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A combinatorial theory of Grünbaum's new regular polyhedra, Part II: Complete enumeration

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Abstract. The new regular polyhedra as defined by Branko Grünbaum in 1977 (cf. [5]) are completely enumerated. By means of a theorem of Bieberbach, concerning the existence of invariant affine subspaces for discrete affine isometry groups (cf. [3], [2] or [1]) the standard crystallographic restrictions are established for the isometry groups of the non finite (Grünbaum-)polyhedra. Then, using an appropriate classification scheme which—compared with the similar, geometrically motivated scheme, used originally by Grünbaum—is suggested rather by the group theoretical investigations in [4], it turns out that the list of examples given in [5] is essentially complete except for one additional polyhedron.

So altogether—up to similarity—there are two classes of planar polyhedra, each consisting of 3 individuals and each class consisting of the Petrie duals of the other class, and there are ten classes of non planar polyhedra: two mutually Petrie dual classes of finite polyhedra, each consisting of 9 individuals, two mutually Petrie dual classes of infinite polyhedra which are contained between two parallel planes with each of those two classes consisting of three one-parameter families of polyhedra, two further mutually Petrie dual classes each of which consists of three one parameter families of polyhedra whose convex span is the whole 3-space, two further mutually Petrie dual classes consisting of three individuals each of which span $\mathbb{E}^3$ and two further classes which are closed with respect to Petrie duality, each containing 3 individuals, all spanning $\mathbb{E}^3$, two of which are Petrie dual to each other, the remaining one being Petrie dual to itself.

In addition, a new classification scheme for regular polygons in $\mathbb{E}^n$ is worked out in §9.

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§1 Grünbaum systems

In the following note we are going to use the results proved in [4] to classify in detail all of the new regular polyhedra, defined by Branko Grünbaum in [5]. Let us recall the basic results of [4] in a form which is suitable for our further considerations: if π is a regular polyhedron according to Grünbaum with vertex set $V = V_0 \subseteq \mathbb{E} := \mathbb{E}^3$, edge set $V_1 \subseteq \mathscr{P}(V_0) =: \{T \mid T \subseteq V_0\}$ and face set $V_2 \subseteq \mathscr{P}(V_1)$ and if $(v_0, v_1, v_2) \in V_0 \times V_1 \times V_2$ is a "π-flag" (i.e. satisfies $v_0 \in v_1 \in v_2$) then for each $i = 0, 1, 2$ there exists a unique isometry α_i of the affine subspace $\langle V_0 \rangle$ generated by V_0 with the following properties: $\alpha_i(V_0) = V_0$, the induced map $\alpha_i : \mathscr{P}(V_0) \to \mathscr{P}(V_0)$ maps V_1 onto V_1 and the consequently induced map $\mathscr{P}(V_1) \to \mathscr{P}(V_1)$ maps V_2 onto V_2, $\alpha_i(v_j) = v_j$ for $j \in \{0, 1, 2\} \setminus \{i\}$ and $\alpha_i(v_i) \neq v_i$. Moreover, these isometries satisfy the relations $\alpha_0^2 = \alpha_1^2 = \alpha_2^2 = (\alpha_0\alpha_2)^2 = 1$, the group $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$, generated by $\alpha_0, \alpha_1, \alpha_2$ is discrete, coincides with the automorphism group of π and acts transitively on V_0, V_1, V_2 and on the "flag space" $\mathscr{F}_\pi := \{(w_0, w_1, w_2) \in V_0 \times V_1 \times V_2 \mid w_0 \in w_1 \in w_2\}$, so one has $V_0 = \langle \alpha_0, \alpha_1, \alpha_2 \rangle \cdot v_0 := \{\alpha v_0 \mid \alpha \in \langle \alpha_0, \alpha_1, \alpha_2 \rangle\}$, $v_1 = \langle \alpha_0 \rangle \cdot v_0 = \{v_0, \alpha_0 v_0\}$, $V_1 = \langle \alpha_0, \alpha_1, \alpha_2 \rangle \cdot v_1$, $v_2 = \langle \alpha_0, \alpha_1 \rangle \cdot v_1$, $V_2 = \langle \alpha_0, \alpha_1, \alpha_2 \rangle \cdot v_2$, i.e., the knowledge of $v = v_0$, α_0, α_1 and α_2 allows to reconstruct the polyhedron π. In particular, if (w_0, w_1, w_2) is another π-flag and if β_0, β_1 and β_2 are the corresponding isometries of V_0, then the isometry $\alpha \in \langle \alpha_0, \alpha_1, \alpha_2 \rangle$ with $\alpha(v_0, v_1, v_2) = (w_0, w_1, w_2)$ satisfies $\alpha\alpha_i\alpha^{-1} = \beta_i$ – more generally, two polyhedra, π, π' are isometric/similar, if and only if for some (or—equivalently—for all) flags $(v_0, v_1, v_2) \in \mathscr{F}_\pi$ and $(v_0', v_1', v_2') \in \mathscr{F}_{\pi'}$ with corresponding isometries $\alpha_0, \alpha_1, \alpha_2$ and $\alpha_0', \alpha_1', \alpha_2'$ there exists some isometry/similarity $\gamma : \langle V_0 \rangle \to \langle V_0' \rangle$ with $\gamma v_0 = v_0'$ and $\gamma\alpha_i\gamma^{-1} = \alpha_i'$ $(i = 0, 1, 2)$.

Hence, to classify all Grünbaum polyhedra it is enough to classify all "discrete Grünbaum systems" up to isometry or similarity, where a Grünbaum system is defined to be a system $(v; \alpha_0, \alpha_1, \alpha_2) \in \mathbb{E} \times Iso(\mathbb{E})^3$ satisfying the conditions

$$\alpha_0^2 = \alpha_1^2 = \alpha_2^2 = (\alpha_0\alpha_2)^2 = Id_{\mathbb{E}}, \tag{G1}$$

$$\alpha_1 v = \alpha_2 v = v, \tag{G2}$$

$$\alpha_i|_{<\alpha_0,\alpha_1,\alpha_2>\cdot v} \neq Id_{<\alpha_0,\alpha_1,\alpha_2>\cdot v}, \tag{G3}$$

such a system is defined to be discrete if $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ is discrete, and two such systems $(v; \alpha_0, \alpha_1, \alpha_2)$ and $(w; \alpha_0, \alpha_1, \alpha_2)$ are defined to be isometric/similar if there exists an isometry/similarity $\gamma : \mathbb{E} \to \mathbb{E}$ with $\gamma v = w$ and $\gamma\alpha_i\gamma^{-1}|_{<\beta_0,\beta_1,\beta_2>\cdot w} = \beta_i|_{<\beta_0,\beta_1,\beta_2>\cdot w}$

For an isometry $\alpha \in Iso(\mathbb{E})$ we denote by $\dim\alpha$ the dimension of the fixspace $\mathbb{E}^\alpha := \{x \in \mathbb{E} \mid \alpha x = x\}$ and for $\alpha, \beta \in Iso(\mathbb{E})$ we define $\sphericalangle(\alpha, \beta)$ to be 0 unless $1 \leqq \dim\alpha, \dim\beta \leqq 2$ in which case we define $\sphericalangle(\alpha, \beta)$ as the angle between $\mathbb{E}^\alpha$ and

$\mathbb{E}^{\beta}$ or—in case $\mathbb{E}^{\alpha} \cap \mathbb{E}^{\beta} = \emptyset$—as the angle between properly intersecting parallel transforms of these two spaces. To avoid ambiguities, we assume always $0° \leqq \sphericalangle(\alpha, \beta) \leqq 90°$.

Note that for two involutions $\alpha, \beta \in Iso(\mathbb{E})$ one has $\alpha \cdot \beta = \beta \cdot \alpha$ if and only if $\mathbb{E}^{\alpha} \cap \mathbb{E}^{\beta} \neq \emptyset$ and either $\sphericalangle(\alpha, \beta) = 0°$ or $\sphericalangle(\alpha, \beta) = 90°$ and one has $\alpha = \beta$ if and only if $\mathbb{E}^{\alpha} \cap \mathbb{E}^{\beta} \neq \emptyset$, $\dim \alpha = \dim \beta$ and $\sphericalangle(\alpha, \beta) = 0°$. Note also that $\{(\dim \alpha, \dim \beta, \dim \gamma) \mid \alpha, \beta, \gamma \in Iso(\mathbb{E}) \backslash \{Id_{\mathbb{E}}\}$ and $\alpha^2 = \beta^2 = \gamma^2 = \alpha\beta\gamma = \{Id_{\mathbb{E}}\}\} = \{(a, b, c) \in \mathbb{N}^3 \mid 0 \leqq a, b, c \leqq 2$ and $a + b + c \in \{3, 5\}\}$.

Now, we define two Grünbaum systems $(v; \alpha_0, \alpha_1, \alpha_2)$ and $(w; \beta_0, \beta_1, \beta_2)$ to have the same dimensionality if $\dim \alpha_0 = \dim \beta_0$, $\dim \alpha_1 = \dim \beta_1$, $\dim \alpha_2 = \dim \beta_2$ and $\dim \alpha_0\alpha_2 = \dim \beta_0\beta_2$ or—equivalently—$\dim \alpha_i = \dim \beta_i$ for $i = 0, 1, 2$ and $\sphericalangle(\alpha_0, \alpha_2) = \sphericalangle(\beta_0, \beta_2)$ and we define $(v; \alpha_0, \alpha_1, \alpha_2)$ and $(w; \beta_0, \beta_1, \beta_2)$ to have the same angularity if, moreover, $\sphericalangle(\alpha_0, \alpha_1) = \sphericalangle(\beta_0, \beta_1)$ and $\sphericalangle(\alpha_1, \alpha_2) = \sphericalangle(\beta_1, \beta_2)$.

We are going to enumerate all discrete Grünbaum systems $(v; \alpha_0, \alpha_1, \alpha_2)$ by first discussing the various possibilities for the sequence of numbers $\dim \alpha_0, \dim \alpha_2, \dim \alpha_0\alpha_2$ and then deriving for each such sequence the various—always finitely many—possibilities for $\sphericalangle(\alpha_0, \alpha_1)$ and $\sphericalangle(\alpha_1, \alpha_2)$. It will turn out that for all but four dimensionality types the angularity determines the system already up to similarity, whereas for those four dimensionality types there is for each admissible angularity a one-parameter family of similarity types.

Each similarity type, of course, consists of a one-parameter family of isometry types.

But before discussing these various types we have to establish some simple properties of Grünbaum systems which we do in the next section.

§2 Simple properties of Grünbaum systems

We continue with our notation. At first we claim the following.

PROPOSITION 1. *If* $(v; \alpha_0, \alpha_1, \alpha_2) \in \mathbb{E} \times Iso(\mathbb{E})^3$ *is a Grünbaum system, then the following holds:*

(i) $\alpha_0 v \neq v$, *in particular* $\alpha_0 \neq \alpha_2$,
(ii) $\alpha_0\alpha_1 \neq \alpha_1\alpha_0$, *in particular* $\alpha_0 \neq \alpha_1$,
(iii) $\alpha_1\alpha_2 \neq \alpha_2\alpha_1$, *in particular* $\alpha_1 \neq \alpha_2$,
(iv) $\dim \alpha_1, \dim \alpha_2 \geqq 1$,
(v) there is no line $\mathbb{F} \subseteq \mathbb{E}$ *with* $\alpha_i(\mathbb{F}) = \mathbb{F}$ *for all* $i = 0, 1, 2$.

Proof. (i) This follows immediately from $\alpha_0|_{<\alpha_0,\alpha_1,\alpha_2>\cdot v} \neq Id_{<\alpha_0,\alpha_1,\alpha_2>\cdot v}$ and $\alpha_1 v = \alpha_2 v = v$.

(ii) If $\alpha_0\alpha_1 = \alpha_1\alpha_0$, then $\alpha_i(\{v, \alpha_0 v\}) = \{v, \alpha_0 v\}$ for all $i = 0, 1, 2$ and, hence, $\langle\alpha_0, \alpha_1, \alpha_2\rangle\cdot v = \{v, \alpha_0 v\}$ in contradiction to $\alpha_2|_{\{v,\alpha_0 v\}} = Id_{\{v,\alpha_0 v\}}$ and $\alpha_2|_{<\alpha_0,\alpha_1,\alpha_2>\cdot v} \neq Id_{<\alpha_0,\alpha_1,\alpha_2>\cdot v}$.

(iii) If $\alpha_1\alpha_2 = \alpha_2\alpha_1$, then $\alpha_2|_{<\alpha_0,\alpha_1>\cdot v} = Id_{<\alpha_0,\alpha_1>\cdot v}$ and, hence, $\langle\alpha_0, \alpha_1, \alpha_2\rangle\cdot v = \langle\alpha_0, \alpha_1\rangle\cdot v$, again in contradiction to $\alpha_2|_{<\alpha_0,\alpha_1,\alpha_2>\cdot v} \neq Id_{<\alpha_0,\alpha_1,\alpha_2>\cdot v}$.

(iv) Since $v\in\mathbb{E}^{\alpha 1}\cap\mathbb{E}^{\alpha 2} \neq \emptyset$ and $\alpha_1\alpha_2 \neq \alpha_2\alpha_1$, we have necessarily $\dim\alpha_1 \neq 0 \neq \dim\alpha_2$.

