Aequationes Mathematicae 25 (1982) 83–89
University of Waterloo

0001-9054/82/001083-7 $1.50 + 0.20/0

Multiple tilings by cubes with no shared faces

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I. We consider a family of translates of a unit n-dimensional closed cube and assume that any point lies in only a finite number of the cubes. If every point which is not on the boundary of any cube lies in exactly k cubes, then we say that the given family of cubes furnishes a k-fold tiling of n-dimensional space. The number k is the multiplicity of the tiling. If the translations of the cubes form a lattice, then the tiling is called a lattice tiling.

Lattice cube tilings arise in connection with a famous conjecture of Minkowski. In a 1-fold lattice cube tiling of n-dimensional space, two of the cubes must share a complete $(n-1)$-dimensional face. Hajós confirmed Minkowski's conjecture [3].

There were two different generalizations of Minkowski's conjecture: Furtwängler's and Keller's conjectures.

Furtwängler conjectured that in a k-fold lattice cube tiling of n-dimensional space, two of the cubes must share a complete $(n-1)$-dimensional face. Furtwängler proved this statement for $n \leq 3$ [1], while Hajós proved that it was false for $n > 3$ [2].

Consider the following question: For which k does there exist a k-fold lattice cube tiling of n-dimensional space such that no two cubes have a common face?

Robinson proved the following [9]:

If $n = 4$, then $p^2 \mid k$, where p is an odd prime; if $n = 5$, then $k = 3$ or $k \geq 5$; if $n \geq 6$, then $k \geq 2$.

Keller conjectured that in a 1-fold cube tiling of n-dimensional space two cubes must share a complete $(n-1)$-dimensional face. Perron proved this statement for $n \leq 6$ [7], [8].

AMS (1980) subject classification: Primary 10E30. Secondary 20K01.

Manuscript received March 17, 1982, and in final form, October 24, 1982.

Consider the following question: For which k does there exist a k-fold cube tiling of n-dimensional space in which no two cubes have a common face?

Robinson proved that this is not possible for any k for $n \leq 2$ but $k = 25, 49, 50, 74, 75, 81, 98, 100$ and every $k > 313$ is possible for $n = 3$ [9]. He wrote [9]: "But the most interesting question is whether 25 is the smallest possible multiplicity".

The main result of this paper is that every $k > 1$ is possible for $n > 2$. Thus $k = 2$ is the smallest possible multiplicity for $n = 3, 4, 5, 6$.

II. Let $\mathscr{E}^n$, $\mathbb{R}$, $\mathbb{Z}$ be n-dimensional Euclidean space, the real number field, and the integer number ring, respectively. The translations of $\mathscr{E}^n$ belong to the n-dimensional vector space $\boldsymbol{E}^n$ over $\mathbb{R}$. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be an orthonormal basis in $\boldsymbol{E}^n$ and O a fixed point in $\mathscr{E}^n$. The set

$$\mathscr{C}_O := \{P : \overrightarrow{OP} = c_1\boldsymbol{e}_1 + \cdots + c_n\boldsymbol{e}_n \,;\, 0 \leq c_1 \leq 1, \ldots, 0 \leq c_n \leq 1\}$$

is called a cube with preferential vertex O. Denote the interior of $\mathscr{C}_O$ by int $\mathscr{C}_O$. The linear transformation $\boldsymbol{\alpha} : \boldsymbol{E}^n \to \boldsymbol{E}^n$ is defined by $(x_1\boldsymbol{e}_1 + \cdots + x_n\boldsymbol{e}_n)\boldsymbol{\alpha} = (x_1/q_1)\boldsymbol{e}_1 + \cdots + (x_n/q_n)\boldsymbol{e}_n$; $x_1, \ldots, x_n \in \mathbb{Z}$, where $q_1, \ldots, q_n$ are fixed positive integers. The mapping $\alpha : \mathscr{E}^n \to \mathscr{E}^n$ belongs to the linear transformation $\boldsymbol{\alpha}$. Let $\boldsymbol{X}$ be a free abelian group with generators $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. If $\boldsymbol{L}$ is a subgroup of $\boldsymbol{X\alpha}$, then we say that $\boldsymbol{L}$ is a lattice. We shall use the following notation $(\mathscr{C}_O, \boldsymbol{L}) = \{\mathscr{C}_P : \overrightarrow{OP} \in \boldsymbol{L}\}$. The set $\mathscr{C}_O\boldsymbol{\alpha}$ will be called a cell. Obviously $(\mathscr{C}_O, \boldsymbol{L})\alpha = (\mathscr{C}_O\alpha, \boldsymbol{L\alpha})$.

