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There Exist $6n/13$ Ordinary Points*

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Abstract. In 1958 L. M. Kelly and W. O. J. Moser showed that apart from a pencil, any configuration of n lines in the real projective plane has at least $3n/7$ ordinary or simple points of intersection, with equality in the Kelly–Moser example (a complete quadrilateral with its three diagonal lines). In 1981 S. Hansen claimed to have improved this to $n/2$ (apart from pencils, the Kelly–Moser example and the McKee example). In this paper we show that one of the main theorems used by Hansen is false, thus leaving $n/2$ open, and we improve the $3n/7$ estimate to $6n/13$ (apart from pencils and the Kelly–Moser example), with equality in the McKee example. Our result applies also to arrangements of pseudolines.

1. Introduction

In 1893 Sylvester posed the following problem in a column of mathematical problems and solutions in the *Educational Times* [Sy]:

(1.1) "Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third, unless they all lie in the same right line."

The following incorrect solution was advanced by a reader in a subsequent issue ([W1]—see also [W2]): "Suppose that a set of n points are so situated as to fulfil the condition (but not collinear). Now abstract one of them and note its position; then the $\frac{1}{2}(n-1)(n-2)$ lines joining the remaining points must pass through the position of the abstracted point, which is absurd." The problem

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remained unsolved until 1933 when it was independently raised by Erdös and solved shortly after by Gallai (see [E1], [E2], and [E3]). Other solutions have been given by Melchior [Me] (where the dual (1.2) is proved), Kelly (see p. 28 of [C1] and p. 65 of [C2]), Robinson (see [Mot]), Steinberg [St], Coxeter [C1, p. 27], [C2, p. 181], Lang [L]), and others. Melchior's solution uses Euler's formula, and the others, while they differ slightly in character, are based on the same clever idea of Gallai, most transparent in Lang's proof. Before giving these proofs, we note that (1.1), which is the same problem in either the Euclidean or real projective plane, can be dualized to

(1.2) Let there be given a finite set of lines in the real projective plane, with not all the lines passing through one point. Show that among all the intersection points of these lines, at least one is incident with exactly two of the lines.

We remark that the principle of duality for the real projective plane was first stated explicitly by Gergonne in 1826 (see p. 29 of [VY]). In the following we discuss only (1.2) and, for convenience, we translate results on (1.1) into this dual setting. A point that is incident with exactly two of the lines is called ordinary (see [Mot] where this terminology is introduced).

Melchior's proof of (1.2) reduces to the observation that if no point were ordinary, then every vertex would be of degree at least six, and so $6V \leq 2E$. Since every region is bounded by at least three edges, $3F \leq 2E$. Combining these inequalities yields $6(V + F - E) \leq 0$, contradicting Euler's formula in the projective plane. It is interesting to note that the map dual of this observation was known in the Euclidean plane 14 years before Sylvester posed his problem. In his attempt to prove the four-colour theorem, Kempe deduced from Euler's formula that [K, bottom of p. 198] "every map drawn on a simply connected surface must have a district with less than six boundaries." (Applied to the dual map of a configuration in the Euclidean plane, this shows the existence of a point with less than six edges, i.e., an ordinary point.)

Here is Lang's proof of (1.2) (which is the dual of Gallai's proof of (1.1)). Choose a line l and then a point (of intersection) P that is off l but closest to it. If P itself is not ordinary, then there are lines l_1, l_2, and l_3 through P meeting l at points P_1, P_2, and P_3 which we may assume appear left to right along the horizontal line l. Then P_2 is necessarily ordinary as an additional line m through P_2 would intersect either l_1 or l_3 at a point M closer to l than P (see Fig. 1).

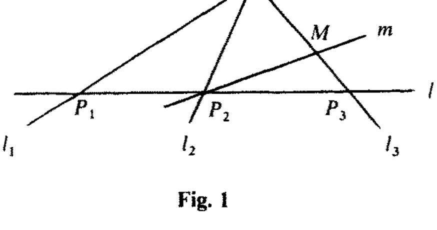

Fig. 1

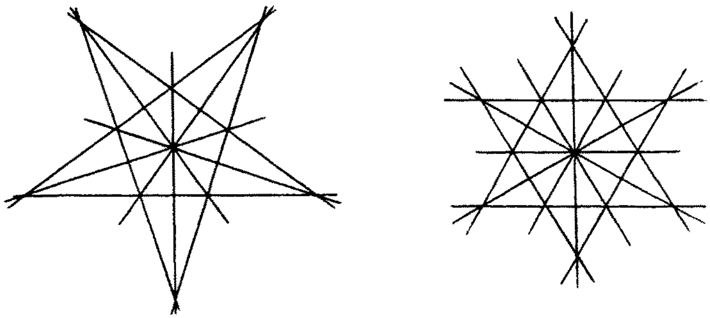

Fig. 2. Examples of Böröczky.

