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Convergence Properties of Ordinal Comparison in the Simulation of Discrete Event Dynamic Systems[1]

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Abstract. Recent research has demonstrated that ordinal comparison has fast convergence despite the possible presence of large estimation noise in the design of discrete event dynamic systems. In this paper, we address the fundamental problem of characterizing the convergence of ordinal comparison. To achieve this goal, an indicator process is formulated and its properties are examined. For several performance measures frequently used in simulation, the rate of convergence for the indicator process is proven to be exponential for regenerative simulations. Therefore, the fast convergence of ordinal comparison is supported and explained in a rigorous framework. Many performance measures of averaging type have asymptotic normal distributions. The results of this paper show that ordinal comparison converges monotonically in the case of averaging normal random variables. Such monotonicity is useful in simulation planning.

Key Words. Stochastic optimization, simulation, discrete event dynamic systems.

1. Introduction

In this paper, we address a fundamental design problem: Characterizing convergence of ordinal comparison in the simulation of discrete event dynamic systems (DEDS). By DEDS, we mean systems where changes in states are triggered by occurrences of events at some discrete (often random) time instants. Such systems include queueing systems, computer networks, manufacturing systems, communication networks, transportation systems.

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DEDS form an important class of man-made systems that are being intensively investigated. By ordinal comparison, we mean comparing the relative goodness (rank) of different designs without knowing the exact values of corresponding performance measures.

Difficulties arise in the design of DEDS from several sources. First, most DEDS are very complex in nature. For example, analytical solutions are available only for some special cases of queueing systems. It is difficult to evaluate the performance of a design. As a result, computer simulation is often used for the analysis and design of DEDS. Second, it is costly and time consuming to obtain an accurate estimate of a performance measure. To see this point, consider estimating $E[X]$ of a random $X \in R$ using $S_t = (1/t) \sum_{i=1}^{t} X_i$, where $\{X_i, i \geq 1\}$ is a sequence of i.i.d. samples of X. If $E[X] < \infty$, $\mathrm{Var}[X] < \infty$, the strong law of large numbers guarantees that $\lim_{t \to \infty} S_t = E[X]$, a.s. However, it is also well known that the preceding limit converges with rate $O(1/\sqrt{t})$, in the sense that

$$(E[(S_t - E[X])^2])^{1/2} = ((1/t)\mathrm{Var}[X])^{1/2} = O(1/\sqrt{t}).$$

Such rate is unsatisfactorily slow. In order to see that the dynamics may not accelerate the convergence, consider an irreducible and aperiodic finite-state discrete-time Markov chain $\{X_i, i \geq 0\}$ with a performance measure of additive type, $(1/t) \sum_{i=1}^{t} f(X_i)$, where $f(x) \in R$. Markov chains are special DEDS. It has been proven that such additive performance measures converge with rate $O(1/\sqrt{t})$ in time t; see Ref. 1, p. 228. Finally, search spaces in the design of DEDS are usually very large. Consider the problem of arranging 10 machines and 50 buffer spaces to form a serial production line. A buffer (possibly of zero size) is required in front of every machine. Then, the total number of different combinations is

$$10! \binom{59}{9} \approx 1.26 \times 10^{11}.$$

It is difficult to find the best design in such a large (discrete) search space. Therefore, an important problem in the design of DEDS is to allocate quickly a good satisficing design (Ref. 2, p. 69).

Optimization is a natural approach to design. Since randomness is involved in evaluating performances of DEDS, stochastic optimization is a conventional method for allocating the best design (the best parameter value) when different designs can be represented by a continuous parameter $\theta \in R$. This is usually done via stochastic approximation algorithms of the following generic form:

$$\theta_{t+1} = \theta_t - a_t h_t, \tag{1}$$

where θ_t is the parameter value at the tth iteration, $a_t \geq 0$ is a stepsize, and $h_t \in R$ is a noisy estimate of the gradient of the performance measure at θ_t. Algorithms of the form (1) have been extensively studied (Ref. 3–4). Once again, the best possible rate of convergence for (1) is $O(1/\sqrt{t})$ in time t in the sense of $(E[(\theta_t - \theta^*)^2])^{1/2} = O(1/\sqrt{t})$, where θ^* is the true optimum (Refs. 3–4). For discrete search space, the situation is no better. In addition to the previously mentioned difficulties, we have to overcome the curse of the discrete nature of parameter values where no gradient information is available.

In reality, we often face a decision-making problem of choosing one relatively better design from all possible alternatives. As long as we can single out the good and satisficing designs, the accuracy of the performance measure is of secondary importance (Ref. 2, p. 69). Such goal relaxation can bring great saving of effort. For example, it is known that (1) converges exponentially to a fixed neighborhood of the optimum (Ref. 5). In the context of recently proposed ordinal optimization methods, ordinal comparison of different designs by their relative ranks is very efficient, as observed in many experiments. Ordinal comparison is able to discern quickly the good designs (Refs. 6–11). Recent research has revealed a more interesting phenomenon: Ordinal comparison is beneficial when used in conjunction with conventional methods such as stochastic approximation and simulated annealing as reported in Refs. 12–15.

To further clarify the issue involved here, consider the problem of finding the best or a good design among all N possibilities. Let $\Theta = \{\theta_1, \theta_2, \ldots \theta_N\}$ denote the set of all designs, and let $J(\theta) \in R$ denote the performance measure of a particular design $\theta \in \Theta$. In general DEDS, the exact form of $J(\theta)$ is not available. We can only use the noisy estimate of $J(\theta)$ in our decision making. Let us consider the dynamics of the following experiment. We simultaneously simulate N parallel DEDS with designs θ_i, $i = 1, 2, \ldots, N$. This can be done, for example, using the standard clock (Ref. 16), or the augmented system analysis (Refs. 17–18), or some other techniques (Ref. 19). As the simulation continuous, we collect data and output an estimate of $J(\theta)$, denoted by $L(\theta, t)$, for every $\theta \in \Theta$. Convergence of $L(\theta, t)$ to $J(\theta)$ is generally slow with rate at most $O(1/\sqrt{t})$ according to the law of large numbers (assuming the validity of convergence). However, it has been observed that the observed order of performance measures $L(\theta_i, t)$, $i = 1, 2, \ldots, N$, can quickly converge to an order very close to the true order of performance measures $J(\theta_i)$, $i = 1, 2, \ldots, N$, despite the possible presence of large noises (Refs. 6–8 and 10–11).

Therefore, there naturally arise questions of what is the meaning of the convergence of ordinal comparison and why ordinal comparison converges fast. To answer these questions, it is necessary to provide the theoretical framework of characterizing the mechanism of ordinal comparison. As the first step, in this paper we formulate an indicator process to characterize the dynamical behavior of ordinal comparison, enabling us to provide a rigorous investigation of its convergence properties. By examining the indicator process, we prove that, in several important situations, ordinal comparison indeed converges very fast—at an exponential rate. It should be emphasized that we are studying the time-varying behavior of ordinal comparison as opposed to the conventional (static with possible replicas) order statistics; see, for example, Ref. 20. Correlation in time is often present in the dynamics of DEDS. Transient behavior also affects the estimation of steady-state performance measures.

The rest of this paper is arranged as follows. The problem of ordinal comparison is stated in Section 2. An indicator process is defined to characterize the ordinal comparison in parallel simulation. General properties of the indicator process and a lower bound on its rate of convergence are provided in Section 3. Many important DEDS are regenerative. For regenerative simulations, Section 4 proves the exponential convergence rate for the indicator process, thus providing an explanation for the fast convergence of ordinal comparison. Some results on averaging a sequence of i.i.d. random variables are presented in Section 5. A detailed formula for the rate of convergence for random variables with normal distributions is given. Monotonicity of the indicator process in the simulation time is proven. Finally, Section 6 contains several concluding remarks.

2. Problem Statement

Throughout the paper, the DEDS under simulation is described by a right-continuous, piecewise-constant state process $\{X(\theta, \xi, t) \in \mathscr{X}, t \geq 0\}$, parameterized by $\theta \in \Theta$ and defined on a common probability space. Here, $\mathscr{X} \subset R$ is the state space, θ is used simply to indicate a design, and ξ represents all the randomness involved. For each design $\theta \in \Theta$, let $L(\theta, t) \in R$ be an estimate of a performance measure $J(\theta) \in R$ based on a particular sample trajectory of $\{X(\theta, \xi, t)\}$ over $[0, t]$. In DEDS, $J(\theta)$ is often a steady-state performance and $L(\theta, t)$ is a sample performance over $[0, t]$. We assume that:

(A1) the DEDS under simulation is ergodic, in the sense that

$$\lim_{t \to \infty} L(\theta, t) = J(\theta), \qquad \text{a.s.} \tag{2}$$

This assumption guarantees that $J(\theta)$ can be estimated with arbitrary accuracy by increasing the simulation duration of any one sample trajectory. Note that, because of the law of large numbers, $L(\theta, t)$ converges slowly with rate at most $O(1/\sqrt{t})$. Long and time-consuming simulation has to be performed in order to have a good estimate of the (steady-state) performance measure $J(\theta)$.