(v) Assume $\alpha_0(\mathbb{F}) = \alpha_1(\mathbb{F}) = \alpha_2(\mathbb{F}) = \mathbb{F}$ for some line $\mathbb{F}\subseteq\mathbb{E}$. If $v\notin\mathbb{F}$, then the intersection w of $\mathbb{F}$ and the line $\mathbb{F}'$ through v which is perpendicular to $\mathbb{F}$ is fixed by α_1 and α_2, so we have $\mathbb{F}'\subseteq\mathbb{E}^{\alpha 1}\cap\mathbb{E}^{\alpha 2}$. But the group $\{\alpha\in Iso(\mathbb{E})|\alpha(\mathbb{F}) = \mathbb{F}, \mathbb{F}'\subseteq\mathbb{E}^{\alpha}\}$ is isomorphic to the Klein four-group, consisting of the identity, the 180°-degree rotation around $\mathbb{F}'$, the reflection at the plane $\langle\mathbb{F}\,\mathbb{F}'\rangle$ and the reflection at the plane containing $\mathbb{F}'$ and perpendicular to $\mathbb{F}$. So $\alpha_1\alpha_2 \neq \alpha_2\alpha_1$ implies that α_1 and α_2 cannot both be contained in this group, a contradiction.

If $v\in\mathbb{F}$, then $\langle\alpha_0, \alpha_1, \alpha_2\rangle\cdot v\subseteq\mathbb{F} = \langle v, \alpha_0 v\rangle\subseteq\mathbb{F}^{\alpha_2}$ in contradiction to $\alpha_2|_{<\alpha_0,\alpha_1,\alpha_2>\cdot v} \neq Id_{<\alpha_0,\alpha_1,\alpha_2>\cdot v}$. □

It follows easily from $\alpha_0 v \neq v$ that a quadrupel $(v; \alpha_0, \alpha_1, \alpha_2)\in\mathbb{E}\times Iso(\mathbb{E})^3$ is a (discrete) Grünbaum system if its "Petrie dual" $P(v;\alpha_0,\alpha_1,\alpha_2) := (v;\alpha_0\alpha_2,\alpha_1,\alpha_2)$ is a (discrete) Grünbaum system. So, since obviously $P(P(v; \alpha_0, \alpha_1, \alpha_2)) = (v; \alpha_0, \alpha_1, \alpha_2)$, it follows that Grünbaum systems come in pairs of mutually Petrie dual Grünbaum systems. Moreover, since two Grünbaum systems $(v; \alpha_0, \alpha_1, \alpha_2)$ and $(w; \beta_0, \beta_1, \beta_2)$ are isometric/ similar if and only if the systems $P(v; \alpha_0, \alpha_1, \alpha_2) = (v; \alpha_0\alpha_2, \alpha_1, \alpha_2)$ and $P(w; \beta_0, \beta_1, \beta_2) = (w; \beta_0\beta_2, \beta_1, \beta_2)$ are isometric/similar, it follows that this pairing induces a pairing of the isometry/similarity classes of Grünbaum systems—though in this situation it may happen that an isometry/similarity class is Petrie dual to itself.

Note that the Petrie dual of a Grünbaum system $(v; \alpha_0, \alpha_1, \alpha_2)$ has the same dimensionality as $(v; \alpha_0, \alpha_1, \alpha_2)$ if and only if $\dim\alpha_0 = \dim\alpha_0\alpha_2$ if and only if $\dim\alpha_0 = \dim\alpha_2 = 1$, since $\dim\alpha_0 + \dim\alpha_0\alpha_2 + \dim\alpha_2\in\{3, 5\}$, so $\dim\alpha_0 = \dim\alpha_0\alpha_2 \neq 1$ would imply $\dim\alpha_2 = 1$, $\dim\alpha_0 = \dim\alpha_0\alpha_2 = 2$ and $\mathbb{E}^{\alpha_2} = \mathbb{E}^{\alpha_0}\cap\mathbb{E}^{\alpha_0\alpha_2}\subseteq\mathbb{E}^{\alpha_0}$ in contradiction to $v\in\mathbb{E}^{\alpha_2}\backslash\mathbb{E}^{\alpha_0}$.

§3 Discrete Grünbaum systems

Continuing with our notation let us now consider discrete Grünbaum systems. We claim Proposition 2.

PROPOSITION 2. *If* $(v; \alpha_0, \alpha_1, \alpha_2)\in\mathbb{E}\times Iso(\mathbb{E})^3$ *is a discrete Grünbaum system, then the intersection of* $\langle\alpha_0, \alpha_1, \alpha_2\rangle$ *with the group* $T = T_{\mathbb{E}} \leq Iso(\mathbb{E})$ *of translations of* $\mathbb{E}$ *has finite index in* $\langle\alpha_0, \alpha_1, \alpha_2\rangle$ *and rank* $\neq 1$.

Proof. According to Bieberbach (cf. [3], [2] or [1]) for any discrete subgroup $G \leqq Iso(\mathbb{E})$ there exists a subspace $\mathbb{F} \subseteq \mathbb{E}$ with $\alpha(\mathbb{F}) = \mathbb{F}$ for all $\alpha \in G$ and $(G|_{\mathbb{F}}:(G|_{\mathbb{F}} \cap T_{\mathbb{F}})) < \infty$.

Choose such a subspace $\mathbb{F}$ for $G = \langle \alpha_0, \alpha_1, \alpha_2 \rangle$. W. l. o. g. assume $\mathbb{F} \neq \mathbb{E}$. If $\dim \mathbb{F} = 0$, it follows that the discrete group G has a fixed point and so G itself is finite. $\operatorname{Dim} \mathbb{F} = 1$ is excluded by Proposition 1, (v), and $\dim \mathbb{F} = 2$ implies $(G : G \cap T) < \infty$, since in this case $G \to G|_{\mathbb{F}}$ has finite kernel and $\alpha|_{\mathbb{F}} \in T_{\mathbb{F}}$ implies $\alpha^2 \in T$.

So we have always $(G : G \cap T) < \infty$. Now assume $rk(G \cap T) = 1$ and $\tau \in G \cap T, \tau \neq 1$. Consider the set of all lines $\mathbb{F} \subseteq \mathbb{E}$ in the direction of τ, i.e. with $\tau(\mathbb{F}) = \mathbb{F}$. The action of G on $\mathbb{E}$ induces an action of the finite group $G/(G \cap T)$ on this set which inherits the structure of an affine euclidean plan from $\mathbb{E}$. Moreover, $G/(G \cap T)$ acts isometrically on this euclidean plane, so it has a fixed point. But this contradicts once again Proposition 1, (v). □

COROLLARY 1. *If $(v; \alpha_0, \alpha_1, \alpha_2)$ is a discrete Grünbaum system and if $G = \langle \alpha_0, \alpha_1, \alpha_2 \rangle$ is not finite, then $\measuredangle(\alpha_1, \alpha_2) \in \{30°, 45°, 60°\}$ and $\measuredangle(\alpha_0, \alpha_1) \in \{0°, 30°, 45°, 60°, 90°\}$.*

Proof. This follows from $rk(G \cap T) \geqq 2$ and the standard crystallographic restrictions together with the fact that $\mathbb{E}^{\alpha 1} \cap \mathbb{E}^{\alpha 2} \neq \emptyset$ and $\alpha_1\alpha_2 \neq \alpha_2\alpha_1$ which excludes $\measuredangle(\alpha_1, \alpha_2) \in \{0°, 90°\}$. □

COROLLARY 2. *If $(v; \alpha_0, \alpha_1, \alpha_2)$ is a discrete Grünbaum system, then $rk(\langle \alpha_0, \alpha_1, \alpha_2 \rangle \cap T) = 2$ if and only if there is a plane $\mathbb{F} \subseteq \mathbb{E}$ with $\alpha_0(\mathbb{F}) = \alpha_1(\mathbb{F}) = \alpha_2(\mathbb{F}) = \mathbb{F}$, in which case we say that $(v; \alpha_0, \alpha_1, \alpha_2)$ has an invariant plane.*

Proof. If $\langle \alpha_0, \alpha_1, \alpha_2 \rangle \cap T$ is spanned by τ_1 and τ_2, then the isometric action of the finite group $G/G \cap T$ on the "euclidean line" consisting of all planes $\mathbb{F} \subseteq \mathbb{E}$ with $\tau_1(\mathbb{F}) = \tau_2(\mathbb{F}) = \mathbb{F}$ necessarily has a fixed point $\mathbb{F}_0$.

Vice versa, if $\alpha_i(\mathbb{F}) = \mathbb{F}$ for $i = 0, 1, 2$ for some plane $\mathbb{F} \subseteq \mathbb{E}$ then we have necessarily $rk(\langle \alpha_0, \alpha_1, \alpha_2 \rangle \cap T) \leqq 2$. But $rk(\langle \alpha_0, \alpha_1, \alpha_2 \rangle \cap T) < 2$ implies $\langle \alpha_0, \alpha_1, \alpha_2 \rangle \cap T = \{Id_{\mathbb{E}}\}$, so $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$—being finite— has a fixed point w and so it also leaves invariant the line through w which is perpendicular to $\mathbb{F}$, once again contradicting Proposition 1, (v). □

§4. The planar case

We define a Grünbaum system $(v; \alpha_0, \alpha_1, \alpha_2)$ to be planar, if $\langle \alpha_0, \alpha_1, \alpha_2 \rangle \cdot v$ is contained in a plane $\mathbb{F}$, in which case $\mathbb{F}$ must be spanned by $\langle\alpha_0, \alpha_1, \alpha_2\rangle\cdot v$, since otherwise $\alpha_0 v \neq v$ would imply that $\langle\alpha_0, \alpha_1, \alpha_2\rangle\cdot v$ spans a line which then would be invariant under $\alpha_0, \alpha_1, \alpha_2$, contradicting Proposition 1, (v).

Now assume $(v; \alpha_0, \alpha_1, \alpha_2)$ to be planar. We may replace α_0, α_1 and α_2 by those—uniquely determined—isometries β_0, β_1 and β_2, respectively, which satisfy $\beta_0^2 = \beta_1^2 = \beta_2^2 = Id_{\mathbb{E}}$ and $\mathbb{E}^{\beta_i} = \mathbb{F}\cap\mathbb{E}^{\alpha_i}$ $(i = 0, 1, 2)$ without changing the isometry type of our system, since $\beta_i|_{\mathbb{F}} = \alpha_i|_{\mathbb{F}}$ implies that the necessary relations hold for $\gamma = Id_{\mathbb{E}}$. So we may assume $\dim\alpha_1 = \dim\alpha_2 = 1$ and $\dim\alpha_0 \leqq 1$.

Moreover, Corollary 2 of Proposition 2 implies $rk(\langle\alpha_0, \alpha_1, \alpha_2\rangle\cap T) = 2$ and, hence, $\mathbb{F}^{\alpha_0}\cap\mathbb{F}^{\alpha_1}\cap\mathbb{F}^{\alpha_2} = \emptyset$ and $\sphericalangle(\alpha_1, \alpha_2)\in\{30°, 45°, 60°\}$. Thus, in case $\dim\alpha_0 = 0$, we have at most three angularity types and, similarly, in case $\dim\alpha_0 = 1$ which—up to isometry—is Petrie dual to the first case, we have also at most three angularity types in view of $\sphericalangle(\alpha_0, \alpha_1) + \sphericalangle(\alpha_1, \alpha_2) = \sphericalangle(\alpha_0, \alpha_2) = 90°$.

Moreover, one verifies easily that all these 6 angularity types determine the corresponding Grünbaum system up to similarity and that all these exist and correspond to the six well known types of planar polyhedra, exhibited in [5].

§5. The non planar cases

From now on let us restrict our attention to non planar Grünbaum systems. In principle, for a Grünbaum system $(v; \alpha_0, \alpha_1, \alpha_2)$ the quadrupel $(\dim\alpha_2, \dim\alpha_1 | \dim\alpha_0, \dim\alpha_0\alpha_2)$ can be any one of the quadrupels $(n_2, n_1 | n_0, n_0')\in\mathbb{N}^4$ with $1 \leqq n_1, n_2 \leqq 2$, $0 \leqq n_0, n_0' \leqq 2$ and $n_0 + n_0' + n_2 \in \{3, 5\}$. But we know already that $n_0 = n_0'$ can hold only for $n_0 = n_0' = n_2 = 1$ and we see easily that in case $n_0 = 0$ and $n_2 = 1$ (and thus $n_0' = 2$) as well as in the Petrie dual case $n_0 = 2$, $n_2 = 1$, $n_0' = 0$ our system is planar since in case $n_1 = 1$ the plane $\langle\mathbb{E}^{\alpha_1}\cup\mathbb{E}^{\alpha_2}\rangle$ contains v and is invariant under $\langle\alpha_0, \alpha_1, \alpha_2\rangle$ whereas in the case $n_1 = 2$ the plane $\mathbb{F}$ which contains $\mathbb{E}^{\alpha_2}$ and is perpendicular to $\mathbb{E}^{\alpha_1}$ is invariant under $\langle\alpha_0, \alpha_1, \alpha_2\rangle$ and contains v.

So we are left to study non planar Grünbaum systems $(v; \alpha_0, \alpha_1, \alpha_2)$ with $(\dim\alpha_2, \dim\alpha_1 | \dim\alpha_0, \dim\alpha_0\alpha_2)$ being one of the following quadrupels, grouped into pairwise Petrie dual cases: (2, 2|2, 1), (2, 2|1, 2); (2, 2|1, 0), (2, 2|0, 1); (2, 1|2, 1), (2, 1|1, 2); (2, 1|1, 0); (2, 1|0, 1); (1, 2|1, 1); (1, 1|1, 1).

