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Conditioning Convex and Nonconvex Problems[1]

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Abstract. Two ways of defining a well-conditioned minimization problem are introduced and related, with emphasis on the quantitative aspects. These concepts are used to study the behavior of the solution sets of minimization problems for functions with connected sublevel sets, generalizing results of Attouch–Wets in the convex case. Applications to continuity properties of subdifferentials and to projection mappings are pointed out.

Key Words. Best approximations, conditioners, conditioning functions, gages, hemicontinuity, inf-connected functions, marginal functions, performance functions, projections, quasi-inverses, sensitivity analysis, solution sets, value functions.

1. Introduction

The notion of a well-posed optimization problem is important and has been given several variants; see Refs. 1 to 6, and in particular Refs. 1 and 6 for comprehensive treatments. Besides the qualitative results presented in the just quoted papers, one may wish to dispose of quantitative estimates. This may be useful for the study of the speed of convergence of algorithms. Generally speaking, it is well known that the notion of conditioning is important in numerical analysis and statistics, see for instance Ref. 7.

We introduce here two functions which describe the conditioning of a minimization problem. It appears that they are quasi-inverse functions in the sense of Refs. 8 and 9; the precise meaning of this inversion property is recalled in Section 2.

[1]We are grateful to M. Valadier for pointing out, during a lecture by the author in Montpellier in October 1990 presenting the main results of the present paper, that existence results in Section 2 of the present paper can be dissociated from estimates.

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Using these tools, we are able to devise estimates for the variation of the infimum of a function and for the variation of its set of minimizers (Section 3) when the function varies in the set of inf-connected functions, i.e., functions whose sublevel sets are connected. Such a class of functions has gained wide interest as it is a natural generalization of the class of quasiconvex functions (Refs. 10–12). Here, we get a true Lipschitzian property for the variation of the infimum in terms of the function, whereas a Stepanov property or stability property in the sense described below was obtained in Ref. 13. We show by an example that the situation is different for the sets of minimizers: their Stepanov property cannot be improved to a Lipschitz property. Recall that a mapping F: $M \to M'$ between two metric spaces M, M' is said to be *stable* (or Stepanoff) at $x_0 \in M$ (Refs. 14, 15) if there exist $c > 0$ and a neighborhood N of x_0 in M such that, for any $x \in N$, one has

$$d(F(x), F(x_0)) \leq cd(x, x_0).$$

It is stable if it is stable at any point of M. Such a property is weaker than a Lipschitzian property, but is still useful; see Ref. 14, for instance. Note that here as in Ref. 13 we do not use a true metric on the space M of extended real-valued functions on the normed vector space (n.v.s.) X; we prefer to use the more natural family of polymetrics $(d_r)_{r>0}$ of bounded hemiconvergence (see Section 2) described or used in Refs. 16–25, which is more convenient than the genuine metric inducing the same topology (Ref. 21).

Section 4 is devoted to an application to the hemicontinuity (continuity in the sense of the Hausdorff metric or Hausdorff hemimetrics) of subdifferentials of convex functions. In Section 5, we consider another application: we estimate the variation of the projection of a fixed given point on a variable closed convex subset of a uniformly convex Banach space. It has been shown in Ref. 13 that, in the Hilbert case, the behavior is of Hölderian type rather than of Lipschitzian type. Here, we relate the estimate that we present to a geometrical characteristic of the space, its modulus of uniform convexity.

2. Conditioning Gages and Conditioning Modulus

In the sequel, f, g are extended real-valued functions on a metric space (X, d). We denote by P [resp. R_+] the set of positive [resp. nonnegative] real numbers, and we denote by

$$m_f := \inf_X f,$$

$$S_f := \{x \in X : f(x) = m_f\},$$

the infimum of f and the set of minimizers of f, respectively.

For $\epsilon \in \mathrm{R}_+$, the ϵ-approximate solution set of f is

$$S_f(\epsilon) := \{x \in X : f(x) \leq m_f + \epsilon\}.$$

If A is a subset of X and $x \in X$, $\epsilon \in R_+$, we set

$$d(x, A) := \inf_{a \in A} d(x, a),$$

$$U(A, \epsilon) := \{x \in X : d(x, A) < \epsilon\},$$

$$B(A, \epsilon) := \{x \in X : d(x, A) \leq \epsilon\}.$$

If $A = \{a\}$, we write $U(a, \epsilon)$ [resp. $B(a, \epsilon)$] instead of $U(A, \epsilon)$ [resp. $B(A, \epsilon)$]. When X has a fixed base point 0, in particular when X is a normed vector space (n.v.s.), we set $B_\epsilon = B(0, \epsilon)$.

For two subsets C, D of X and $r \in \mathrm{R}_+$, we set

$$e_r(C, D) := \sup\{d(x, D) : x \in C \cap B_r\},$$

$$d_r(C, D) := \max(e_r(C, D), e_r(D, C)),$$

with the usual convention

$$\inf \varnothing = \infty, \qquad \sup \varnothing = 0 \text{ in } \mathrm{R}_+.$$

These quantities have been introduced by Kato to study the variation of vector subspaces; they are also mentioned in the work of Mosco (Ref. 16). The use of these local excesses (or hemimetrics) and distances for dealing with analytical and geometrical operations has been pointed out in Ref. 17; subsequently, they received a great deal of attention; see for instance Refs. 18–25.

