le t\le\tau\le T$,

$$\|\mathscr{B}^*P_i(\tau)Y_0\|_U\le C_T\|Y\|_{H_1\times H_0},\qquad i=1,2. \tag{157}$$

Using the above estimate and taking the norm of both sides of Eq. (155) yields

$$\|V(t)Y_0\|\le\int_t^T [C_T/(\tau-t)^\theta]\|V(\tau)\|\,\|Y_0\|_{H_1\times H_0} \tag{158a}$$
$$\le\int_t^T [C_T/(\tau-t)^\theta]\left\{\sup_{t\le\tau\le T}\|V(\tau)\|\right\}\|Y_0\|_{H_1\times H_0} \tag{158b}$$
$$\le C_T(T-t)^{1-\theta}\left\{\sup_{t\le\tau\le T}\|V(\tau)\|\right\}\|Y_0\|_{H_1\times H_0}. \tag{158c}$$

Note that (158b) ensues from (158a) with $0<\theta<1/2$, after using the norm estimate (60). Thus, for $t_0\le t\le T$, we obtain

$$\sup_{t\le\tau\le T}\|V(t)\|\le C_T(T-t_0)^{1-\theta}\left\{\sup_{t\le\tau\le T}\|V(\tau)\|\right\}, \tag{159}$$

and for $T-t_0$ small enough, we have that $C_T(T-t_0)^{1-\theta}$ is less than one, and uniqueness can be deduced within the class of all self-adjoint

$\hat{P}(t) \in \mathscr{L}(H_1 \times H_0, H_1^* \times H_0^*)$ such that

$$\mathscr{B}^* \hat{P}(\cdot) Y_0 \in C([t_0, T]; U),$$
$$\forall Y_0 \in H_1 \times H_0 \text{ and } t \in (t_0, T). \tag{160}$$

Iterating this argument and establishing uniqueness within each specified class will yield uniqueness for the entire interval after a finite number of steps. □

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