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On orthogonally additive mappings

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Dedicated to Professor Janos Aczél on his 60th birthday

0. Introduction

Most papers on orthogonally additive mappings contain investigations under certain regularity conditions such as continuity, weaker forms of continuity, or certain kinds of boundedness (cf., e.g., [7], [8], [13], [17]). It is the purpose of the present note to avoid these regularity conditions in the beginning and to determine the most general orthogonally additive mapping in some situations. Our main idea is to make extensive use of the properties of the domain space and, accordingly, to impose a minimum of conditions on the mappings. Later we shall derive as corollaries some of the results mentioned above.

Throughout the paper, R, R_+, R_+^*, Q, Z, N denote the sets of real, nonnegative real, positive real, rational numbers, integers, positive integers, respectively. For a subset A of a vector space, *lin* A stands for the linear hull (span) of A. We use o for the zero vector and 0 for both the real number zero and the identity element of the abelian group $(Y, +)$; it will always be clear from the context what is meant. Finally, $\underline{c}$ is the symbol for the constant mapping with value c, and $:=$ means that the right hand side defines the left hand side.

1. Orthogonality spaces

DEFINITION 1. Let X be a real vector space of dimension ≥ 2 and $\perp$ a binary relation on X with the properties

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(01) $x \perp o$, $o \perp x$ for every $x \in X$;
(02) if $x, y \in X \backslash \{o\}$, $x \perp y$, then x, y are linearly independent;
(03) if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in R$;
(04′) if P is a 2-dimensional subspace of X, $x \in P$, $\lambda \in R_+$, then there exists $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

Then $(X, \perp)$ is called an *orthogonality space.*

This definition is a modification of the one in [7, pp. 427–428]. Although our axioms (01), (02), (03), (04′) are more restrictive than the ones in [7], none of the examples given there is eliminated here. Indeed, let X be a real vector space with $\dim_R X \geqslant 2$. Then the following relations $\perp$ make X into an orthogonality space:

EXAMPLE A. The "trivial" orthogonality on X, defined by (01) and $x, y \in X \backslash \{o\} \Rightarrow [x \perp y \Leftrightarrow x, y$ linearly independent$]$.

EXAMPLE B. The ordinary orthogonality on the inner product space $(X, \langle \cdot, \cdot \rangle)$, defined by $x \perp y \Leftrightarrow \langle x, y \rangle = 0$. For brevity, we then call $(X, \perp)$ also an inner product space.

EXAMPLE C. The Birkhoff–James orthogonality on the normed vector space $(X, \| \cdot \|)$, given by $x \perp y \Leftrightarrow \|x + \beta y\| \geqslant \|x\|$ $(\forall \beta \in R)$.

The crucial point in Example C is (04′). We briefly sketch a refinement of the argument of the proof of [17, p. 188, Lemma 1]. If $\lambda x = o$, then $y = o$ has the properties required in (04′). In the following let $\lambda x \neq o$. By [9, p. 268, Corollary 2.2], there exists a $v \in P$ such that $\|v\| = 1$ and $x \perp v$, and for $u := (1/\|x\|).x$ we have $\|u\| = 1$ and $u \perp v$. Now let $\varphi : P \to R^2$ be the linear bijection which transforms u, v into $(1,0), (0,1)$, respectively. By $|\varphi(z)| := \|z\|$ $(z \in P)$, a norm $|.|$ is defined on R^2, and φ is an isometry. Clearly $L := \{(1, \alpha); \alpha \in R\}$ is a line of support of the unit ball T in $(R^2, |.|)$. Taking into account that T may fail to be smooth or strictly convex, we form the set

$$\Gamma := \left\{(\alpha, m) \in R_+ \times R;\ m \text{ is the slope of a line of support of } T \text{ at } \frac{(1, \alpha)}{|(1, \alpha)|}\right\}.$$

Γ turns out to be connected, and the continuous mapping ψ given by $\psi(\alpha, m) = \alpha + \lambda m$ takes positive and negative values on Γ; here $\lambda > 0$ is important. Therefore there exists $(\tilde{\alpha}, m) \in \Gamma$ such that $\tilde{\alpha} + \lambda m = 0$, i.e., $m = -\tilde{\alpha}/\lambda$. In other words, there exists $\tilde{\alpha} \in R_+$ such that a line of support at $(1, \tilde{\alpha})/|(1, \tilde{\alpha})|$ and the ray from $(0,0)$ to $(\lambda, -\tilde{\alpha})$ are parallel, i.e., $(1, \tilde{\alpha}) \perp (\lambda, -\tilde{\alpha})$, i.e., $(1,0) + (0, \tilde{\alpha}) \perp (\lambda, 0) + (0, -\tilde{\alpha})$.

Therefore, by transforming back to P, $u + \tilde{\alpha} v \perp \lambda u - \tilde{\alpha} v$, and for $y := \tilde{\alpha} \|x\| v$ we get $x \perp y$ and $x + y \perp \lambda x - y$. (Alternative arguments would consist in using approximation techniques for convex sets, or the fact that, as the graph of a special subdifferential, Γ is maximal monotonic ([16, p. 32, Theorem B]), and the existence of $\tilde{\alpha}$ then follows from a result of Minty ([12, p. 343, Theorem 3]).)