In particular, it has been shown that the Legendre–Young–Fenchel transform is continuous for the topology which they induce (Refs. 22, 23), and it has been shown in Ref. 24 that they satisfy the following collective triangle inequality which justifies the name "polymetrics": given nonempty subsets C, D, E of X, there exists $r_0 \in \mathrm{R}_+$ such that, for $r \geq r_0$, one has

$$d_r(C, E) \leq d_{3r}(C, D) + d_{3r}(D, E).$$

Identifying a function $f: X \to \bar{\mathrm{R}}$ with its epigraph,

$$E(f) := \{(x, r) \in X \times \mathrm{R} : r \geq f(x)\},$$

the preceding excesses and polymetrics induce hemimetrics and polymetrics on the space $\bar{R}^X$ of extended real-valued functions on X,

$$e_r(f, g) := e_r(E(f), E(g)),$$

$$d_r(f, g) := d_r(E(f), E(g)).$$

Then, one can define a topology, called the topology of bounded hemiconvergence or bounded Hausdorff convergence, on $\bar{R}^X$ by taking as a base of open sets the open balls

$$V_r(f, \epsilon) := \{g \in \bar{R}^X : d_r(f, g) < \epsilon\}.$$

This topology is metrizable and, if X is complete, the space of closed proper functions is complete in the associated metric (Ref. 21). Moreover, when X is a Hilbert space, the topology of bounded hemiconvergence just described coincides with the topology of Attouch–Wets defined through infimal convolution (Ref. 26).

Given an extended real-valued function $f: X \to \bar{R}$ such that $m_f \in R$ and S_f is nonempty, a function $\varphi: R_+ \to \bar{R}_+ := R_+ \cup \{\infty\}$ is said to be a *conditioner* for f if

$$d(x, S_f) \leq \varphi(f(x) - m_f), \qquad \forall x \in X.$$

Such a function is of interest for numerical purposes, since it yields an estimate of the error on the current estimate of an algorithm in terms of the error on the value, assuming that m_f is known. In general, it is not easy to determine a conditioner; it is usually easier to determine a function $\gamma: R_+ \to \bar{R}_+$ such that

$$\gamma(d(x, S_f)) \leq f(x) - m_f, \qquad \forall x \in X.$$

Such a function will be called a *growth function* (Ref. 13). In Ref. 13, the largest one is called the radial regularization of f. We prefer to use the largest nondecreasing function γ_f satisfying this inequality. We call it the canonical growth function of f. It is obviously given by

$$\gamma_f(r) = \inf\{f(x) - m_f : x \in X, d(x, S_f) \geq r\}.$$

When γ_f is a gage, i.e., is positive on $P :=]0, \infty[$, we call it the canonical growth gage of f. Let us recall that a function γ is a gage if it is nondecreasing and *firm* (or admissible or forcing) in the following sense: any sequence (r_n) of R_+ such that $\lim_n \gamma(r_n) = 0$ converges to 0. Here, we note that, for a nondecreasing function γ, firmness is equivalent to the property that γ is positive on P.

Similarly, one can introduce the smallest conditioner and the smallest nondecreasing conditioner μ_f of f, owing to the fact that the infimum of a family of conditioners is still a conditioner. Let us observe that the latter is given by

$$\mu_f(r) := e(S_f(r), S_f)$$
$$:= \sup\{d(x, S_f): x\in X, f(x)\leq m_f + r\}.$$

We call it the canonical conditioner. We are interested in situations in which μ_f is a modulus, i.e., when μ_f is continuous at 0 with value 0 at 0.

A direct proof of the following result is easy; however, in the proof below, we will use some material from Ref. 8 in order to get connections as precise as possible.

Proposition 2.1. For any function $f: X\to \mathrm{R}^{\bullet} := \mathrm{R}\cup\{\infty\}$ with finite infimum m_f and nonempty set of minimizers S_f, the following conditions are equivalent:

(a) f has a conditioner which is a modulus;
(b) the canonical conditioner μ_f of f is a modulus;
(c) the canonical growth function γ_f of f is a gage;
(d) there exists a growth function which is a gage;
(e) f is metrically well set in the sense of Ref. 8:

$$\lim_{\epsilon\to 0} e(S_f(\epsilon), S_f) = 0.$$

Clearly, (b) and (e) are the same statement with a different phrasing. In order to prove the other equivalences, it will be useful to recall from Refs. 8 and 9 some facts about the generalized inversion of nondecreasing functions. If $F: X \rightrightarrows Y$ is a relation, we denote by

$$F^{-1} = \{(y, x)\in Y\times X: (x, y)\in F\}$$

its inverse relation.

Given a nondecreasing function $\varphi: \mathrm{R}_+ \to \bar{\mathrm{R}}_+$, let us denote by $E(\varphi)$ and $H(\varphi)$ its epigraph and its hypograph, respectively,

$$E(\varphi) = \{(r, s)\in \mathrm{R}\times\mathrm{R}: \varphi(r)\leq s\},$$
$$H(\varphi) = \{(r, s)\in \mathrm{R}\times\mathrm{R}: \varphi(r)\geq s\}.$$

We observe that $H(\varphi)^{-1}$ is not an epigraph but is a pseudo-epigraph in the sense that, if $(r, s)\in H(\varphi)^{-1}$ and if $t\geq s$, then $(r, t)\in H(\varphi)^{-1}$. Thus, its vertical closure is an epigraph. More precisely, $H(\varphi)^{-1}$ is the epigraph of the function φ^e given by

$$\varphi^e(s) = \inf\{r\in \mathrm{R}_+ : (r, s)\in H(\varphi)^{-1}\}.$$

Similarly $E(\varphi)^{-1}$ is a pseudo-hypograph for the function φ^h given by

$$\varphi^h(s)=\sup\{r\in \mathrm{R}_+ : (r, s)\in E(\varphi)\}.$$

Let us recall (Refs. 8, 9) that $\psi: \mathrm{R}_+ \to \bar{\mathrm{R}}_+$ is a semi-epi-inverse [resp. a semi-hypo-inverse] of φ if $E(\psi)\subset H(\varphi)^{-1}$ [resp. $H(\psi)\subset E(\varphi)^{-1}$] iff $\psi \geq \varphi^h$ [resp. $\psi \leq \varphi^e$]; see Ref. 9, Proposition 2.6. Recall (Refs. 8 and 9, Proposition 2.8) that, if φ is a gage, then φ^e and φ^h are moduli, a modulus being a nondecreasing function $\mu: \mathrm{R}_+ \to \bar{\mathrm{R}}_+$ such that $\lim_{r\to 0} \mu(r)=0$.