With the existence of an ordinary point established, Dirac [D1] turned to the question of how many. If Γ is a collection of nonconcurrent lines and Σ is the set of intersection points, let n be the number of lines in Γ and let s be the number of ordinary points in Σ. Dirac showed $s \geq 3$ and conjectured $s \geq [\frac{1}{2}n]$ in general. In fact, for n even, examples by Böröczky (see [CM]) have exactly $\frac{1}{2}n$ ordinary points. Simply consider a regular $(n/2)$-gon and let Γ consist of the $n/2$ lines containing the edges together with the $n/2$ lines of reflective symmetry. The midpoints of the edges are the only ordinary points. The cases $n = 10$, 12 are shown in Fig. 2. Thus the optimal asymptotic result is (at least for n even) Dirac's conjecture. In [Mot] Motzkin showed $s > \sqrt{2n} - 2$ and in [KM] Kelly and Moser showed that $s \geq \frac{3}{7}n$ with equality occurring in the "Kelly–Moser configuration" (see Fig. 3), a complete quadrilateral with its three diagonal lines, having three ordinary points in a configuration of seven lines. In fact, this was the only known example with $s < \frac{1}{2}n$ until the "McKee configuration" [CM] of six ordinary points in a configuration of 13 lines (two of the ordinary points are at infinity in Fig. 4).

In his 1981 dissertation for the habilitation, Hansen [H] claims that, except for pencils and the special configurations of Figs. 3 and 4, $s \geq \frac{1}{2}n$. However, in both [EP, p. 6] and [BM, p. 121] the authors indicated that Hansen's argument, which is long (96 pp.) and difficult to read, had not yet been independently checked. As we show in Section 3 below, one of the main subtheorems in Hansen's thesis is actually false, thus leaving Dirac's conjecture open. We also mention here Hansen's conjecture that the configuration of 12 lines (other than the line at

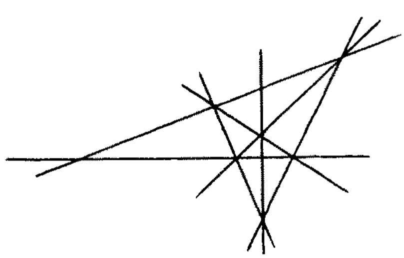

Fig. 3. The "Kelly–Moser configuration."

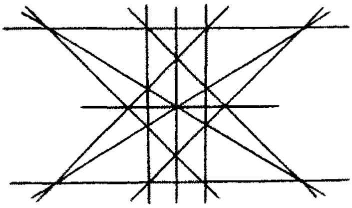

Fig. 4. The "McKee configuration." The 13th line is at infinity.

infinity) shown in Fig. 4 and the examples of Böröczky, as shown in Fig. 2, are the only configurations for which $s = \frac{1}{2}n$.

The purpose of this paper is to improve on the $3n/7$ estimate of Kelly and Moser by showing that, except for pencils and the Kelly–Moser configuration (Fig. 3), the number s of ordinary points in a configuration of n lines is at least $\frac{6}{13}n$, with equality of course occurring for the McKee configuration (Fig. 4). Our proof readily extends to arrangements of pseudolines, improving on the previous result of $s \geq \frac{3}{7}n$, due to Kelly and Rottenberg [KR]. Grünbaum has conjectured that $s \geq \frac{1}{2}n$ for $n \neq 7, 13$ [G] in an arrangement of pseudolines (see also the discussion in [H] near the end of Chapter 1).