§6. The finite case

We define a Grünbaum system $(v; \alpha_0, \alpha_1, \alpha_2)$ to be finite if $\langle\alpha_0, \alpha_1, \alpha_2\rangle$ is finite. This implies obviously the finiteness of $\langle\alpha_0, \alpha_1, \alpha_2\rangle\cdot v$ and in case of discrete systems it is equivalent with the finiteness of $\langle\alpha_0, \alpha_1, \alpha_2\rangle\cdot v$ (since for a discrete group $G \subseteq Iso(\mathbb{E})$

and $v \in \mathbb{E}$ the stabilizer group $G_v = \{\alpha \in G \mid \alpha \cdot v = v\}$ is necessarily finite) as well as with $\mathbb{E}^{\alpha_0} \cap \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2} \neq \emptyset$, i.e. $\mathbb{E}^{<\alpha_0,\alpha_1,\alpha_2>} \neq \emptyset$. In general, a Grünbaum system $(v; \alpha_0, \alpha_1, \alpha_2)$ is defined to be bounded if $\mathbb{E}^{<\alpha_0,\alpha_1,\alpha_2>} \neq \emptyset$. We claim

PROPOSITION 3. *A non planar Grünbaum system* $(v; \alpha_0, \alpha_1, \alpha_2)$ *is bounded if and only if* $\dim \alpha_2 = \dim \alpha_1 = 2$ *and* $\dim \alpha_0 + \dim \alpha_0\alpha_2 = 3$.

Proof. If $\dim \alpha_0 = \dim \alpha_1 = \dim \alpha_2 = 2$, then either the three planes $\mathbb{F}_i := \mathbb{E}^{\alpha_i}$ $(i = 0, 1, 2)$ have a non empty intersection in which case $(v; \alpha_0, \alpha_1, \alpha_2)$ is bounded or there exists a plane $\mathbb{F}$ containing v which is perpendicular to $\mathbb{F}_0$, $\mathbb{F}_1$ and $\mathbb{F}_2$ in which case $\langle \alpha_0, \alpha_1, \alpha_2 \rangle \cdot v \subseteq \langle \alpha_0, \alpha_1, \alpha_2 \rangle \cdot \mathbb{F} = \mathbb{F}$ implies that our system is necessarily planar.

Similarly, $\dim \alpha_0\alpha_2 = \dim \alpha_1 = \dim \alpha_2 = 2$ implies also the boundedness of $(v; \alpha_0, \alpha_1, \alpha_2)$ if this system is non planar.

Vice versa, if $(v; \alpha_0, \alpha_1, \alpha_2)$ is bounded and, hence, $\mathbb{E}^{<\alpha_0,\alpha_1,\alpha_2>} \neq \emptyset$, say $w \in \mathbb{E}^{<\alpha_0,\alpha_1,\alpha_2>}$, then v, $w \in \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2}$ and $v \neq w$ implies $\dim(\mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2}) \geqq 1$. But $\alpha_1\alpha_2 \neq \alpha_2\alpha_1$ implies $\mathbb{E}^{\alpha_1} \neq \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2} \neq \mathbb{E}^{\alpha_2}$, so we have necessarily $\dim \alpha_1 = \dim \alpha_2 = 2$. Similarly, $\alpha_0\alpha_1 \neq \alpha_1\alpha_0$ implies $\mathbb{E}^{\alpha_0} \neq \{w\}$ and, hence, $\dim \alpha_0 \geqq 1$, whereas $(\alpha_0\alpha_2)\alpha_1 \neq \alpha_1(\alpha_0\alpha_2)$ implies $\mathbb{E}^{\alpha_0\alpha_2} \neq \{w\}$ and hence, $\dim \alpha_0\alpha_2 \geqq 1$. Since moreover $\dim \alpha_0 + \dim \alpha_0\alpha_2 + \dim \alpha_2 \in \{3, 5\}$, we are left with $\dim \alpha_0 + \dim \alpha_0\alpha_2 = 3$, i.e. $\dim \alpha_0 = 2$ and $\dim \alpha_0\alpha_2 = 1$ or $\dim \alpha_0 = 1$ and $\dim \alpha_0\alpha_2 = 2$. □

Let us now study in detail the bounded case $\dim \alpha_2 = \dim \alpha_1 = \dim \alpha_0 = 2$. Since the line $\mathbb{E}^{\alpha_0\alpha_2} = \mathbb{E}^{\alpha_0} \cap \mathbb{E}^{\alpha_2}$ cannot be contained in $\mathbb{E}^{\alpha_1}$, we have necessarily $\mathbb{E}^{\alpha_0} \cap \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2} = \{u\}$ for some $u \in \mathbb{E} \setminus \{v\}$. Moreover, it is easily seen that for two tripels of planes $(\mathbb{F}_0, \mathbb{F}_1, \mathbb{F}_2)$ and $(\mathbb{F}'_0, \mathbb{F}'_1, \mathbb{F}'_2)$ and two points u, $u' \in \mathbb{E}$ with $u \in \mathbb{F}_0 \cap \mathbb{F}_1 \cap \mathbb{F}_2$, $u' \in \mathbb{F}'_0 \cap \mathbb{F}'_1 \cap \mathbb{F}'_2$, $\measuredangle(\mathbb{F}_0, \mathbb{F}_2) = 90^\circ$ and $\measuredangle(\mathbb{F}'_0, \mathbb{F}'_2) = 90^\circ$ there exists an isometry $\alpha \in Iso(\mathbb{E})$ with $\alpha u = u'$ and $\alpha(\mathbb{F}_i) = \mathbb{F}'_i$ for $i = 0, 1, 2$ if and only if $\measuredangle(\mathbb{F}_0, \mathbb{F}_1) = \measuredangle(\mathbb{F}'_0, \mathbb{F}'_1)$ and $\measuredangle(\mathbb{F}_1, \mathbb{F}_2) = \measuredangle(\mathbb{F}'_1, \mathbb{F}'_2)$, since for two perpendicular planes $\mathbb{F}_0$ and $\mathbb{F}_2$, a given point $u \in \mathbb{F}_0 \cap \mathbb{F}_2$ and two given angles φ_0 and φ_2 with $0^\circ \leqq \varphi_0$, $\varphi_2 < 90^\circ$ there exist at most four planes $\mathbb{F}$ with $u \in \mathbb{F}$, $\measuredangle(\mathbb{F}_0, \mathbb{F}) = \varphi_0$ and $\measuredangle(\mathbb{F}_2, \mathbb{F}) = \varphi_1$ on which the Klein four group generated by the reflections at $\mathbb{F}_0$ and $\mathbb{F}_2$ acts transitively.

More precisely, if $\varphi_0 + \varphi_1 < 90^\circ$, there exists no such plane, if $\varphi_0 + \varphi_1 = 90^\circ$, there exist two such planes both of which contain $\mathbb{F}_0 \cap \mathbb{F}_2$, and if $\varphi_0 + \varphi_1 > 90^\circ$, there exist four such planes none of which contains $\mathbb{F}_0 \cap \mathbb{F}_2$.

So, for a bounded Grünbaum system $(v; \alpha_0, \alpha_1, \alpha_2)$ with $\dim \alpha_0 = 2$ we have $\measuredangle(\alpha_0, \alpha_1) + \measuredangle(\alpha_1, \alpha_2) > 90^\circ$ and for two bounded Grünbaum systems $(v; \alpha_0, \alpha_1, \alpha_2)$ and $(w; \beta_0, \beta_1, \beta_2)$ there exists an isometry $\gamma \in Iso(\mathbb{E})$ with $\gamma\alpha_i\gamma^{-1} = \beta_i$ for $i = 0, 1, 2$ if and only if both systems have the same angularity in which case we can compose γ with a (unique!) dilatation with center $\mathbb{E}^{\beta_0} \cap \mathbb{E}^{\beta_1} \cap \mathbb{E}^{\beta_2}$ to get a similarity between $(v; \alpha_0, \alpha_1, \alpha_2)$ and $(w; \beta_0, \beta_1, \beta_2)$. Moreover, if the distance from v to $\mathbb{E}^{\alpha_0} \cap \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2}$

coincides with the distance from w to $\mathbb{E}^{\beta_0} \cap \mathbb{E}^{\beta_1} \cap \mathbb{E}^{\beta_2}$, then this dilatation is either the identity or the inversion at w, so—for similar systems—this condition is necessary and sufficient for isometry.

Finally, using Petrie duality, we can deduce corresponding results in case $\dim \alpha_0 = 1$. So, altogether we have proved Proposition 4.

PROPOSITION 4. *Two bounded Grünbaum systems $(v; \alpha_0, \alpha_1, \alpha_2)$ and $(w; \beta_0, \beta_1, \beta_2)$ are similar if and only if they have the same angularity and they are isometric if and only if in addition the distance from v to $\mathbb{E}^{\alpha_0} \cap \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2}$ coincides with the distance from w to $\mathbb{E}^{\beta_0} \cap \mathbb{E}^{\beta_1} \cap \mathbb{E}^{\beta_2}$.*

Let us now study the angularity of finite Grünbaum systems $(v; \alpha_0, \alpha_1, \alpha_2)$ with $\dim \alpha_0 = 2$. Since $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ is finite and since there is no line which is left invariant by $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$, this group must leave invariant either a cube or a dodecahedron with center $\mathbb{E}^{\alpha_0} \cap \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2}$. In the first case this leads to $\sphericalangle(\alpha_0, \alpha_1)$, $\sphericalangle(\alpha_1, \alpha_2) \in \{30°, 45°, 60°\}$, so in view of $\sphericalangle(\alpha_0, \alpha_1) + \sphericalangle(\alpha_1, \alpha_2) > 90°$ we have the three possibilities $\sphericalangle(\alpha_0, \alpha_1) = 45°$ and $\sphericalangle(\alpha_1, \alpha_2) = 60°$, $\sphericalangle(\alpha_0, \alpha_1) = 60°$ and $\sphericalangle(\alpha_1, \alpha_2) = 45°$ or $\sphericalangle(\alpha_0, \alpha_1) = \sphericalangle(\alpha_1, \alpha_2) = 60°$ (which, by the way, correspond to the cube, the octahedron and the tetrahedron—cf. §10). In the second case we get from an analysis of the reflection planes of a dodecahedron $\sphericalangle(\alpha_0, \alpha_1)$, $\sphericalangle(\alpha_1, \alpha_2) \in \{36°, 60°, 72°\}$ and $\sphericalangle(\alpha_0, \alpha_1) \neq \sphericalangle(\alpha_1, \alpha_2)$—the last inequality being a consequence of the fact that the angle φ between $\mathbb{E}^{\alpha_1}$ and the plane which is perpendicular to $\mathbb{E}^{\alpha_0}$ and to $\mathbb{E}^{\alpha_2}$ satisfies also $\varphi \in \{36°, 60°, 72°\}$ and $\sin^2 \varphi + \sin^2(\sphericalangle(\alpha_0, \alpha_1)) + \sin^2(\sphericalangle(\alpha_1, \alpha_2)) = 2 = \sin^2 36° + \sin^2 60° + \sin^2 72°$, so one has necessarily $\{\varphi, \sphericalangle(\alpha_0, \alpha_1), \sphericalangle(\alpha_1, \alpha_2)\} = \{36°, 60°, 72°\}$. This can also be deduced from the fact that there is no isometry of a dodecahedron which switches two perpendicular ones among its reflection planes into each other since otherwise its isometry would contain a rotation of order 4.

So we are left with the 6 possibilities (36°, 60°), (36°, 72°), (60°, 36°), (60°, 72°), (72°, 36°), (72°, 60°) for $(\sphericalangle(\alpha_0, \alpha_1), \sphericalangle(\alpha_1, \alpha_2))$, (which, by the way, correspond to the dodecahedron, the great dodecahedron, the icosahedron, the great icosahedron, the small stellated dodecahedron and the great stellated dodecahedron—cf. §10).

If $(v; \alpha_0, \alpha_1, \alpha_2)$ is a finite Grünbaum system with $\dim \alpha_0 = 1$, then we may apply Petrie duality to conclude that, again, there are altogether at most 9 possibilities for the compatible angularities. More precisely, the Petrie duals of the nine cases (45°, 60°), (60°, 45°), (60°, 60°), (36°, 60°), (36°, 72°), (60°, 36°), (60°, 72°), (72°, 36°), (72°, 60°) have angularity (30°, 60°), (30°, 45°), (45°, 60°), (18°, 60°), (30°, 72°), (18°, 36°), (54°, 72°), (30°, 36°), (54°, 60°)—in respective order. This follows once again from the observation that for three pairwise perpendicular planes $\mathbb{F}_1$, $\mathbb{F}_2$, $\mathbb{F}_3$ and one additional line $\mathbb{L}$/plane $\mathbb{F}$ one has

$$\sin^2(\mathbb{F}_1, \mathbb{L}) + \sin^2(\mathbb{F}_2, \mathbb{L}) + \sin^2(\mathbb{F}_3, \mathbb{L}) = 1$$

and

$$\sin^2(\mathbb{F}_1, \mathbb{F}) + \sin^2(\mathbb{F}_2, \mathbb{F}) + \sin^2(\mathbb{F}_3, \mathbb{F}) = 2,$$

whereas for two lines $\mathbb{L}_1, \mathbb{L}_2$ and two planes $\mathbb{F}_1, \mathbb{F}_2$ with $\sphericalangle(\mathbb{L}_1, \mathbb{F}_1) = \sphericalangle(\mathbb{L}_2, \mathbb{F}_2) = 90°$ one has $\sphericalangle(\mathbb{L}_1, \mathbb{L}_2) = 90° - \sphericalangle(\mathbb{L}_1, \mathbb{F}_2) = 90° - \sphericalangle(\mathbb{F}_1, \mathbb{L}_2) = \sphericalangle(\mathbb{F}_1, \mathbb{F}_2)$. So, in the three cubic cases with $\dim\alpha_0 = 2$ the angle between $\mathbb{E}^{\alpha_1}$ and $\mathbb{E}^{\alpha_0\alpha_2} = \mathbb{E}^{\alpha_0} \cap \mathbb{E}^{\alpha_2}$ is determined by the fact that—counting multiplicities—one must have $\{\sphericalangle(\alpha_0, \alpha_1), \sphericalangle(\alpha_1, \alpha_2), 90° - \sphericalangle(\alpha_1, \alpha_0\alpha_2)\} = \{45°, 60°, 60°\}$ whereas in the 6 dodecahedral cases one has $\{\sphericalangle(\alpha_0, \alpha_1), \sphericalangle(\alpha_1, \alpha_2), 90° - \sphericalangle(\alpha_1, \alpha_0\alpha_2)\} = \{36°, 60°, 72°\}$.