The core of the proof of Proposition 2.1 is contained in the following lemma.

Lemma 2.1.

(a) Suppose that γ is a growth gage for f. Then, any modulus $\mu \geq \gamma^h$, in particular γ^h itself, is a conditioning modulus for f, i.e., a conditioner which is a modulus.

(b) Suppose that μ is a conditioning modulus for f. Then, any gage $\gamma \leq \mu^e$, in particular μ^e itself, is a growth gage for f.

Proof. Let us introduce

$$F=\{(r, s)\in \mathrm{R}_+ \times \mathrm{R}_+ : \exists x\in X, r=f(x)-m_f, s=d(x, S_f)\}.$$

Then, a gage γ is a growth gage for f iff $F^{-1}\subset E(\gamma)$, and a modulus μ is a conditioning modulus for f iff $F\subset H(\mu)$. Therefore, assuming that γ is a growth gage for f, if $\mu \geq \gamma^h$, i.e., if γ is a semi-epi-inverse of μ or if $E(\gamma)\subset H(\mu)^{-1}$, we get $F^{-1}\subset E(\gamma)\subset H(\mu)^{-1}$ and μ is a conditioning modulus of f: $F\subset H(\mu)$. The proof of the second assertion is similar: $\gamma \leq \mu^e$ iff $H(\mu)\subset E(\gamma)^{-1}$, hence $F^{-1}\subset E(\gamma)$ when $F\subset H(\mu)$. □

The following lemma supplements our study of the relationships between the canonical conditioning and the growth functions μ_f and γ_f of f, even when f is not metrically well set.

Lemma 2.2. The function γ_f and μ_f are quasi-inverses: for any r, s in R_+, one has $r \leq \mu_f(s)$, whenever $\gamma_f(r)<s$, and $s\leq \gamma_f(r)$, whenever $\mu_f(s)<r$.

Proof. When $\gamma_f(r)<s$, we can find $x\in X$ with $d(x, S_f)\geq r$ such that $f(x)-m_f<s$, so that $x\in S_f(s)$ and $\mu_f(s)\geq d(x, S_f)\geq r$.

When $\mu_f(s)<r$, for any $x\in X$ with $d(x, S_f)\geq r$, we cannot have $x\in S_f(s)$, so that $f(x)\geq m_f+s$ and $\gamma_f(r)\geq s$. □

Although the preceding lemmas bring an appealing symmetry between conditioning modulus and growth gages, we note a degradation of information when using quasi-inverses: we have only (in view of the definitions of μ_f and γ_f)

$$(\gamma_f)^h \geq \mu_f, \qquad (\mu_f)^e \leq \gamma_f.$$

However, we observe that $(\gamma_f)^h$ is u.s.c. and $(\mu_f)^e$ is l.s.c. These properties are often very useful, as we will see. Moreover, when μ_f [resp. γ_f] is u.s.c. [resp. l.s.c.] the first [resp. second] of the preceding inequalities is an equality (Ref. 8, Proposition 2.2).

Let us observe that, in view of Proposition 2.2 of Ref. 8, Lemma 2.2 implies that the functions γ_f and μ_f are related by the inequalities

$$(\gamma_f)^e \leq \mu_f \leq (\gamma_f)^h,$$

$$(\mu_f)^e \leq \gamma_f \leq (\mu_f)^h.$$

When μ_f is (strictly) increasing, by Proposition 2.2 and Corollary 2.3 of Ref. 8, its quasi-inverses all coincide and are continuous, so that the last inequalities above are equalities. This happens under convexity and uniqueness assumptions (Ref. 13, Proposition 5.2). In fact these assumptions can be relaxed: uniqueness is not needed and starshapedness at each point of S_f can replace convexity. Recall that f is said to be *starshaped* at u if, for any $v \in X$ and any $t \in [0, 1]$, one has

$$f((1-t)u + tv) \leq (1-t)f(u) + tf(v);$$

it is said to be starshaped if it is starshaped at 0 with $f(0) = 0$.

The following result completes and makes more precise Ref. 13, Proposition 5.2; here, S_f is not supposed to be a singleton and the convexity assumption is relaxed.

Proposition 2.2. Suppose that X is a n.v.s., S_f is nonempty, m_f is finite, and f is starshaped at any $x \in S_f$. Then, the radial regularization of f,

$$\psi_f(r) := \inf\{f(x) - m_f : x \in X, d(x, S_f) = r\},$$

is starshaped. Therefore, if $\psi_f(r_0) > 0$ for some $r_0 > 0$, then ψ_f is strictly increasing on $[r_0, \infty[\cap \operatorname{dom} \psi_f$ and coincides with γ_f on this interval. In particular, if ψ_f is a gage, $\gamma_f = \psi_f$.