Let $\mathfrak{A}$ be a finite abelian group, which is written multiplicatively. We shall use the group ring $\mathbb{Z}[\mathfrak{A}]$ with integer coefficients over $\mathbb{Z}$. The sum in the group ring of the elements of $\mathfrak{A}$ is denoted by $\Sigma[\mathfrak{A}]$. If A is an element of $\mathfrak{A}$ and q is a positive integer, then $S = 1 + A + A^2 + \cdots + A^{q-1}$ is called a series.

III. THEOREM 1 ([2], [4]). *If there is a finite abelian group $\mathfrak{A}$ and series $S_1, \ldots, S_n$ such that*

$$S_1 \cdots S_n = k\Sigma[\mathfrak{A}], \tag{1}$$

then there exists a k-fold lattice tiling $(\mathscr{C}_O, \boldsymbol{L})$ of n-dimensional space. The lattice $\boldsymbol{L}$ is the kernel of the homomorphism $\psi : \boldsymbol{X\alpha} \to \mathfrak{A}$, which is defined as

$$\left(\frac{x_1}{q_1}\boldsymbol{e}_1 + \cdots + \frac{x_n}{q_n}\boldsymbol{e}_n\right)\psi = A_1^{x_1} \cdots A_n^{x_n}; \qquad x_1, \ldots, x_n \in \mathbb{Z}.$$

Robinson [9] gave some solutions of (1) and we shall use two of them.

The first example of a solution of equation (1) is the following [9, p. 253]: If $\mathfrak{A}$ is defined by $U^6 = V^4 = 1$ and $A_1 = U$, $A_2 = V$, $A_3 = UV^2$, $A_4 = U^3V$, $A_5 = U^2V^2$; $q_1 = 3$, $q_2 = 2$, $q_3 = 3$, $q_4 = 2$, $q_5 = 2$, then the series $S_1 = 1 + U + U^2$, $S_2 = 1 + V$, $S_3 = 1 + UV^2 + (UV^2)^2$, $S_4 = 1 + U^3V$, $S_5 = 1 + U^2V^2$ and $S_1S_2S_3S_4S_5 = 3\Sigma[\mathfrak{A}]$. According to Theorem 1, there is a 3-fold lattice tiling $(\mathscr{C}_O, \boldsymbol{L})$ of 5-dimensional space. For later use, construct the lattice $\boldsymbol{L}$ by exhibiting a basis for it.