Now, consider the problem of comparing performance measures corresponding to a set Θ of N, $1 \leq N < \infty$, designs. We say that design θ_i is better than θ_j if $J(\theta_i) > J(\theta_j)$. Without loss of generality, we assume that the N designs are indexed in such a way that

$$\infty > J(\theta_1) > J(\theta_2) > J(\theta_3) > \cdots > J(\theta_N) > -\infty.$$

Particularly, we are interested in finding a design that is one of the M, $1 \leq M \leq N$, true best designs in Θ. For convenience, let us define

$$\Theta_g = \{\theta_i, i = 1, 2, \ldots, M\},$$

$$\Theta_b = \{\theta_i, i = M+1, M+2, \ldots, N\},$$

the sets of good and bad designs, respectively. For the problem of finding one of the M best designs based on the simulation over $[0, t]$, we choose the design with the largest sample performance, i.e.,

$$\theta^* = \arg\max_{\theta \in \Theta} L(\theta, t). \tag{3}$$

Experimental results have shown that (3) can quickly find the true desired design (Refs. 7–8, 10). Intuitively, this implies that the relative order of performance measures converges very fast.

In order to characterize the rate of convergence for ordinal comparison, we define the following indicator process:

$$I_{M,N}(t) = \begin{cases} 1, & \text{if } \max_{\sigma \in \Theta_g} L(\sigma, t) \geq \max_{\theta \in \Theta_b} L(\theta, t), \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Then, $I_{M,N}(t)$ is equal to 1 if the observed best design is among the true good designs and equal to 0 otherwise. Since $\max_{\sigma \in \Theta_g} L(\sigma, t)$ is the maximum of the observed true good designs and $\max_{\theta \in \Theta_b} L(\theta, t)$ is the maximum of the observed true bad designs, $I_{M,N}(t)$ is a function of t indicating when the observed best design determined by (3) is one of the M true best designs.

The main goal of this paper is to examine the behavior of

$$\text{Prob}[I_{M,N}(t) = 1] \text{ and } \text{Prob}[I_{M,N}(t) = 0]$$

as functions of simulation time t. The convergence rate of $I_{M,N}(t)$ will be taken as that of

$$\text{Prob}[I_{M,N}(t) = 1] \text{ or } \text{Prob}[I_{M,N}(t) = 0].$$

3. General Cases

3.1. Basic Properties. In this section, we list several simple properties of the indicator process.

Proposition 3.1.

(a) $\text{Prob}[I_{M,N}(t) = 1]$ is nondecreasing in M and nonincreasing in N.
(b) $\text{Prob}[I_{M,N}(t) = 0]$ is nonincreasing in M and nondecreasing in N.

Proof. This proof was suggested by a reviewer. Denote

$$\Theta_g(M) = \{\theta_i, i = 1, 2, \ldots, M\},$$

$$\Theta_b(M) = \{\theta_i, i = M+1, M+2, \ldots, N\}.$$

For $M_1 > M_2$, we have $\Theta_g(M_2) \subset \Theta_g(M_1)$ and $\Theta_b(M_1) \subset \Theta_b(M_2)$. Thus,

$$\left\{\max_{\sigma \in \Theta_g(M_2)} L(\sigma, t) \geq \max_{\theta \in \Theta_b(M_2)} L(\theta, t)\right\}$$

$$\subset \left\{\max_{\sigma \in \Theta_g(M_1)} L(\sigma, t) \geq \max_{\theta \in \Theta_b(M_1)} L(\theta, t)\right\}, \tag{5}$$

which shows that $\text{Prob}[I_{M,N}(t) = 1]$ is nondecreasing in M. A similar argument shows that it also is nonincreasing in N. □

Proposition 3.2. If Assumption (A1) is satisfied,

$$\lim_{t \to \infty} \text{Prob}[I_{M,N}(t) = 1] = 1, \qquad \lim_{t \to \infty} \text{Prob}[I_{M,N}(t) = 0] = 0,$$

for any M, N.

Proof. Under Assumption (A1), $I_{M,N}(t)$ converges to 1, a.s., implying convergence in probability. The proposition is another statement of convergence in probability, since $I_{M,N}(t)$ takes only 1 or 0. □

Based on simulation observation over $[0, t]$, Proposition 3.1 states that a larger design space with M fixed reduces the probability of finding the correct design and that a looser design requirement with larger M increases this possibility. Proposition 3.2 simply says that we can eventually find the correct design if the DEDS is ergodic in the sense of (A1).

Proposition 3.3. The following inequality always holds:

$$\text{Prob}[I_{M,N}(t)=1] \geq \max_{\sigma\in\Theta_g} \left\{1 - \sum_{\theta\in\Theta_b} (1 - \text{Prob}[L(\sigma,t) \geq L(\theta,t)])\right\}^+, \qquad (6)$$

where $\{x\}^+ = \max\{x, 0\}$.

Proof. First, the following inequality is true:

$$\begin{aligned}\text{Prob}[I_{M,N}(t)=1] &= \text{Prob}\left[\max_{\sigma\in\Theta_g} L(\sigma,t) \geq \max_{\theta\in\Theta_b} L(\theta,t)\right]\\ &\geq \max_{\sigma\in\Theta_g}\left(1 - \text{Prob}\left[L(\sigma,t) < \max_{\theta\in\Theta_b} L(\theta,t)\right]\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\text{Prob}\left[L(\sigma,t) < \max_{\theta\in\Theta_b} L(\theta,t)\right]\\ &= \text{Prob}\left[\bigcup_{\theta\in\Theta_b} \{L(\sigma,t) < L(\theta,t)\}\right]\\ &\leq \sum_{\theta\in\Theta_b} \text{Prob}[L(\sigma,t) < L(\theta,t)]\\ &= \sum_{\theta\in\Theta_b} (1 - \text{Prob}[L(\sigma,t) \geq L(\theta,t)]).\end{aligned}$$

Therefore, a lower bound of $\text{Prob}[I_{M,N}(t)=1]$ is

$$\max_{\sigma\in\Theta_g} \left\{1 - \sum_{\theta\in\Theta_b} (1 - \text{Prob}[L(\sigma,t) \geq L(\theta,t)])\right\}^+,$$

which is exactly Inequality (6). □

The validity of (6) does not require the independence of $\{X(\theta, \xi, t)\}$ for different designs. It will be used repeatedly later.

3.2. Lower Bound. In many cases, such as finite-state Markov chains and averaging i.i.d. random variables, the variance of an estimate $L(\theta, t)$ decays to zero with rate $O(1/t)$. Such delay rate of the variance can be used to derive a lower bound on the rate of convergence for the indicator process, which is precisely stated in the following theorem.

Theorem 3.1. Assume that (A1) holds, that $\lim_{t\to\infty} E[L(\theta, t)] = J(\theta)$ for any $\theta\in\Theta$, and that the variance of the estimate $L(\theta, t)$ decays to zero with rate $O(1/t)$ for all $\theta\in\Theta$, i.e.,

$$\mathrm{Var}[L(\theta, t)] = O(1/t). \tag{7}$$

Then,

$$\mathrm{Prob}[I_{M,N}(t) = 1] = 1 - O(1/t),$$

$$\mathrm{Prob}[I_{M,N}(t) = 0] = O(1/t).$$

In other words, the indicator process $I_{M,N}(t)$ converges to 1 with rate at least $O(1/t)$.