Proof. Let $r > 0$ and let $t \in]0, 1[$. Suppose we have $\psi(tr) > t\psi(r)$ for $\psi := \psi_f$. Then, we can find $q \in]\psi(r), t^{-1}\psi(tr)[$ and $x \in X$, with $d(x, S_f) = r$, such that $f(x)-m_f<q$. As $\inf_{u\in S_f}(1-\|x-u\|^{-1}(1-t)r)=t$, we can pick $u\in S_f$ such that $(1-\|x-u\|^{-1}(1-t)r)\,(f(x)-m_f)<tq$. Then, for $s\in[0,1]$, we have

$$\begin{aligned}\alpha(s) &:= d((1-s)u+sx,\, S_f)\\ &\geq d(x, S_f)-\|(1-s)(x-u)\|=r-(1-s)\|x-u\|,\end{aligned}$$

and the intermediate value theorem yields some $s_u\in[0, 1-\|x-u\|^{-1}(1-t)r]$ such that $d(y_u, S_f)=tr$ for $y_u:=(1-s_u)u+s_ux$. Then, as f is starshaped at u and $f(u)=m_f$, we have

$$\psi(tr)\leq f(y_u)-m_f\leq s_u(f(x)-m_f)<tq<\psi(tr),$$

a contradiction. The last assertions follow easily. □

Remark 2.1. If we suppose moreover that the closed balls of X are compact for some topology σ for which f is l.s.c. and S_f is compact, then ψ_f is a gage and $\gamma_f=\psi_f$. In general, it is not so. To see that, take an infinite-dimensional space X and a sequence (a_n) of the unit sphere S_X without limit point, and set $f(a_n)=\alpha_n$ with $\alpha_n\in\,]0,1[$, $(\alpha_n)\to 0_+$. Since $A:=\{a_n: n\in N\}$ is closed, we can extend f to a continuous function on S_X with values in $[0,1]$. Adding $d(\,.\,, A)$, if necessary, we may assume that f is positive on S_X and extend it by positive homogeneity to X, with $f(0)=0$. Then, $S_f=\{0\}$, f is starshaped, but $\psi_f(r)=0$ for each $r\in\mathrm{R}_+$.

In the following lemma, we identify a class of functions which will be important for the sequel.

Lemma 2.3. For any $f: X\to\mathrm{R}\cup\{\infty\}$, one has the implications (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) among the following assertions. If X can be endowed with a topology σ for which f is l.s.c., and if the closed balls of X are compact, the three assertions are equivalent:

(a) S_f is bounded and nonempty, and γ_f is not identically 0;
(b) for any $0\in X$, there exists β, $\rho\in\mathrm{P}$ such that $S_f(\beta)\subset B(0,\rho)$;
(c) f is pseudo-coercive: for any $0\in X$, $\liminf_{d(0,x)\to\infty} f(x)>m_f$.

Proof.

(a) $\Rightarrow$ (b) Given $0\in X$, and $\alpha,\beta,\xi\in\mathrm{P}$ such that $S_f\subset B(0,\xi)$, $\gamma_f(\alpha)>\beta>0$, for any $x\in S_f(\beta)$ we have

$$\gamma_f(d(x, S_f))\leq f(x)-m_f\leq\beta;$$

hence, $d(x, S_f)<\alpha$ and $S_f(\beta)$ is contained in $B(0,\xi+\alpha)$.

(b) $\Rightarrow$ (c) For $r>\rho$, we have $\inf\{f(x): x\in X,\, d(0,x)>r\}\geq m_f+\beta$ when $S_f(\beta)\subset B(0,\rho)$.

(c)$\Rightarrow$(b) Taking $\beta>0$ such that $m_f+\beta<\liminf_{d(0,x)\to\infty} f(x)$, we can find $\rho>0$ such that $f(x)>m_f+\beta$ for $x\in X\backslash B(0,\rho)$, so that $S_f(\beta)\subset B(0,\rho)$.

When f is l.s.c. for a topology σ for which $B(0,\rho)$ is compact, f attains its infimum on $B(0,\rho)$, hence on X when $S_f(\beta)\subset B(0,\rho)$. Moreover, then $S_f\subset B(0,\rho)$ and, for $\alpha>2\rho$ and $x\in X$, with $d(x,S_f)\geq\alpha$, we have $x\notin B(0,\rho)$, hence $f(x)-m_f\geq\beta$ and $\gamma_f(\alpha)\geq\beta>0$. □

3. Continuity of Values and Upper Hemicontinuity of the Solution Set

It is well known that epiconvergence is precisely the type of convergence which is suited to obtain semicontinuity properties of the value function (Refs. 27–28). It is desirable to obtain quantitative versions of results of this kind. Such an aim has been pursued with great success in the convex case (Refs. 13, 19). Here, we complete the study made in these articles and we treat some nonconvex cases.

We consider the class of inf-connected functions: $f\in\bar{\mathrm{R}}^X$ is said to be *inf-connected* if, for each $\lambda\in\mathrm{R}$, the sublevel set $[f\leq\lambda]:=\{x\in X: f(x)\leq\lambda\}$ is connected. Obviously, when X is a n.v.s. and f is quasiconvex, f is inf-connected.

It is easy to obtain an upper semicontinuity property for the infimal value function $f\mapsto m_f$. The following one is in the vein of Ref. 13, Theorem 3.1, but here there is no restriction of the type $\min f=f(0)=0$. For another result of this sort, see Ref. 30, Proposition 3.4.

Proposition 3.1. Let $r>\max(|m_f|, d(0,S_f))$. Then, for any $g: X\to R\cup\{\infty\}$, one has

$$m_g\leq m_f+d_r(f,g).$$

More generally, if for some $\epsilon\in\mathrm{R}_+$ one has $r>\max(|m_f+\epsilon|, d(0,S_f(\epsilon)))$, then

$$m_g\leq m_f+\epsilon+d_r(f,g). \tag{1}$$

Proof. Since $r>d(0,S_f(\epsilon))$, we can find $x\in S_f(\epsilon)\cap B_r$. Then, for any $\delta>d_r(f,g)$, there exists $(y,s)\in E_g$, with $d(y,x)<\delta$, $|s-(m_f+\epsilon)|<\delta$. It follows that

$$m_g\leq g(y)\leq s\leq m_f+\epsilon+\delta.$$

Taking the infimum over δ, we get the result. □

Upper hemicontinuity results for the solution multifunction are more subtle and involve a kind of well-posedness property. Note however that assumption (3) below is weaker than well-posedness. The next proposition prepares for upper hemicontinuity results; it also incorporates a Stepanov property for the value function.