The linear transformation $\boldsymbol{\alpha} : \boldsymbol{E}^5 \to \boldsymbol{E}^5$ and the homomorphism $\psi : \boldsymbol{X\alpha} \to \mathfrak{A}$ are defined by $\boldsymbol{e}_1\boldsymbol{\alpha} = \frac{1}{3}\boldsymbol{e}_1$, $\boldsymbol{e}_2\boldsymbol{\alpha} = \frac{1}{2}\boldsymbol{e}_2$, $\boldsymbol{e}_3\boldsymbol{\alpha} = \frac{1}{3}\boldsymbol{e}_3$, $\boldsymbol{e}_4\boldsymbol{\alpha} = \frac{1}{2}\boldsymbol{e}_4$, $\boldsymbol{e}_5\boldsymbol{\alpha} = \frac{1}{2}\boldsymbol{e}_5$; $\frac{1}{3}\boldsymbol{e}_1\psi = U$, $\frac{1}{2}\boldsymbol{e}_2\psi = V$, $\frac{1}{3}\boldsymbol{e}_3\psi = UV^2$, $\frac{1}{2}\boldsymbol{e}_4\psi = U^3V$, $\frac{1}{2}\boldsymbol{e}_5\psi = U^2V^2$. The vectors $\boldsymbol{l}_1 = 2\boldsymbol{e}_1$, $\boldsymbol{l}_2 = 2\boldsymbol{e}_2$, $\boldsymbol{l}_3 = \frac{1}{3}\boldsymbol{e}_1 + \boldsymbol{e}_2 - \frac{1}{3}\boldsymbol{e}_3$, $\boldsymbol{l}_4 = \boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{e}_2 - \frac{1}{2}\boldsymbol{e}_4$, $\boldsymbol{l}_5 = \frac{2}{3}\boldsymbol{e}_1 + \boldsymbol{e}_2 - \frac{1}{2}\boldsymbol{e}_5$ span the lattice $\boldsymbol{L} = \operatorname{Ker}\psi$. Indeed, $\boldsymbol{l}_1\psi = 2\boldsymbol{e}_1\psi = 6\frac{1}{3}\boldsymbol{e}_1\psi = U^6 = 1$, $\boldsymbol{l}_2\psi = 2\boldsymbol{e}_2\psi = 4\frac{1}{2}\boldsymbol{e}_2\psi = V^4 = 1$, $\boldsymbol{l}_3\psi = (\frac{1}{3}\boldsymbol{e}_1 + \boldsymbol{e}_2 - \frac{1}{3}\boldsymbol{e}_3)\psi = UV^2(UV^2)^{-1} = 1$, $\boldsymbol{l}_4\psi = (\boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{e}_2 - \frac{1}{2}\boldsymbol{e}_4)\psi = U^3V(U^3V)^{-1} = 1$, $\boldsymbol{l}_5\psi = (\frac{2}{3}\boldsymbol{e}_1 + \boldsymbol{e}_2 - \frac{1}{2}\boldsymbol{e}_5)\psi = U^2V^2(U^2V^2)^{-1} = 1$ and $\det(\boldsymbol{l}_1, \ldots, \boldsymbol{l}_5) = -\frac{1}{3}$.

The second example of a solution of (1) is the following [9, p. 255]: Let $\mathfrak{A}^*$ be defined by $U^6 = V^6 = W^3 = 1$ and $A_1 = U$, $A_2 = V$, $A_3 = U^2V^3$, $A_4 = U^3V^2$, $A_5 = UV^3W$, $A_6 = U^4V^3W$; $q_1 = 3$, $q_2 = 3$, $q_3 = 2$, $q_4 = 2$, $q_5 = 3$, $q_6 = 2$. Then the series $S_1 = 1 + U + U^2$, $S_2 = 1 + V + V^2$, $S_3 = 1 + U^2V^3$, $S_4 = 1 + U^3V^2$, $S_5 = 1 + UV^3W + (UV^3W)^2$, $S_6 = 1 + U^4V^3W$ satisfy $S_1 \cdots S_6 = 2\Sigma[\mathfrak{A}^*]$. According to Theorem 1 there is a 2-fold tiling $(\mathscr{C}_O, \boldsymbol{L}^*)$ of 6-dimensional space. We shall construct the lattice $\boldsymbol{L}^*$.

