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On convex triangle functions

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Abstract. We prove that the strongest (largest convex) solution of the functional inequality

$$\tau\left(\frac{F+G}{2},\frac{H+K}{2}\right)\le\frac{\tau(F,H)+\tau(G,K)}{2},$$

where F, G, H and K are arbitrary distribution functions, is the triangle function $\tau(F,G)(x)=\mathrm{Max}(F(x)+G(x)-1,0)$.

Our chief concern in this paper is to find the strongest solution of the functional inequality

$$\tau\left(\frac{F+G}{2},\frac{H+K}{2}\right)\le\frac{\tau(F,H)+\tau(G,K)}{2}, \tag{*}$$

where F, G, H and K are arbitrary distribution functions in Δ^+ and τ is a triangle function on Δ^+. This inequality arises in the investigation of countable products of probabilistic metric spaces ([1]) and it has been studied in detail in the case $\tau=\tau_T$ ([1], [2]). In this paper we generalize the techniques introduced in [1] in order to study (*) in a more general context. Specifically, let Δ^+ be the space of distribution functions vanishing at the origin, i.e.,

$$\Delta^+=\{F \mid F:\mathbb{R}\to[0,1],\ F(0)=0,\ F \text{ is non-decreasing and left-continuous on } \mathbb{R}\}.$$

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Among the elements of Δ^+ are functions ε_a and A_b^a defined, for $a \geq 0$ and $b \in [0,1]$, by

$$\varepsilon_a(x) = \begin{cases} 0, & x \leq a, \\ 1, & x > a, \end{cases} \qquad A_b^a(x) = \begin{cases} 0, & x \leq 0, \\ b, & 0 < x \leq a \\ 1, & x > a. \end{cases}$$

respectively. We denote by ε_∞ the null function ($\varepsilon_\infty(x) = 0$ for all x in $\mathbb{R}$) in Δ^+.

DEFINITION 1. A two-place function τ from $\Delta^+ \times \Delta^+$ into Δ^+ is a *triangle function* if the following conditions are satisfied for all F, G, H and K in Δ^+:

(i) $\tau(F, \varepsilon_0) = F$;
(ii) $\tau(F, G) \leq \tau(H, K)$ whenever $F \leq H$ and $G \leq K$;
(iii) $\tau(F, G) = \tau(G, F)$;
(iv) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$.

Any triangle function satisfying (*) will be called *convex*.

LEMMA 1. *A continuous triangle function τ is convex if and only if τ satisfies the inequality*

$$\tau\left(\sum_{i=1}^{\infty} \frac{1}{2^i} F_i, \sum_{i=1}^{\infty} \frac{1}{2^i} G_i\right) \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \tau(F_i, G_i), \tag{**}$$

for all sequences (F_i), (G_i) in Δ^+.

Proof. Obviously (**) implies (*). Conversely, if (*) holds, then an easy induction shows that

$$\tau\left(\sum_{i=1}^{n} \frac{1}{2^i} F_i, \sum_{i=1}^{n} \frac{1}{2^i} G_i\right) \leq \sum_{i=1}^{n} \frac{1}{2^i} \tau(F_i, G_i), \tag{1}$$

for all natural numbers n. The continuity of τ and (1) yield (**).

Since addition dominates Max (see [3, p. 209]), we immediately have

LEMMA 2. *The continuous triangle function π_W defined by*

$$\pi_W(F, G)(x) = \operatorname{Max}(F(x) + G(x) - 1, 0),$$

is convex.

THEOREM 1. *If a continuous triangle function* τ *is convex, then* $\tau \le \pi_W$, *i.e.,* π_W *is the strongest solution of the inequality* (*).

Proof. Given F and G in Δ^+ and $x > 0$, we want to show that

$$\tau(F, G)(x) \le \mathrm{Max}(F(x) + G(x) - 1, 0). \tag{2}$$

We will divide the proof into two cases (corresponding to the nullity or strict positivity of the right-hand side of (2)).

Case 1. Suppose x is such that $F(x) + G(x) \le 1$. Let

$$F(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} F(x)_i \quad \text{and} \quad G(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} G(x)_i,$$

where $F(x)_i$, $G(x)_i \in \{0, 1\}$ for each i, be binary expansions of $F(x)$ and $G(x)$, respectively. Next, for each i we consider the functions f_i and g_i in Δ^+ given by

$$f_i = \begin{cases} \varepsilon_0, & \text{if } G(x)_i = 0, \\ \varepsilon_x, & \text{if } G(x)_i = 1, \end{cases} \qquad g_i = \begin{cases} \varepsilon_0, & \text{if } G(x)_i = 1, \\ \varepsilon_x, & \text{if } G(x)_i = 0. \end{cases}$$

Then

$$G \le A^x_{G(x)} = \sum_{i=1}^{\infty} \frac{1}{2^i} g_i, \tag{3}$$

and, using the fact that

$$F(x) \le 1 - G(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} [1 - G(x)_i],$$

it follows that

$$F \le A^x_{1-G(x)} = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i. \tag{4}$$

Thus, since τ is non-decreasing and convex, it follows from (3), (4) and Lemma 1 that

$$\tau(F, G) \le \tau\left(\sum_{i=1}^{\infty} \frac{1}{2^i} f_i, \sum_{i=1}^{\infty} \frac{1}{2^i} g_i\right) \le \sum_{i=1}^{\infty} \frac{1}{2^i} \tau(f_i, g_i). \tag{5}$$

However, by the construction of f_i and g_i, for each i we have $\tau(f_i, g_i) = \varepsilon_x$, whence (5) yields

$$\tau(F, G)(x) \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \tau(f_i, g_i)(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} \varepsilon_x(x) = 0,$$

i.e., $\tau(F, G)(x) = 0$ and (2) holds when $F(x) + G(x) \leq 1$.