Proposition 3.2. Let $f: X \to \bar{\mathrm{R}}$ be such that, for some $\alpha, \beta, \delta, \rho$ in P with $2\delta \le \beta$ and some nonempty subset S of X, one has $U(S, \alpha+\delta) \subset B_\rho$ and

$$\inf f(B_\rho) = m_f := \inf f(X) \in \mathrm{R}, \tag{2}$$

$$x \in B_\rho,\ d(x, S) > \alpha \Rightarrow f(x) \ge m_f + \beta. \tag{3}$$

Let $\epsilon \in \mathrm{R}_+$ and $r \in \mathrm{R}_+$ be such that $2\delta + \epsilon \le \beta$, $r \ge \max(\rho, |m_f| + \beta)$. Then, for any inf-connected function $g \in V_r(f, \delta)$, one has

(a) $\inf g(B_\rho) = m_g := \inf g(X) \in \mathrm{R}$,
(b) $e(S_g(\epsilon), S) \le \alpha + d_r(f, g)$,
(c) $|m_g - m_f| \le d_r(f, g)$.

Proof. We first observe that, by (2) and (3), we have

$$\inf f(B(S, \alpha)) = m_f.$$

It follows from the proof of the preceding proposition that

$$m_g \le \inf g(U(S, \alpha+\delta)) \le m_f + d_r(f, g).$$

In fact, taking $\delta' \in]d_r(f, g), \delta[$, $\beta' \in]0, \beta[$, and $x \in U(S, \alpha)$, with $f(x) < m_f + \beta'$, we can find $(y, s) \in E_g$, with $y \in x + \delta' B_X$, $g(y) \le m_f + \beta' + \delta'$, so that $y \in U(S, \alpha+\delta)$ and $m_g \le m_f + \beta' + \delta'$; and taking the infimum on β' and δ', the preceding inequality ensues. Moreover, since $2\delta \le \beta$, we see that the sublevel set $[g \le m_f + \beta - \delta']$ meets $U(S, \alpha+\delta')$.

Now, let us show that, for any $\delta' \in]d_r(f, g), \delta[$ and any $y \in B_\rho \backslash U(S, \alpha+\delta')$, we have $g(y) > m_f + \beta - \delta'$. Otherwise, we could find $x \in U(y, \delta')$, with $f(x) < m_f + \beta$, so that by (3), we would have $d(x, S) \le \alpha$, a contradiction with $d(x, y) < \delta'$ and $d(y, S) \ge \alpha + \delta'$.

It follows that the sublevel set $[g \le m_f + \beta - \delta']$ of g is contained in the union of the disjoint open sets $X \backslash B_\rho$ and $U(S, \alpha+\delta')$; moreover, we have seen that it meets $U(S, \alpha+\delta')$. Since this sublevel set is connected, it is contained in $U(S, \alpha+\delta')$. As

$$m_g + \epsilon \le m_f + \delta' + \epsilon \le m_f + \beta - \delta',$$

we get

$$S_g(\epsilon) := [g \le m_g + \epsilon] \subset [g \le m_f + \beta - \delta'] \subset U(S, \alpha + \delta') \subset B_\rho.$$

Since δ' is arbitrarily close to $d_r(f, g)$, we get $\inf g(B_\rho) = m_g$ and (b).

It remains to show that $m_g \ge m_f - \delta'$ for any δ' as above. If we had $m_g < m_f - \delta'$, there would exist some $y \in [g \le m_f - \delta'] \subset [g \le m_f + \beta - \delta'] \subset U(S, \alpha + \delta') \subset B_\rho$ and we could find $x \in B(y, \delta')$ with $f(x) < (m_f - \delta') + \delta'$, an impossibility. Therefore,

$$m_g - m_f \ge -d_r(f, g). \qquad \square$$

Taking $\epsilon = 0$ and $S = S_f$, we get a more striking result.

Corollary 3.1. Suppose that, for $f \in \bar{\mathrm{R}}^X$ with m_f finite and for some α, β, ρ in P, we have $S_f \ne \emptyset$, $\|S_f\| := \sup\{\|x\| : x \in S_f\} < \rho - \alpha$, and

$$x \in B_\rho,\ d(x, S_f) > \alpha \Rightarrow f(x) > m_f + \beta.$$

Let $r \ge \max(\rho, |m_f| + \beta)$. Then for $\delta \in [0, (1/2)\beta]$, with $\|S_f\| + \delta \le \rho - \alpha$, and for any inf-connected function $g \in V_r(f, \delta)$, one has

$$e(S_g, S_f) \le \alpha + d_r(f, g),$$

$$|m_g - m_f| \le d_r(f, g).$$

Note that, in Corollary 3.1, the set S_g may be empty. In the following theorem, the existence of solutions is part of the conclusions; in Ref. 13, Theorem 3.5, it was an assumption. Moreover, in this last result, a localization of S_g was needed in the assumptions; here, it is part of the conclusions too, and the hypothesis that f attains a unique firm (or strong) minimum at 0 is not required. Observe that our assumption on X is satisfied if X is a reflexive Banach space and if σ is the weak topology, or more generally if X is a dual Banach space and if σ is its weak* topology.

Theorem 3.1. Suppose that X is endowed with a topology σ such that, for each $r \in \mathrm{R}_+$, the ball B_r is compact for σ. Suppose that $f, \alpha, \beta, \delta, \rho, r$ are as in the preceding corollary. Then, for any lower semicontinuous (for σ), inf-connected function g such that $d_r(f, g) < \delta$, the set S_g of minimizers of g is nonempty and

$$e(S_g, S_f) \le \alpha + d_r(f, g),$$

$$|m_g - m_f| \le d_r(f, g).$$

Proof. This follows from the fact that the proof of Proposition 3.2

shows that the level set $[g \le m_f + \delta]$ is nonempty and contained in B_ρ. Since it is closed for σ, it is compact for σ and $S_g = \arg\min\,(g, [g \le m_f + \delta])$ is nonempty. □

The preceding results are stability results of the Stepanov type, as in Ref. 13. The following result has a true Lipschitzian character.