Proof. Note that we do not assume the independence of simulations for different designs. For any $\sigma \in \Theta_g$, $\theta \in \Theta_b$, define

$$e_{\sigma,\theta}(t) = L(\sigma, t) - L(\theta, t).$$

Since

$$\lim_{t \to \infty} E[L(\theta, t)] = J(\theta),$$

we have

$$\lim_{t \to \infty} E[e_{\sigma,\theta}(t)] = \lim_{t \to \infty} (E[L(\sigma, t)] - E[L(\theta, t)])$$
$$= J(\sigma) - J(\theta) > 0.$$

Choose t_0 sufficiently large so that $E[e_{\sigma,\theta}(t)] \geq c$ for some constant $c > 0$, for any $\sigma \in \Theta_g$, $\theta \in \Theta_b$, and for all $t \geq t_0$. Then, when $t \geq t_0$,

$$\mathrm{Prob}[L(\sigma, t) \geq L(\theta, t)]$$
$$\geq 1 - \mathrm{Prob}[|e_{\sigma,\theta}(t) - E[e_{\sigma,\theta}(t)]| \geq E[e_{\sigma,\theta}(t)]].$$

Using the Chebyshev inequality, we can further bound the previous inequality as

$$\mathrm{Prob}[L(\sigma, t) \geq L(\theta, t)]$$
$$\geq 1 - (E[e_{\sigma,\theta}(t)])^{-2}\mathrm{Var}[e_{\sigma,\theta}(t)]$$
$$\geq 1 - 2c^{-2}(\mathrm{Var}[L(\theta, t)] + \mathrm{Var}[L(\theta, t)]) \tag{8}$$
$$= 1 - O(1/t).$$

The claim of the theorem follows from combining the previous inequality with (6). □

Remark 3.1. Choice of Metric. We cannot conclude from Theorem 3.1 that $I_{M,N}(t)$ converges faster than $O(1/\sqrt{t})$. We use $\mathrm{Prob}[I_{M,N}(t) = 1]$ to characterize the convergence of $I_{M,N}(t)$. If we change the criterion to the square root of the mean square error, then the rate would be $O(1/\sqrt{t})$, since

$$(E[(I_{M,N}(t)-1)^2])^{1/2} = (\text{Prob}[I_{M,N}(t)=0])^{1/2} = O(t^{-1/2}). \tag{9}$$

However, what Theorem 3.1 gives is a general lower bound. It demonstrates that convergence of the indicator process is not slower than that of the performance measure.

We prefer to use $\text{Prob}[I_{M,N}(t)=1]$, rather than (9), to measure the convergence of the indicator process, since it is intuitive and convenient. When we observe that $I_{M,N}(t)$ takes the value 1 much more often than 0 in the simulation process, we are pretty sure that $I_{M,N}(t)$ should be 1. If we want to find the true good design with 95% confidence, we simply terminate the simulation when $\text{Prob}[I_{M,N}(t)=1]$ reaches and stays within [0.95, 1]. This is often easily done.

Tightness of the Bound. The rate cannot be improved in general. To see this point, consider two independent random variables $X(\theta_i, \xi, t)=\theta_i$, with probability $1-1/t$, and $X(\theta_i, \xi, t)=0$, with probability $1/t$, $i=1, 2$. Assume that $\theta_1 > \theta_2 > 0$ and $M=1$. Define $L(\theta, t)=X(\theta, \xi, t)$. Then $\Theta_g=\{\theta_1\}$, $\Theta_b=\{\theta_2\}$. Direct computation confirms that the assumptions of Theorem 3.1 are satisfied. For this problem, we have

$$\text{Prob}[I_{M,N}(t)=0]=(1/t)(1-1/t)=O(1/t).$$

The lower bound is achieved.

Effect of Variance Reduction. It is possible to reduce the variance of $e_{\sigma,\theta}(t)$ by introducing a positive correlation between $L(\sigma, t)$ and $L(\theta, t)$. Inequality (8) shows that such variance reduction increases the lower bound on $\text{Prob}[I_{M,N}(t)=1]$. It is reasonable to expect that positive correlation between $L(\sigma, t)$ and $L(\theta, t)$ should accelerate the convergence of the indicator process, since the variance of $e_{\sigma,\theta}(t)$ is reduced. Unfortunately, this may not be necessarily true. Smaller variance of $e_{\sigma,\theta}(t)$ does not automatically imply a better discrimination between two designs. To illustrate this point, consider the following two pairs of random variables:

$$X(\theta_1)=\begin{cases}3, & \text{w.p.0.9},\\ 0, & \text{w.p.0.1},\end{cases} \qquad X(\theta_2)=\begin{cases}5, & \text{w.p.0.2},\\ 1, & \text{w.p.0.8},\end{cases}$$

$$(\tilde{X}(\theta_1), \tilde{X}(\theta_2))=\begin{cases}(3, 5), & \text{w.p.0.2},\\ (3, 1), & \text{w.p.0.7},\\ (0, 5), & \text{w.p.0.0},\\ (0, 1), & \text{w.p.0.1},\end{cases}$$

where $X(\theta_1)$ and $X(\theta_2)$ are independent. Then, $\tilde{X}(\theta_i)$ has the marginal distribution of $X(\theta_i)$, $i=1, 2$,

$$E[X(\theta_1)]=E[\tilde{X}(\theta_1)]=2.7 > E[X(\theta_2)]=E[\tilde{X}(\theta_2)]=1.8.$$

Let

$$e_{\theta_1,\theta_2} = X(\theta_1) - X(\theta_2), \qquad \tilde{e}_{\theta_1,\theta_2} = \tilde{X}(\theta_1) - \tilde{X}(\theta_2).$$

Then,

$$\mathrm{Var}[e_{\theta_1,\theta_2}] = 3.37 > \mathrm{Var}[\tilde{e}_{\theta_1,\theta_2}] = 2.89.$$

However,

$$\mathrm{Prob}[X(\theta_1) \geq X(\theta_2)] = 0.72 > \mathrm{Prob}[\tilde{X}(\theta_1) \geq \tilde{X}(\theta_2)] = 0.7;$$

in other words, smaller variance of e_{θ_1,θ_2} may not increase the accuracy of the comparison.

Nevertheless, there exist cases where smaller variance (or positive correlation) does increase the accuracy of ordinal comparison. In the case of averaging a sequence of i.i.d random variables with normal distributions, Ho, Deng, and Hu show in Ref. 21 that positive correlation is always helpful for ordinal comparison. In fact, they prove that the probability of correct selection is monotone increasing in correlation coefficient. The random variables $X_1, X_2, \ldots, X_n$ are said to be associated if

$$\mathrm{Cov}[f(X_1, X_2, \ldots, X_n), g(X_1, X_2, \ldots, X_n)] \geq 0,$$

for any nondecreasing functions $f(\cdot)$, $g(\cdot)$: $R^n \to R$. It is clear that association is stronger than positive correlation. In the case of comparing two designs of a finite-state Markov chain with additive performance measures, Glasserman and Vakili show in Ref. 22 that association increases the rate of convergence of $\mathrm{Prob}[L(\theta_1, t) \geq L(\theta_2, t)]$ as t goes to infinity.

4. Regenerative Simulation

Theorem 3.1 gives a very conservative bound for general cases. In many situations, we know much more about the structure and distribution of $\{X(\theta, \xi, t)\}$. Quite often, the rate of convergence can be proven faster than that in Theorem 3.1. For finite-state Markov chains with additive performance measures, Glasserman and Vakili show in Ref. 22 that the convergence of ordinal comparison of two designs is exponential. In this section, we consider the case of simulating a general regenerative DEDS.

Definition 4.1. See Ref. 23, p. 25. A nonnegative random variable $\tau \geq 0$ is said to be periodic with period $\tau_p > 0$ if $\tau \in \{0, \tau_p, 2\tau_p, \ldots\}$ a.s. and τ_p is the largest such number. If there is no such τ_p, then τ is said to be aperiodic.

Definition 4.2. See Ref. 24, p. 19. A DEDS $\{X(\theta, \xi, t)\}$ is said to be regenerative in a classic sense if there is an increasing sequence of nonnegative finite random times $\{\tau_i(\theta), i \geq 0\}$ such that, for each $i \geq 0$,

(i) $\{X(\theta, \xi, \tau_i(\theta)+t), \tau_k(\theta)-\tau_i(\theta), t \geq 0, k \geq i\}$ are identically distributed;
(ii) $\{X(\theta, \xi, \tau_i(\theta)+t), \tau_k(\theta)-\tau_i(\theta), t \geq 0, k \geq i\}$ does not depend on $\{X(\theta, \xi, t), \tau_j(\theta), t \leq \tau_i, 0 \leq j \leq i\}$.