We now sketch the main ideas in our proof. Lang's (Gallai's) proof of (1.2) associates to each line l an ordinary point either on l itself or attached to l in a certain minimal sense (as indicated above and made precise in Definition 2.7 below). Since any given ordinary point P is on exactly two lines and attached to at most four lines (the corresponding minimal triangles in Definition 2.7 must lie in distinct sectors at P), lower bounds on s in terms of n can be obtained by showing that, on average, each line is associated to some positive number of ordinary points. For example, since every line is associated to at least one ordinary point by Lang's (Gallai's) result, we clearly have $s \geq \frac{1}{6}n$ (compare [Mos] where it is shown that $s > (n + 11)/6$ for n even). A simple way to picture this is to imagine an $n \times s$ matrix whose rows are labeled by the lines l and whose columns are labeled by the ordinary points P. The entry in the lth row and Pth column is 1 if P is on l or attached to l, and 0 otherwise. The row sums are at least 1 and the column sums at most 6, and so $n \leq 6s$. It is not hard to modify this approach to yield $s \geq \frac{1}{4}n$ by observing that Lang's argument actually shows that any line without ordinary points must have at least two attached—one from each "side." With an (l, P)-entry of 2 if P is on l, 1 if P is attached to l, and 0 otherwise, the row sums are at least 2 and the column sums at most 8, yielding $2n \leq 8s$. However, further improvements require additional geometric information about configurations of lines in the plane.

Kelly and Moser provided such information in their 1958 paper to prove $s \geq \frac{3}{7}n$. They showed that any line having no ordinary points must have at least three attached, while any line having just one ordinary point must have at least two attached (see [D2] for a correction). If we now set the (l, P)-entry of our matrix to be $\frac{3}{2}$ if P lies on l, 1 if P is attached to l, and 0 otherwise, then the row sums are at least 3 and the column sums at most 7, thus showing $3n \leq 7s$.

In this paper we first give a short and simple proof of a generalization of the dual of one of Hansen's results, namely, that, apart from the Kelly–Moser configuration (Fig. 3), if two lines intersect in an ordinary point, then at least one of them has three or more ordinary points associated to it. We then give a counting argument to show that, on average, the row sums are at least 3 when we assign to the (l, P)-entry of our matrix $\frac{5}{4}$ if P lies on l, 1 if P is attached to l, and 0 otherwise. Since the column sums are at most $\frac{5}{4} \cdot 2 + 4$ or $\frac{13}{2}$, we obtain $3n \leq \frac{13}{2}s$.

2. The Theorems

A common realization of the real projective plane is the two-dimensional sphere with antipodal points identified and where the lines are great circles. We can identify the projective plane, with a great circle removed, as the Euclidean plane via stereographic projection from the center of the sphere to a tangent plane parallel to the great circle removed. These "windows" into the projective plane allow us to describe the order relations there in terms of the familiar concepts of left, right, up, and down. We use the following notation when viewing the projective plane through one of these windows. If l is a nonvertical line containing points A and B, the closed segment $[A, B]$ will denote all points on l obtained by moving to the right along l from A to B (see Fig. 5). For a vertical line l, replace right by up. To prevent confusion, vertical lines will be avoided whenever possible. Denote the open segment by (A, B) and the half-open, half-closed segments by $[A, B)$ and $(A, B]$. Note that any three nonconcurrent lines determine four triangles whose edges are $[A, B]$ or $[B, A]$, $[B, C]$ or $[C, B]$, and $[A, C]$ or $[C, A]$ (the latter choice is determined by the first two) where A, B, and C are the intersection points of the lines. Finally, these windows permit the use of Euclidean distance when convenient in certain projective arguments.

Definitions. Suppose PAB is a closed triangle with $[A, B]$ on a line l. The point P is the l-vertex and the segment $[A, B]$ is the l-base.