Proposition 3.3. Let $f \in \bar{\mathrm{R}}^X$ be such that, for some α, β, δ, ρ in P, with $2\delta \le \beta$, and some nonempty subset S of X, one has $U(S, \alpha + \delta) \subset B_r$ and

$$\inf f(B_\rho) = m_f := \inf f(X) \in \mathrm{R},$$

$$x \in B_\rho,\ d(x, S) > \alpha \Rightarrow f(x) \ge m_f + \beta.$$

Let $r \ge \max(\rho, |m_f| + \beta)$ and let $\eta = (1/2)\delta$. Then, for any $g, h \in V_r(f, \eta)$ with $d_r(g, h) < \eta$, one has

$$|m_g - m_h| \le d_r(g, h).$$

Proof. Taking $\epsilon = \beta - 2\eta \ge 2\eta$, $g \in V_r(f, \eta)$ in Proposition 3.2, we get by conclusion (b) that

$$x \in B_\rho,\ d(x, S) > \alpha + \eta \Rightarrow g(x) \ge m_g + \epsilon.$$

Using conclusion (a), it follows that we may replace f, α, β, δ in the preceding proposition by g, $\alpha + \eta$, ϵ, η respectively, and we get that, for any $h \in V_r(g, \eta)$ with the same r,

$$|m_h - m_g| \le d_r(g, h).$$ □

Although the roles of g and h are symmetric, we cannot assert that

$$e(S_h, S_g) \le \alpha + \eta + d_r(g, h),$$

because here S cannot be taken to be S_g. See Remark 3.1 below in this connection.

Let us reformulate Theorem 3.1 in a more synthetic and more usable way.

Theorem 3.2. Suppose that the set S_f of minimizers of f is nonempty and bounded. Suppose that f is metrically well set, with an u.s.c. conditioner φ. Then, there exists $r > 0$ and $\delta > 0$ such that, for any inf-connected function $g: X \to \mathrm{R} \cup \{\infty\}$ satisfying $d_r(f, g) < \delta$, one has

$$|m_g - m_f| \le d_r(f, g),$$

$$e(S_g, S_f) \le d_r(f, g) + \varphi(2d_r(f, g)).$$

If moreover g is l.s.c. for a topology σ for which the balls B_r are compact, then S_g is nonempty.

Let us note that, in this statement, we can take for φ the u.s.c. regularization μ_f^U of μ_f, given by

$$\mu_f(r) = \inf\{\mu_f(s) : s > r\}.$$

It is not easy to compare the estimate for $e(S_g, S_f)$ with the one given in Ref. 13, Theorem 3.5: there, a Lipschitz regularization of a growth gage of f is used, and the estimate is indirect; here, we have an explicit estimate.

Proof. Since $\varphi(0)=0$, $\lim_{t\to 0} \varphi(t)=0$, we can find α_0, $\beta_0 > 0$ such that $\varphi(\beta_0) < \alpha_0$. Then, taking $\alpha = \alpha_0$, $\beta = \beta_0$, $\delta = (1/2)\beta_0$, $\rho = \|S_f\| + \alpha_0$, and choosing $r \geq \max(\rho, |m_f| + \beta)$, we get that, for any inf-connected function $g: X \to \mathrm{R} \cup \{\infty\}$, with $d_r(f, g) < \delta$, the inequality $|m_g - m_f| \leq d_r(f, g)$ holds and S_g is nonempty if g is l.s.c. for σ.

For such a g, let us take $\alpha_1 > \varphi(2d_r(f, g))$, $\alpha_1 < \alpha_0$; note that $\varphi(2d_r(f, g)) \leq \varphi(\beta_0) < \alpha_0$. As φ is u.s.c., we can find $\beta_1 > 2d_r(f, g)$ such that $\beta_1 \leq \beta_0$ and $\varphi(t) < \alpha_1$ for $t \in [0, \beta_1]$. Then, if $x \in X$ is such that $d(x, S_f) \geq \alpha_1$, we have $t_x := f(x) - m_f > \beta_1$, since otherwise $d(x, S_f) \leq \mu_f(t_x) < \alpha$. Since the chosen r is still valid for the new $\alpha = \alpha_1$ and $\beta = \beta_1$, and since $d_r(f, g) < \beta/2$, a new application of Proposition 3.2 yields

$$e(S_g, S_f) \leq d_r(f, g) + \alpha_1.$$

Since α_1 is arbitrary in $]\varphi(2d_r(f, g)), \alpha_0[$, the result follows. □

Corollary 3.2. Suppose that the set S_f of minimizers of f is nonempty and bounded, suppose that m_f is finite, and suppose that, for some gage $\gamma := \mathrm{R}_+ \to \bar{\mathrm{R}}_+$, one has, for each $x \in X$,

$$f(x) \geq m_f + \gamma(d(x, S_f)).$$

Then, there exists $r > 0$ and $\delta > 0$ such that, for any inf-connected function $g: X \to \mathrm{R} \cup \{\infty\}$ satisfying $d_r(f, g) \leq \delta$, one has

$$|m_g - m_f| \leq d_r(f, g),$$
$$e(S_g, S_f) \leq d_r(f, g) + \gamma^h(2d_r(f, g)),$$

where

$$\gamma^h(t) = \sup\{r \in \mathrm{R}_+ : t \geq \gamma(r)\}.$$

Proof. Since γ is a growth gage for f, γ^h is a conditioning modulus for f by Lemma 2.1; moreover, γ^h is u.s.c. by Ref. 8, Proposition 2.3. □

In the following example, we prove that the estimate of Corollary 3.2 is sharp.