The sequence $\{\tau_i(\theta), i \geq 1\}$ is a sequence of regenerative points. We consider two kinds of performance measures,

$$L_1(\theta, t) = (1/t) \int_0^t l(X(\theta, \xi, u))\, du \tag{10}$$

and

$$L_2(\theta, t) = \sum_{i=1}^{K(t,\theta)} \int_{\tau_{i-1}(\theta)}^{\tau_i(\theta)} l(X(\theta, \xi, u))\, du \Big/ \sum_{i=1}^{K(t,\theta)} T_i(\theta), \tag{11}$$

where

$$l(x) \in R, \qquad T_i(\theta) = \tau_i(\theta) - \tau_{i-1}(\theta), \quad i \geq 1,$$

is the length of the ith regeneration cycle,

$$K(t, \theta) = \max_i \{i: \tau_i(\theta) \leq t, i \geq 0\}$$

is the number of regeneration cycles on $[0, t]$, and then

$$\bar{K}(t, \theta) = K(t, \theta) + 1$$

is the first passage time process. For convenience, we define $T_0(\theta) = \tau_0(\theta)$. Note that

$$\left\{\left(\int_{\tau_{i-1}(\theta)}^{\tau_i(\theta)} l(X(\theta, \xi, u))\, du, T_i(\theta)\right), i \geq 1\right\}$$

is a sequence of i.i.d. random variables. Performance measures of the form (10) and (11) frequently appear in simulation. The performance measures $L_1(\theta, t)$ and $L_2(\theta, t)$ are essentially identical except for tail terms. However, these tail terms typically converge at the rate of $O(1/t)$, slower than the rate of ordinal comparison as we shall show. Therefore, such tail terms have to be taken into account.

For every $\theta \in \Theta$ and $i = 0, 1$, we make the following assumptions in this section.

(A2) the cycle time $T_i(\theta)$ has finite, continuous moment generating function $E[e^{sT_i(\theta)}]$ in a neighborhood of $s = 0$, $s \geq 0$;
(A3) the cycle time $T_1(\theta)$ is not degenerate in the sense of $\lim_{t\to 0} \text{Prob}[T_1(\theta) \leq t] < 1$;
(A4) the function $l(x) \in R$ is bounded and $|l(x)| \leq B$, $0 < B < \infty$.

Assumptions (A2) and (A4) are technical and will be needed in the proof of a number of results. As we shall explain later, they are satisfied in many cases. Assumption (A3) is very mild. Note that (A2) and (A3) imply $0 < E[T_1(\theta)] < \infty$. Assumption (A2) guarantees that the neighborhood can be chosen small enough such that $E[T_i^2(\theta)\, e^{sT_i(\theta)}]$ is finite and continuous in it.

For both the performance measures $L_1(\theta, t)$ and $L_2(\theta, t)$, the following lemma gives an expression of the limit of the sample performance measures; see Ref. 23, pp. 25–26, and Ref. 24, p. 54.

Lemma 4.1. If $T_i(\theta)$ is aperiodic and Assumptions (A2)–(A4) are satisfied, then

$$\lim_{t\to\infty} L_1(\theta, t) = \lim_{t\to\infty} L_2(\theta, t) = J(\theta), \qquad \text{a.s.,}$$

where the steady-state performance measure $J(\theta) \in R$ exists, is finite, and

$$J(\theta) = E\left[\int_{\tau_0(\theta)}^{\tau_1(\theta)} l(X(\theta, \xi, u))\, du\right] \Big/ E[T_1(\theta)].$$

The following is a result on the existence of the moment generating function of $K(t, \theta)$; see Ref. 24, pp. 34–41.

Lemma 4.2. If Assumption (A3) is satisfied, then there exists a $\delta > 0$ such that $\text{Prob}[T_1(\theta) \leq \delta] < 1$, and for any such δ,

$$E[z^{K(t,\theta)}] \leq (z(1 - \text{Prob}[T_1(\theta) \leq \delta])/(1 - z\, \text{Prob}[T_1(\theta) \leq \delta]))^{1 + t/\delta},$$

for all $t \geq 0$ and all $0 \leq z < (\text{Prob}[T_1(\theta) \leq \delta])^{-1}$.

The following Theorems 4.1 and 4.2 are the main results of this section.

Theorem 4.1. Suppose that Assumptions (A2)–(A4) hold. Then, there exists a positive number $\alpha > 0$ such that

$$\text{Prob}[I_{M,N}(t) = 1] = 1 - O(e^{-\alpha t}), \qquad \text{Prob}[I_{M,N}(t) = 0] = O(e^{-\alpha t}), \tag{12}$$

for the performance measure $L_1(\theta, t)$; in other words, the indicator process converges at an exponential rate. It follows from (12) that

$$(E[(I_{M,N}(t)-1)^2])^{1/2} = (\text{Prob}[I_{M,N}(t)=1])^{1/2} = O(e^{-(\alpha/2)t}).$$

Proof. We only need to prove that there exists a positive constant $\alpha > 0$ such that

$$\text{Prob}[L_1(\sigma, t) \geq L_1(\theta, t)] = 1 - O(e^{-\alpha t}), \tag{13}$$

for any $\sigma \in \Theta_g$, $\theta \in \Theta_b$, since if (13) holds, according to Inequality (6), we have

$$\text{Prob}[I_{M,N}(t)=1] \geq \max_{\sigma \in \Theta_g} \left\{1 - \sum_{\theta \in \Theta_b} (1 - \text{Prob}[L_1(\sigma, t) \geq L_1(\theta, t)])\right\}^+$$
$$= 1 - O(e^{-\alpha t}),$$

from which the conclusion follows since $P[I_{M,N}(t)=1] \leq 1$.

Using the Chernoff bound (Ref. 25 and Ref. 26, p. 391),

$$\text{Prob}[X > a] \leq E[e^{s(X-a)}], \qquad \forall s \geq 0, \tag{14}$$

for any random variable $X \in R$, we have

$$\text{Prob}[L_1(\sigma, t) \geq L_1(\theta, t)] \geq 1 - E[e^{st(L_1(\theta,t) - L_1(\sigma,t))}], \qquad \text{for any } s \geq 0. \tag{15}$$

Note that

$$\tau_{K(t,\theta)}(\theta) \leq t \leq \tau_{K(t,\theta)+1}(\theta), \qquad \text{for each } \theta \in \Theta.$$

For $v = \theta$ or σ, define

$$m_0(v) = \int_0^{\tau_0(v)} (l(X(v, \xi, u)) - (J(\sigma) - J(\theta))/2)\, du,$$

$$m_i(v) = \int_{\tau_{i-1}(v)}^{\tau_i(v)} (l(X(v, \xi, u)) - (J(\sigma) - J(\theta))/2)\, du, \qquad i = 1, 2, \ldots, K(t, v),$$

$$m_t^-(v) = \int_{\tau_{K(t,v)}(v)}^{t} (l(X(v, \xi, u)) - (J(\sigma) - J(\theta))/2)\, du.$$

Then, we have

$$E[m_i(\theta)] = -E[T_1(\theta)](J(\sigma) - J(\theta))/2 < 0, \tag{16a}$$

$$E[m_i(\sigma)] = E[T_1(\sigma)](J(\sigma) - J(\theta))/2 > 0. \tag{16b}$$

The boundedness of the function $l(x)$ in (A4) implies that $|J(\theta)| \leq B$ for any $\theta \in \Theta$. Further, according to Assumption (A2), there exist positive constants $0 < C_1 < \infty$, $d_1 > 0$ such that

$$E[e^{\pm s m_0(\theta)}] \leq E[e^{2BsT_0(\theta)}] \leq C_1 < \infty, \qquad \text{on } s \in [0, d_1], \text{ for all } \theta \in \Theta. \tag{17}$$

Next, we derive an upper bound for $E[e^{\pm s m_t^-(\theta)}]$. Define

$$R(t, \theta) = t - \tau_{K(t,\theta)}(\theta),$$

which is the backward recurrence time. Then,

$$E[e^{\pm s m_t^-(\theta)}] = E[e^{\pm s \int_{\tau_{K(t,\theta)}(\theta)}^{t} (J(X(\theta,\xi,u)) - J(\theta))\, du}] \leq E[e^{2BsR(t,\theta)}]. \tag{18}$$

Consider the distribution of $R(t, \theta)$. Let $F_{T_1(\theta)}(u)$, $F_{Z_n}(u)$ be respectively the cumulative probability distribution of $T_1(\theta)$ and

$$Z_n = T_0(\theta) + \sum_{i=1}^{n} T_i(\theta), \qquad n \geq 0.$$

Then, for any $0 \leq r \leq t$,

$$\text{Prob}[R(t, \theta) \leq r] = \sum_{i=0}^{\infty} \text{Prob}[R(t, \theta) \leq r, K(t, \theta) = i]$$

$$= \sum_{i=0}^{\infty} \text{Prob}[Z_n \leq t, Z_n + T_1(\theta) > t, t - Z_n \leq r]$$

$$= 1 - \int_0^{t-r} (1 - F_{T_1(\theta)}(t-u))\, dH(u),$$

where

$$H(u) = \sum_{i=0}^{\infty} F_{Z_i}(u)$$

is the renewal function. Therefore,

$$E[e^{2BsR(t,\theta)}] = \int_0^t Q(t-u)\, dH(u), \tag{19}$$

where

$$Q(v) = e^{2Bsv}(1 - F_{T_1(\theta)}(v)).$$

By applying the key renewal theorem (Ref. 24, p. 100), we know that

$$\lim_{t \to \infty} \int_0^t Q(t-u)\, dH(u) = (1/E[T_1(\theta)]) \int_0^{\infty} Q(v)\, dv.$$