The following statement will be useful later.

LEMMA 1. *If* $(X, \perp)$ *is an orthogonality space,* $x \in X \setminus \{o\}$, *and* P *a 2-dimensional subspace of* X *with* $x \in P$, *then there exists* $y \in P$ *with the property* $x \perp y$, $\text{lin}\{x, y\} = P$.

Proof. By putting $\lambda = 1$ in (04′) we get $y \in P$ such that $x \perp y$ and $x + y \perp x - y$. $y = o$ would lead to $x \perp x$ which is impossible by (02). Hence $y \neq o$, and (02) implies linear independence of x, y, i.e., $\text{lin}\{x, y\} = P$.

2. Orthogonally additive mappings

DEFINITION 2. If $(X, \perp)$ is an orthogonality space and $(Y, +)$ an abelian group, then a mapping $f: X \to Y$ is called *orthogonally additive* if

$$f(x_1 + x_2) = f(x_1) + f(x_2) \quad \text{for all } x_1, x_2 \in X \text{ with } x_1 \perp x_2. \tag{*}$$

Thus the orthogonally additive mappings $f: X \to Y$ are the solutions of the conditional Cauchy functional equation (*). The set of all solutions of (*) is denoted by $\text{Hom}_\perp(X, Y)$. The relation $\perp$ is not symmetric in general, but for $f \in \text{Hom}_\perp(X, Y)$ we have nevertheless

$$f(x_1 + x_2) = f(x_1) + f(x_2) \quad \text{for all } x_1, x_2 \in X \text{ with } x_2 \perp x_1 \tag{*'}$$

since the groups $(X, +)$, $(Y, +)$ are abelian. Here a generalization to the symmetric completion of $\perp$ and to the corresponding homomorphisms would be possible. The facts in the next two lemmas are straightforward.

LEMMA 2. a) *If* $f \in \text{Hom}_\perp(X, Y)$, *then* $f(o) = 0$. b) *If* $f, g \in \text{Hom}_\perp(X, Y)$, *then* $f + g \in \text{Hom}_\perp(X, Y)$. c) *If* $f \in \text{Hom}_\perp(X, Y)$ *and* $h: X \to Y$, $h(x) := f(-x)$ *for all* $x \in X$, *then* $h \in \text{Hom}_\perp(X, Y)$.

REMARK 1. In Definition 1, we required $\dim_R X \geq 2$ for an orthogonality space. This is now well motivated by the fact that for $\dim_R X = 0$, $\dim_R X = 1$ we would have the uninteresting situations $\mathrm{Hom}_\perp(X, Y) = \{\underset{\sim}{0}\}$, $\mathrm{Hom}_\perp(X, Y) = \{f : X \to Y; f(o) = 0\}$, respectively, which do not need further consideration.

DEFINITION 3. We say that the abelian group $(Y, +)$ is *uniquely 2-divisible*, if the mapping $\omega : Y \to Y$, $\omega(y) = 2y$ $(\forall y \in Y)$ is bijective. Then both ω and ω^{-1} are automorphisms of $(Y, +)$, and we write $y/2$ for $\omega^{-1}(y)$.

LEMMA 3. *If* $(Y, +)$ *is uniquely 2-divisible and* $f \in \mathrm{Hom}_\perp(X, Y)$, $g(x) := \frac{1}{2}[f(x) + f(-x)]$, $h(x) := \frac{1}{2}[f(x) - f(-x)]$ $(\forall x \in X)$, *then* $g, h \in \mathrm{Hom}_\perp(X, Y)$, *$g$ is even, h is odd, and* $f = g + h$.

LEMMA 4. *If* $u, v \in X$, $u \perp v$ *and* $f \in \mathrm{Hom}_\perp(X, Y)$ *is additive on* $\mathrm{lin}\{u\}$ *and on* $\mathrm{lin}\{v\}$, *then f is additive on* $\mathrm{lin}\{u, v\}$.

Proof. Let $x_1, x_2 \in \mathrm{lin}\{u, v\}$ arbitrary, $x_1 = \alpha_1 u + \beta_1 v$, $x_2 = \alpha_2 u + \beta_2 v$. Then $x_1 + x_2 = (\alpha_1 + \alpha_2)u + (\beta_1 + \beta_2)v$ with $(\alpha_1 + \alpha_2)u \perp (\beta_1 + \beta_2)v$, by (03). Hence, by hypothesis and commutativity of $(Y, +)$,

$$\begin{aligned} f(x_1 + x_2) &= f[(\alpha_1 + \alpha_2)u] + f[(\beta_1 + \beta_2)v] = f(\alpha_1 u) + f(\alpha_2 u) + f(\beta_1 v) + f(\beta_2 v) \\ &= f(\alpha_1 u) + f(\beta_1 v) + f(\alpha_2 u) + f(\beta_2 v) = f(\alpha_1 u + \beta_1 v) + f(\alpha_2 u + \beta_2 v) \\ &= f(x_1) + f(x_2). \end{aligned}$$

3. General solutions

We next determine the general odd and the general even solution of (*). The corresponding continuous real-valued solutions in the framework of Example C were characterized in [17, Lemmas 2 and 3]; see also [5, p. 4.73 ff]. In those proofs, continuity is used for establishing certain homogeneity properties. It is clear that our way of procedure must be completely different.

