Aequationes Mathematicae 28 (1985) 199–207
University of Waterloo
0001–9054/85/010199–09$1.50+0.20/0

Integral inequalities related to Hardy's inequality

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Dedicated to Professor Janos Aczél on his 60th birthday

Abstract. Some generalizations are given of Hardy's inequality relating to L^p-spaces. The results include many existing integral inequalities.

1. Introduction and Statement of the Main Results

Hardy's Integral Inequality (see [5], or [6, p. 240, Theorem 327]) is well-known and states the following:

Let $f(t) \geqslant 0$ *for* $t \in \mathbb{R}_+ := (0, \infty)$ *and* $F(x) := \int_0^x f(t)dt$. *If* $1 < p < \infty$ *then*

$$\int_0^\infty [x^{-1}F(x)]^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty [f(x)]^p dx \tag{1.1}$$

the constant being best-possible.

There is a companion theorem for $0 < p < 1$ [6, Theorem 337] and an analogue for series in place of integrals [6, Theorem 326], together with generalizations in several different directions (e.g. [1], [2], [3], [4], [7]). The theorems in the present note are essentially the integral analogues of the series inequalities of J. Németh [7], but they include, for example, all the inequalities given by E. T. Copson [2], and (for $p > 0$) those given by P. R. Beesack [1]. We have not considered the case $p < 0$.

We suppose throughout that all our functions are non-negative and measurable on their domains of definition, and we now state our main results. We remark that Hardy's inequality (1.1) is the case $g(x) = x^{-p}(x > 0)$, $a(x, t) = 1(0 < t < x)$, $m = \infty$, of Theorem 1(a) below, though we shall give a number of other corollaries in §§3, 4, 5 below.

AMS (1980) subject classification: Primary 26D15.

Manuscript received January 21, 1984.

THEOREM 1. *Assume that* $a(\cdot,\cdot)$ *is defined on* $\mathbb{R}_+ \times \mathbb{R}_+$, *with* $a(x,t) \geqslant 0$ *for* $0 < t < x$, $a(x,t) = 0$ *for* $t > x$, *and suppose that, for some constant* $K_1 \geqslant 1$,

$$a(x,t) \leqslant K_1 a(y,t) \text{ for } x > y > t. \tag{1.2}$$

Let $g(x) \geqslant 0$ $(x \in \mathbb{R}_+)$ *and* $g(\cdot)a(\cdot,t) \in L(t,\infty)$ *for each* $t > 0$, *and write*

$$G_2(t) := \int_t^\infty g(x)a(x,t)dx \qquad (t > 0). \tag{1.3}$$

Let $f(t) \geqslant 0$ $(t \in \mathbb{R}_+)$ *and* $a(x,\cdot)f(\cdot) \in L(0,x)$ *for each* $x > 0$, *and write*

$$F_1(x) := \int_0^x a(x,t)f(t)dt \qquad (x > 0). \tag{1.4}$$

(*a*) *If* $1 < p < \infty$, $0 < m \leqslant \infty$, $g(x) > 0$ *on* $(0,m)$, *then*

$$\int_0^m gF_1^p dx \leqslant (pK_1^{p-1})^p \int_0^m g^{1-p}(G_2 f)^p dx. \tag{1.5}$$

(*b*) *If* $0 < p < 1$, $0 \leqslant r < \infty$, $F_1(x) > 0$ *on* $\mathbb{R}_+$, *then*

$$\int_r^\infty gF_1^p \, dx \geqslant (pK_1^{p-1})^p \int_r^\infty g^{1-p}(G_2 f)^p dx. \tag{1.6}$$

(*c*) *If* $p = 1$ *then hypothesis* (1.2) *is not required:* (1.5) $(0 < m < \infty)$ *and* (1.6) $(0 < r < \infty)$ *hold, with equality in* (1.5)$(m = \infty)$ *and* (1.6)$(r = 0)$.