An elementary computation shows that

$$\int_0^{\infty} Q(v)\, dv = \int_0^{\infty} ((e^{2Bsv} - 1)/2Bs)\, dF_{T_1(\theta)}(v).$$

For any $s > 0$ and x, a two-term Taylor-series expansion yields

$$e^{sx} \leq 1 + xs + (1/2)x^2 e^{s|x|} s^2. \tag{20}$$

Hence, we can choose C_1 sufficiently large and d_1 sufficiently small so that

$$\lim_{t\to\infty}\int_0^t Q(t-u)\,dH(u)$$

$$\leq (E[T_1(\theta)]+BsE[T_1^2(\theta)\,e^{2BsT_1(\theta)}])/E[T_1(\theta)]\leq (1/2)C_1,$$

when $s\in[0, d_1]$. This inequality, together with (18)–(19), implies that, for such a d_1, there exists a t_0 such that

$$E[e^{\pm sm_t^-(\theta)}]\leq C_1, \tag{21}$$

when $t\in[t_0,\infty)$ and $s\in[0, d_1]$.

In order to obtain a desired bound for $E[e^{st(L_1(\theta,t)-L_1(\sigma,t))}]$, we rewrite it as

$$E[e^{st(L_1(\theta,t)-L_1(\sigma,t))}]=E[e^{s(m_0(\theta)+m_t^-(\theta)+\sum_{i=1}^{K(t,\theta)}m_i(\theta)-m_0(\sigma)-m_t^-(\sigma)-\sum_{i=1}^{K(t,\sigma)}m_i(\sigma))}].$$

Using the Cauchy–Schwarz inequality, also noticing that $m_0(\theta)$, $m_t^-(\theta)$ are independent, we obtain

$$\begin{aligned}E[e^{st(L_1(\theta,t)-L_1(\sigma,t))}]&\leq (E[e^{4sm_0(\theta)}]E[e^{4sm_t^-(\theta)}])^{1/4}(E[e^{-4sm_0(\sigma)}]E[e^{-4sm_t^-(\sigma)}])^{1/4}\\&\quad\times(E[e^{4s\sum_{i=1}^{K(t,\theta)}m_i(\theta)}])^{1/4}(E[e^{-4s\sum_{i=1}^{K(t,\sigma)}m_i(\sigma)}])^{1/4}\\&\leq C_1(E[e^{4s\sum_{i=1}^{K(t,\theta)}m_i(\theta)}])^{1/4}(E[e^{-4s\sum_{i=1}^{K(t,\sigma)}m_i(\sigma)}])^{1/4},\end{aligned} \tag{22}$$

when $t\in[t_0,\infty)$ and $s\in[0, d_1/4]$. The last inequality comes from (17) and (21).

Further, we know from Inequality (20), also noting (16), that

$$\begin{aligned}E[e^{sm_i(\theta)}]&\leq 1+sE[m_i(\theta)]+s^2E[m_i^2(\theta)\,e^{s|m_i(\theta)|}]\\&\leq 1-sE[T_1(\theta)](J(\sigma)-J(\theta))/2+(2Bs)^2E[T_1^2(\theta)\,e^{2BsT_1(\theta)}].\end{aligned}$$

Therefore, Inequality (16), together with Assumption (A2) and the remark following it, guarantees the existence of a positive constant $d_2>0$ such that

$$E[e^{4sm_i(\theta)}]\leq 1-sC_2,\qquad \text{on } s\in[0, d_2], \tag{23}$$

where

$$C_2=E[T_1(\theta)](J(\sigma)-J(\theta))/4.$$

To bound $E[e^{4s\sum_{i=1}^{K(t,\theta)}m_i(\theta)}]$, we calculate

$$E[e^{4s\sum_{i=1}^{K(t,\theta)} m_i(\theta)}] = \sum_{n \leq \epsilon t} E[e^{4s\sum_{i=1}^{n} m_i(\theta)} I_{\{K(t,\theta)=n\}}]$$

$$+ \sum_{n > \epsilon t} [e^{4s\sum_{i=1}^{n} m_i(\theta)} I_{\{K(t,\theta)=n\}}],$$

where $I_{\{A\}} = 1$ is the indicator function and $\epsilon > 0$ is a small constant to be chosen. $I_{\{A\}} \leq 1$. Since $|l(x)| \leq B$, we have

$$E\left[e^{4s} \sum_{i=1}^{K(t,\theta)} m_i(\theta)\right] \leq e^{8Bst}.$$

Then,

$$E[e^{4s\sum_{i=1}^{K(t,\theta)} m_i(\theta)}] \leq \sum_{n \leq \epsilon t} e^{8Bst} E[I_{\{K(t,\theta)=n\}}] + \sum_{n > \epsilon t} E[e^{4s\sum_{i=1}^{n} m_i(\theta)}]$$

$$= e^{8Bst} \operatorname{Prob}[K(t, \theta) \leq \epsilon t] + \sum_{n > \epsilon t} (E[e^{4sm_1(\theta)}])^n.$$

Using the Chernoff bound (14) and (23), we know that

$$E[e^{4s\sum_{i=1}^{K(t,\theta)} m_i(\theta)}] \leq e^{8Bst} E[e^{\epsilon t - K(t,\theta)}] + (E[e^{4sm_1(\theta)}])^{\epsilon t - 1}/1(1 - E[e^{4sm_1(\theta)}])$$

$$\leq e^{(8Bs+\epsilon)t} E[e^{-K(t,\theta)}] + (sC_2)^{-1}(1 - sC_2)^{\epsilon t - 1},$$

when $t \in [t_0, \infty]$, $s \in [0, \min\{d_1/4, d_2\}]$. Let

$$C_3 = e^{-1}\{1 - \operatorname{Prob}[T_1(\theta) \leq \delta]\}/\{1 - e^{-1} \operatorname{Prob}[T_1(\theta) \leq \delta]\} < 1.$$

According to Lemma 4.2 with $z = e^{-1}$, we have

$$E[e^{4s\sum_{i=1}^{K(t,\theta)} m_i(\theta)}] \leq C_3\, e^{(8Bs+\epsilon+\delta^{-1}(\log C_3))t}$$

$$+ (sC_2)^{-1}(1 - sC_2)^{-1}\, e^{\epsilon t \log(1 - sC_2)}. \tag{24}$$

Choose $s \in [0, \min\{d_1/4, d_2\}]$ and ϵ sufficiently small such that

$$0 < 8Bs + \epsilon \leq -0.5\delta^{-1} \log C_3.$$

Then, from (24),

$$E[e^{4s\sum_{i=1}^{K(t,\theta)} m_i(\theta)}] \leq (C_3 + (sC_2)^{-1}(1 - sC_2)^{-1})\, e^{-C_4(\theta,\sigma)t}, \tag{25}$$

where

$$C_4(\theta, \sigma) = \min\{-0.5\delta^{-1} \log C_3,\ -\epsilon \log(1 - sE[T_1(\theta)](J(\sigma) - J(\theta))/4)\} > 0.$$

Similarly, C_3 and $C_4(\theta, \sigma)$ can be chosen such that

$$E[e^{-4s\sum_{i=1}^{K(t,\sigma)} m_i(\sigma)}] \leq (C_3 + (sC_2)^{-1}(1 - sC_2)^{-1})\, e^{-C_4(\theta,\sigma)t}. \tag{26}$$

The combination of (25)–(26) with (22) yields (13) with

$$\alpha = \min_{\sigma\in\Theta_g, \theta\in\Theta_b} C_4(\theta, \sigma)/2 > 0.$$

Thus, the proof is complete. □

Theorem 4.2. Assume that $\{X(\theta, \xi, t)\}$ and $\{X(\sigma, \xi, t)\}$ are independent for any $\theta \neq \sigma\in\Theta$. Then, the conclusion of Theorem 4.1 is also valid for the performance measure $L_2(\theta, t)$.