THEOREM 5. *If* $(X, \perp)$ *is an orthogonality space and* $(Y, +)$ *an abelian group, then* $h : X \to Y$ *is an odd solution of* (*) *if and only if h is additive* ([14, Theorem 1]).

Proof. The "if" part is obvious. Let $h \in \mathrm{Hom}_\perp(X, Y)$, h odd. Then we have
(i) $x \in X$, $\lambda \in R_+$ implies $h(x + \lambda x) = h(x) + h(\lambda x)$,
(ii) $x_1, x_2 \in X$ linearly dependent implies $h(x_1 + x_2) = h(x_1) + h(x_2)$.

(i) Let P be a 2-dimensional subspace of X containing x. By (04′) there exists $y \in P$ such that $x \perp y$, $x + y \perp \lambda x - y$, hence, by (03), $x \perp (-y)$. Therefore

$$h(x + \lambda x) = h(x + y + \lambda x - y) = h(x + y) + h(\lambda x - y)$$
$$= h(x) + h(y) + h(\lambda x) + h(-y) = h(x) + h(\lambda x).$$

(ii) For $x_1 = o$, the assertion is clear by Lemma 2a. If $x_1 \neq o$, then there exists $\mu \in R$ with $x_2 = \mu x_1$.

Case 1. $\mu \geqslant 0$. Then by (i) $h(x_1 + x_2) = h(x_1 + \mu x_1) = h(x_1) + h(\mu x_1) = h(x_1) + h(x_2)$.

Case 2. $-1 < \mu < 0$. We put $y := (1 + \mu)x_1$, $\lambda := -\mu/(\mu + 1)$ and get $\lambda > 0$, $\lambda y = -\mu x_1$ hence, by (i), $h(y + \lambda y) = h(y) + h(\lambda y)$, i.e., $h[(1 + \mu)x_1 - \mu x_1] = h[(1 + \mu)x_1] + h(-\mu x_1)$, i.e., $h(x_1) = h(x_1 + \mu x_1) - h(\mu x_1) = h(x_1 + x_2) - h(x_2)$, i.e., again $h(x_1 + x_2) = h(x_1) + h(x_2)$.

Case 3. $\mu \leqslant -1$. Here we put $\lambda := -1 - \mu$, $y := -x_1$ and get $\lambda \geqslant 0$, $\lambda y = (1 + \mu)x_1 = x_1 + x_2$, hence, by (i), $h(y + \lambda y) = h(y) + h(\lambda y)$, i.e., $h(-x_1 + x_1 + x_2) = h(-x_1) + h(x_1 + x_2)$, i.e, $h(x_2) = -h(x_1) + h(x_1 + x_2)$, and (ii) is proved.

Now let $x_1, x_2 \in X$ arbitrary and P a 2-dimensional subspace containing x_1, x_2. By Lemma 1, there are $x, y \in P$ with the properties $x \perp y$, $\mathrm{lin}\{x, y\} = P$. By (ii), h is additive on $\mathrm{lin}\{x\}$ and on $\mathrm{lin}\{y\}$, therefore by Lemma 4 also additive on P, and we have $h(x_1 + x_2) = h(x_1) + h(x_2)$, i.e., additivity of h on X.

REMARK 2. It is clear that the set $\mathrm{Hom}(X, Y)$ of all additive mappings from $(X, +)$ into $(Y, +)$ also strongly depends on $(Y, +)$. If, for instance, the orders of the elements of Y form a finite subset of N, then $\mathrm{Hom}(X, Y) = \{\underline{0}\}$. On the other hand, there are torsion groups $(Y, +)$ such that $\mathrm{Hom}(X, Y) \neq \{\underline{0}\}$: Consider X as a Q-vector space and a surjective additive mapping $k : X \to Q$ (definition via Hamel base). Then the composite of k with the mapping $\alpha \mapsto e^{2\pi i \alpha}$ $(\alpha \in Q)$ is a homomorphism onto a nontrivial torsion group.

THEOREM 6. *If* $(X, \perp)$ *is an orthogonality space,* $(Y, +)$ *an abelian group and* $g : X \to Y$ *an even solution of* $(*)$, *then* g *is a quadratic mapping, i.e.,*

$$g(x_1 + x_2) + g(x_1 - x_2) = 2g(x_1) + 2g(x_2) \quad \textit{for all } x_1, x_2 \in X \tag{Q}$$

([15, Theorem 1′]).

Proof. We proceed in several steps. If g is an even solution of $(*)$,

(iii) $u, v \in X$; $u + v \perp u - v$ implies $g(u) = g(v)$,

(iv) $u \in X$ implies $g(2u) = 4g(u)$,

(v) $x \in X$, $\lambda \in R_+$ implies $g(x+\lambda x)+g(x-\lambda x)=2g(x)+2g(\lambda x)$,
(vi) $x \in X$; $\alpha, \beta \in R$ implies $g(\alpha x+\beta x)+g(\alpha x-\beta x)=2g(\alpha x)+2g(\beta x)$.