THEOREM 2. *Assume that* $a(\cdot,\cdot)$ *is defined on* $\mathbb{R}_+ \times \mathbb{R}_+$, *with* $a(x,t) \geqslant 0$ *for* $0 < x < t$, $a(x,t) = 0$ *for* $x > t$, *and suppose that, for some constant* $K_2 \geqslant 1$,

$$a(x,t) \leqslant K_2 a(y,t) \text{ for } x < y < t. \tag{1.7}$$

Let $g(x) \geqslant 0$ $(x \in \mathbb{R}_+)$ *and* $g(\cdot)a(\cdot,t) \in L(0,t)$ *for each* $t > 0$, *and write*

$$G_1(t) := \int_0^t g(x)a(x,t)\,dx \qquad (t > 0). \tag{1.8}$$

Let $f(t) \geqslant 0$ $(t \in \mathbb{R}_+)$ *and* $a(x,\cdot)f(\cdot) \in L(x,\infty)$ *for each* $x > 0$, *and write*

$$F_2(x) := \int_x^\infty a(x,t)f(t)dt \qquad (x > 0). \tag{1.9}$$

(*a*) *If* $1 < p < \infty$, $0 \leqslant r < \infty$, $g(x) \geqslant 0$ *on* (r,∞), *then*

$$\int_r^\infty gF_2^p dx \leqq (pK_2^{p-1})^p \int_r^\infty g^{1-p}(G_1 f)^p dx. \tag{1.10}$$

(*b*) *If* $0 < p < 1$, $0 < m \leqslant \infty$, $F_2(x) > 0$ *on* $\mathbb{R}_+$, *then*

$$\int_0^m gF_2^p dx \geqslant (pK_2^{p-1})^p \int_0^m g^{1-p}(G_1 f)^p dx. \tag{1.11}$$

(*c*) *If* $p = 1$ *then hypothesis* (1.7) *is not required:* (1.10)($0 < r < \infty$) *and* (1.11) ($0 < m < \infty$) *hold, with equality in* (1.10)($r = 0$) *and* (1.11)($m = \infty$).

2. Proofs of the Theorems

We first require the following elementary result.

LEMMA 1. (*a*) *Let* $1 \leqslant p < \infty$ *and* $z(\cdot)$ *be non-negative and integrable over* $(0, x)$. *Then*

$$\left(\int_0^x z(t)dt\right)^p = p\int_0^x z(t)\left(\int_0^t z(u)du\right)^{p-1}dt; \tag{2.1}$$

the result holds for $0 < p < 1$ *provided that* $\int_0^t z(u)du > 0$ *for* $0 < t < x$.

(*b*) *Let* $1 \leqslant p < \infty$ *and* $z(\cdot)$ *be non-negative and integrable over* (x, ∞). *Then*

$$\left(\int_x^\infty z(t)dt\right)^p = p\int_x^\infty z(t)\left(\int_t^\infty z(u)du\right)^{p-1}dt; \tag{2.2}$$

the result holds for $0 < p < 1$ *provided that* $\int_t^\infty z(u)du > 0$ *for* $x < t < \infty$.

Proof. (2.1) is proved by Davies and Petersen [2, Lemma 2] (their proof, stated for $p > 1$, clearly holds also for $0 < p < 1$ under the given positivity hypothesis). In a similar way, for (2.2) we write $F(t) := \int_t^\infty z(u)du$ and then $F'(t) = -z(t)$ a.e. on (x, ∞); then (2.2) follows from

$$[F(x)]^p = -p\int_x^\infty [F(t)]^{p-1}F'(t)dt. \qquad \square$$

Proof of Theorem 1(*a*) ($1 < p < \infty$). By (1.4) and (2.1),

$$\begin{aligned}[F_1(x)]^p &= p\int_0^x a(x,t)f(t)\left(\int_0^t a(x,u)f(u)du\right)^{p-1}dt \\ &\leqq pK_1{}^{p-1}\int_0^x a(x,t)f(t)[F_1(t)]^{p-1}dt, \text{ by (1.2).}\end{aligned} \tag{2.3}$$

Hence on multiplying through by $g(x)$, integrating over $(0, m)$, and inverting the order of integration (the integrand being non-negative),

$$\begin{aligned}\int_0^m gF_1^p &\leqq pK_1^{p-1}\int_0^m f(t)[F_1(t)]^{p-1}dt\int_t^m g(x)a(x,t)dx\\ &\leqq pK_1^{p-1}\int_0^m fF_1^{p-1}G_2, \quad \text{by (1.3)}\\ &= pK_1^{p-1}\int_0^m g^{(1-p)/p}G_2 f(g^{1/p}F_1)^{p-1}, \quad g>0 \text{ on } (0,m)\\ &\leqq pK_1^{p-1}\left(\int_0^m g^{1-p}(G_2 f)^p\right)^{1/p}\left(\int_0^m gF_1^p\right)^{1/p'}, \quad \frac{1}{p}+\frac{1}{p'}=1 \end{aligned} \tag{2.4}$$

by Hölder's inequality. If $\int_0^m gF_1^p = 0$ then (1.5) is trivially true; otherwise we may divide both sides of (2.4) by $(\int_0^m gF_1^p)^{1/p'}$, and then raise both sides to the power p, to get (1.5). □