Proof. According to the proof of Theorem 4.1, we again only need to prove that

$$\text{Prob}[L_2(\sigma, t) \geq L_2(\theta, t)] = 1 - O(e^{-\alpha t}), \tag{27}$$

for any $\sigma\in\Theta_g$, $\theta\in\Theta_b$, and for some $\alpha > 0$. Define

$$M(\theta) = \sum_{i=1}^{K(t,\theta)} \int_{\tau_{i-1}(\theta)}^{\tau_i(\theta)} l(X(\theta, \xi, s))\, ds, \qquad T(\theta) = \sum_{i=1}^{K(t,\theta)} T_i(\theta).$$

Then,

$$0 \leq T(\sigma) \leq t, \qquad 0 \leq T(\theta) \leq t.$$

Using the Chernoff bound (14), we have

$$\begin{aligned}\text{Prob}[L_2(\sigma, t) \geq L_2(\theta, t)] &= \text{Prob}[M(\sigma)T(\theta) \geq M(\theta)T(\sigma)]\\ &\geq 1 - E[e^{s(M(\theta)T(\sigma) - M(\sigma)T(\theta))}],\end{aligned} \tag{28}$$

for any $s \geq 0$. Note that, if $\sigma \neq 0$ and $\sigma, \theta\in\Theta$,

$$(M(\theta) - T(\theta)(J(\sigma) - J(\theta))/2)T(\sigma) = T(\sigma) \sum_{i=1}^{K(t,\theta)} m_i(\theta),$$

$$(M(\sigma) - T(\theta)(J(\sigma) - J(\sigma))/2)T(\theta) = T(\theta) \sum_{i=1}^{K(t,\theta)} m_i(\sigma),$$

and

$$\begin{aligned}&E[e^{s(M(\theta)T(\sigma) - M(\sigma)T(\theta))}]\\ &= E[e^{s[(M(\theta) - T(\theta)(J(\sigma) - J(\theta))/2)T(\sigma) - (M(\sigma) - T(\theta)(J(\sigma) - J(\sigma))/2)T(\theta)]}].\end{aligned}$$

By applying the Cauchy–Schwarz inequality, we get

$$E[e^{s(M(\theta)T(\sigma)-M(\sigma)T(\theta))}]$$
$$=(E[e^{2sT(\sigma)\sum_{i=1}^{K(t,\theta)} m_i(\theta)}])^{1/2}(E[e^{2sT(\theta)\sum_{i=1}^{K(t,\sigma)} m_i(\sigma)}])^{1/2}. \tag{29}$$

Define

$$\tilde{m}_i(\theta)=2sT(\sigma)m_i(\theta), \qquad \tilde{m}_i(\sigma)=2sT(\theta)m_i(\sigma).$$

Then, the independence of $\{X(\theta, \xi, t)\}$ and $\{X(\sigma, \xi, t)\}$ shows that

$$E[\tilde{m}_i(\theta)] = -s(t-E[T_0(\sigma)]-E[R(t,\sigma)])$$
$$\times E[T_1(\theta)](J(\sigma)-J(\theta))<0.$$

By replacing $e^{2BsT(t,\theta)}$ with $R(t, \sigma)$ in (19), a direct computation yields

$$\lim_{t\to\infty} E[R(t,\sigma)] = E[T_1^2(\sigma)]/2E[T_1(\sigma)].$$

Therefore, there is a $t_0>0$ such that

$$E[T_0(\sigma)]+E[R(t,\sigma)]\le 0.5t, \qquad \text{whenever } t\in[t_0,\infty).$$

Consequently,

$$E[\tilde{m}_i(\theta)] = -0.5stE[T_1(\theta)](J(\sigma)-J(\theta))<0,$$

for all $t\in[t_0, \infty)$. A procedure similar to that used in obtaining (23) reveals that there are constants $C_5>0$, $s_0>0$ such that

$$E[e^{\tilde{m}_i(\theta)}]\le 1-s_0C_5, \tag{30}$$

whenever $t\in[t_0,\infty)$ and $st\in[0, s_0]$. Consider

$$E[e^{2sT(\sigma)\sum_{i=1}^{K(t,\theta)} m_i(\theta)}] = E[e^{\sum_{i=1}^{K(t,\theta)} \tilde{m}_i(\theta)}].$$

Using Inequality (30), the same argument as that used in obtaining (26) shows that the constant s_0 can be chosen sufficiently small such that

$$E[e^{\sum_{i=1}^{K(t,\theta)} \tilde{m}_i(\theta)}]\le C_6(\theta,\sigma)\, e^{-C_7(\theta,\sigma)t},$$

for some constants $C_6(\theta,\sigma)>0$, $C_7(\theta,\sigma)>0$. Thus,

$$E[e^{2sT(\sigma)\sum_{i=1}^{K(t,\theta)} m_i(\theta)}]\le C_6(\theta,\sigma)\, e^{-C_7(\theta,\sigma)t}. \tag{31}$$

Similarly, we can prove that $C_6(\theta,\sigma)>0$, $C_7(\theta,\sigma)>0$ can be chosen such that

$$E[e^{2sT(\theta)\sum_{i=1}^{K(t,\sigma)} m_i(\sigma)}]\le C_6(\theta,\sigma)\, e^{-C_7(\theta,\sigma)t}. \tag{32}$$

The combination of (31)–(32) with (29) gives

$$E[e^{s(M(\theta)T(\sigma)-M(\sigma)T(\theta))}]\le C_6(\theta,\sigma)\, e^{-C_7(\theta,\sigma)t}, \tag{33}$$

which shows that (26) is true with

$$\alpha = \min_{\sigma \in \Theta_g, \theta \in \Theta_b} C_7(\theta, \sigma).$$

Hence, the statement of the theorem follows from (28)–(29) and (33). □

A random variable X is said to be exponentially bounded if there exist two positive numbers C, $\rho > 0$ such that

$$\text{Prob}[X \geq x] \leq C\, e^{-\rho x}.$$

Assumption (A2) is satisfied if the duration of each regeneration cycle is exponentially bounded such as that in the M/M/1 queue. Moreover, if the interarrival times and serivce times are exponentially bounded, then the busy periods of any stable GI/G/1 queue are also exponentially bounded (Ref. 27).

For regenerative simualtion, both $L_1(\theta, t)$ and $L_2(\theta, t)$ converge to $J(\theta)$ with rate $O(1/\sqrt{t})$; see Ref. 28. Theorems 4.1 and 4.2 show that, in such situations, the indicator process converges exponentially, much faster than that of the performance measure estimation.

Example 4.1. See Ref. 29, p. 155. Consider a closed queueing network with two single servers, as depicted in Fig. 1. The service times at the two servers are exponential. There is a total of N customers in the network. The buffer size in front of each server is larger than N. The routing probabilities are $0 < p < 1$, $0 < q < 1$. This system can be described by a finite-state continuous-time Markov chain. Moreover, such a Markov chain is regenerative with exponentially bounded regenerative cycle times (Ref. 23, pp. 37–41). Therefore, Assumption (A2) is satisfied. Therorems 4.1 and 4.2 are applicable for appropriate $l(x)$.

5. Averaging i.i.d. Random Variables

In many simulations, performance measures are of the form of averaging i.i.d. random variables, i.e.,

$$L_3(\theta, t) = (1/t) \sum_{i=1}^{t} X_i(\theta, \xi), \tag{34}$$

where $\{X_i(\theta, \xi) \in R, i \geq 1\}$ is a sequence of i.i.d. random variables. If $E[X_1(\theta, \xi)] < \infty$ and $\text{Var}[X_1(\theta, \xi)] < \infty$, the strong law of large numbers guarantees that

$$\lim_{t \to \infty} L_3(\theta, t) = J(\theta), \qquad \text{a.s.},$$

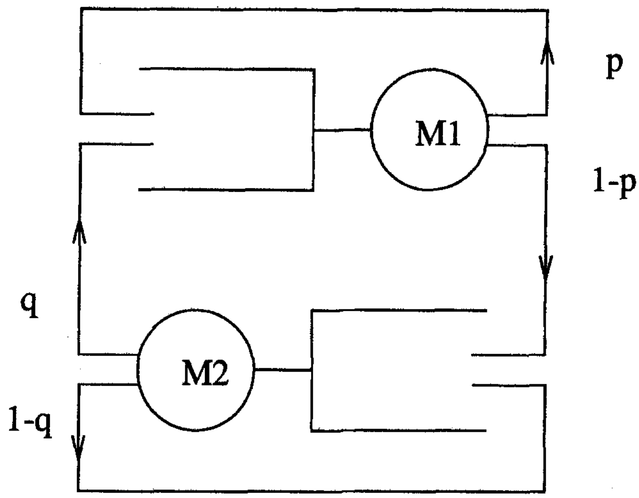

Fig. 1. Closed queueing network with N customers.

with

$$J(\theta) = E[X_1(\theta, \xi)].$$

In such cases, the assumptions of Theorem 4.1 can be weakened.