Since moreover altogether 18 similarity types of finite Grünbaum polyhedra are known to exist (cf. [5]), we have proved the following.

PROPOSITION 5. *A Grünbaum system* $(v; \alpha_0, \alpha_1, \alpha_2)$ *is finite if and only if* $\dim\alpha_2 = \dim\alpha_1 = 2$ *and either* $\dim\alpha_0 = 2$ *and* $(\sphericalangle(\alpha_0, \alpha_1), \sphericalangle(\alpha_1, \alpha_2)) \in \{(45°, 60°), (60°, 45°), (60°, 60°), (36°, 60°), (36°, 72°), (60°, 36°), (60°, 72°), (72°, 36°), (72°, 60°)\}$ *or* $\dim\alpha_0\alpha_2 = 2$ *and* $(\sphericalangle(\alpha_0, \alpha_1), \sphericalangle(\alpha_1, \alpha_2)) \in \{(30°, 60°), (30°, 45°), (45°, 60°), (18°, 60°), (30°, 72°), (18°, 36°), (54°, 72°), (30°, 36°), (54°, 60°)\}$. *Moreover, all these* 18 *possibilities do occur and each one is characterized up to similarity by these data.*

§7. The non planar rank 2 case

Let us now study non planar Grünbaum systems $(v; \alpha_0, \alpha_1, \alpha_2)$ for which there exists an invariant plane, i.e. a plane $\mathbb{F} \subseteq \mathbb{E}$ with $\alpha_i(\mathbb{F}) = \mathbb{F}$ for all $i = 0, 1, 2$. Recall that for a discrete system this is equivalent $rk(\langle\alpha_0, \alpha_1, \alpha_2\rangle \cap T) = 2$. We claim the following.

PROPOSITION 6. *A non planar Grünbaum system* $(v; \alpha_0, \alpha_1, \alpha_2)$ *has an invariant plane if, and only if,* $\dim\alpha_2 = \dim\alpha_1 = 2$ *and* $\dim\alpha_0 + \dim\alpha_0\alpha_2 = 1$.

Proof. If $(v; \alpha_0, \alpha_1, \alpha_2)$ satisfies $\dim\alpha_2 = \dim\alpha_1 = 2$ and $\dim\alpha_0 = 0$ or $\dim\alpha_0\alpha_2 = 0$, then the plane $\mathbb{F}$ which is perpendicular to $\mathbb{E}^{\alpha_2}$ and $\mathbb{E}^{\alpha_1}$ and contains $\mathbb{E}^{\alpha_0}$ or $\mathbb{E}^{\alpha_0\alpha_2}$ is necessarily an invariant plane.

Vice versa, assume $(v; \alpha_0, \alpha_1, \alpha_2)$ to be a non planar Grünbaum system and to possess an invariant plane $\mathbb{F}$. Since our system is non planar, we have $v \notin \mathbb{F}$, so the line through v which is perpendicular to $\mathbb{F}$ is necessarily pointwise invariant under

α_1 and under α_2, i.e., we have $\dim(\mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2}) \geqq 1$ which implies (as above) $\dim \alpha_1 = \dim \alpha_2 = 2$ in view of $\alpha_1\alpha_2 \neq \alpha_2\alpha_1$.

Since, moreover, one has always $\dim \alpha_2 + \dim \alpha_0 + \dim \alpha_0\alpha_2 \in \{3, 5\}$ and since $\dim \alpha_1 = \dim \alpha_2 = 2$ and $\dim \alpha_0 + \dim \alpha_0\alpha_2 = 3$ together with non planarity is known to lead to bounded Grünbaum systems which can never have an invariant plane (otherwise, as pointed out already above, the line through a fixed point which is perpendicular to an invariant plane would be invariant, too), we are indeed left with the case $\dim \alpha_0 + \dim \alpha_0\alpha_2 = 1$. □

Let us now consider the isometry and the similarity problem for non planar Grünbaum systems $(v; \alpha_0, \alpha_1, \alpha_2)$ with $\dim \alpha_2 = \dim \alpha_1 = 2$ and $\dim \alpha_0 = 0$. Obviously, if $(w; \beta_0, \beta_1, \beta_2)$ is another such system with $\sphericalangle(\alpha_1, \alpha_2) = \sphericalangle(\beta_1, \beta_2)$, then we can always find an isometry γ with $\gamma v = w$, $\gamma\alpha_1\gamma^{-1} = \beta_1$ and $\gamma\alpha_2\gamma^{-1} = \beta_2$. Moreover, we can compose γ with one of the four isometries δ with $\delta w = w$, $\delta\beta_1\delta^{-1} = \beta_1$, $\delta\beta_2\delta^{-1} = \beta_2$ so that $\delta\gamma\alpha_0\gamma^{-1}\delta^{-1} = \beta_0$ if and only if the distances $d(\mathbb{E}^{\alpha_0}, v)$ and $d(\mathbb{E}^{\alpha_0}, \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2})$ of $\mathbb{E}^{\alpha_0}$ to v and to the line $\mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2}$ coincide with the corresponding distances in the system $(w; \beta_0, \beta_1, \beta_1)$. Note that $d(\mathbb{E}^{\alpha_0}, v) > d(\mathbb{E}^{\alpha_0}, \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2}) > 0$ since otherwise the plane containing v and perpendicular to $\mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2}$ would contain $\mathbb{E}^{\alpha_0}$ and so it would be invariant.

So we have a 2 parameter family of isometry classes of such systems for any given angularity. Moreover, we can find a similarity δ with $\delta w = w$, $\delta\beta_1\delta^{-1} = \beta_1$, $\delta\beta_2\delta^{-1} = \beta_2$ (so δ is the composition of an isometry with the same properties and a dilatation with center w) and with $\delta\gamma\alpha_0\gamma^{-1}\delta^{-1} = \beta_0$ if and only if the quotient $\dfrac{d(\mathbb{E}^{\alpha_0}, v)}{d(\mathbb{E}^{\alpha_0}, \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2})}$ which always exceeds 1 coincides with the corresponding quotient for the system $(w; \beta_0, \beta_1, \beta_2)$. So we have a one parameter family of similarity classes of such systems for any given angularity.

Using Petrie duality and $\sphericalangle(\alpha_1, \alpha_2) \in \{30°, 45°, 60°\}$ for discrete and non finite Grünbaum systems as well as the existence of altogether 6 one parameter families of similarity classes of Grünbaum polyhedra whose isometry group leaves a plane invariant (cf. [5]) the following proposition results.

PROPOSITION 7. *A non planar, discrete Grünbaum system* $(v; \alpha_0, \alpha_1, \alpha_2)$ *has an invariant plane if and only if* $\dim \alpha_1 = \dim \alpha_2 = 2$, $\dim \alpha_0 + \dim \alpha_0\alpha_2 = 1$ *and* $\sphericalangle(\alpha_1, \alpha_2) \in \{30°, 45°, 60°\}$, *in which case* $\sphericalangle(\alpha_1, \alpha_0) + \sphericalangle(\alpha_1, \alpha_0\alpha_2) + \sphericalangle(\alpha_1, \alpha_2) = 90°$.

If $(w; \beta_0, \beta_1, \beta_2)$ *in another such system with the same angularity, then* $(v; \alpha_0, \alpha_1, \alpha_2)$ *is isometric/similar to* $(w; \beta_0, \beta_1, \beta_2)$ *if and only if* $d(\mathbb{E}^{\alpha_0}, v) = d(\mathbb{E}^{\beta_0}, w)$ *and* $d(\mathbb{E}^{\alpha_0}, \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2}) = d(\mathbb{E}^{\beta_0}, \mathbb{E}^{\beta_1} \cap \mathbb{E}^{\beta_2}) / \dfrac{d(\mathbb{E}^{\alpha_0}, v)}{d(\mathbb{E}^{\alpha_0}, \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2})} = \dfrac{d(\mathbb{E}^{\beta_0}, w)}{d(\mathbb{E}^{\beta_0}, \mathbb{E}^{\beta_1} \cap \mathbb{E}^{\beta_2})}$, *respectively.*

Moreover, for any choice of $(\dim\alpha_0, \dim\alpha_0\alpha_2)\in\{(0, 1), (1, 0)\}$, $\sphericalangle(\alpha_1,\alpha_2)\in\{30°, 45°, 60°\}$, $\dfrac{d(\mathbb{E}^{\alpha_0}, v)}{d(\mathbb{E}^{\alpha_0}, \mathbb{E}^{\alpha}{}_1\cap\mathbb{E}^{\alpha}{}_2)}>1$ *and* $d(\mathbb{E}^{\alpha_0}, v)>0$ *there exists a corresponding non planar, discrete Grünbaum system with these values.*

§8. The rank 3 case

Let us now consider the remaining non planar and non bounded Grünbaum systems, i.e. those systems $(v;\alpha_0,\alpha_1,\alpha_2)$ for which there exists neither a fixed point nor an invariant plane. According to the last remark in §5, their dimensionalities $(\dim\alpha_2, \dim\alpha_1|\dim\alpha_0, \dim\alpha_0\alpha_2)$ can attain any one of the following values $(2, 1|2, 1)$, $(2, 1|1, 2)$, $(2, 1|1, 0)$, $(2, 1|0, 1)$, $(1, 2|1, 1)$, $(1, 1|1, 1)$.

So, in particular, v is always determined as the intersection of $\mathbb{E}^{\alpha_1}$ and $\mathbb{E}^{\alpha_2}$.

We claim at first Proposition 8.

PROPOSITION 8. *If* $(v;\alpha_0,\alpha_1,\alpha_2)$ *is a non planar Grünbaum system without a fixed point or an invariant plane, then* $\sphericalangle(\alpha_0,\alpha_1)\in\{0°,90°\}$ *if and only if* $\dim\alpha_0=0$ *(and, hence,* $\sphericalangle(\alpha_0,\alpha_1)=0°$*).*

Proof. If $\dim\alpha_0=0$, then, of course, $\sphericalangle(\alpha_0,\alpha_1)=0°$ and $\sphericalangle(\alpha_0\alpha_2,\alpha_1)=90°-\sphericalangle(\alpha_2,\alpha_1)\notin\{0°, 90°\}$. So—using Petrie duality—we have $\sphericalangle(\alpha_0,\alpha_1)\notin\{0°, 90°\}$ if $\dim\alpha_0\alpha_2=0$. If $\dim\alpha_0=\dim\alpha_0\alpha_2=\dim\alpha_2=1$ and $\sphericalangle(\alpha_0,\alpha_1)\in\{0°, 90°\}$, then the plane containing v which is perpendicular to $\mathbb{E}^{\alpha_0}$ would be invariant.

In the remaining cases we have $\dim\alpha_2=2$, $\dim\alpha_1=1$ and $\dim\alpha_0+\dim\alpha_0\alpha_2=3$. But then $\dim\alpha_0=1$ and $\sphericalangle(\alpha_0,\alpha_1)=0°$ is ruled out, since it implies $\mathbb{E}^{\alpha_1}\subseteq\mathbb{E}^{\alpha_2}$, $\dim\alpha_0=1$ and $\sphericalangle(\alpha_0,\alpha_1)=90°$ is ruled out, since in this case the plane containing v which is perpendicular to $\mathbb{E}^{\alpha_0}$ would be invariant, $\dim\alpha_0=2$ and $\sphericalangle(\alpha_0,\alpha_1)=0°$ is ruled out, since in this case the line through v which is perpendicular to $\mathbb{E}^{\alpha_0}$ would be invariant, and the last possibility $\dim\alpha_0=2$ and $\sphericalangle(\alpha_0,\alpha_1)=90°$ is also ruled out, since in this case $\mathbb{E}^{\alpha_0}\cap\mathbb{E}^{\alpha_1}\neq\emptyset$ and therefore $\alpha_0\alpha_1=\alpha_1\alpha_0$, contradicting Proposition 1. □

Next, let us observe that in case $\dim\alpha_2=2$, $\dim\alpha_1=1$ the classification of isometry/similarity types for given angularity can be derived from the corresponding classification in case $\dim\alpha_2=\dim\alpha_1=2$ since—with respect to this question—we may always replace α_1 by the product β_1 of α_1 with the inversion at v which is a plane reflection. Hence in case $\dim\alpha_0+\dim\alpha_0\alpha_2=3$ the angularity determines the similarity type and the angularity together with the distance $d(v, \mathbb{E}^{\alpha_0})$ from v to $\mathbb{E}^{\alpha_0}$ determines the isometric type, whereas in case $\dim\alpha_0+\dim\alpha_0\alpha_2=1$ the isometry type is determined by the angularity and $d(\mathbb{E}^{\alpha_0}, v)$ and $d(\mathbb{E}^{\alpha_0}, \mathbb{E}^{\alpha_2}\cap\mathbb{E}^{\beta_1})$ and the similarity type by the angularity and the quotient of these two distances.