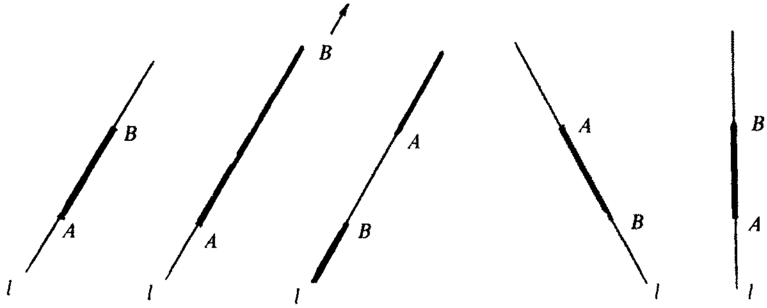

Fig. 5

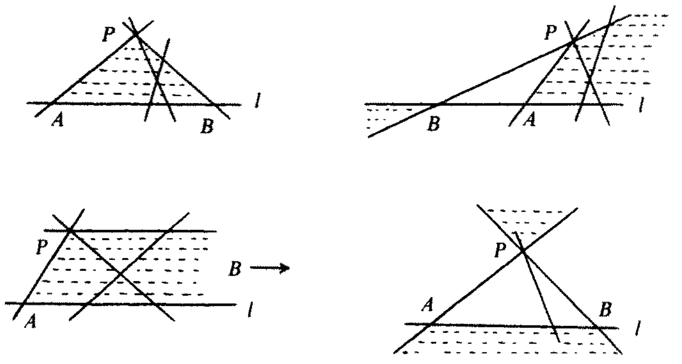

Fig. 6. Examples of l-wide triangles.

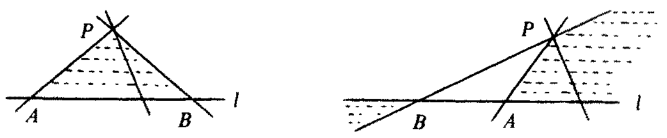

Fig. 7. Examples of l-minimal triangles.

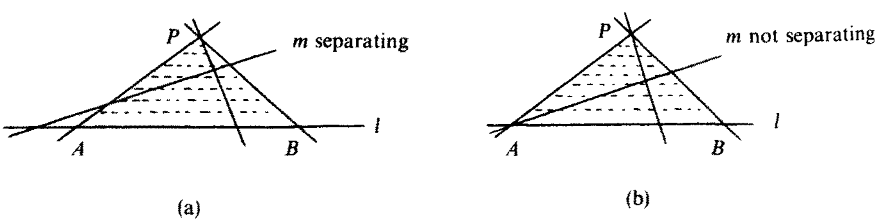

Fig. 8. (a) PAB is not l-solid. (b) PAB is l-solid.

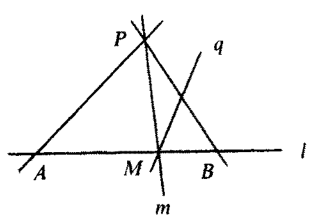

Fig. 9

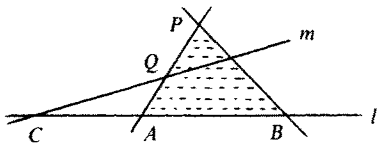

Fig. 10

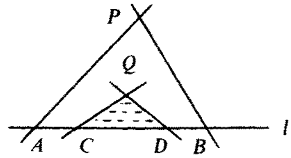

Fig. 11

(2.1) PAB is l-wide if there is no line through P that meets l outside the closed segment $[A, B]$, i.e., in (B, A) (see Fig. 6).

(2.2) PAB is l-minimal if it is l-wide and contains no points off l other than P itself (see Fig. 7).

(2.3) PAB is l-solid if no line through PAB separates P from the closed segment $[A, B]$. In other words, PAB fails to be l-solid if there is a line m such that $PAB \backslash m$ consists of two components, one containing P and the other $[A, B]$ (see Fig. 8).

Note that an l-minimal triangle is l-solid.

(2.4) PAB is l-blemished if it contains a point off l other than P.

Lemma 2.5. *Suppose PAB is l-minimal. Then either P is an ordinary point or there is at least one point in the open segment (A, B) and every such point is ordinary.*

Proof. If P is not ordinary, there is a line m through P meeting l at M in (A, B). The point M is ordinary since another line q through M would intersect either (A, P) or (P, B), contradicting the l-minimality of PAB (see Fig. 9). □

Lemma 2.6. *Suppose PAB is l-solid. If PAB is l-blemished or l-wide, then PAB contains an l-minimal triangle.*

Proof. Choose a window in which the triangle PAB is bounded. Suppose first that PAB is l-solid and l-blemished. Choose a point Q in PAB off l that is closest to l. Then $Q \neq P$. Choose C and D on l as far apart as possible so that QCD is l-wide. Then QCD is contained in PAB since if, for example, C is not in $[A, B]$, then QC lies on a line m that separates P from $[A, B]$, contradicting PAB is l-solid (see Fig. 10). Clearly, QCD is l-minimal since any point other than Q in $QCD \backslash l$ would be in PAB and closer to l than Q (see Fig. 11).