Theorem 5.1. Assume that $E[e^{sX_1(\theta,\xi)}]$ exists and is continuous in a neighborhood of $s=0$, $s \geq 0$. Then, there exists a positive number $\beta > 0$ such that

$$\text{Prob}[I_{M,N}(t) = 1] = 1 - O(e^{-\beta t}),$$

$$\text{Prob}[I_{M,N}(t) = 0] = O(e^{-\beta t}).$$

Proof. The proof is similar to that of Theorem 4.1; thus, it is omitted. It can also be proved by the large deviation principle, since (34) is a classical example where the large deviation principle applies (Ref. 30, pp. 3–5). □

Example 5.1. Consider an M/M/1 queue with $X_i(\theta, \xi)$ being the number of customers served in the ith busy period. Then, $\{X_i(\theta, \xi)\}$ is an i.i.d. sequence with distribution

$$f(x) = \sum_{n=1}^{\infty} p_n \delta(x-n),$$

where

$$p_n = (1/n)\binom{2n-2}{n-1}[1 + 1/\rho(\theta)][\rho(\theta)/(1+\rho(\theta))^2]^n, \qquad n = 1, 2, \ldots,$$

and $\rho(\theta)$ is the traffic intensity. Similar distribution exists for M/D/1 queue. In this case, the performance measure is the mean number of customers served in a busy period in steady-state. For this type of distribution, the assumptions of Theorem 5.1 are satisfied, implying the exponential rate of convergence for the indicator process. Simulation is of course unnecessary in this problem, since the performance measure is monotone in $\rho(\theta)$. We use this simple example to illustrate Theorem 5.1.

Remark 5.1. For the performance measure $L_3(\theta, t)$, it is not necessary that more information be helpful for choosing the correct design, as illustrated by the following example.

Consider a case where, for each $i \geq 1$, $X_i(\theta, \xi)$ is distributed according to

$$X_i(\theta, \xi) = \begin{cases} \theta + 1, & \text{w.p.} p, \\ \theta - 1, & \text{w.p.} 1 - p, \end{cases}$$

where $p = 0.5$. Let

$$N = 2, \qquad M = 1, \qquad \Theta = \{0.8, 0\}, \qquad J(\theta) = \theta.$$

We assume that $X_i(\sigma, \xi)$ and $X_j(\theta, \xi)$ for $i \neq j$ or $\sigma \neq \theta$ are independent. For this problem we can obtain the closed-form of $\text{Prob}[I_{M,N}(t) = 1]$. Figure 2 illustrates that $\text{Prob}[I_{M,N}(t) = 1]$ is not nondecreasing in t, suggesting that more information (larger t) may not be helpful for making the right decision.

Normal distribution is often an asymptotic form of performance measures in DEDS. The following Theorem 5.2 establishes an important property of the indicator process—monotonicity in simulation time. It also provides a detailed formula.

Theorem 5.2. Assume that, in $L_3(\theta, t)$, $\{(X_i(\theta_1, \xi), X_i(\theta_2, \xi), \ldots, X_i(\theta_N, \xi)), i \geq 1\}$ is a sequence of i.i.d. random variables with (nondegenerate) normal distribution. Then:

(a) $\text{Prob}[I_{M,N}(t) = 1]$ is nondecreasing and $\text{Prob}[I_{M,N}(t) = 0]$ is nonincreasing in t;

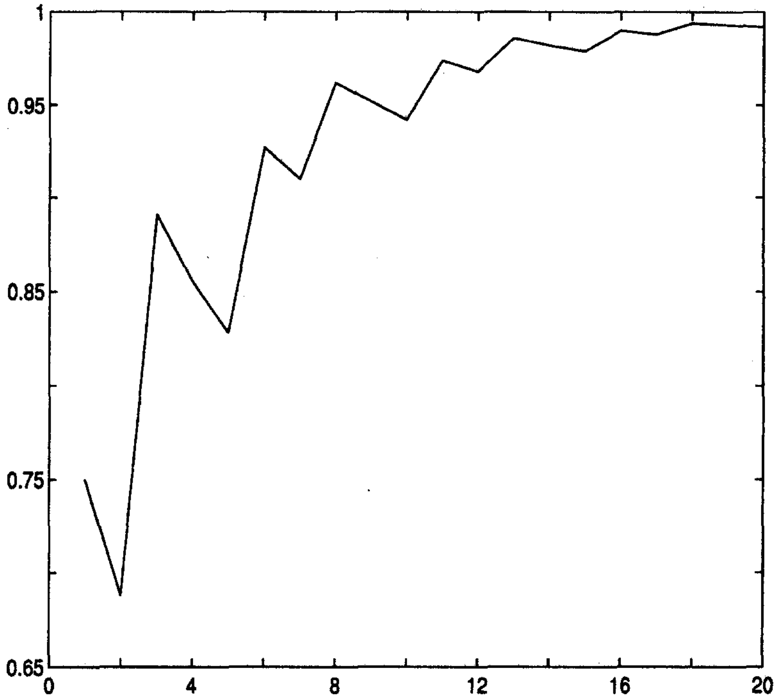

Fig. 2. $\text{Prob}[I_{M,N}(t)=1]$ vs t.

(b) there exists a positive number $\gamma > 0$ such that

$$\text{Prob}[I_{M,N}(t)=1] = 1 - O(t^{-1/2}\, e^{-\gamma t}),$$

$$\text{Prob}[I_{M,N}(t)=0] = O(t^{-1/2}\, e^{-\gamma t}).$$

Proof.

(a) Let the joint density function of $(X_1(\theta_1, \xi), X_1(\theta_2, \xi), \ldots, X_1(\theta_N, \xi))$ be

$$p(x_1, x_2, \ldots, x_N) = [1/\sqrt{(2\pi)^N |A|}]\, e^{-(1/2)(x-\mu)^T A^{-1}(x-\mu)},$$

where

$$x^T = [x_1, x_2, \ldots, x_N], \qquad \mu^T = [J(\theta_1), J(\theta_2), \ldots, J(\theta_N)],$$

and A is the autocovariance matrix. Denote

$$Y(\theta) = (1/\sqrt{t}) \sum_{i=1}^{t} (X_i(\theta, \xi) - J(\theta)), \qquad \theta \in \Theta.$$

The density function of $(Y(\theta_1), Y(\theta_2), \ldots, Y(\theta_N))$ is

$$p(y_1, y_2, \ldots, y_N) = [1/\sqrt{(2\pi)^N |A|}]\, e^{-(1/2)y^T A^{-1} y},$$
$$y = [y_1, y_2, \ldots, y_N]^T.$$

It is independent of t. Therefore,

$$\text{Prob}[I_{M,N}(t) = 1] = \text{Prob}\left[\max_{\sigma \in \Theta_g} \left(\sum_{i=1}^{t} X_i(\sigma, \xi)\right) \geq \max_{\sigma \in \Theta_b} \left(\sum_{i=1}^{t} X_i(\theta, \xi)\right)\right]$$
$$= \text{Prob}\left[\bigcup_{\theta \in \Theta_b} \left\{\max_{\sigma \in \Theta_g} (Y(\sigma) + t^{1/2}(J(\sigma) - J(\theta))) \geq Y(\theta)\right\}\right],$$

which is nondecreasing in t, since

$$J(\sigma) - J(\theta) > 0, \qquad \text{for any } \sigma \in \Theta_g,\ \theta \in \Theta_b,$$

and since the distribution of $Y(\theta)$ is independent of t.

(b) In light of (6), we only need to prove that

$$\text{Prob}[L_3(\sigma, t) \geq L_3(\theta, t)] = 1 - O(t^{-1/2}\, e^{-\gamma t}),$$

for any $\sigma \in \Theta_g$, $\theta \in \Theta_b$. Using the same transformation as that used in the proof of (a), we know that

$$\text{Prob}[L_3(\sigma, t) \geq L_3(\theta, t)]$$
$$= \text{Prob}[Y(\sigma) - Y(\theta) \geq -t^{1/2}(J(\sigma) - J(\theta))].$$

Since both $Y(\sigma)$ and $Y(\theta)$ have (nondegenerate) normal distribution with zero means, so is $Y(\sigma) - Y(\theta)$. Let us denote the distribution of $Y(\sigma) - Y(\theta)$ as $\Phi(x/d)$, where $\Phi(x)$ is the standard normal distribution function and

$$d^2 = \text{Var}[Y(\sigma) - Y(\theta)].$$

Then, we know that

$$\text{Prob}[L_3(\sigma, t) \geq L_3(\theta, t)] = \Phi(t^{1/2} d^{-1}(J(\sigma) - J(\theta))).$$

For normal distributions, we have

$$1 - \Phi(x) = \phi(x) R_x,$$

where $\phi(x)$ is the standard density function of normal distribution and R_x is the Mill's ratio satisfying

$$1/(x + x^{-1}) \leq R_x \leq 1/x,$$

for any $x > 0$; see Ref. 31, p. 505. Thus,

$$\text{Prob}[L_3(\sigma, t) \geq L_3(\theta, t)] = \Phi(t^{1/2}\, d^{-1}(J(\sigma) - J(\theta)))$$
$$= 1 - O(t^{-1/2} \phi(t^{1/2}\, d^{-1}(J(\sigma) - J(\theta)))).$$

By choosing

$$\gamma = (1/2) \min_{\sigma \in \Theta_g, \theta \in \Theta_b} d^{-2}(J(\sigma) - J(\theta))^2 > 0,$$

we know that (b) holds. □

Remark 5.2. Theorem 5.2 shows that the rate of convergence for the indicator process in the case of averaging i.i.d. normal random variables is still exponential, since we can vary the value of $\gamma > 0$. In many simulations, the performance measures have the asymptotic normal distribution. In this sense, Theorem 5.2 is useful for deriving the confidence level. Since $\text{Prob}[I_{M,N}(t) = 1]$ is monotone in t, we only need to allocate the first t such that $\text{Prob}[I_{M,N}(t) = 1]$ reaches the desired confidence level.