Case 2. Suppose x is such that $F(x) + G(x) > 1$. If either $F(x) = 1$ or $G(x) = 1$, then (2) is trivial. So we may suppose that both $F(x)$ and $G(x)$ are in $(0, 1)$. Let $B_0 = 0$, and, for any $n \geq 1$, let $B_n = 1/2 + \cdots + 1/2^n$ and consider the partition of the set

$$R = \{(u, v) \mid 0 < u < 1, 0 < v < 1, u + v > 1\}$$

given by $R = \bigcup_{n=2}^{\infty} R_n$, where

$$R_n = \{(u, v) \mid 0 < u < 1, 0 < v < 1, 1 + B_{n-2} < u + v \leq 1 + B_{n-1}\}.$$

There exists $n \geq 2$ such that $(F(x), G(x)) \in R_n$. Thus we can write

$$F(x) + G(x) = 1 + B_{n-2} + a,$$

where $0 < a \leq 1/2^{n-1}$, so that at least one of $F(x)$, $G(x)$ must be greater than B_{n-1}. Suppose $F(x) = B_{n-1} + \sum_{i=n}^{\infty} F(x)_i/2^i$, where $F(x)_i \in \{0, 1\}$ for each i. Then, since $1 - B_{n-1} = 1/2^{n-1}$, we have

$$\begin{aligned} G(x) = 1 + B_{n-2} + a - F(x) &= B_{n-2} + a + (1/2^{n-1}) - \sum_{i=n}^{n} F(x)_i/2^i \\ &= B_{n-2} + 2^{n-1}a/2^{n-1} + \sum_{i=n}^{\infty} (1 - F(x)_i)/2^i, \end{aligned} \tag{6}$$

and we can also write

$$F(x) = B_{n-2} + 1/2^{n-1} + \sum_{i=n}^{\infty} F(x)_i/2^i. \tag{7}$$

Now we introduce two sequences (f_i) and (g_i) in Δ^+ where, for each i, f_i and g_i are defined by

$$f_i = \begin{cases} \varepsilon_0, & \text{if } 1 \le i \le n-1, \\ A^x_{F(x)_i}, & \text{if } i \ge n, \end{cases} \qquad g_i = \begin{cases} \varepsilon_0, & \text{if } 1 \le i \le n-2, \\ A^x_{2^{n-1}a}, & \text{if } i = n-1, \\ A^x_{1-F(x)_i}, & \text{if } i \ge n. \end{cases}$$

Then, using (6) and (7), it is easily proved that

$$F \le A^x_{F(x)} = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i, \qquad G \le A^x_{G(x)} = \sum_{i=1}^{\infty} \frac{1}{2^i} g_i.$$

Thus, by Lemma 1,

$$\tau(F, G) \le \tau\left(\sum_{i=1}^{\infty} \frac{1}{2^i} f_i, \sum_{i=1}^{\infty} \frac{1}{2^i} g_i\right) \le \sum_{i=1}^{\infty} \frac{1}{2^i} \tau(f_i, g_i), \tag{8}$$

i.e.,

$$\begin{aligned} \tau(F, G)(x) &\le \sum_{i=1}^{\infty} \frac{1}{2^i} \tau(f_i, g_i)(x) \\ &= \sum_{i=1}^{n-2} \frac{1}{2^i} \tau(\varepsilon_0, \varepsilon_0)(x) + \frac{1}{2^{n-1}} \tau(\varepsilon_0, A^x_{2^{n-1}a})(x) \\ &\quad + \sum_{i=n}^{\infty} \frac{1}{2^i} \tau(A^x_{1-F(x)_i}, A^x_{F(x)_i})(x) \\ &= B_{n-2} + a + 0 = F(x) + G(x) - 1 \\ &= \pi_W(F(x), G(x)). \end{aligned}$$

If $F(x) < B_{n-1}$, then reversing the roles of $F(x)$ and $G(x)$ yields the same conclusion, and this completes the proof. □

COROLLARY 1. *If T is a continuous t-norm and τ_T is convex then $T \le W$. Thus τ_W is the strongest convex τ_T-operation.*

Proof. In view of the above theorem, if τ_T is convex, then $\tau_T \le \pi_W$, but this yields, for any a, b in $[0, 1]$,

$$T(a, b) = \tau_T(A^1_a, A^1_b)(1) \le \pi_W(A^1_a, A^1_b)(1) = W(a, b).$$

An easy computation shows that τ_W is convex. □

REFERENCES

[1] ALSINA, C., *On countable products and algebraic convexifications of probabilistic metric spaces.* Pacific J. Math. *76* (1978), 291–300.

[2] ALSINA, C., *On a family of functional inequalities.* In *General Inequalities 2,* Birkhäuser, Basel 1981, pp. 419–427.

[3] SCHWEIZER, B. and SKLAR, A., *Probabilistic metric spaces.* Elsevier, North Holland–New York, 1983.

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