Example 3.1. Let $f: \mathrm{R} \to \mathrm{R}$ be given by $f(x) = c|x|^p$, with $c > 0$, $p > 1$. Let us define a one-parameter family g_a of functions on R for $a \in [0, 1]$ such that, for any $r > 0$, one has $d_r(f, g_a) \leq (1/2)ca^p$ while $e(S_{g_a}, S_f) = a$. Therefore, as $\gamma^h(t) = (c^{-1}t)^{1/p}$ when $\gamma(s) = cs^p$,

$$\limsup_{a \to 0} e(S_{g_a}, S_f)^{-1}[d_r(f, g_a) + \gamma^h(2d_r(f, g_a))] \leq 1.$$

It suffices to define g_a by

$$g_a(x) = (c/2)a^p, \qquad \text{for } x \in [-a, a],$$

$$g_a(x) = f(x) - (c/2)(2 - a^{-1}|x|)_+, \qquad \text{for } |x| \geq a,$$

where $r_+ = \max(r, 0)$.

Remark 3.1. Without further assumptions, one cannot have an estimate of the Hausdorff distance $d(S_g, S_f)$ in terms of $d_r(f, g)$. To see this, take for instance $X = \mathrm{R}$, $f(x) = \max(|x| - 1, 0)$; and, for $\epsilon \in]0, 1[$, let g be given by $g(x) = \max(|x| - 1, \epsilon|x| - \epsilon)$. Then, one has $d(S_g, S_f) = 1$ and, for each $r \geq 0$, one has $d_r(f, g) \leq \epsilon$. □

Finally, we show that the inf-connected assumption on g is crucial in Theorem 3.2.

Remark 3.2. Let $f: \mathrm{R} \to \mathrm{R}$ be given by

$$f(x) = |x|, \qquad \text{for } x \in [-1, 1],$$

$$f(x) = |\sin \pi x| + |x|^{-1}, \qquad \text{for } |x| \geq 1.$$

Then, for any $r > 0$, $\delta > 0$, we can find $g: \mathrm{R} \to \mathrm{R}$ such that $d_r(f, g) < \delta$ and $e(S_g, S_f) = +\infty$. It suffices to take g such that

$$g(x) = |\sin \pi x|, \qquad \text{for } |x| \geq \max(\delta^{-1}, r),$$

$$g(x) = f(x), \qquad \text{otherwise.}$$

4. Application to Upper Hemicontinuity of Subdifferentials

In this section, we deduce from the preceding result a quantitative form of the Asplund-Rockafellar theorem about the continuity of the subdifferential ∂f of a convex function satisfying a smoothness condition.

We need the following lemma whose proof is obvious.

Lemma 4.1. Let $f: X \to \mathrm{R} \cup \{+\infty\}$ and let $g: X \to \mathrm{R}$ be bounded on bounded sets. Then, for each $r \in \mathrm{R}_+$, one has

$$d_r(f, f+g) \leq \sup |g(rB_X)|.$$

In the following theorem, we use the Legendre–Fenchel conjugate of the function f, given by the usual formula $f^*(y) := \sup_{x \in X}(\langle y, x\rangle - f(x))$.

Theorem 4.1. Let $f: X \to \mathrm{R} \cup \{+\infty\}$ be a proper convex l.s.c. function. Suppose that, for some $x_0 \in X$, the set $\partial f(x_0)$ is nonempty and bounded and that, for some gage γ, one has, for each $y \in X^*$,

$$f^*(y) \geq \langle x_0, y\rangle - f(x_0) + \gamma(d(y, \partial f(x_0))).$$

Then, there exists some $r > 0$ and some $\epsilon > 0$ such that, for $x \in B(x_0, \epsilon)$, one has

$$e(\partial f(x), \partial f(x_0)) \leq r\|x - x_0\| + \gamma^h(2r\|x - x_0\|).$$

Proof. Let us observe that the set S_h of minimizers of the function $h := f^* - x$ on X^*, given by $h(y) = f^*(y) - \langle x, y\rangle$, satisfies

$$y \in \partial f(x) \Leftrightarrow x \in \partial f^*(y) \Leftrightarrow y \in S_h.$$

Thus using Corollary 3.2, and observing that the assumption can be written as

$$f^*(y) - \langle x_0, y\rangle \geq f^*(y_0) - \langle x_0, y_0\rangle + \gamma(d(y, \partial f(x_0))),$$

for any $y_0 \in \partial f(x_0)$, we get, for some $r > 0$, $\delta > 0$,

$$\begin{aligned} e(\partial f(x), \partial f(x_0)) &= e(S_{f^* - x}, S_{f^* - x_0}) \\ &\leq d_r(f^* - x, f^* - x_0) + \gamma^h(2d_r(f^* - x, f^* - x_0)), \end{aligned}$$

provided $d_r(f^* - x, f^* - x_0) \leq \delta$, which occurs when $\|x - x_0\| \leq \epsilon := r^{-1}\delta$, since $d_r(f^* - x, f^* - x_0) \leq r\|x - x_0\|$ by the preceding lemma. □

Given a modulus μ, one defines f to be μ-smooth at x_0 if there exists some $y_0 \in X^*$ such that

$$f(x) \leq f(x_0) + \langle x - x_0, y_0\rangle + \mu(\|x - x_0\|), \qquad \forall x \in X.$$

Then, one can show (see Ref. 32, Theorem 2.1, Proposition 3.1) that

$$f^*(y) \geq f^*(y_0) + \langle x_0, y - y_0\rangle + \mu^*(\|y - y_0\|), \qquad \forall y \in Y$$

and if $\hat{\mu}$ given by $\hat{\mu}(t)=t^{-1}\mu(t)$ is a modulus, one sees easily that f is Fréchet differentiable at x_0 with $f'(x_0)=y_0$. Since $f^*(y_0)-\langle x_0, y_0\rangle=-f(x_0)$, one can apply the preceding Corollary 3.2 with $\gamma=\mu^*$. Using the results of Refs. 31 and 32, we get a quantitative form of a well-known upper semicontinuity result. For other results obtained independently along this line, see Ref. 33.