However, Theorem 5.2 also reveals a potential problem. If we choose the correct $\gamma > 0$, the ordinal comparison for normal distributions converges faster than $O(e^{-\gamma t})$. This means that it is possible to underestimate the confidence level if we use normal distribution as an approximation to the distribution of general random variables.

6. Conclusions

In this paper, we have formulated an indicator process to characterize the rate of convergence for ordinal comparison. For several forms of performance measures that are common in simulation, we prove that the rate of convergence is exponential. The results of this paper provide a theoretical explanation on the fast convergence of ordinal comparison that has been observed experimentally.

This research represents a first step in the study of the convergence for ordinal comparison. The following directions warrant further research.

(i) Although regenerative assumption is satisfied in the simulation of many DEDS such as queueing networks and Markov chain models, it would be nice if we can give a bound on the rate of convergence for the simulation of general DEDS.

(ii) Monotonicity in t is very useful for determining the confidence level and for simulation planning. So far, we can only prove the monotonicity for the case of averaging i.i.d. normal random variables. It is important to find out conditions under which $\text{Prob}[I_{M,N}(t) = 1]$ is monotone in the simulation time t.

(iii) Simulation planning can affect the convergence of the indicator process. Therefore, it is desirable to find out how to design a simulation such that $\mathrm{Prob}[I_{M,N}(t)=1]$ is maximized. If this is possible, we can expect a quick separation of good designs from bad ones.

References

1. DOOB, J. L., *Stochastic Processes*, John Wiley and Sons, New York, New York, 1953.
2. SIMON, H., *Economics, Bounded Rationality, and the Cognitive Revolution*, Edward Elgar Publisher, Brookfield, Vermont, 1992.
3. KUSHNER, H. J., and CLARK, D. S., *Stochastic Approximation for Constrained and Unconstrained Systems*, Springer Verlag, New York, New York, 1978.
4. FABIAN, V., *Stochastic Approximation*, Optimizing Methods in Statistics, Edited by J. S. Rustagi, Academic Press, New York, New York, pp. 439–470, 1971.
5. DUPUIS, P., and KUSHNER, H. J., *Stochastic Approximation and Large Deviations: Upper Bounds and w.p.1 Convergence*, SIAM Journal on Control and Optimization, Vol. 27, pp. 1108–1135, 1989.
6. BARNHART, C. M., WIESELTHIER, J. E., and EPHREMIDES, A., *Ordinal Optimization by Means of Standard Clock Simulation and Crude Analytical Models*, Proceedings of the 33rd Conference on Decision and Control, Lake Buena Vista, Florida, pp. 2645–2647, 1994.
7. HO, Y. C., *Heuristics, Rule of Thumb, and the 80/20 Proposition*, IEEE Transactions on Automatic Control, Vol. 39, pp. 1025–1027, 1994.
8. HO, Y. C., *Overview of Ordinal Optimization*, Proceedings of the 33rd Conference on Decision and Control, Lake Buena Vista, Florida, pp. 1975–1977, 1994.
9. HO, Y. C., and DENG, M., *The Problem of Large Search Space in Stochastic Optimization*, Proceedings of the 33rd Conference on Decision and Control, Lake Buena Vista, Florida, pp. 1470–1475, 1994.
10. HO, Y. C., SREENIVAS, R. S., and VAKILI, P., *Ordinal Optimization in DEDS*, Discrete Event Dynamic Systems: Theory and Applications, Vol. 2, pp. 61–88, 1992.
11. PATSIS, N., CHEN, C. H., and LARSON, M., *Parallel Simulations of DEDS*, IEEE Transactions on Control Technology, 1996 (to appear).
12. CASSANDRAS, C. G., and BAO, G., *A Stochastic Comparison Algorithm for Continuous Optimization with Estimations*, Proceedings of the 33rd Conference on Decision and Control, Lake Buena Vista, Florida, pp. 676–677, 1994.
13. CASSANDRAS, C. G., and JULKA, V., *Descent Algorithms for Discrete Resource Allocation Problems*, Proceedings of the 33rd Conference on Decision and Control, Lake Buena Vista, Florida, pp. 2639–2644, 1994.
14. GONG, W. B., HO, Y. C., and ZHAI, W., *Stochastic Comparison Algorithm for Discrete Optimization with Estimations*, Discrete Event Dynamic Systems: Theory and Applications, 1996 (to appear).
15. YAN, D., and MUKAI, H., *Optimization Algorithm with Probabilistic Estimation*, Journal of Optimization Theory and Applications, Vol. 79, pp. 345–371, 1993.

16. VAKILI, P., MOLLAMUSTAFLAGLU, L., and HO, Y. C., *Massively Parallel Simulation of a Class of Discrete Event Systems*, Proceedings of the IEEE Frontiers MPC-92 Symposium, Washington, DC, 1992.
17. CASSANDRAS, C. G., and STRICKLAND, S., *Observable Augmented Systems for Sensitivity Analysis of Markov and Semi-Markov Processes*, IEEE Transactions on Automatic Control, Vol. 34, pp. 1026–1037, 1989.
18. PAN, J., and CASSANDRAS, C. G., *Parallel Sample Path Construction Techniques for Discrete Event Systems: K-Constructability and Applications*, Proceedings of the 33rd Conference on Decision and Control, Lake Buena Vista, Florida, pp. 1978–1983, 1994.
19. FUJIMOTO, R. M., *Parallel Discrete Event Simulation*, Communications of the ACM, Vol. 33, pp. 31–53, 1990.
20. SANTNER, T. J., and TAMHANE, A. C., *Design of Experiments*, Marcel Dekker, New York, New York, 1984.
21. HO, Y. C., DENG, M., and HU, J., *Effect of Correlated Estimation Error in Ordinal Optimization*, Proceedings of the 1992 Winter Simulation Conference, pp. 466–475, 1992.
22. GLASSERMAN, P., and VAKILI, P., *Comparing Markov Chains Simulated in Parallel*, Probability in the Engineering and Information Sciences, Vol. 8, pp. 309–326, 1994.
23. SHEDLER, G. S., *Regeneration and Networks of Queues*, Springer Verlag, New York, New York, 1987.
24. KALASHNIKOV, V. V., *Topics on Regenerative Processes*, CRC Press, Boca Raton, Florida, 1994.
25. STRICKLAND, S. G., *Importance Sampling for Quick Simulation of Rare Events*, Unpublished Manuscript, University of Virginia, 1993.
26. KLEINROCK, L., *Queueing Systems, Vol. 1*, John Wiley and Sons, New York, New York, 1975.
27. THORISSON, H., *The Queue GI/G/1: Finite Moments of the Cycle Variables and Uniform Rates of Convergence*, Stochastic Processes and Their Applications, Vol. 19, pp. 85–99, 1985.
28. IGLEHART, D. L., and SHEDLER, G. S., *Statistical Efficiency of Regenerative Simulation Methods for Networks of Queues*, Advances in Applied Probability, Vol. 15, pp. 183–197, 1983.
29. HO, Y. C., and CAO, X. R., *Perturbation Analysis of Discrete Event Dynamic Systems*, Kluwer Academic Publishers, Boston, Massachusetts, 1991.
30. DEUSCHEL, J. D., and STROOK, D. W., *Large Deviations*, Academic Press, Boston, Massachusetts, 1989.
31. KOTZ, S., and JOHNSON, N. L., *Encyclopedia of Statistical Sciences*, John Wiley, New York, New York, Vol. 5, 1985.