The linear transformation $\boldsymbol{\alpha}^* : \boldsymbol{E}^6 \to \boldsymbol{E}^6$ and the homomorphism $\psi^* : \boldsymbol{X\alpha}^* \to \mathfrak{A}^*$ are defined by $\boldsymbol{e}_1\boldsymbol{\alpha}^* = \frac{1}{3}\boldsymbol{e}_1$, $\boldsymbol{e}_2\boldsymbol{\alpha}^* = \frac{1}{3}\boldsymbol{e}_2$, $\boldsymbol{e}_3\boldsymbol{\alpha}^* = \frac{1}{2}\boldsymbol{e}_3$, $\boldsymbol{e}_4\boldsymbol{\alpha}^* = \frac{1}{2}\boldsymbol{e}_4$, $\boldsymbol{e}_5\boldsymbol{\alpha}^* = \frac{1}{3}\boldsymbol{e}_5$, $\boldsymbol{e}_6\boldsymbol{\alpha}^* = \frac{1}{2}\boldsymbol{e}_6$; $\frac{1}{3}\boldsymbol{e}_1\psi^* = U$, $\frac{1}{3}\boldsymbol{e}_2\psi^* = V$, $\frac{1}{2}\boldsymbol{e}_3\psi^* = U^2V^3$, $\frac{1}{2}\boldsymbol{e}_4\psi^* = U^3V^2$, $\frac{1}{3}\boldsymbol{e}_5\psi^* = UV^3W$, $\frac{1}{2}\boldsymbol{e}_6\psi^* = U^4V^3W$. The vectors $\boldsymbol{l}_1^* = 2\boldsymbol{e}_1$, $\boldsymbol{l}_2^* = 2\boldsymbol{e}_2$, $\boldsymbol{l}_3^* = \frac{2}{3}\boldsymbol{e}_1 + \boldsymbol{e}_2 - \frac{1}{2}\boldsymbol{e}_3$, $\boldsymbol{l}_4^* = \boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 - \frac{1}{2}\boldsymbol{e}_4$, $\boldsymbol{l}_5^* = \boldsymbol{e}_1 + 3\boldsymbol{e}_2 - \boldsymbol{e}_5$, $\boldsymbol{l}_6^* = \boldsymbol{e}_1 + \frac{1}{3}\boldsymbol{e}_5 - \frac{1}{2}\boldsymbol{e}_6$ span the lattice $\boldsymbol{L} = \operatorname{Ker}\psi^*$. Indeed, $\boldsymbol{l}_1^*\psi^* = 2\boldsymbol{e}_1\psi^* = 6\frac{1}{3}\boldsymbol{e}_2\psi^* = U^6 = 1$, $\boldsymbol{l}_2^*\psi^* = 2\boldsymbol{e}_2\psi^* = 6\frac{1}{3}\boldsymbol{e}_2\psi^* = V^6 = 1$, $\boldsymbol{l}_3^*\psi^* = (\frac{2}{3}\boldsymbol{e}_1 + \boldsymbol{e}_2 - \frac{1}{2}\boldsymbol{e}_3)\psi^* = U^2V^3(U^2V^3)^{-1} = 1$, $\boldsymbol{l}_4^*\psi^* = (\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 - \frac{1}{2}\boldsymbol{e}_4)\psi^* = U^3V^2(U^3V^2)^{-1} = 1$, $\boldsymbol{l}_5^*\psi^* = (\boldsymbol{e}_1 + 3\boldsymbol{e}_2 - \boldsymbol{e}_5)\psi^* = U^3V^9(UV^3W)^{-3} = 1$, $\boldsymbol{l}_6\psi^* = (\boldsymbol{e}_1 + \frac{1}{3}\boldsymbol{e}_5 - \frac{1}{2}\boldsymbol{e}_6)\psi^* = U^3UV^3W(U^4V^3W)^{-1} = 1$ and $\det(\boldsymbol{l}_1^*, \ldots, \boldsymbol{l}_6^*) = \frac{1}{2}$.