Step (iii). By (03), $\frac{1}{2}(u+v) \perp \pm\frac{1}{2}(u-v)$, so

$$g(u)=g\left(\frac{u+v}{2}+\frac{u-v}{2}\right)=g\left(\frac{u+v}{2}\right)+g\left(\frac{u-v}{2}\right)$$
$$=g\left(\frac{u+v}{2}\right)+g\left(\frac{v-u}{2}\right)=g\left(\frac{u+v}{2}+\frac{v-u}{2}\right)=g(v).$$

Step (iv). By (04′) there exists $v \in X$ with $u \perp v$ and $u+v \perp u-v$, hence by (03) also $u \perp (-v)$, and the hypothesis and (iii) imply

$$g(2u)=g(u+v+u-v)=g(u+v)+g(u-v)$$
$$=g(u)+g(v)+g(u)+g(-v)=2g(u)+2g(v)=4g(u).$$

Step (v). By (04′) there exists $y \in X$ such that $x \perp y$, $x+y \perp \lambda x-y$. Application of (*), (03), evenness of g, and (iv) yields

$$\begin{aligned} g(x+\lambda x)+g(x-\lambda x)+g(2y) &= g(x+y+\lambda x-y)+g(x-\lambda x)+g(2y)\\ &= g(x+y+\lambda x-y)+g(x-\lambda x+2y)\\ &= g(x+y+\lambda x-y)+g(x+y-\lambda x+y)\\ &= g(x+y)+g(\lambda x-y)+g(x+y)+g(-\lambda x+y)\\ &= 2g(x+y)+2g(\lambda x-y)\\ &= 2g(x)+2g(y)+2g(\lambda x)+2g(-y)\\ &= 2g(x)+2g(\lambda x)+4g(y)\\ &= 2g(x)+2g(\lambda x)+g(2y). \end{aligned}$$

Subtraction of $g(2y)$ on both sides leads to the assertion.

Step (vi). For $\alpha=0$, the assertion is clear by Lemma 2a. Let $\alpha \neq 0$, $\lambda := |\beta/\alpha|$, $z := \alpha x$.

Case 1. $\beta/\alpha \geq 0$, i.e., $\lambda = \beta/\alpha$, $\lambda z = \beta x$. Then (v) implies

$$g(\alpha x+\beta x)+g(\alpha x-\beta x)=g(z+\lambda z)+g(z-\lambda z)$$
$$=2g(z)+2g(\lambda z)=2g(\alpha x)+2g(\beta x).$$

Case 2. $\beta/\alpha < 0$, i.e., $\lambda = -\beta/\alpha$, $\lambda z = -\beta x$, and we may continue as in Case 1, and (vi) is proved.

Now let $x_1, x_2 \in X$ arbitrary. If $x_1 = o$, then (Q) holds by Lemma 2a. Let $x_1 \neq o$ and P a 2-dimensional subspace of X containing x_1 and x_2. By Lemma 1, there exists $y \in P$ such that $x_1 \perp y$ and $\text{lin}\{x_1, y\} = P$. Therefore $x_2 = \alpha x_1 + \beta y$ for suitable $\alpha, \beta \in R$. (03) ensures $(1+\alpha)x_1 \perp \beta y$, $(1-\alpha)x_1 \perp (-\beta y)$, and from the hypotheses and (vi) we obtain

$$\begin{aligned} g(x_1+x_2)+g(x_1-x_2) &= g[(1+\alpha)x_1+\beta y]+g[(1-\alpha)x_1-\beta y] \\ &= g[(1+\alpha)x_1]+g(\beta y)+g[(1-\alpha)x_1]+g(-\beta y) \\ &= g(x_1+\alpha x_1)+g(x_1-\alpha x_1)+2g(\beta y) \\ &= 2g(x_1)+2g(\alpha x_1)+2g(\beta y) \\ &= 2g(x_1)+2g(\alpha x_1+\beta y) \\ &= 2g(x_1)+2g(x_2), \end{aligned}$$

so (Q) holds again.

COROLLARY 7. *If $(Y, +)$ is uniquely 2-divisible, then every solution f of $(*)$ has the form $f = g + h$ with g quadratic and h additive.*

This follows immediately from Lemma 3 and Theorems 5 and 6.

REMARK 3. If every solution of $(*)$ is additive, then the only even solution g of $(*)$ is $\underline{0}$. In fact, for every $x \in X$ we have

$$g(x) = g\left(\frac{x}{2}+\frac{x}{2}\right) = g\left(\frac{x}{2}\right)+g\left(\frac{x}{2}\right) = g\left(\frac{x}{2}\right)+g\left(-\frac{x}{2}\right) = g(o) = 0.$$

Here $(Y, +)$ need not be uniquely 2-divisible.

4. Special elements of orthogonality spaces

Following an idea in [11], we prepare here the background for dealing with some of our examples.

DEFINITION 4. If $(X, \perp)$ is an orthogonality space, then an element u of X is called

a) a *ρ-element* if there exists $v \in X$ such that ($u \perp v$ and $u + v \perp u$) or ($v \perp u$ and $u \perp u + v$),

b) a *σ-element* if there exists $w \in X$ such that ($w \perp u$ and $u + w \perp u$) or ($u \perp w$ and $u \perp u + w$).