Now suppose that PAB is l-solid and l-wide. If PAB is l-minimal, there is nothing left to prove. Otherwise PAB contains a point off l other than P, making PAB l-blemished, a case we have already considered. □

Definition 2.7. If PAB is l-minimal and P is ordinary, then P is attached to l.

An immediate corollary of Lemma 2.5 and Lemma 2.6 is

Lemma 2.8. *Suppose PAB is l-solid. If PAB is l-blemished or l-wide, then there is a point Q in PAB other than A or B that is ordinary and either on l or attached to l.*

Definition 2.9. A line l has type $T(l) = (\mu, v)$ if there are exactly μ ordinary points on l and v ordinary points attached to l.

Definition 2.10. If a line l has type (μ, v) and $1 \leq \alpha \leq 2$, the α-weight of l, denoted $w_\alpha(l)$, is $\alpha\mu + v$.

The following corollary of Lemma 2.8 is used repeatedly in the next theorem.

Lemma 2.11. *Suppose Q is an ordinary point on a line l. Suppose T_1 and T_2 are l-solid triangles with at most one point in common, neither containing Q except possibly as a vertex. Suppose moreover that T_1 is either l-wide or l-blemished, and that T_2 is either l-wide or l-blemished. Then l cannot have type* (2, 0).

Proof. By Lemma 2.8, $w_1(l) \geq 3$. □

The next theorem is a slight generalization of the dual of subtheorem 19 of [H] (where the lines are more restricted than type (2, 0)).

Theorem 2.12 *Suppose Γ is a finite configuration of lines in the real projective plane having two lines of type* (2, 0) *that intersect in an ordinary point. Then Γ is the Kelly–Moser configuration* (*Fig.* 3).

Proof. First observe that Γ cannot be a near-pencil, i.e., a configuration in which all lines but one pass through a single point. Indeed, if $n = 3$, then each line has type (2, 1), while if $n \geq 4$, one line has type $(n - 1, 0)$ and the rest have type (1, 2). Suppose l_1 and l_2 are horizontal lines of type (2, 0) that intersect in an ordinary point Q at infinity. Let A and B (resp. C and D) be the extreme left and right points on l_1 (resp. l_2). We have $A \neq B$ and $C \neq D$ since Γ is not a near-pencil. By applying an appropriate projective transformation, we may assume that $ABCD$ is a rectangle (see Fig. 12). We now complete the proof in four steps by eliminating, with the aid of Lemmas 2.8 and 2.11, successively more and more configurations until only (Fig. 3) remains.

Step 1. None of the extreme points A, B, C, D is ordinary.

Proof. Suppose on the contrary that A is ordinary, m is the other line through A, and that m meets l_2 at P. If T is the triangle bounded by $[A, P]$, $[A, Q]$, and $[P, Q]$, then T is l_1-solid and l_1-wide since no line can intersect the open segment (Q, A). Lemma 2.8 now shows that l_1 cannot have type (2, 0), a contradiction (see Fig. 13).

We may now assume that A, B, C, D are not ordinary points.

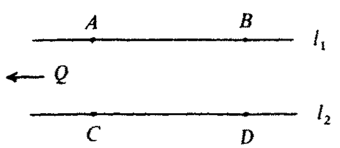

Fig. 12

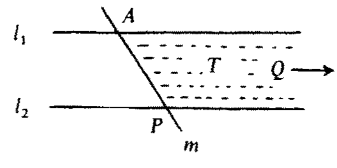

Fig. 13

Step 2. A line contains either none of the extreme points A, B, C, D or exactly two of them.

Proof. Suppose on the contrary that l passes through A but not through C or D. Let m and n be lines through C and D respectively but not through A (which exist since C and D are not ordinary). Then if l intersects m, n, and l_2 at O, P, and R, respectively, the triangles OCR and PRD are l_2-solid and l_2-wide. For example, any separating line of OCR would have to intersect one of the forbidden segments $[Q, A)$ or $[Q, C)$ (drawn dashed in Fig. 14), and similarly for any line through O that meets l_2 outside $[C, R]$. Lemma 2.11 now yields a contradiction (see Fig. 14).