Proof of Theorem 1(*b*) ($0 < p < 1$). The proof begins as in Theorem 1(*a*), denoting F_1 as in (1.4) and using (2.1), and proceeding as in (2.3) since, by hypothesis, $F_1 > 0$ on $\mathbb{R}_+$; but the inequality sign in (2.3) is now reversed since $p - 1 < 0$. If $g = 0$ a.e. on $\mathbb{R}_+$ then (1.6) holds trivially. Hence suppose that $g > 0$ on a set E of positive measure, $g = 0$ on $\mathbb{R}_+ \setminus E$. Then multiplying the (reversed) inequality (2.3) by $g(x)$ and integrating over E, we shall obtain

$$\begin{aligned}\int_E gF_1^p &\geqq pK_1^{p-1}\int_E g(x)dx\int_0^x a(x,t)f(t)[F_1(t)]^{p-1}dt\\ &= pK_1^{p-1}\int_0^\infty f(t)[F_1(t)]^{p-1}dt\int_{E\cap(t,\infty)} g(x)a(x,t)dx\\ &= pK_1^{p-1}\int_0^\infty fF_1^{p-1}G_2, \quad \text{by (1.3)}\\ &\geqq pK_1^{p-1}\int_E fF_1^{p-1}G_2\\ &= pK_1^{p-1}\int_E g^{(1-p)/p}G_2 f(g^{1/p}F_1)^{p-1}, \quad g>0 \text{ on } E\\ &\geqq pK_1^{p-1}\left(\int_E g^{1-p}(G_2 f)^p\right)^{1/p}\left(\int_E gF_1^p\right)^{1/p'}, \quad \frac{1}{p}+\frac{1}{p'}=1\end{aligned}$$

by Hölder's inequality for $0 < p < 1$ ($p' < 0$) (e.g. [6, (6.9.3)]). Dividing through by $(\int_E gF_1^p)^{1/p'}$ and raising both sides to the (positive) power p, we get (1.6) with the integrals $\int_r^\infty$ on both sides replaced by $\int_E$. However, since $g = g^{1-p} = 0$ on $\mathbb{R}_+ \setminus E$, and by taking $g(x) = 0$ for $0 < x < r$ (since (1.6) is not affected by those values) we can write our result in the form (1.6). □

Proof of Theorem 1(*c*) ($p = 1$). This is an obvious special case of (a) and (b); only Fubini's Theorem is required. □

Proof of Theorem 2. With F_2 defined as in (1.9), we begin by using (2.2) and (1.7), instead of (2.1) and (1.2). The modifications required, to the proof of Theorem 1 set out above, will be immediately evident. □

3. A Convolution Inequality

THEOREM 3. *Let* $s(t) \geqq 0(t > 0)$, $s(t) = 0(t < 0)$, *and for some constant* $K \geqq 1$, $s(x) \leqq Ks(y)$ *for* $x > y > 0$; *suppose also that*

$$S(x) := \int_0^x s(t)dt > 0 \text{ for } x > 0.$$

If $f(t) \geqq 0$ *on* $\mathbb{R}_+$ *and* $1 < p < \infty$, *then*

$$\int_0^\infty \left(\frac{1}{S(x)} \int_0^x s(x-t)f(t)dt \right)^p dx \leqq \left(\frac{p^2 K}{p-1} \right)^p \int_0^\infty [f(t)]^p dt. \tag{3.1}$$

Proof. For any $p > 0$, denote

$$G(t) := \int_t^\infty [S(x)]^{-p} s(x-t)dx \ (t > 0). \tag{3.2}$$

Now take $a(x, t) = s(x-t)$, $g(x) = [S(x)]^{-p}$, $r = 0$, $m = \infty$, in Theorem 1 and we get, for $1 < p < \infty$,

$$\int_0^\infty ([S(x)]^{-1} \int_0^x s(x-t)f(t)dt)^p dx \leqq (pK^{p-1})^p \int_0^\infty (S^{p-1}Gf)^p dt. \tag{3.3}$$