IV. THEOREM 2. *If n, $k \in \boldsymbol{Z}$; $n > 2$, $k > 1$, then there exists a k-fold cube tiling of n-dimensional space in which no two cubes have a common face.*

Proof. First we prove a lemma that will enable us to construct a 3-fold tiling of $\mathscr{E}^3$ and a 2-fold tiling of $\mathscr{E}^3$ in which no cubes share a common face. This will be accomplished by taking a 3-dimensional cross section of tilings of higher dimensional spaces, in particular, the tilings of Robinson discussed in Section III.

Once that is done, Theorem 2 follows almost immediately. First of all, any integer $k \geq 2$ can be expressed as a sum of 2's and 3's. So, by superposing 2-fold tilings and 3-fold tilings of $\mathscr{E}^3$, none of which have cubes sharing a common face, we can build a k-fold tiling of $\mathscr{E}^3$ with the same property. Then, to construct a k-fold tiling of $\mathscr{E}^4$, take the product of a k-fold tiling of $\mathscr{E}^3$ with the unit interval. This produces a tiling of layer of $\mathscr{E}^4$ without cubes sharing a common face. By taking copies of this tiling, translated to avoid cubes with common faces, we produce a tiling of $\mathscr{E}^4$. By induction on n, there is a k-fold tiling of $\mathscr{E}^n$, $k \geq 2$, $n \geq 3$, where no cubes share a common face.

Let I be a subset of $\{1, \ldots, n\}$, $\boldsymbol{r} \in \boldsymbol{E}^n$ and

$$\mathscr{P}_I^{(r)} := \{P : \overrightarrow{OP} = \boldsymbol{r} + \Sigma \lambda_i \boldsymbol{e}_i,\ \lambda_i \in \mathbf{R},\ i \in I\}.$$

This is an $|I|$-dimensional plane in $\mathscr{E}^n$. We consider the n-dimensional cubes $\mathscr{C}_P$, $\mathscr{C}_Q$, where $\overrightarrow{PQ} = x_1\boldsymbol{e}_1 + \cdots + x_n\boldsymbol{e}_n$, $\mathscr{C}_P \cap \mathscr{P}_I^{(r)} = \mathscr{C}'_{P'}$, $\mathscr{C}_Q \cap \mathscr{P}_I^{(r)} = \mathscr{C}'_{Q'}$. Assume that

$$(\operatorname{int} \mathscr{C}_P) \cap \mathscr{P}_I^{(r)} \neq \emptyset \quad \text{and} \quad (\operatorname{int} \mathscr{C}_Q) \cap \mathscr{P}_I^{(r)} \neq \emptyset. \tag{2}$$

LEMMA 1. *If the $|I|$-dimensional cubes $\mathscr{C}'_{P'}$, $\mathscr{C}'_{Q'}$ have a common $(|I|-1)$-dimensional face (i.e., there is a $t \in I$ such that $\overrightarrow{P'Q'} = \pm \boldsymbol{e}_t$), then there exists a $t \in I$ such that $|x_t| = 1$ and $|x_j| < 1$ for $j \in \{1, \ldots, n\} \setminus I$ and $|x_i| = 0$ for $i \in I \setminus \{t\}$.*

Proof. Indeed, $\overrightarrow{PQ} = \overrightarrow{PP'} + \overrightarrow{P'Q'} + \overrightarrow{Q'Q}$ and $\overrightarrow{PP'} = \Sigma \mu_j \boldsymbol{e}_j$, $\overrightarrow{QQ'} = \Sigma \nu_j \boldsymbol{e}_j$, $j \in \{1, \ldots, n\} \setminus I$. Thus $\overrightarrow{PQ} = \pm \boldsymbol{e}_t + \Sigma(\mu_j - \nu_j)\boldsymbol{e}_j$ so $|x_t| = 1$ and $|x_i| = 0$ for $i \in I \setminus \{t\}$. By virtue of (2), $|x_j| < 1$ for $j \in \{1, \ldots, n\} \setminus I$.