We may now assume that there are lines l_3 through A and C, l_4 through B and D, l_5 through B and C, and l_6 through A and D, and that no lines other than $l_1, \ldots, l_6$ pass through A, B, C, or D. Let P be the intersection of l_5 and l_6. Note that l_3 and l_4 are vertical (see Fig. 15).

Step 3. All additional lines are vertical.

Proof. Suppose on the contrary that there is an additional nonvertical line. Then there is at least one point on l_3 not at infinity and other than A or C. Let E be such a point with the property that either (E, C) or (A, E) is empty of points. There is then a similar such point F on l_4. We now derive a contradiction. Suppose E lies below l_2. Then (E, C) is empty of points since (A, E) already contains a point at infinity. If there is a line l through E passing above P, and if l intersects l_6, l_1, and l_4 at O, R, and S, respectively, then OAR and SRB are l_1-solid and l_1-wide. For example, any separating line of OAR would have to intersect one of the forbidden segments $[Q, A)$, $[Q, C)$, or $(E, C]$, and similarly for any line through O meeting l_1 outside $[A, R]$. Lemma 2.11 now yields a contradiction (see Fig. 16).

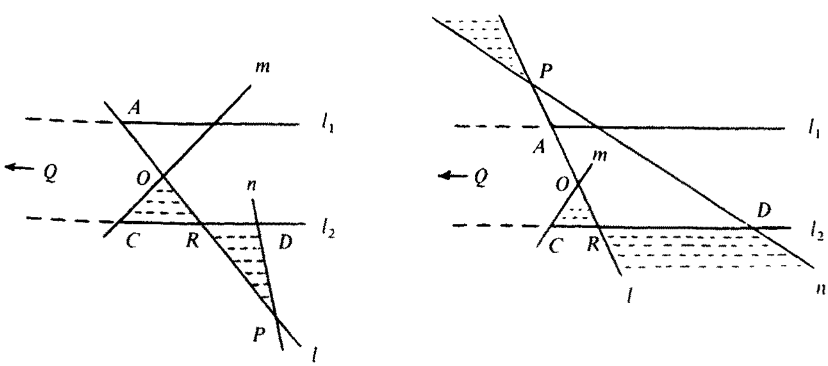

Fig. 14

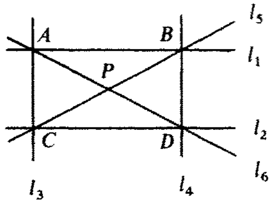

Fig. 15

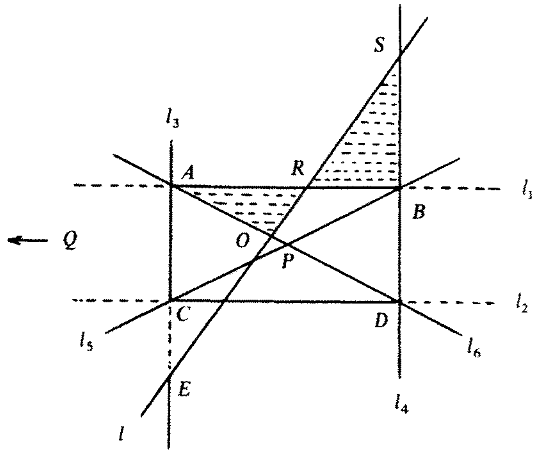

Fig. 16

On the other hand, if l passes below P and is chosen to intersect l_2 at M as far to the right as possible (so that ECM is l_2-wide), and if l intersects l_6 at O, then, as above, triangles ECM and OMD are l_2-solid and l_2-wide and again Lemma 2.11 yields a contradiction (see Fig. 17).

The case when E lies above l_1 is symmetrical. Thus E must be ordinary and the other line through E must pass through P. Since the same holds for F, we may now assume that both E and F are ordinary points and that there is a line through E, P, and F.

Suppose E lies below l_2. Since $T(l_2) = (2, 0)$, E cannot be attached to l_2 and since no additional lines intersect $(E, C]$, there is a line m through both (E, M) and (C, M) and so also (A, P). Thus PAN is l_1-solid and l_1-blemished while FNB is l_1-solid and l_1-wide. Lemma 2.11 now yields a contradiction (see Fig. 18).

Step 4. Γ is the Kelly–Moser configuration (Fig. 3).