LEMMA 8. *If $(X, \perp)$ is an orthogonality space, $(Y, +)$ an abelian group, u a ρ-element or a σ-element of X, and $f \in \mathrm{Hom}_\perp(X, Y)$, then f is additive on* $\mathrm{lin}\{u\}$.

Proof. We put $z := v$ if u is a ρ-element, $z := w$ if u if a σ-element. Let $\alpha, \beta \in R$ arbitrary. From the conditions on z, from (03), (*), and (*′) we get

$$\begin{aligned} f(\alpha u) + f(\beta u) + f(\beta z) &= f(\alpha u) + f(\beta u + \beta z) = f(\alpha u) + f[\beta(u + z)] \\ &= f[\alpha u + \beta(u + z)] = f(\alpha u + \beta u + \beta z) \\ &= f[(\alpha + \beta)u + \beta z] = f[(\alpha + \beta)u] + f(\beta z) \\ &= f(\alpha u + \beta u) + f(\beta z), \end{aligned}$$

and subtraction of $f(\beta z)$ on both sides completes the proof.

REMARK 4. By (01), the zero vector o always is a ρ-element and a σ-element (choose $v = o$, $w = o$, respectively). If $\perp$ stems from an inner product (Example B), then o is the only ρ-element and the only σ-element of $(X, \perp)$. Thus ρ-elements and σ-elements are instruments relevant only for non inner product orthogonality spaces, a fact which is illustrated for the first time as follows.

REMARK 5. Let $\perp$ be the trivial orthogonality on the vector space X (Example A). Since $\perp$ is symmetric, ρ-elements and σ-elements coincide. Moreover we have:

a) Every element of X is a ρ-element.

b) For an arbitrary abelian group $(Y, +)$, every $f \in \mathrm{Hom}_\perp(X, Y)$ is additive.

Proof. a) Let $u \neq o$ be arbitrary (the case $u = o$ is settled by Remark 4). Choose $v \in X$ so that u, v are linearly independent. Hence $v \neq o$, $u \perp v$, $u + v \neq o$, and $u + v$, u are linearly independent, i.e., $u + v \perp u$, i.e., u is a ρ-element.

b) Let $x_1, x_2 \in X$ arbitrary and u, v linearly independent such that x_1, $x_2 \in \mathrm{lin}\{u, v\}$. Then u, v are ρ-elements and $u \perp v$. By Lemma 8, f is additive on $\mathrm{lin}\{u\}$ and on $\mathrm{lin}\{v\}$, hence by Lemma 4 additive on $\mathrm{lin}\{u, v\}$, and we get $f(x_1 + x_2) = f(x_1) + f(x_2)$, i.e., additivity of f on X.

REMARK 6. Let $\perp$ be the trivial orthogonality on $X = R^2$. By Remarks 5b and 3, the only even solution $g : R^2 \to R$ of $(*)$ is $\underline{0}$. The square of the euclidean norm on R^2 is a quadratic mapping $\neq \underline{0}$, hence not a solution of $(*)$. This shows that there is no converse of Theorem 6, a contrast to the situation in Theorem 5.

5. Solutions of $(*)$ on inner product spaces

For Example A, the even solutions of $(*)$ could be specified in Remark 6. The same is possible for Example B, as we now show.

THEOREM 9. *If $(X, \perp)$ is an inner product space and $(Y, +)$ an abelian group, then $g : X \to Y$ is an even solution of $(*)$ if and only if there exists an additive mapping $l : R \to Y$ such that $g(x) = l(\|x\|^2)$ for every $x \in X$* ([14], Theorem 2).

Proof. If l exists, then g is even and, by Pythagoras' theorem, a solution of $(*)$. Conversely, assume that g be an even solution of $(*)$. If $u, v \in X$, $\|u\| = \|v\|$, then $u + v \perp u - v$, and (iii) in the proof of Theorem 6 implies $g(u) = g(v)$. Thus g is constant on each sphere around o, and $\tilde{l} : R_+ \to Y$ is well-defined by $\tilde{l}(\|x\|^2) := g(x)$ $(\forall x \in X)$. If $\lambda, \mu \in R_+$ then there exist $x, y \in X$ such that $x \perp y$ and $\|x\|^2 = \lambda$, $\|y\|^2 = \mu$, hence $\|x + y\|^2 = \|x\|^2 + \|y\|^2 = \lambda + \mu$, i.e., $\tilde{l}(\lambda + \mu) = \tilde{l}(\|x + y\|^2) = g(x + y) = g(x) + g(y) = \tilde{l}(\|x\|^2) + \tilde{l}(\|y\|^2) = \tilde{l}(\lambda) + \tilde{l}(\mu)$. As an additive mapping, $\tilde{l}$ has an additive extension $l : R \to Y$ ([2, p. 265, Theorem 2]), and $g(x) = l(\|x\|^2)$ $(\forall x \in X)$ holds.

COROLLARY 10. *If $(X, \perp)$ is an inner product space and $(Y, +)$ a uniquely 2-divisible abelian group, then $f : X \to Y$ is a solution of $(*)$ if and only if there exist additive mappings $l : R \to Y$, $h : X \to Y$ such that $f(x) = l(\|x\|^2) + h(x)$ for every $x \in X$* ([14, Corollary 3]).