LEMMA 2. *For $k = 2$ and 3 there is a k-fold tiling of $\mathscr{E}^3$ in which no two cubes share a common face.*

Proof. Case $k = 3$. Let $I = \{1, 2, 3\}$ and $\boldsymbol{r} = \frac{1}{4}\boldsymbol{e}_4 + \frac{1}{4}\boldsymbol{e}_5$, whereby $(\operatorname{int} \mathscr{C}) \cap \mathscr{P}_I^{(r)} \neq \emptyset$ for $\mathscr{C} \in (\mathscr{C}_O, \boldsymbol{L})$, where $(\mathscr{C}_O, \boldsymbol{L})$ is the first example considered in Section III. Consider the system

$$(\mathscr{C}_O, \boldsymbol{L}) \cap \mathscr{P}_I^{(r)} := \{\mathscr{C}_P \cap \mathscr{P}_I^{(r)} : \mathscr{C}_P \in (\mathscr{C}_O, \boldsymbol{L})\}.$$

Obviously this system is a 3-fold cube tiling of $\mathscr{E}^3$. We prove that there are no two cubes with a common face in this system. Assume that $\mathscr{C}_P$, $\mathscr{C}_Q \in (\mathscr{C}_O, \boldsymbol{L})$; $\mathscr{C}_P \cap \mathscr{P}_I^{(r)} = \mathscr{C}'_{P'}$, $\mathscr{C}_Q \cap \mathscr{P}_I^{(r)} = \mathscr{C}'_{Q'}$ and $\mathscr{C}'_{P'}$, $\mathscr{C}'_{Q'}$ have a common 2-dimensional face.

Since $\overrightarrow{PQ} = t_1 l_1 + \cdots + t_5 l_5$; $t_1, \ldots, t_5 \in \mathbb{Z}$; $(t_1, \ldots, t_5) \neq (0, \ldots, 0)$, one of the following systems has a solution which differs from $(0, \ldots, 0)$:

$$\left.\begin{aligned} |6t_1 + t_3 + 3t_4 + 2t_5| &= 3, \\ |4t_2 + 2t_3 + t_4 + 2t_5| &= 0, \\ |t_3| &= 0, \\ |t_4| &< 2, \\ |t_5| &< 2; \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} |6t_1 + t_3 + 3t_4 + 2t_5| &= 0, \\ |4t_2 + 2t_3 + t_4 + 2t_5| &= 2, \\ |t_3| &= 0, \\ |t_4| &< 2, \\ |t_5| &< 2; \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} |6t_1 + t_3 + 3t_4 + 2t_5| &= 0, \\ |4t_2 + 2t_3 + t_4 + 2t_5| &= 0, \\ |t_3| &= 3, \\ |t_4| &< 2, \\ |t_5| &< 2. \end{aligned}\right\} \tag{5}$$

System (3) is not possible: $4t_2 + t_4 + 2t_5 = 0$ and $|6t_1 + 3t_4 + 2t_5| = 3$ so $|6t_1 + 3(-4t_2 - 2t_5) + 2t_5| = 3$, so the left-hand side is even and the right-hand side is odd.

System (4) is not possible either: $6t_1 + 3t_4 + 2t_5 = 0$, so $3 \mid t_5$. But $|t_5| < 2$; thus $t_5 = 0$ and $6t_1 = -3t_4$, that is, $-2t_1 = t_4$. Since $|t_4| < 2$, we have $|2t_1| < 2$, that is, $|t_1| < 1$, hence $t_1 = 0$. Thus $(t_1, \ldots, t_5) = (0, \ldots, 0)$.

Finally, system (5) is not possible: $6t_1 \pm 3 + 3t_4 + 2t_5 = 0$, so $3 \mid t_5$ and $4t_2 \pm 6 + t_4 + 2t_5 = 0$, so $2 \mid t_4$. From $3 \mid t_5$ and $|t_5| < 2$, it follows that $t_5 = 0$. From $2 \mid t_4$ and $|t_4| < 2$, it follows that $t_4 = 0$, and $6t_1 = \pm 3$. However, ± 3 is not a multiple of 6.