Proof. The previous steps show that Γ contains the lines $l_1, \ldots, l_6$ and that any additional lines are vertical. Since l_1 and l_2 have type $(2, 0)$ and every additional vertical line meets both l_1 and l_2 in an ordinary point, there must be exactly one additional vertical line in Γ and it must pass through P. Then Γ is the Kelly–Moser configuration (see Fig. 19). □

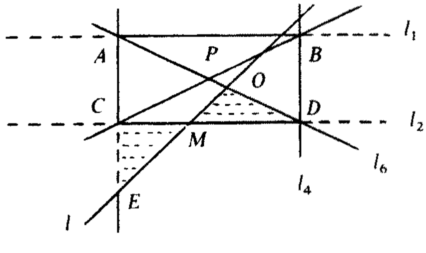

Fig. 17

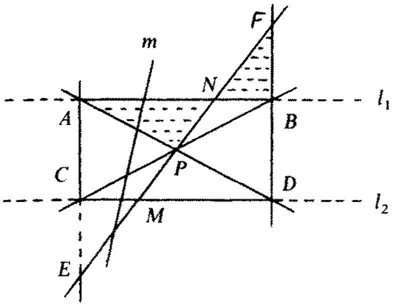

Fig. 18

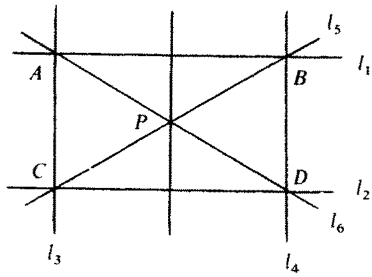

Fig. 19

We will need the following dual of Theorem 3.3 of [KM].

Theorem 2.13 [KM]. *Apart from pencils, if $T(l) \neq (2, 0)$, then $w_1(l) \geq 3$.*

An immediate corollary of Theorem 2.12 and Theorem 2.13 is

Corollary 2.14. *If l_1 and l_2 have ordinary intersection in any configuration other than pencils and* (Fig. 3), *then $w_1(l_1) + w_1(l_2) \geq 5$.*

Theorem 2.15. *Except for pencils and the Kelly–Moser configuration, $s \geq \frac{6}{13}n$.*

Proof. We partition the ordinary points into the sets

$$\sigma = \{\text{ordinary points that lie on a line of type } (2, 0)\},$$

$$\tau = \{\text{ordinary points that do not lie on a line of type } (2, 0)\},$$

and we partition the lines into sets of bad, good, and fair lines,

$$\mathscr{B} = \{\text{lines } l \text{ of type } (2, 0)\},$$

$$\mathscr{G} = \{\text{lines } l \text{ that contain a point in } \sigma \text{ but } l \notin \mathscr{B}\},$$

$$\mathscr{F} = \{\text{lines } l \text{ that do not contain a point in } \sigma\}.$$

We further partition $\mathscr{G}$ into the sets

$$\mathscr{G}_j = \{\text{lines } l \text{ in } \mathscr{G} \text{ that contain exactly } j \text{ points of } \sigma\}.$$

Suppose the intersection P of l and m is in σ. Then one of l or m is in $\mathscr{B}$ with 1-weight 2 and so by Corollary 2.14, the other has 1-weight at least 3 and lies in $\mathscr{G}$. Thus each point in σ appears on exactly one line from $\mathscr{B}$ and exactly one line from $\mathscr{G}$. Thus if $B = \#\mathscr{B}$, $G = \#\mathscr{G}$, $F = \#\mathscr{F}$, and $G_j = \#\mathscr{G}_j$, we have

$$G = \sum_j G_j, \tag{2.1}$$

$$\sum_{j \geq 1} jG_j = \#\sigma = 2B. \tag{2.2}$$

Now if $l \in \mathscr{G}_1$, then $T(l) = (\mu, \nu) \geq (1, 0)$ and $w_1(l) = \mu + \nu \geq 3$ (by Theorem 2.13) and since $\alpha \geq 1$, we have $w_\alpha(l) = \alpha\mu + \nu \geq \alpha + 2$. If $l \in \mathscr{G}_2$, then $T(l) \geq (2, 0)$ and $w_1(l) \geq 3$ and so $w_\alpha(l) \geq 2\alpha + 1$. If $l \in \mathscr{G}_j$ for $j \geq 3$, then of course $w_\alpha(l) \geq j\alpha$. Finally, if $l \in \mathscr{B}$, then $w_\alpha(l) = 2\alpha$ and if $l \in \mathscr{F}$, then $w_\alpha(l) \geq w_1(l) \geq 3$ by Theorem 2.13. Thus