Moreover, in case $\dim\alpha_2 = \dim\alpha_0 = 1$, $\dim\alpha_1 = 2$ and in case $\dim\alpha_2 = \dim\alpha_0 = \dim\alpha_1 = 1$, the angularity determines the similarity type and the angularity together with the distance $d(\mathbb{E}^{\alpha_0}, v)$ determines the isometric type, since—as above—for the given pair of orthogonal lines $\mathbb{E}^{\alpha_0}$ and $\mathbb{E}^{\alpha_2}$ there are at most four planes (or lines) $\mathbb{F}$ with $\mathbb{E}^{\alpha_0} \cap \mathbb{E}^{\alpha_2} \subseteq \mathbb{F}$ and with given angles $\sphericalangle(\mathbb{F}, \mathbb{E}^{\alpha_0})$ and $\sphericalangle(\mathbb{E}, \mathbb{F}^{\alpha_2})$ and the Klein four group $\langle\alpha_0, \alpha_2\rangle$ acts transitively on those planes (or lines), so there are at most eight such planes (or lines) through v and the group generated by α_0, α_2 and the inversion at $\mathbb{E}^{\alpha_0} \cap \mathbb{E}^{\alpha_1}$ acts transitively on those planes (or lines). Finally, we know that the discreteness of $(v; \alpha_0, \alpha_1, \alpha_2)$ implies $\sphericalangle(\alpha_1, \alpha_2) \in \{30°, 45°, 60°\}$ and—using Proposition 8—it implies $\sphericalangle(\alpha_0, \alpha_1) \in \{30°, 45°, 60°\}$ or $\dim\alpha_0 = 0$ as well as (using Petrie duality) $\sphericalangle(\alpha_0\alpha_2, \alpha_1) \in \{30°, 45°, 60°\}$ or $\dim\alpha_0\alpha_2 = 0$.

Hence in case $\dim\alpha_2 = 2$, $\dim\alpha_1 = 1$, $\dim\alpha_0 = 2$, $\dim\alpha_0\alpha_2 = 1$, in which case we have necessarily $(90° - \sphericalangle(\alpha_1, \alpha_2)) + (90° - \sphericalangle(\alpha_1, \alpha_0)) > \sphericalangle(\alpha_0, \alpha_2) = 90°$, i.e. $\sphericalangle(\alpha_0, \alpha_1) + \sphericalangle(\alpha_0, \alpha_1) < 90°$, we are left with the three possibilities $\sphericalangle(\alpha_0, \alpha_1) = \sphericalangle(\alpha_1, \alpha_2) = 30°$, $\sphericalangle(\alpha_0, \alpha_1) = 30°$ and $\sphericalangle(\alpha_1, \alpha_2) = 45°$ or $\sphericalangle(\alpha_0, \alpha_1) = 45°$ and $\sphericalangle(\alpha_1, \alpha_2) = 30°$.

From $\sin^2(\sphericalangle(\alpha_0, \alpha_1)) + \sin^2(\sphericalangle(\alpha_1, \alpha_2) + \sin^2(90° - \sphericalangle(\alpha_1, \alpha_0\alpha_2)) = 1$ one derives easily that the corresponding Petrie dual cases are characterized up to similarity by $\dim\alpha_2 = 2$, $\dim\alpha_1 = \dim\alpha_0 = 1$, $\dim\alpha_0\alpha_2 = 2$ and $(\sphericalangle(\alpha_0, \alpha_1), \sphericalangle(\alpha_1, \alpha_2)) = (45°, 30°), (60°, 45°)$ or $(60°, 30°)$, respectively.

In case $\dim\alpha_2 = 2$, $\dim\alpha_1 = 1$ and $\dim\alpha_0 = 0$, the condition $\sphericalangle(\alpha_1, \alpha_2) = 30°$, $45°$ or $60°$ is the only condition which has to be specified and in the Petrie dual case $\dim\alpha_2 = 2$, $\dim\alpha_1 = \dim\alpha_0 = 1$, $\dim\alpha_0\alpha_2 = 0$, the corresponding conditions are $(\sphericalangle \mp \alpha_0, \alpha_1), \sphericalangle \mp \alpha_1, \alpha_2)) = (60°, 30°), (45°, 45°)$ or $(30°, 60°)$, respectively.

In case $\dim\alpha_2 = \dim\alpha_0 = \dim\alpha_0\alpha_2 = 1$ and $\dim\alpha_1 = 2$, one has necessarily $\sin^2(\sphericalangle(\alpha_0, \alpha_1)) + \sin^2(\sphericalangle(\alpha_1, \alpha_2)) + \sin^2(\sphericalangle(\alpha_1, \alpha_2\alpha_0)) = 1$, so one has $(\sphericalangle(\alpha_0, \alpha_1), \sphericalangle(\alpha_1, \alpha_2), \sphericalangle(\alpha_1, \alpha_2\alpha_0)) = (30°, 30°, 45°), (30°, 45°, 30°)$ or $(45°, 30°, 30°)$, the first case being Petrie dual to the last one and the second case being Petrie dual to itself.

And finally, in case $\dim\alpha_2 = \dim\alpha_1 = \dim\alpha_0 = \dim\alpha_0\alpha_2 = 1$, one has necessarily $\sin^2(\sphericalangle(\alpha_0, \alpha_1)) + (\sin^2(\sphericalangle(\alpha_1, \alpha_2)) + \sin^2(\sphericalangle(\alpha_1, \alpha_2\alpha_0)) = 2$, so one has $(\sphericalangle(\alpha_0, \alpha_1), \sphericalangle(\alpha_1, \alpha_2), \sphericalangle(\alpha_1, \alpha_2\alpha_0)) = (45°, 60°, 60°), (60°, 45°, 60°)$ or $(60°, 60°, 45°)$, again the first one being Petrie dual to the last one and the second one being Petrie dual to itself. Actually, it was this last configuration which has been overlooked in [5].

So we have proved the following.

PROPOSITION 9. *If* $(v; \alpha_0, \alpha_1, \alpha_2)$ *is a non planar and non finite discrete Grünbaum system without an invariant plane then the 7-tupel* $(\dim\alpha_2, \dim\alpha_1, \dim\alpha_0, \dim\alpha_0\alpha_2; \sphericalangle(\alpha_0, \alpha_1), \sphericalangle(\alpha_1, \alpha_2), \sphericalangle(\alpha_1, \alpha_2\alpha_0))$ *can assume the following values, only:*

$(2, 1, 2, 1; 30°, 30°, 45°)$, $(2, 1, 2, 1; 30°, 45°, 60°)$, $(2, 1, 2, 1; 45°, 30°, 60°)$,

(2, 1, 1, 2; 45°, 30°, 30°), (2, 1, 1, 2; 60°, 45°, 30°), (2, 1, 1, 2; 60°, 30°, 45°),
(2, 1, 0, 1; 0°, 30°, 60°), (2, 1, 0, 1; 0°, 45°, 45°), (2, 1, 0, 1; 0°, 60°, 30°),
(2, 1, 1, 0; 60°, 30°, 0°), (2, 1, 1, 0; 45°, 45°, 0°), (2, 1, 1, 0; 30°, 60°, 0°),
(1, 2, 1, 1; 30°, 30°, 45°), (1, 2, 1, 1; 30°, 45°, 30°), (1, 2, 1, 1; 45°, 30°, 30°),
(1, 1, 1, 1; 45°, 60°, 60°), (1, 1, 1, 1; 60°, 45°, 60°), (1, 1, 1, 1; 60°, 60°, 45°).

Moreover, these values determine the Grünbaum system up to similarity except in case $\dim\alpha_0 + \dim\alpha_0\alpha_2 = 1$, *in which case the similarity type is determined by these values together with the quotient* $d(\mathbb{E}^{\alpha_0}, v)/d(\mathbb{E}^{\alpha_0}, \mathbb{E}^{\alpha_2}\cap\mathbb{E}^{\beta_1})$ *which can assume any value larger than* 1 *where* β_1 *denotes the product of* α_1 *with the inversion at* v.

In the last section, where we are going to compare this list to the list of polyhedra given by Branko Grünbaum in [5], we will also see that all these values can indeed be attained by discrete Grünbaum systems. To prepare this comparison, we will discuss regular polygons in the next section which we will study, quite generally, in the euclidean n-space rather than in 3-space, not only because it can be done easily in this generality, but also because things will become clearer this way.

§9. Regular polygons

According to [5], a (regular) polygon in the euclidean n-space $\mathbb{E} = \mathbb{E}^n$ consists of a non empty set $V \subseteq \mathbb{E}$ of vertices and a set $E \subseteq \mathscr{P}_2(V) := \{e \subseteq V \mid \#e = 2\}$ of edges such that for each vertex $v \in V$ one has $\#\{e \mid v \in e\} = 2$ and for any two vertices $v, w \in V$ there exists a finite sequence $v = v_0, v_1, \ldots, v_k = w \in V$ of vertices with $\{v_0, v_1\}$, $\{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\} \in E$ (and, moreover, for any two pairs (v, e) and (w, f) in $V \times E$ with $v \in e$ and $w \in f$ (i.e. for any two "flags") there exists an isometry $\alpha \in \mathbb{E}$ with $\alpha(V) = V$, the induced map α: $\mathscr{P}_2(V) \to \mathscr{P}_2(V)$ maps E onto E, $\alpha(v) = w$ and $\alpha(e) = f$).

Two regular polygons (V, E) and (W, F) are defined to be isometric/similar if there exists an isometry/similarity α: $\mathbb{E} \to \mathbb{E}$ with $\alpha(V) = W$ such that the induced map α: $\mathscr{P}_2(V) \to \mathscr{P}_2(W)$ maps E onto F, in which case one can find such an α for any $(v, e) \in V \times E$ with $v \in e$ and any $(w, f) \in W \times F$ with $w \in f$ which in addition satisfies $\alpha(v) = w$ and $\alpha(e) = f$.

Now assume (V, E) to be a regular polygon and pick some $v \in V$. Let $e = \{v, w\}$ and $e' = \{v, w'\}$ denote the two edges containing v and let α denote an isometry with $\alpha(V) = V$, $\alpha(E) = E$, $\alpha(w) = v$ and $\alpha(e) = e'$, in particular $\alpha(v) = w'$ and $\alpha^2(v) \neq v$, since otherwise $w' = \alpha(v) = \alpha^{-1}(v) = w$ in contradiction to $e \neq e'$. Then we have $\langle\alpha\rangle v =: \{\alpha^n(v) \mid n \in \mathbb{Z}\} = V$ and $\langle\alpha\rangle e =: \{\alpha^n(e) \mid n \in \mathbb{Z}\} = E$, since at least $\alpha^n(v) \in V$ and $\alpha^n(e) \in E$ for each $n \in \mathbb{Z}$, so in case $\langle\alpha\rangle v \subsetneq V$ we may find some vertex $u \in \langle\alpha\rangle v$ and some vertex $u \in V \setminus \langle\alpha\rangle v$ with $\{u, u'\} \in E$. But for $u = \alpha^n(v)$ we have the two edges $\alpha^n(e) = \{\alpha^n(v), \alpha^n(w)\}$

and $\alpha^n(e') = \{\alpha^n(v), \alpha^n(w')\}$ with $u \in \alpha^n(e)$ and $u \in \alpha^n(e')$. Since moreover $\alpha^n(e) \neq \alpha^n(e')$, we have necessarily either $\{u, u'\} = \alpha^n(e) = \{\alpha^n(v), \alpha^n(w)\} = \{\alpha^n(v), \alpha^{n-1}(v)\}$ or $\{u, u'\} = \alpha^n(e') = \{\alpha^n(v), \alpha^n(w')\} = \{\alpha^n(v), \alpha^{n+1}(v)\}$, so in any case we have $u' \in \langle\alpha\rangle v$.

PROPOSITION 10. *(i) A set* $V \subseteq \mathbb{E}$ *together with a set* $E \subseteq \mathscr{P}_2(V)$ *is a regular polygon if and only if there exists some* $\alpha \in \mathrm{Iso}(\mathbb{E})$ *and some* $v \in \mathbb{E}\backslash\mathbb{E}^{\alpha^2}$ *with* $\langle\alpha\rangle v = V$ *and* $\langle\alpha\rangle\{v, \alpha v\} = E$.

(ii) Moreover, if $(v, \alpha), (w, \beta) \in \mathbb{E} \times \mathrm{Iso}(\mathbb{E})$, $v \notin \mathbb{E}^{\alpha^2}$, $w \notin \mathbb{E}^{\beta^2}$, *then* $(\langle\alpha\rangle v, \langle\alpha\rangle\{v, av\})$ *and* $(\langle\beta\rangle w, \langle\beta\rangle\{w, \beta w\})$ *are isometric/similar if and only is there exists an isometry/similarity* γ *with* $\gamma v = w$ *and* $\gamma\alpha\gamma^{-1}|_{<b>w} = \beta|_{<b>w}$.