This was also found independently by R. Ger and by Gy. Szabó. In [6], an application to ideal gas theory is indicated. The "if" part follows from Theorem 9, Theorem 5, and Lemma 2b, the "only if" part from Lemma 3 and the same two theorems.

We now turn to deriving from Corollary 10 results under various regularity conditions.

COROLLARY 11. *If $(X, \perp)$ is an inner product space and $(Y, +)$ a separated topological R-vector space then, for any continuous solution $f : X \to Y$ of $(*)$, there exist $a \in Y$ and $h : X \to Y$ linear and continuous such that $f(x) = \|x\|^2 \cdot a + h(x)$ for every $x \in X$.*

Continuity may be replaced by hemicontinuity ([7, p. 427]); see also [7, p. 429, Corollary 2.3b] and [17, p. 190, first part of Theorem 1]. For $X = L_2[a, b]$, $Y = R$, we obtain the main result of [13].

Proof. If f is continuous, so are its even and odd parts $x \mapsto l(\|x\|^2)$ and h. Let $\lambda \in R_+^*$ and $\alpha_n \in Q$, $\alpha_n > 0$ such that $\alpha_n \to \lambda$ $(n \to \infty)$. Furthermore, $x \in X$ such that $\|x\|^2 = \lambda$ and $x_n := (\alpha_n/\lambda)^{1/2} \cdot x$ $(\forall n \in N)$. Then $l(\|x_n\|^2) \to l(\|x\|^2) = l(\lambda)$ $(n \to \infty)$. On the other hand, $l(\|x_n\|^2) = l[(\alpha_n/\lambda)\|x\|^2] = l(\alpha_n) = \alpha_n l(1) \to \lambda l(1)$, and separatedness of Y implies $l(\lambda) = \lambda l(1)$. Thus l is positive-homogeneous, and oddness of l leads to $l(\lambda) = \lambda l(1)$ for all $\lambda \in R$. If we put $a := l(1)$, we get $l(\|x\|^2) = \|x\|^2 \cdot a$. Linearity of h is obtained similarly.

For the next two results see [7, p. 429, Corollary 2.4], and [8]. We present them here with different approaches.

COROLLARY 12. *If $(X, \perp)$ is an inner product space and if $f \in \mathrm{Hom}_\perp(X, R)$ has the property $|f(x)| \leq m\|x\|$ for all $x \in X$ and a fixed $m \in R_+$, then f is a continuous linear functional.*

Proof. By Corollary 10, $f(x) = l(\|x\|^2) + h(x)$ $(\forall x \in X)$ with additive mappings $l: R \to R$, $h: X \to R$, and we have $|l(\|x\|^2)| = |\frac{1}{2}[f(x) + f(-x)]| \leq m\|x\|$. Assume that there exist $z \in X$ such that $l(\|z\|^2) \neq 0$. Then for every $n \in N$ we get $|l(\|nz\|^2)| \leq m\|nz\|$, i.e., $n^2|l(\|z\|^2)| \leq nm\|z\|$, i.e., $n|l(\|z\|^2)| \leq m\|z\|$, a contradiction. Therefore $l(\|x\|^2) = 0$ for all $x \in X$, thus $f = h$, i.e., $|h(x)| \leq m\|x\|$ $(\forall x \in X)$. Hence h is continuous at o, and additivity of h implies that h is continuous on the whole of X. A standard argument leads to linearity of h.

COROLLARY 13. *If $(X, \perp)$ is an inner product space and if $f \in \mathrm{Hom}_\perp(X, R)$ has the property $f(x) \geq 0$ for all $x \in X$, then there exists $c \in R_+$ with $f(x) = c\|x\|^2$ $(\forall x \in X)$.*

Proof. By Corollary 10, $f(x) = l(\|x\|^2) + h(x)$ $(\forall x \in X)$ with additive mappings $l: R \to R$, $h: X \to R$, and we have $l(\|x\|^2) = \frac{1}{2}[f(x) + f(-x)] \geq 0$, i.e., $l(R_+) \subset R_+$, which guarantees the existence of $c \in R_+$ with $l(\alpha) = c\alpha$ $(\forall \alpha \in R)$ ([1, p. 34, Theorem 1]). Consequently, $f(x) = c\|x\|^2 + h(x)$ $(\forall x \in X)$, and, from $c\|x\|^2 + h(x) \geq 0$, $c\|-x\|^2 - h(x) \geq 0$ we get $|h(x)| \leq c\|x\|^2$ for every $x \in X$. Suppose the existence of $z \in X$ such that $h(z) \neq 0$. Then for every $n \in N$ we get

$$\left|h\left(\frac{1}{n}z\right)\right| \leq c\left\|\frac{1}{n}z\right\|^2,$$

i.e.,

$$\frac{1}{n}|h(z)| \leq \frac{c}{n^2}\|z\|^2,$$

i.e., $n|h(z)| \leq c\|z\|^2$, a contradiction. Therefore $h = \underset{\sim}{0}$, i.e., $f(x) = c\|x\|^2$ $(\forall x \in X)$.