By (3.2), and since $S(x)\uparrow$,

$$\begin{aligned} G(t) &\leqq [S(t)]^{-p} \int_t^{2t} s(x-t)dx + \int_{2t}^\infty [S(x-t)]^{-p} s(x-t)dx \\ &= [S(t)]^{1-p} + (p-1)^{-1}\{[S(t)]^{1-p} - [S(\infty)]^{1-p}\} \\ &\leqq p(p-1)^{-1}[S(t)]^{1-p} \quad \text{for } p > 1. \end{aligned}$$

With this estimate on the right of (3.3), we now obtain (3.1). □

EXAMPLE. $s(t) = 1(t > 0)$, $K = 1$, in (3.3) gives Hardy's inequality (1.1) again.

REMARK. Inequality (3.3) is reversed if $0 < p < 1$.

COROLLARY. *If* $f(t) \geqq 0$ *on* $\mathbb{R}_+$ *and* $1 < p < \infty$, *then*

$$\int_0^\infty \left(\frac{1}{\log(x+1)} \int_0^x \frac{f(t)}{x-t+1} dt \right)^p dx \leqq \left(\frac{p^2}{p-1} \right)^p \int_0^\infty [f(t)]^p dt. \tag{3.4}$$

Proof. Theorem 3 with $s(t) = (t+1)^{-1} (t > 0)$, $K = 1$. □

4. Corollaries from Hardy, Littlewood and Polya [6]

We select, as corollaries of our Theorems 1 and 2, some examples from [6] which are directly related to Hardy's inequality. Given $f(t) \geqq 0$ on $\mathbb{R}_+$, then whenever the integral exists finitely, we write

$$F_1(x) := \int_0^x f(t) dt, \quad F_2(t) := \int_x^\infty f(t) dt, \quad x > 0.$$

(4.1) [6, Th. 330]. *If* $1 < p < \infty$, $c > 1$, *then*

$$\int_0^\infty x^{-c} F_1{}^p dx \leqq \left(\frac{p}{c-1} \right)^p \int_0^\infty x^{-c} (xf)^p dx.$$

(4.2) [6, Th. 347]. *If* $0 < p < 1$, $c > 1$ *the inequality in* (4.1) *is reversed.*

Proofs. Theorem 1: $g(x) = x^{-c}$ $(x > 0)$, $a(x, t) = 1$ $(0 < t < x)$, $r = 0$, $m = \infty$. □

REMARK. For a generalization of (4.1) see Fehér [4].

(4.3) [6, Th. 330]. *The case* $1 < p < \infty$, $c < 1$ *of* (4.1).
(4.4) [6, Th. 347]. *The case* $0 < p < 1$, $c < 1$ *of* (4.2).

Proofs. Theorem 2: $g(x) = x^{-c}$ $(x > 0)$, $a(x, t) = 1$ $(0 < x < t)$, $r = 0$, $m = \infty$. □

(4.5) [6, Th. 328]. Choose $g(x) = 1$ $(x > 0)$, $a(x, t) = 1$ $(0 < x < t)$ in Theorem 2(*a*). □
(4.6) [6, (9.9.9)]. *If* $1 < p < \infty$, $\alpha < 1/p$, *then*

$$\int_0^\infty x^{-\alpha p} \left(\int_x^\infty t^{\alpha - 1} f(t) dt \right)^p dx \leqq \left(\frac{p}{1 - \alpha p} \right)^p \int_0^\infty [f(t)]^p dt.$$

Proof. Theorem 2(*a*): $g(x) = x^{-\alpha p}$ $(x > 0)$, $a(x, t) = t^{\alpha - 1}$ $(0 < x < t)$, $K_2 = 1$, $r = 0$.

(4.7) [6, (9.9.8)]. Choose $g(x) = x^{p(\alpha-1)}(x > 0)$, $a(x,t) = t^{-\alpha}(0 < t < x)$ in Theorem 1(*a*).