Proof. (i): We have to show that for $\alpha \in \mathrm{Iso}\,(\mathbb{E})$ and $v \in \mathbb{E}\backslash\mathbb{E}^{\alpha^2}$ the set $V := \langle\alpha\rangle v$ forms the set of vertices of a regular polygon with edge set $E := \langle\alpha\rangle\{v, \alpha(v)\}$.

Since for any $n \in \mathbb{Z}$ one has $\alpha^n(v) \in \{\alpha^n(v), \alpha^{n+1}(v)\} = \alpha^n(\{v, \alpha(v)\})$, $\alpha^n(v) \in \{\alpha^{n-1}(v), \alpha^n(v)\} = \alpha^{n-1}(\{v, \alpha(v)\})$ and $\alpha^{n-1}(v) \neq \alpha^{n+1}(v)$ any element in V is indeed contained in at least two different sets in E. Moreover, $\alpha^n(v) \in \alpha^k(\{v, \alpha(v)\})$ for some $k \in \mathbb{Z}$ implies $\alpha^n(v) = \alpha^k(v)$ or $\alpha^n(v) = \alpha^{k+1}(v)$ and hence $\alpha^k(\{v, \alpha(v)\}) = \{\alpha^k(v), \alpha^{k+1}(v)\} = \{\alpha^n(v), \alpha^{n+1}(v)\} = \alpha^n(\{v, \alpha(v)\})$ or $\alpha^k(\{v, \alpha(v)\}) = \alpha^{n-1}(\{v, \alpha(v)\})$, so there is no further set in E which contains $\alpha^n(v)$.

Finally, for $\alpha^n(v), \alpha^m(v) \in V$ and, say, $n < m$, the sequence $v_0 = \alpha^n(v), v_1 = \alpha^{n+1}(v), \ldots,$ $\{v_{m-n} = \alpha^m(v)$ satisfies $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{m-n-1}, v_{m-n}\} \in E$, so (V, E) is a polygon.

To show regularity, we observe at first that $\langle\alpha\rangle$ acts transitively on the "flags" of the form $(\alpha^n(v), \{\alpha^n(v), \alpha^{n+1}(v)\})$ and on the flags of the form $(\alpha^n(v), \{\alpha^n(v), \alpha^{n-1}(v)\})$. So it is enough to show that there exists an isometry β: $\mathbb{E} \to \mathbb{E}$ with $\beta(\alpha^n(v)) = \alpha^{-n}(v)$ for each $n \in \mathbb{Z}$ which follows from the fact that the euclidean distance between $\alpha^n(v)$ and $\alpha^m(v)$ coincides with the euclidean distance between $\alpha^{-n}(v) = \alpha^{-m-n}(\alpha^m(v))$ and $\alpha^{-m}(v) = \alpha^{-m-n}(\alpha^n(v))$.

(ii) If $(v, \alpha), (w, \beta) \in \mathbb{E} \times \mathrm{Iso}\,(\mathbb{E})$, $v \notin \mathbb{E}^{\alpha^2}$, $w \notin \mathbb{E}^{\beta^2}$, $V = \langle\alpha\rangle v$, $E = \langle\alpha\rangle\{v, \alpha(v)\}$, $W = \langle\beta\rangle w$ and $F = \langle\beta\rangle\{w, \beta(w)\}$, and if the polygons (V, E) and (W, F) are isometric/similar, then there exists an isometry/similarity γ with $\gamma(v) = w, \gamma(\alpha(v)) = \beta(w), \gamma(V) = W$ and $\gamma(E) = F$. It follows that $\gamma(\alpha^n(v)) = \beta^n(w)$, so for $\beta^n(w) \in W$ we have $(\gamma\alpha\gamma^{-1})(\beta^n(w)) = (\gamma\alpha\gamma^{-1})(\gamma(\alpha^n(v))) = \gamma(\alpha^{n+1}(v)) = \beta^{n+1}(w) = \beta(\beta^n(w))$, i.e. we have $\gamma\alpha\gamma^{-1}|_W = \beta|_W$.

Vice versa, if $\gamma v = w$ and $\gamma\alpha\gamma^{-1}|_w = \beta|_w$ for some isometry/similarity γ, then $\gamma(\alpha^n(v)) = (\gamma\alpha^n\gamma^{-1})(\gamma v) = (\gamma\alpha\gamma^{-1})^n(w) = \beta^n(w)$, so γ establishes an isometry/similarity between (V, E) and (W, F). □

We are now going to associate with each regular polygon (V, E) a certain mapping $f = f_{(V,E)}$ from $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ into $\mathbb{R}_+$, the set of non negative real numbers, which determines (V, E) up to isometry.

We begin by representing (V, E) in the form $(\langle\alpha\rangle v, \langle\alpha\rangle\{v, \alpha(v)\})$ according to Proposition 10. Then we consider the (linear) action of α on the real vectorspace T of translations of $\mathbb{E}$ given by $\bar{\alpha}: T \to T: \vartheta \mapsto \bar{\alpha}(\vartheta) := \alpha\vartheta\alpha^{-1}$. Let $\langle|\rangle: T \times T \to \mathbb{R}: (\vartheta_1, \vartheta_2) \mapsto \langle\vartheta_1|\vartheta_2\rangle$ denote the canonical positive definite form on T which is induced from the euclidean metric on $\mathbb{E}$. Since $\bar{\alpha}: T \to T$ preserves this form, the action of $\bar{\alpha}$ is diagonizable once we extend it to an action, also denoted by $\bar{\alpha}$, on the complexification $\mathbb{C} \otimes T$ of T, i.e. $\mathbb{C} \otimes T$ splits canonically into the direct sum of its $\bar{\alpha}$-eigenspaces $(\mathbb{C} \otimes T)_z =: \{\vartheta \in \mathbb{C} \otimes T | \bar{\alpha}(\vartheta) = z \cdot \vartheta\}$, i.e. $\mathbb{C} \otimes T = \bigoplus_z (\mathbb{C} \otimes T)_z$. Moreover, $(\mathbb{C} \otimes T)_z = 0$ unless $z \in S^1$, complex conjugation swaps $(\mathbb{C} \otimes T)_z$ and $(\mathbb{C} \otimes T)_{\bar{z}}$ and the above decomposition is a decomposition into pairwise perpendicular subspaces with respect to the canonical extension of $\langle|\rangle$ to a hermitian form on $\mathbb{C} \otimes T$ which will also be denoted by $\langle|\rangle$.

Now consider the translation $\tau = \tau_v^{\alpha(v)}$ which maps v onto $\alpha(v)$ and its decomposition $\tau = \bigoplus_{z \in S^1} \tau_z \in \bigoplus_{z \in S^1} (\mathbb{C} \oplus T)_z$ into $\bar{\alpha}$-eigenvectors and define $f := f_{(\alpha,v)}: S^1 \to \mathbb{R}_+$ by $f(z) := \langle\tau_z|\tau_z\rangle^{1/2}$.

At first, we observe that $f_{(\alpha,v)} = f_{(\beta,w)}$ whenever there exists some isometry γ with $\gamma v = w$ and $\gamma\alpha\gamma^{-1}|_{<\beta>w} = \beta|_{<\beta>w}$, since these conditions imply $\gamma(\alpha^n(v)) = (\gamma\alpha^n\gamma^{-1})(\gamma(v)) = (\gamma\alpha\gamma^{-1})^n(w) = \beta^n(w)$, so $\bar{\gamma}$ maps the $\bar{\alpha}$-invariant subspace T_v^α of translations which preserve the affine subspace $\langle\langle\alpha\rangle v\rangle$, spanned by $\langle\alpha\rangle v$, isometrically onto the correspondingly defined $\bar{\beta}$-invariant subspace T_w^β, it maps $\tau = \tau_v^{\alpha(v)} \in T_v^\alpha$ onto $\tau' := \tau_w^{\beta(w)} \in T_w^\beta$ and it satisfies $\bar{\gamma}(\bar{\alpha}(\vartheta)) = \bar{\beta}(\bar{\gamma}(\vartheta))$ for $\vartheta \in T_v^\alpha$, so the decomposition $\tau = \bigoplus_{z \in S^1} \tau_z$ of τ into a sum of $\bar{\alpha}$-eigenvectors is mapped by $\bar{\gamma}$ into the corresponding decomposition $\tau' = \bigoplus_{z \in S^1} \tau'_z$ of τ' into a sum of $\bar{\beta}$-eigenvectors, in particular, $f_{(\alpha,v)}(z) = \langle\tau_z|\tau_z\rangle^{1/2} = \langle\tau'_z|\tau'_z\rangle^{1/2} = f_{(\beta,w)}^{(z)}$ for all $z \in S1$, i.e. $f_{(\alpha,v)} = f_{(\beta,w)}$.

It follows, that for $(V, E) = (\langle\alpha\rangle v, \langle\alpha\rangle\{v, \alpha(v)\})$ the map $f_{(\alpha,v)}$ depends only on the polygon (V, E) and that, moreover, it depends only on the isometry class of (V, E). It is easy to see that – vice versa – it also determines this isometry class, since for any $n, m \in \mathbb{Z}$ we can compute the distance $d(\alpha^n(v), \alpha^m(v))$ in terms of $f = f_{(\alpha,v)}$: since $d(\alpha^n(v), \alpha^n(v)) = d(\alpha^{n-m}(v), v) = d(v, \alpha^{am-n}(v))$, we may assume w.l.o.g. that $n > 0 = m$. Consider the translation ϑ_n which maps v into $\alpha^n(v)$ and, hence, satisfies $\langle\vartheta_n|\vartheta_n\rangle = \mathrm{d}(\alpha^n(v), v)^2$. Since ϑ_n is the sum of the translations $\bar{\alpha}^i(\tau)$ $(i = 0, \ldots, n-1)$ which map $\alpha^i(v)$ onto $\alpha^{i+1}(v)$, we have $\vartheta_n = \sum_{i=0}^{n-1} \bar{\alpha}^i(\tau) = \sum_{i=0}^{n-1} \sum_{z \in S^1} z^i \tau_z = n \cdot \tau_1 + \sum_{z \in S^1 \setminus \{1\} v04n} \frac{1 - z^n}{1 - z} \tau_z$ and therefore $d(\alpha^n(v), v)^2 = \langle\vartheta_n|\vartheta_n\rangle = n \cdot f(1)^2 + \sum_{z \in S^1 \setminus \{1\} v04n} \frac{1 - Re(z^n)}{1 - Re(z)} f(z)^2$.

It is also easily seen that the dimension of the affine subspace $\langle V\rangle$, spanned by $V = \langle\alpha\rangle v$, coincides with the cardinality of the support $\mathrm{supp}(f) := \{z \in S^1 | f(z) \neq 0\}$ of f since both coincide with the dimension of the α-invariant subspace $\langle\bar{\alpha}^n(\tau) | n \in \mathbb{Z}\rangle \subseteq T$ of T, generated by $\tau = \tau_v^{\alpha(v)}$.

So we have proved parts of Proposition 11.

PROPOSITION 11. *There is a one to one correspondence between isometry classes of regular polygons in* $\mathbb{E}^n$ *and mappings* $f: S^1 \to \mathbb{R}_+$ *of the unit circle into the set of non negative real numbers which satisfy* $f(z) = f(\bar{z})$ *for all* $z \in S^1$, $\mathrm{supp}(f) \nsubseteq \{-1\}$ *and* $\#\mathrm{supp}(f) \leqslant n$.

If $f = f_{(V,E)}: S^1 \to \mathbb{R}_+$ *is associated to the regular polygon* (V, E) *with respect to this correspondence, then the following holds:*

(i) *if* $v_0, v_1, \ldots, v_n \in V$; $e_1 = :\{v_0, v_1\}, \ldots, e_n = :\{v_{n-1}, v_n\} \in E$ *and* $e_1 \neq e_2 \neq \ldots \neq e_n$, *then*

$$d(v_0, v_n)^2 = n \cdot f(1)^2 + \sum_{z \in S^1 \setminus \{1\}} \frac{1 - Re(z^n)}{1 - Re(z)} f(z)^2;$$

(ii) *V is bounded if and only if* $f(1) = 0$;
(iii) *V is finite if and only if* $f(1) = 0$ *and* $\mathrm{supp}(f) \subseteq \{\exp(2\pi i \cdot q \,|\, q \in \mathbb{Q}\}$,
(iv) *V is discrete and infinite if and only if* $f(1) \neq 0$, *so a regular polygon is either discrete or bounded;*
(v) (V, E) *is similar to* (W, F) *if and only if there is some* $\lambda > 0$ *with* $f_{(V,E)} = \lambda f_{(W,F)}$;
(vi) *the dimension of the affine subspace* $\langle V \rangle$, *spanned by V, coincides with* $\#\mathrm{supp}(f)$.