5. Application to Best Approximation

Recall that a n.v.s. X with unit ball B_X is said to be uniformly convex if there exists a function γ: $[0, 1]\to[0, 1]$ satisfying $\gamma(0)=0$, $\gamma(t)>0$ for $t>0$, such that, for u, $v\in B_X$, $\epsilon\in[0, 1]$,

$$\|(1/2)(u-v)\|\geq\epsilon\Rightarrow 1-\|(1/2)(u+v)\|\geq\gamma(\epsilon);$$

see Ref. 34 for the existence of such a function and its relation to other uniform convexity gages. Let γ_E be given by

$$\gamma_X(\epsilon):=\inf\{1-\|(1/2)(u+v)\|: u, v\in B_X, \|(1/2)(u-v)\|\geq\epsilon\}.$$

Then, the canonical modulus of uniform convexity, given by

$$\mu_X(\delta):=\sup\{\|(1/2)(u-v)\|: u, v\in B_X, 1-\|(1/2)(u+v)\|\leq\delta\},$$

is easily seen to be a quasi-inverse of the canonical gage γ_X, in the sense that $\gamma_X^e\leq\mu_X\leq\gamma_X^h$ (see Ref. 8, Proposition 2.2 and Lemma 4.1). In particular, μ_X is a modulus.

The qualitative part of the following result is well known, but its quantitative part seems to be new.

Lemma 5.1. Let C be a nonempty closed convex subset of X, and let $p\in C$ be the projection of some point $w\in X\backslash C$. Then, for each $x\in C$, one has

$$\begin{aligned}\|p-x\|&\leq 2\|x-w\|\mu_X\left(1-\frac{\|p-w\|}{\|x-w\|}\right)\\&\leq 2\|x-w\|\mu_X\left(\frac{\|x-w\|-\|p-w\|}{d(w, C)}\right).\end{aligned}$$

Proof. Without loss of generality, we may suppose that $w=0$. Let $r=\|x\|$. Then, $u=r^{-1}p$ and $v=r^{-1}x$ belong to B_X and $\|(1/2)(u+v)\|=r^{-1}\|(1/2)(p+x)\|\geq r^{-1}\|p\|$, as C is convex and $\|c\|\geq\|p\|$ for each $c\in C$. By the definition of μ_X, setting $\delta = 1 - r^{-1}\|p\|$, we have

$$(1/2)\|u-v\| \leq \mu_X(1 - r^{-1}\|p\|).$$

The second inequality follows from the fact that μ_X is nondecreasing. □

The preceding result shows that the minimization problem of $f(x) = \|x - v\|$ on C is well set. This allows us to estimate the variation of the solution set, which is a singleton $\{p_C(w)\}$, as C varies in the family $\mathscr{C}(X)$ of nonempty closed convex subsets of X.

Proposition 5.1. Let X be an uniformly convex Banach space with modulus of uniform convexity μ_X, and let $C \in \mathscr{C}(X)$. Given $w \in X \backslash C$, one can find $r > 0$, $\delta > 0$ such that, for any $D \in \mathscr{C}(X)$ satisfying $d_r(C, D) < \delta$, one has

$$|d_C(w) - d_D(w)| \leq d_r(C, D),$$

$$\|p_C(w) - p_D(w)\| \leq d_r(C, D) + \varphi(2d_r(C, D)),$$

where $p_C(w)$ and $p_D(w)$ are the best approximations of w in C and D respectively, $d_C(w) = \|w - p_C(w)\|$, $d_D(w) = \|w - p_D(w)\|$, and $\varphi(t) = 2(d_C(w) + t)\mu_X(d_C(w)^{-1}t)$.

Proof. This ensues from Theorem 3.2 by setting

$$f(x) = \|x - w\| + i_C(x),$$

$$g(x) = \|x - w\| + i_D(x),$$

where i_C and i_D are the indicator functions of C and D respectively [(0 on C, $+\infty$ on $X \backslash C$) and (0 on D, $+\infty$ on $X \backslash D$) respectively]. Here, we see easily that

$$d_r(f, g) \leq d_r(C, D),$$

and the preceding lemma shows that φ is a conditioning modulus for f. □

Remark 5.1. The preceding result is not exactly a locally Hölderian property, even when μ is known to be Hölderian, as it is the case with L_p spaces.

6. Conclusions

We have examined the Lipschitzian and the Stepanovian behavior of the infima and of the sets of minimizers of convex functions and of generalized convex functions. The quantitative estimates that we have obtained can be used to get information about the rate of convergence of algorithms which use approximations of the sets or the functions involved in a given minimization problem. Our results have been completed recently in Ref. 35 by further studies in which a connection with the notion of an asymptotically well-behaved function is pointed out.

On the other hand, our study of the behavior of the metric projection can be seen as a special case of the sensitivity analysis of solutions to parametrized variational inequalities. In this direction, we are pleased to point out the recent work of Yen (Ref. 36), who tackles the case of a parametrized convex set. By considering the whole space $\mathscr{C}(X)$ of closed nonempty convex subsets of X as a parameter space, our setting could be reduced to his framework. Conversely, his framework is encompassed by our setting, at least for the metric projection, since the parametrization can be factorized through $\mathscr{C}(X)$.

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