□

5. Corollaries from Copson [2] and Beesack [1]

Copson [2] has given six inequalities which generalize those which we have mentioned in §4 above. Beesack [1] has also given six similar inequalities, two of which provide alternative proofs for [2, Theorems 5 and 6]; a further two are additional cases which complete Copson's list. Beesack's final two inequalities [1, (30) and (31)] deal with the case $p < 0$.

Let f, ϕ be positive and measurable on $\mathbb{R}_+$, and suppose that

$$\Phi(x) := \int_0^x \phi(t)dt$$

exists for $0 < x < \infty$. Whenever the integral exists finitely, we write

$$F_1(x) := \int_0^x f\phi dt, \qquad F_2(x) := \int_x^\infty f\phi dt, \qquad x > 0.$$

(5.1) [2, Th. 1]. *If* $1 < p < \infty$, $c > 1$, $0 < m \leqq \infty$, *then*

$$\int_0^m F_1{}^p \Phi^{-c}\phi dx \leqq \left(\frac{p}{c-1}\right)^p \int_0^m f^p \Phi^{p-c}\phi dx.$$

(5.2) [2, Th. 2]. *If* $0 < p < 1$, $c > 1$, $0 \leqq r < \infty$, $\Phi(\infty) = \infty$, *then*

$$\int_r^\infty F_1{}^p \Phi^{-c}\phi dx \geqq \left(\frac{p}{c-1}\right)^p \int_r^\infty f^p \Phi^{p-c}\phi dx.$$

Proofs. In our Theorem 1 choose $a(x, t) = \phi(t)$ $(0 < t < x)$, $K_1 = 1$, $g(x) = \phi(x)[\Phi(x)]^{-c}(x > 0)$, so that (1.3) gives

$$G_2(t) = (c-1)^{-1}\phi(t)\{[\Phi(t)]^{1-c} - [\Phi(\infty)]^{1-c}\}.$$

In case (5.1) it is enough to use $G_2 \leqq (c-1)^{-1}\phi\Phi^{1-c}$; for the opposite inequality (5.2) we need $\Phi(\infty) = \infty$ to give us $G_2 = (c-1)^{-1}\phi\Phi^{1-c}$. □

(5.3) [2, Th. 3]. *The case* $1 < p < \infty$, $c < 1$, $0 \leqq r < \infty$.
(5.4) [2, Th. 4]. *The case* $0 < p < 1$, $c < 1$, $0 < m \leqq \infty$.

Proofs. These companions to (5.1) and (5.2) come from our Theorem 2 by choosing $a(x, t) = \phi(t)\ (0 < x < t)$, $K_2 = 1$, $g(x) = \phi(x)[\Phi(x)]^{-c}(x > 0)$. □

(5.5) [1, (29)]. *If* $1 < p < \infty$, $0 < r < \infty$, *then*

$$\int_r^\infty F_2{}^p\Phi^{-1}\phi dx \leqq p^p \int_r^\infty f^p\Phi^{p-1}\left\{\log\frac{\Phi(x)}{\Phi(r)}\right\}^p \phi dx.$$

(5.6) [2, Th. 6] *and* [1, (33)]. *If* $0 < p < 1$, $0 < r < \infty$, *then the inequality in* (5.5) *is reversed.*

Proofs. In our Theorem 2 take $a(x, t) = \phi(t)(0 < x < t)$, $K_2 = 1$, $m = \infty$, and $g(x) = 0\ (0 < x < r)$, $g(x) = \phi(x)[\Phi(x)]^{-1}(x > r)$. □

(5.7) [2, Th. 5] *and* [1, (28)]. *If* $1 < p < \infty$, $0 < m < \infty$, *then*

$$\int_0^m F_1{}^p\Phi^{-1}\phi dx \leqq p^p \int_0^m f^p\Phi^{p-1}\left\{\log\frac{\Phi(m)}{\Phi(x)}\right\}^p \phi dx.$$

(5.8) [1, (32)]. *If* $0 < p < 1$, $0 < m < \infty$, *the inequality in* (5.7) *is reversed.*

Proofs. In our Theorem 1 take $a(x, t) = \phi(t)(0 < t < x)$, $K_1 = 1$, $r = 0$, and $g(x) = \phi(x)[\Phi(x)]^{-1}(0 < x < m)$, $g(x) = 0\ (x > m)$. □

Acknowledgement

This work was supported by an Operating Grant from the Natural Sciences and Engineering Research Council of Canada, and was written while the first author was visiting York University from the American University of Beirut.

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