Proof. As above, let T denote the real vectorspace of translations of $\mathbb{E}$. We have shown already how to associate to a regular polygon (V, E) a map $f = f_{(V,E)}: S^1 \to \mathbb{R}_+$ which characterizes (V, E) up to isometry by representing (V, E) in the form $(V, E) = (\langle \alpha \rangle v, \langle \alpha \rangle \{v, \alpha(v)\})$ and studying the decomposition of the translation $\tau = \tau_v^{\alpha(v)}$ with $\tau(v) = \alpha(v)$ into $\bar{\alpha}$-eigenvectors $\tau = \bigoplus_{z \in S^1} \tau_z$ with respect to the linear action $\bar{\alpha}: T \mapsto T\vartheta \mapsto \alpha\vartheta\alpha^{-1}$. Since complex conjugation in $\mathbb{C} \otimes T$ fixes τ, it maps τ_z onto $\tau_{\bar{z}}$, so we have $f(z) := \langle \tau_z | \tau_z \rangle^{1/2} = \langle \tau_{\bar{z}} | \tau_{\bar{z}} \rangle^{1/2} = f(\bar{z})$. Since $\mathrm{supp}(f)$ is contained in the set Λ_α of eigenvalues of the linear operator $\bar{\alpha}$, we have $\#\mathrm{supp}(f) \leqslant n$. Since $\alpha^2(v) \neq v$, we have $\langle \alpha\tau\alpha^{-1})(\alpha v) = \alpha(\tau(v)) = \alpha^2(v) \neq v = \tau^{-1}(\tau v) = \tau^{-1}(\alpha v)$ and, hence, $\alpha\tau\alpha^{-1} \neq \tau^{-1} = -\tau$, i.e. $\tau \neq \tau_{-1}$ which implies $\tau_z \neq 0$ for some $z \in S^1 \setminus \{-1\}$, i.e. $\mathrm{supp}(f) \not\subset \{-1\}$. So, the map $f = f_{(V,E)}: S^1 \to \mathbb{R}_+$, associated with a regular polygon (V, E) satisfies indeed the special conditions stated in Proposition 11.

Assertions (i) and (vi) have also been established already and the assertions (ii)–(v) follow immediately from (i).

So it remains to show that for any map $f: S^1 \to \mathbb{R}_+$ with $f(z) = f(\bar{z})$, $\mathrm{supp}(f) \not\subset \{-1\}$

and $\#\text{supp}(f) \leqq n$ there exists a regular polygon (V, E) with $f = f_{(V,E)}$. But, at least, there exists some isometry α with $\text{supp}(f) \subseteq \Lambda_\alpha$ and so, this follows from Proposition 12.

PROPOSITION 12. *If* $\alpha \in \text{Iso}(\mathbb{E})$ *and if* $f: \mathbb{R}_+$ *satisfies* $f(z) = f(\bar{z})$ $(z \in S^1)$ *as well as* $\text{supp}(f) \not\subset \{-1\}$, *then there exists some* $v \in E \backslash \mathbb{E}^{\alpha^2}$ *with* $f = f_{(\alpha,v)}$ *if and only if* $\text{supp}(f)$ *is contained in the set* Λ_α *of eigenvalues of* α *and* $f(1) = m(\alpha) := \inf(d(w, \alpha(w)) | w \in \mathbb{E})$.

Moreover, for each $\alpha \in \text{Iso}(\mathbb{E})$ *there exists some* $w \in \mathbb{E}$ *with* $d(w, \alpha(w)) = m(\alpha)$, *in particular one has* $m(\alpha) = 0$ *if and only if* $\mathbb{E}^\alpha \neq \varnothing$, *and if* $\mathbb{E}^\alpha \neq \varnothing$ *and* $\vartheta \in T^\alpha$, *then* $m(\alpha\vartheta) = \langle \vartheta / \vartheta \rangle^{1/2}$, *so, if f satisfies all of the above conditions except perhaps* $f(1) = m(\alpha)$, *one can always replace* α *by the composition* $\alpha\vartheta$ *of* α *with some translation* ϑ *so that* $f(1) = m(\alpha\vartheta)$ *in which case there exists some* $v \in \mathbb{E} \backslash \mathbb{E}^{(\alpha\vartheta)^2}$ *with* $f = f_{(\alpha\vartheta, v)}$.

REMARK. This is, of course, in accordance with the well known fact that $\mathbb{E}^\alpha \neq \varnothing$ if $1 \notin \Lambda_\alpha$ and that α satisfies $\alpha^2 = 1$ if and only if $\Lambda_\alpha = \{-1\}$ or $\Lambda_\alpha \subseteq \{\pm 1\}$ and $\mathbb{E}^\alpha \neq \varnothing$.

Proof. If $v \in \mathbb{E} \backslash \mathbb{E}^{\alpha^2}$, $w \in \mathbb{E}$, $\tau = \tau_v^{\alpha(v)}$, $\vartheta = \tau_v^w$ and $f = f_{(\alpha,v)}$, then $\alpha(w) = \alpha(\vartheta(v)) = (\bar{\alpha}(\vartheta))(\alpha(v)) = (\bar{\alpha}(\vartheta))(\tau(\vartheta^{-1}(w))$ shows that $\tau_w^{\alpha(w)} = \bar{\alpha}(\vartheta) - \vartheta + \tau$, so we have $d(w, \alpha(w)) = \langle \bar{\alpha}(\vartheta) - \vartheta + \tau | \bar{\alpha}(\vartheta) - \vartheta + \tau \rangle^{1/2} = (\langle \bar{\alpha}(\vartheta) - \vartheta + \bigoplus_{z \neq 1} \tau_z | \bar{\alpha}(\vartheta) - \vartheta + \bigoplus_{z \neq 1} \tau_z > + \langle \tau_1 | \tau_1 \rangle)^{1/2} \geqq f(1)$, since $\langle \bar{\alpha}(\vartheta) - \vartheta | \tau_1 \rangle = \langle \bar{\alpha}(\vartheta) | \tau_1 \rangle - \langle \vartheta | \tau_1 \rangle = \langle \vartheta | \bar{\alpha}^{-1}(\tau_1) \rangle - \langle \vartheta | \tau_1) \rangle = \langle \vartheta | \tau_1 \rangle - \langle \vartheta | \tau_1 \rangle = 0$, whereas $d(w, \alpha(w)) = f(1)$ for $\vartheta = \bigoplus_{z \neq 1} \frac{1}{1-z} \tau_z \in T = 1 \otimes T \subseteq \mathbb{C} \otimes T$.

This computation shows also that $m(\alpha) = \inf(d(w, \alpha(w)) | w \in \mathbb{E})$ is always assumed by some $w \in \mathbb{E}$, in particular $m(\alpha) = 0$ if and only if $\mathbb{E}^\alpha \neq \varnothing$, and that $d(w, \alpha(w)) = m(\alpha)$ if and only if $\tau_w^{\alpha(w)}$ is fixed by $\bar{\alpha}$. It is also clear that $\text{supp}(f_{(\alpha,v)}) \subseteq \Lambda_\alpha$.

Vice versa, if $f(1) = \inf(\text{d}(w, \alpha(w)) | w \in \mathbb{E}) = \text{d}(v, \alpha(v))$ for some $v \in \mathbb{E}$ — so we have $\bar{\alpha}(\tau_v^{\alpha(v)}) = \tau_v^{\alpha(v)}$ — and, if $\text{supp}(f) \subseteq \Lambda_\alpha$ for some $f: \text{S}^1 \to \mathbb{R}_+$ with $\text{supp}(f) \nsubseteq \{-1\}$ and $f(z) = f(\bar{z})(z \in S^1)$, then we can find for each $z \in \text{supp}(f) \backslash \{1\}$ some $\vartheta_z \in (\mathbb{C} \otimes T)_z$ with $\langle \vartheta_z | \vartheta_z \rangle = f(z)$ such that ϑ_z coincides with the complex conjugate of $\vartheta_{\bar{z}}$ for all $z \in \text{supp}(f)$, in which case there exists some $\vartheta \in T$ with $\vartheta = 1 \otimes \vartheta = \bigoplus_{z \in \text{supp}(f) \backslash \{1\}} \frac{1}{z-1} \vartheta_z$ and $w = \vartheta(v)$ satisfies $w \in \mathbb{E} \backslash \mathbb{E}^{\alpha^2}$ and $f = f_{(\alpha,w)}$.

Finally, if $w \in \mathbb{E}^\alpha \neq \emptyset$ and $\vartheta \in T^\alpha$, then $\alpha\vartheta = \vartheta\alpha$ and $\alpha(\eta(w)) = \bar{\alpha}(\eta)w$ for $\eta \in T$, so one has

$$m(\alpha\vartheta) = m(\vartheta\alpha) = \inf(d(u, \vartheta(\alpha(u)))\,|\,u \in \mathbb{E}) = \inf(d(\eta(w), \vartheta(\alpha(\eta(w)))))\,|\,\eta \in T) =$$
$$= \inf(\langle -\eta + \vartheta + \bar{\alpha}(\eta)\,|\,-\eta + \vartheta + \bar{\alpha}(\eta)\rangle^{1/2}) =$$
$$= \inf(\langle \bar{\alpha}(\eta) - \eta\,|\,\bar{\alpha}(\eta) - \eta\rangle + \langle \vartheta\,|\,\vartheta\rangle)^{1/2}\,|\,\eta \in T) = \langle \vartheta\,|\,\vartheta\rangle^{1/2}.$$

In particular, if $\mathbb{E} = \mathbb{E}^3$ is the three dimensional euclidean space, then we have one similarity class of one-dimensional polygons, associated with the mappings $f: S^1 \to \mathbb{R}_+$ with support $\{1\}$, we have a one-parameter family of similarity classes of planar polygons associated with the mappings of support $\{+1, -1\}$, all of which are discrete and infinite, for each $z = x + iy \in S^1$ with $y > 0$ there exists one similarity class of planar polygons (V, E) associated with the mappings $f: S^1 \to \mathbb{R}_+$ with support $\{z, \bar{z}\}$ (and (V, E) is finite if and only if $z \in \{\exp(2\pi i \cdot q\,|\,q \in \mathbb{Q}\})$ and there exist two one-parameter families of similarity classes of non-planar polygons (V, E), associated with the mappings $f: S^1 \to \mathbb{R}_+$ with support $\{1, z, \bar{z}\}$ or $\{-1, z, \bar{z}\}$, respectively. In the first case all polygons are discrete and infinite, in the second case all polygons are bounded and they are finite if and only if $z \in \{\exp(2\pi i \cdot q\,|\,q \in \mathbb{Q}\}$.

And, if $\alpha \in \mathrm{Iso}(\mathbb{E})$, $v \in \mathbb{E} \backslash \mathbb{E}^{\alpha^2}$, $(V, E) = (\langle\alpha\rangle v, \langle\alpha\rangle\{v, \alpha(v)\})$, $\dim(\langle V\rangle) \geqslant 2$ and $f = f_{(\alpha,v)} = f_{(V,E)}$, then in case $\mathbb{E}^\alpha = \emptyset$ one has either $\dim(\langle V\rangle) = 2$ and $\Lambda_\alpha = \mathrm{supp}(f) = \{+1, -1\}$ or $\dim(\langle V\rangle) = 3$ and $\Lambda_\alpha = \mathrm{supp}(f) = \{1, z, \bar{z}\}$ for some $z \in S^1 \backslash \{+1, -1\}$, whereas in case $\mathbb{E}^\alpha \neq \emptyset$ there exists some $z \in \Lambda_\alpha \backslash \{+1, -1\}$, and one has either $\dim(\langle V\rangle) = 2$ and $\mathrm{supp}(f) = \{z, \bar{z}\}$ or $\dim(\langle V\rangle) = 3$ and $\mathrm{supp}(f) = \Lambda_\alpha = \{-1, z, \bar{z}\}$ and (V, E) is finite if and only if α is of finite order if and only if $z^n = 1$ for some $n > 0$.

§10. Comparison to Branko Grünbaum's list of regular polyhedra

We are now well prepared to compare our list of Grünbaum systems to the list of regular polyhedra given in [5]. There the classification scheme is in terms of the polygon and the vertex figure associated with a regular polyhedron.

Let us restrict our attention to the non planar case. So let $\Pi = (V_0, V_1, V_2)$ be a regular, non planar polyhedron in the sense of [5], let $(v_0, v_1, v_2) \in V_0 \times V_1 \times V_2$ be a Π-flag and let $(v = v_0; \alpha_0, \alpha_1, \alpha_2)$ denote the associated Grünbaum system. Then $P_0 =: (\bigcup_{w_1 \in v_2} w_1, v_2)$ is a regular polygon which coincides with $(\langle\alpha_0\alpha_1\rangle, \langle\alpha_0\alpha_1\rangle\{v, \alpha_0(v)\})$ and whose isometry class depends only on Π and not on the element $v_2 \in V_2$ and $P_1 := (\{w \in V_0\,|$ there exists a sequence of flags $(v_0, v_1^i, v_2^i) - i = 0, 1, \ldots, n -$ with $v_1^0 = v_1$, $v_2^0 = v_2$, $v_1^n = \{v, w\}$ and either $v_1^i = v_1^{i-1}$ or $v_2^i = v_2^{i-1}$ for $i = 1, \ldots, n\}$, $\{\{w, w'\} \in \mathscr{P}_2(V_0)\,|$there exists a sequence of flags $(v_0, v_1^i, v_2^i) - i = 0, 1, \ldots, n$—with $v_1^0 = v_1$, $v_2^0 = v_2$, $v_1^{n-1} = \{v, w\}$, $v_1^n = \{v, w'\}$ and either $v_1^i = v_1^{i-1}$ or $v_2^i = v_2^{i-1}$ for $i = 1, \ldots, n\})$

is a regular polygon, too, which coincides with $(\langle\alpha_1\alpha_2\rangle(\alpha_0(v)), \langle\alpha_1\alpha_2\rangle\{\alpha_0(v), \alpha_1\alpha_0(v)\})$ and whose isometry class depends also only on Π and not on the flag (v_0, v_1, v_2). The classification of regular polyhedra, given in [5], is in terms of the isometry classes of these two polygons. Let us therefore compute the support of $f_0 := f_{(\alpha_0\alpha_1, v)}$ and of $f_1 := f_{(\alpha_1\alpha_2, \alpha_0(v))}$. Since $\langle\alpha_0\alpha_1\rangle v$ cannot be contained in a straight line $\mathbb{F}$, since otherwise $\langle\alpha_0\alpha_1\rangle v \subseteq \mathbb{F} = \langle v, \alpha_0(v)\rangle \subseteq \mathbb{E}^{\alpha_2}$, which would imply $\langle\alpha_0, \alpha_1, \alpha_2\rangle v \subseteq \mathbb{E}^{\alpha_2}$ in contradiction to $\alpha_2|_{<\alpha_0,\alpha_1,\alpha_2>v} \neq Id|_{<\alpha_0,\alpha_1,\alpha_2>}$, we have necessarily $\mathrm{supp}(f_0) \nsubseteq \{1\}$.