Case $k = 2$. Let $I = \{1, 2, 5\}$, $\boldsymbol{r} = \frac{1}{4}\boldsymbol{e}_3 + \frac{1}{4}\boldsymbol{e}_4 + \frac{1}{4}\boldsymbol{e}_6$, whereby $(\operatorname{int} \mathscr{C}) \cap \mathscr{P}_I^{(r)} \neq \emptyset$ for $\mathscr{C} \in (\mathscr{C}_O, \boldsymbol{L}^*)$, where $(\mathscr{C}_O, \boldsymbol{L}^*)$ is the second example considered in Section III. Consider the system

$$(\mathscr{C}_O, \boldsymbol{L}^*) \cap \mathscr{P}_I^{(r)} := \{\mathscr{C}_P \cap \mathscr{P}_I^{(r)} : \mathscr{C}_P \in (\mathscr{C}_O, \boldsymbol{L}^*)\}.$$

Obviously this system is a 2-fold cube tiling of $\mathscr{E}^3$. We prove that there are no two cubes with a common face in this system. Assume that $\mathscr{C}_P$, $\mathscr{C}_Q \in (\mathscr{C}_O, \boldsymbol{L}^*)$; $\mathscr{C}_P \cap \mathscr{P}_I^{(r)} = \mathscr{C}'_{P'}$, $\mathscr{C}_Q \cap \mathscr{P}_I^{(r)} = \mathscr{C}'_{Q'}$ and $\mathscr{C}'_{P'}$, $\mathscr{C}'_{Q'}$ have a common 2-dimensional face. Since $PQ = t_1\boldsymbol{l}_1^* + \cdots + t_6\boldsymbol{l}_6^*$; $t_1, \ldots, t_6 \in \boldsymbol{Z}$; $(t_1, \ldots, t_6) \neq (0, \ldots, 0)$, one of the following systems has a solution which differs from $(0, \ldots, 0)$:

$$\left.\begin{aligned} |6t_1 \quad + 2t_3 + 3t_4 + 3t_5 + 3t_6| &= 3,\\ |6t_2 + 3t_3 + 2t_4 + 9t_5| &= 0,\\ |t_3| &< 2,\\ |t_4| &< 2,\\ |-3t_5 + t_6| &= 0,\\ |t_6| &< 2; \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} |6t_1 \quad + 2t_3 + 3t_4 + 3t_5 + 3t_6| &= 0,\\ |6t_2 + 3t_3 + 2t_4 + 9t_5| &= 3,\\ |t_3| &< 2,\\ |t_4| &< 2,\\ |-3t_5 + t_6| &= 0,\\ |t_6| &< 2; \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} |6t_1 \quad + 2t_3 + 3t_4 + 3t_5 + 3t_6| &= 0,\\ |6t_2 + 3t_3 + 2t_4 + 9t_5| &= 0,\\ |t_3| &< 2,\\ |t_4| &< 2,\\ |-3t_5 + t_6| &= 3,\\ |t_6| &< 2. \end{aligned}\right\} \tag{8}$$

Obviously $3 \mid t_3$, $3 \mid t_4$, $3 \mid t_6$, $|t_3| < 2$, $|t_4| < 2$, $|t_6| < 2$, so $t_3 = t_4 = t_6 = 0$.

System (6) is not possible: $-3t_5 + t_6 = 0$ and $t_6 = 0$ imply that $t_5 = 0$, so $|6t_1| = 3$, a contradiction.

System (7) is not possible because $-3t_5 + t_6 = 0$ and $t_6 = 0$ imply $t_5 = 0$, so $|6t_2| = 3$, a contradiction.

System (8) is not possible as $6t_2 + 9t_5 = 0$, so $2 \mid t_5$ and $|-3t_5| = 3$. The left-hand side is even and the right-hand side is odd.

This completes the proof of Theorem 2.

The author would like to thank the referee for his valuable suggestions.

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