COROLLARY 14. *If* $(X, \perp)$ *is an inner product space and* $f: X \to R$, *then* f *is orthogonally additive and bounded below if and only if*

(i) $f = \underset{\sim}{0}$ *or*

(ii) *there exist* $c \in R_+^*$ *and* $h: X \to R$ *continuous and linear such that* $f(x) = c\|x\|^2 + h(x)$ $(\forall x \in X)$.

This extension of Corollary 13 was proved independently by P. Fischer (oral communication).

Proof. 1. Let $f \in \mathrm{Hom}_\perp(X, R)$ bounded below, say $f(x) \geq d$ $(\forall x \in X)$. Then, by Lemma 2a, $d \leq 0$, and from a similar argument as in the proof of Corollary 13, we obtain the existence of $c \in R_+$, $h: X \to R$ additive such that

$$f(x) = c\|x\|^2 + h(x) \qquad (\forall x \in Y). \tag{1}$$

Case 1. $c = 0$. Then $f = h$, i.e., $h \geq \underset{\sim}{d}$. Since $h(nx) = nh(x)$ for all $x \in X$ and all $n \in Z$, we necessarily have $h = \underset{\sim}{0}$, i.e., $f = \underset{\sim}{0}$, i.e., variant (i) holds.

Case 2. $c \in R_+^*$. From (1) and $f \geq \underset{\sim}{d}$ we obtain

$$-h(x) \leq c\|x\|^2 - d, \quad h(x) \leq c\|x\|^2 - d, \quad |h(x)| \leq c\|x\|^2 - d \quad (\forall x \in X). \tag{2}$$

Let $x \in X$ arbitrary but fixed. Then $\alpha \mapsto h(\alpha x)$ is an additive mapping from R into R. For $0 \leq \alpha \leq 1$, (2) implies

$$h(\alpha x) \leq c\|\alpha x\|^2 - d \leq c\alpha^2\|x\|^2 - d \leq c\|x\|^2 - d,$$

thus, by [1, p. 34, Theorem 1], $h(\alpha x) = \alpha h(x)$ $(\forall \alpha \in R)$, so h is linear. From (2) we also conclude

$$\sup\{|h(x)|; \|x\| \leq 1\} \leq c - d,$$

i.e., continuity of h. This is variant (ii).

2. If (i) or (ii) holds, then certainly $f \in \mathrm{Hom}_\perp(X, R)$ by Corollary 10. In the case of variant (ii), $|h(x)| \leqslant \|h\| \cdot \|x\|$, i.e., $h(x) \geqslant -\|h\| \cdot \|x\|$, i.e., $f(x) \geqslant c\|x\|^2 - \|h\| \cdot \|x\|$ $(\forall x \in X)$. Since $c \in R_+^*$, the mapping $\alpha \mapsto c\alpha^2 - \|h\|\alpha$ $(\alpha \in R)$ is bounded below, and then so is f. The same fact is trivially true for variant (i).

COROLLARY 15. *For an inner product space* $(X, \perp)$ *the following statements are equivalent*:

(a) *For every* $f \in \mathrm{Hom}_\perp(X, R)$ *which is bounded below, there exists an element* x_0 *in* X *such that* $f(x_0) \leqslant f(x)$ *for all* $x \in X$.

(b) X *is a Hilbert space.*

Proof. (a) implies (b): Let $h : X \to R$ be a continuous linear functional and define $f : X \to R$ by

$$f(x) := \|x\|^2 + h(x) \qquad (\forall x \in X). \tag{3}$$

By Corollary 14, $f \in \mathrm{Hom}_\perp(X, R)$, and f is bounded below. By (a) there exists $x_0 \in X$ such that $f(x_0) \leqslant f(x)$ $(\forall x \in X)$. Moreover $f(o) = 0$. From [7, p. 435, Corollary 3.6], we conclude the existence of $c \in R_+$ with the property

$$f(x) = c\|x - x_0\|^2 + f(x_0) \qquad (\forall x \in X). \tag{4}$$

From (3) and (4) we get

$$h(x) = \tfrac{1}{2}[f(x) - f(-x)] = \tfrac{1}{2}[c\|x - x_0\|^2 - c\|-x - x_0\|^2]$$
$$= \frac{c}{2}[\|x - x_0\|^2 - \|x + x_0\|^2] = -2c\langle x, x_0\rangle = \langle x, -2cx_0\rangle \qquad (\forall x \in X).$$

Let Ω denote the (conjugate)-linear isometry from X into its dual space X^* defined by $\Omega(a) := \langle \cdot, a\rangle$ $(\forall a \in X)$. Since $h \in X^*$ is arbitrary, we see that $\Omega(X) = X^*$, i.e., that X is a Hilbert space (cf., e.g., [3, p. 105, Theorem 2]).