Note also that $\mathbb{E}^{\alpha_0\alpha_1} \neq \emptyset$ if and only if $\mathbb{E}^{\alpha_0} \cap \mathbb{E}^{\alpha_1} \neq \emptyset$, since $\langle\alpha_0, \alpha_1\rangle$ acts as a finite group (of order at most 2) on $\mathbb{E}^{\alpha_0\alpha_1}$. So, if $\dim\alpha_0 = 0$, we have $\Lambda_{\alpha_0\alpha_1} = \{+1, -1\}$ and $\mathrm{supp}(f_0) = \{+1, -1\}$, i.e. P_0 is a zigzag polygon, and if $\dim\alpha_0 = \dim\alpha_1 \geqslant 1$ and $0° < \measuredangle(\alpha_0, \alpha_1) =: \phi < 90°$ we have $\Lambda_{\alpha_0\alpha_1} = \{1, \cos 2\varphi + i\cdot\sin 2\varphi, \cos 2\varphi - i\cdot\sin 2\varphi\}$, so we have $\mathrm{supp}(f_0) = \{\cos 2\varphi \pm \sin 2\varphi\}$ if $\mathbb{E}^{\alpha_0} \cap \mathbb{E}^{\alpha_1} \neq \emptyset$ — which, by the way, is equivalent to $\dim\alpha_0 = \dim\alpha_2 = 2$ — and leads to planar polygons which are "convex" if and only if 2φ divides $360°$, so it leads to convex polygons unless $\varphi = 72°$, whereas in case $\mathbb{E}^{\alpha_0} \cap \mathbb{E}^{\alpha_1} = \emptyset$ — which, by the way, is equivalent to $\dim\alpha_0 = \dim\alpha_1 = 1$ — we have $\mathrm{supp}(f_0) = \{1, \cos 2\varphi \pm i\cdot\sin 2\varphi\}$ which leads to helical polygons. Finally, if $\dim\alpha_0 \neq \dim\alpha_1$ and $0 < \varphi := 90° - \measuredangle(\alpha_0, \alpha_1) < 90°$, then $\Lambda_{\alpha_0\alpha_1} = \{-1, \cos 2\varphi \pm i\cdot\sin 2\varphi\}$ and $-1 \notin \mathrm{supp}(f_0)$ if and only if $v \in \mathbb{E}^{\alpha_1}$ is contained in the plane which contains the one-dimensional subspace among $\mathbb{E}^{\alpha_0}$ and $\mathbb{E}^{\alpha_1}$ and is perpendicular to the two dimensional one. So, if $\dim\alpha_1 = 1 < \dim\alpha_0 = 2$, we have $\mathrm{supp}(f_0) = \{\cos 2\varphi \pm i\cdot\sin 2\varphi\}$, whereas in case $\dim\alpha_1 > \dim\alpha_0 = 1$ we have $\mathrm{supp}(f_0) = \{-1, \cos 2\varphi \pm i\cdot\sin 2\varphi\}$ since in this case the plane containing v and $\mathbb{E}^{\alpha_0}$ is necessarily invariant under α_2, so for a non planar polyhedron this plane cannot be perpendicular to $\mathbb{E}^{\alpha_1}$.

Let us now consider $\mathrm{supp}(f_1)$. Since $v \in \mathbb{E}^{\alpha_1} \cap \mathbb{E}^{\alpha_2} \neq \emptyset$, we have necessarily $f_1(1) = 0$, so in case $\dim\alpha_1 = \dim\alpha_2$ and hence $-1 \notin \Lambda_{\alpha_1\alpha_2}$ we have necessarily $\mathrm{supp}(f_1) = \{\cos 2\psi \pm i\cdot\sin 2\psi\}$ for $\psi := \measuredangle(\alpha_1, \alpha_2)$ whereas in case $\dim\alpha_1 \neq \dim\alpha_2$ we have—with $\psi := 90° - \measuredangle(\alpha_1, \alpha_2) - \{\cos 2\psi \pm i\cdot\sin 2\psi\} \subseteq \mathrm{supp}(f_1) \subseteq \Lambda_{\alpha_1\alpha_2} = \{-1, \cos 2\psi \pm i\cdot\sin 2\psi)$ with $-1 \notin \mathrm{supp}(f_1)$ if and only if $\alpha_0(v) \in \mathbb{E}^{\alpha_2}$ is contained in the plane which contains the one-dimensional subspace among the two subspaces $\mathbb{E}^{\alpha_1}$ and $\mathbb{E}^{\alpha_2}$ and is perpendicular to the two dimensional one. So we have $-1 \notin \mathrm{supp}(f_1)$ if $\dim\mathbb{E}^{\alpha_2} = 1$, whereas in case $\dim\mathbb{E}^{\alpha_1} = 1 < \dim\mathbb{E}^{\alpha_2} = 2$ the plane containing $\mathbb{E}^{\alpha_1}$ and $\alpha_0(v)$ cannot be perpendicular to $\mathbb{E}^{\alpha_2}$ in the non planar case, so we have $\mathrm{supp}(f_1) = \{-1, \cos 2\psi \pm i\cdot\sin 2\psi\}$ in this case.

It is now easy to identify the Grünbaum systems associated to the various classes of regular polyhedra described in [1]. We summarize the result in the following table:

	Grünbaum's description	dim α_0	dim α_1	dim α_2	$\sphericalangle(\alpha_0, \alpha_1)$	$\sphericalangle(\alpha_1, \alpha_2)$
class 1:	Platonic polyhedra	2	2	2	$\neq 72°$	$\neq 72°$
	{3, 3} tetrahedron				60°	60°
	{3, 4} octahedron				60°	45°
	{4, 3} cube				45°	60°
	{3, 5} icosahedron				60°	36°
	{5, 3} dodecahedron				36°	60°
class 2:	planar tesselations	1	1	1	$\sphericalangle(\alpha_0, \alpha_1) + \sphericalangle(\alpha_1, \alpha_2) = 90°$	
	{4, 4}				45°	45°
	{3, 6}				60°	30°
	{6, 3}				30°	60°
class 3:	Kepler-Poinsot polyhedra	2	2	2	$72° \in \sphericalangle(\alpha_0, \alpha_1)$,	$\sphericalangle(\alpha_1, \alpha_2)$
	{5, 5/2} great dodecahedron				36°	72°
	{3, 5/2} great icosahedron				60°	72°
	{5/2, 5} small stellated dodecahedron				72°	36°
	{5/2, 3} great stellated dodecahedron				72°	60°
class 4:	Petrie-Coxeter polyhedra	2	1	2		
	$\{4, 6^{\pi/3}/1\}$				45°	30°
	$\{6, 4^{48°12'}/1\}$				30°	45°
	$\{6, 6^{33°33'}/1\}$				30°	30°
class 5:	Finite regular polyhedra with finite skew polygons	1	2	2		
	$\{4^{\pi/3}/1, 3\}$				45°	60°
	$\{6^{\pi/3}/1, 4\}$				30°	45°
	$\{6^{\pi/2}/1, 4\}$				30°	60°
	$\{10^{\pi/3}/1, 5\}$				18°	36°
	$\{6^{\pi/5}/1, 5\}$				30°	36°
	$\{6^{3\pi/5}/1, 5/2\}$				30°	72°
	$\{10^{\pi/3}/3, 5/2\}$				54°	72°
	$\{10^{3\pi/5}/1, 3\}$				18°	60°
	$\{10^{\pi/5}/3, 3\}$				54°	60°
class 6:	Infinite polyhedra with finite skew polygons	1	2			
	three infinite families of similarity classes	1	2	2		
	$\{4^{\alpha}/1, 4\}$				45°	45°
	$\{6^{\alpha}/1, 3\}$				30°	60°
	$\{2.3^{\alpha}/1, 6\}$				60°	30°
	three individual similarity classes	1	2	1		
	$\{6^{\pi/3}, 6\}$				30°	30°
	$\{4^{\pi/3}/1, 6\}$				45°	30°
	$\{6^{\pi/2}/1, 4\}$				30°	45°

	Grünbaum's description	$\dim\alpha_0$	$\dim\alpha_1$	$\dim\alpha_2$	$\sphericalangle(\alpha_0, \alpha_1)$	$\sphericalangle(\alpha_1, \alpha_2)$
class 7:	Regular polyhedra with					
	zig-zag polygons	0				
	three infinite families with					
	an invariant plane	0	2	2		
	$\{\infty^{\alpha}, 4\}$				0°	45°
	$\{\infty^{\alpha}, 3\}$				0°	60°
	$\{\infty^{\alpha}, 6\}$				0°	30°
	three infinite families without					
	an invariant plane					
	$\{\infty^{\alpha(b)}, 4^{\alpha*(b)}/1\}$	0	1	2	0°	45°
	$\{\infty^{\gamma(b)}, 6^{\gamma*(b)}/1\}$	0	1	2	0°	30°
	$\{\infty^{\delta(b)}, 2.3^{\delta*(b)}/1\}$	0	1	2	0°	60°
class 8:	Polyhedra with helical polygons	1	1			
	three infinite families	1	1	2		
	$\{\infty^{\alpha(b),\pi/2}, 4^{\alpha*(b)}/1\}$				45°	45°
	$\{\infty^{\gamma(b),2\pi/3}, 6^{\gamma*(b)}/1\}$				60°	30°
	$\{\infty^{\delta(b),\pi/3}, 2.3^{\delta*(b)}/1\}$				30°	60°
	three individual similarity types	1	1	2		
	$\{\infty^{\pi/2,2\pi/3}, 6^{\pi/3}/1\}$				60°	30°
	$\{\infty^{2\pi/3,\pi/2}, 6^{\alpha**}/1\}$				45°	30°
	$\{\infty^{2\pi/3,2\pi/3}, 4^{\pi/3}/1\}$				60°	45°
	three individual similarity types	1	1	1		
	$\{\infty^{2\pi/3,\pi/2}, 3\}$				45°	60°
	$\{\infty^{2\pi/3,2\pi/3,}\ 3\}$				60°	60°
	$\{\infty^{\pi/2,2\pi/3}, 4\}$				60°	45°

From this correspondence it follows easily that all Grünbaum systems do indeed correspond to regular polyhedra except perhaps in case $\dim\alpha_2 = \dim\alpha_1 = \dim\alpha_0 = 1$, $\sphericalangle(\alpha_0, \alpha_1) = \sphericalangle(\alpha_0\alpha_2, \alpha_1) = 60°$, $\sphericalangle(\alpha_1, \alpha_2) = 45°$ which has not been considered in [5]. But the cubic lattice $\mathbb{Z}^3 \subseteq \mathbb{R}^3$ is preserved under the 180°-rotations α_0, α_1 and α_2 around the lines $\mathbb{F}_0 = (0, 0, 1/2) + \mathbb{R}\cdot(0, -1, 1)$, $\mathbb{F}_1 = (0, 0, 0) + \mathbb{R}\cdot(-1, 0, 1)$ and $\mathbb{F}_2 = (0, 0, 0) + \mathbb{R}\cdot(1, 0, 0)$ from which fact one can easily deduce that the Grünbaum system $(v; \alpha_0, \alpha_1, \alpha_2)$ with $\dim\alpha_0 = \dim\alpha_1 = \dim\alpha_2 = 1$, $\sphericalangle(\alpha_0, \alpha_1) = 60°$, $\sphericalangle(\alpha_1, \alpha_2) = 45°$ corresponds to a regular polyhedron, too, which then is described by $\{\infty^{\pi/2,2\pi/3}, 4\}$ according to Grünbaum's terminology.

Let us finally mention two open problems in this context.

(1) Does there exist a (finite) subgroup $G \leqq Iso(\mathbb{E})_v$ for some $v \in \mathbb{E} = \mathbb{E}^n$ and some element $\alpha \in Iso(\mathbb{E}) \setminus Iso(\mathbb{E})_v$ (with $\alpha^2 = 1$) such that $\langle G, \alpha\rangle_v \neq G$ (and $\langle G, \alpha\rangle$ is discrete)?

(2) Given finitely many subspaces $\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_k \subseteq \mathbb{E} = \mathbb{E}^n$,

(a) find necessary and sufficient conditions for the discreteness/finiteness of the group, generated by the involutions $\alpha_1, \alpha_2, \ldots, \alpha_k$ with $\mathbb{E}^{\alpha_i} = \mathbb{F}_i$ $(i = 1, 2, \ldots, k)$.

(b) Given a second sequence of subspaces $\mathbb{F}'_1, \mathbb{F}'_2, \ldots, \mathbb{F}'_k \subseteq \mathbb{E}$, find necessary and sufficient conditions for the existence of an isometry/similarity γ with $\gamma(\mathbb{F}_i) = \mathbb{F}'_i$ $(i = 1, \ldots, n)$.

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