(b) implies (a): Let X be a Hilbert space and $f \in \mathrm{Hom}_\perp(X, R)$ bounded below. By Corollary 14, there are two cases. If $f = \underline{0}$, then $x_0 := o$ satisfies the condition in (a). In the remaining case, there are $c \in R_+^*$, $h \in X^*$ such that

$$f(x) = c\|x\|^2 + h(x) \qquad (\forall x \in X). \tag{5}$$

By the Riesz representation theorem there exists $z \in X$ with

$$h(x) = \langle x, z\rangle \qquad (\forall x \in X). \tag{6}$$

From (5) and (6) we get

$$f(x) = c\left[\|x\|^2 + \frac{1}{c}\,h(x)\right] = c\left[\|x\|^2 + \frac{1}{c}\langle x, z\rangle\right]$$
$$= c\left[\|x\|^2 + \left\langle x, \frac{1}{c}\,z\right\rangle\right] = c\left[\|x\|^2 + 2\left\langle x, \frac{1}{2c}\,z\right\rangle + \left\|\frac{1}{2c}\,z\right\|^2 - \left\|\frac{1}{2c}\,z\right\|^2\right]$$
$$= c\left\|x + \frac{1}{2c}\,z\right\|^2 - c\left\|\frac{1}{2c}\,z\right\|^2 \geqslant -c\left\|\frac{1}{2c}\,z\right\|^2 = -\frac{1}{4c}\|z\|^2.$$

For $x_0 := -(1/2c)z$ we therefore have

$$f(x_0) = -\frac{1}{4c}\|z\|^2 \quad \text{and} \quad f(x) \geqslant f(x_0) \qquad (\forall x \in X).$$

So (a) is true.

6. Solutions of (*) on normed vector spaces

We suppose in this section that $(X, \|\cdot\|)$ is furnished with the Birkhoff–James orthogonality (see Example C in Section 1), and we recall the remarks at the beginning of Section 3 about the continuous real-valued solutions of (*). With respect to Theorem 5, we have to focus our interest on the general even solution.

THEOREM 16. *For a normed R-vector space* $(X, \|\cdot\|)$ *with Birkhoff–James orthogonality* $\perp$ *and an arbitrary abelian group* $(Y, +)$, *each of the following conditions is sufficient for the additivity of every* $f \in \mathrm{Hom}_\perp(X, Y)$:

(a) $\dim_R X \geqslant 3$, *and X is not an inner product space.*

(b) $\dim_R X = 2$, *and the unit ball T of X is a polygon.*

(c) $\dim_R X = 2$, *and* $\perp$ *is not symmetric.*

It then follows from Remark 3 that every even $g \in \mathrm{Hom}_\perp(X, Y)$ is $\underset{\sim}{0}$.

Proof. (a) is due to Lawrence [11]. The restriction to the case of $\dim_R X \geqslant 3$ stems from the use of a theorem of Day ([4, p. 333, Theorem 6.4]) and James ([10, p. 560, Theorem 1]) which has no analogue for $\dim_R X = 2$.

(b) Let x and $x + y$ be any two consecutive vertices of the convex polygon T. Then $x \perp y$ and $x + y \perp y$, i.e., any "side vector" y of T is a σ-element in the sense of Definition 4b. For every side vector y of T, the point $(1/\|y\|)y$ lies on the boundary of T. Consequently there exists a side vector w of T such that $(1/\|y\|)y \perp w$, i.e., $y \perp w$. By (02), y and w are linearly independent. Let $f \in \mathrm{Hom}_\perp(X, Y)$ arbitrary. Since y and w are σ-elements, Lemma 8 ensures additivity of f on $\mathrm{lin}\{y\}$ and on $\mathrm{lin}\{w\}$, hence by Lemma 4 also additivity on $\mathrm{lin}\{y, w\} = X$.

(c) Let $u, v \in X$ be such that $u \perp v$ but not $v \perp u$. (01) guarantees that $u \neq o$ and $v \neq o$. By [9, p. 269, Theorem 2.3], there exists $\alpha \in R$ such that $\alpha u + v \perp u$, and our hypothesis makes $\alpha = 0$ impossible. Therefore by (03),

$$u + \frac{1}{\alpha} v \perp u \quad \text{and} \quad u \perp \frac{1}{\alpha} v$$

which shows that u is a ρ-elemeent. In a similar way, using Corollary 2.2 of [9], we see that also v is a ρ-element. Now we proceed as in the proof of (b) by means of Lemmas 8 and 4.

REMARK 7. In the situation of Example C, the existence of non-zero σ-elements in X means that the normed space $(X, \|\cdot\|)$ is not smooth or not strictly convex ([9, pp. 274–275, Theorems 4.2 and 4.3]).

REMARK 8. The l^1-norm and the l^∞-norm on R^2 satisfy conditions (b), (c) of Theorem 16. The mixed l^1–l^∞-norm on R^2 defined by $\|(\xi, \eta)\| := |\xi| + |\eta|$ if $\xi \cdot \eta \geq 0$, $\|(\xi, \eta)\| := \max\{|\xi|, |\eta|\}$ if $\xi \cdot \eta < 0$ satisfies (b) but not (c) ([10, p. 561]). Finally, for $p > 1$, $p \neq 2$, the l^p-norm on R^2 violates (b) but satisfies (c) as the vectors $u = (-1, 2^{1/p})$, $v = (1, 2^{(1/p)-1})$ demonstrate: $u \perp v$ but not $v \perp u$. This shows that conditions (b) and (c) in Theorem 16 are not necessary for $\mathrm{Hom}_\perp(X, Y) \subset \mathrm{Hom}(X, Y)$ and that a good condition for the case $\dim_R X = 2$ is still missing.

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