$$\begin{aligned}
\sum_{l \in \Gamma} w_\alpha(l) &= \sum_{l \in \mathscr{B}} w_\alpha(l) + \sum_j \sum_{m \in \mathscr{G}_j} w_\alpha(m) + \sum_{l \in \mathscr{F}} w_\alpha(l) \\
&\geq 2\alpha B + (\alpha + 2)G_1 + (2\alpha + 1)G_2 + \sum_{j \geq 3} j\alpha G_j + 3F \\
&= 2\alpha B + \alpha\left(\sum_{j \geq 1} jG_j\right) + 2G_1 + G_2 + 3F \\
&= (4\alpha - 2)B + 3G_1 + 3G_2 + \sum_{j \geq 3} jG_j + 3F \qquad \text{(by (2.2))} \\
&\geq (4\alpha - 2)B + 3G + 3F \qquad \text{(by (2.1)).} \qquad (2.3)
\end{aligned}$$

Now choose $\alpha = \frac{5}{4}$ so that $(4\alpha - 2) = 3$. Then (2.3) becomes

$$\sum_{l \in \Gamma} w_{5/4}(l) \geq 3B + 3G + 3F = 3n. \qquad (2.4)$$

However, an ordinary point P lies on exactly two lines, say k and m, and there are at most four minimal triangles with vertex P having sides on k and m. Thus P is attached to at most four lines. If we assign to the (l, P)-entry of our $n \times s$ matrix (see the end of Section 1) $\frac{5}{4}$ if P lies on l, 1 if P is attached to l, and 0 otherwise, then the column sums are at most $2(\frac{5}{4}) + 4 = \frac{13}{2}$. From (2.4) we now obtain

$$3n \leq \sum_{l \in \Gamma} w_{5/4}(l) \leq \tfrac{13}{2}s,$$

and this completes the proof of Theorem 2.15. □

3. The Counterexamples

In this section we show that one of the main results in Hansen's thesis, subtheorem 17 [H], is false, thus leaving Dirac's $n/2$ conjecture open. In order to discuss our counterexamples to subtheorem 17, we need to introduce a notion of "type" more refined than that in Definition 2.9 above. We say that two lines are followers (of each other) if no pair of lines separates them. Suppose that P is an ordinary point not on a line l and not attached to l. We say that P is semiattached to l if there is a line m through P such that l and m are followers and one of the open segments

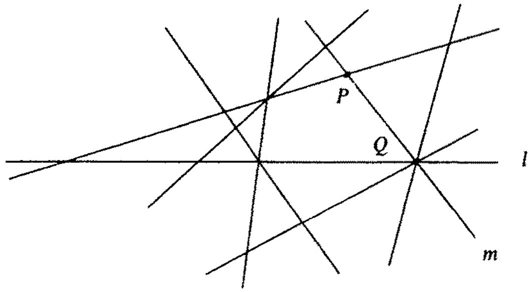

Fig. 20. P is semiattached to l.

from $Q = lm$ to P is empty of points (see Fig. 20). In Hansen's terminology the duals of "attached" and "semiattached" are "genuinely associated" and "ungenuinely associated" respectively. A line l in Γ is said to have Hansen-type (μ, ν, ρ) if there are exactly μ ordinary points on l, ν ordinary points attached to l, and ρ ordinary points semiattached to l.

As subtheorem 17 involves further definitions that are awkward to introduce here, it will be convenient to exhibit an infinite family of counterexamples to the following simply stated consequence of subtheorem 17:

(*) If l_1 and l_2 are followers, they cannot both have Hansen-type (2, 0, 0).

We remark that (*) constitutes the main step in the proof of subtheorem 17 and Hansen's argument for (*) runs from the bottom of p. 85 to the top of p. 91 of [H]. The error occurs at the top of p. 88 where it is falsely assumed that a certain hexagon can be projectively transformed into a regular hexagon.

The configuration of 14 lines in Fig. 21 is the smallest counterexample to (*). The two horizontal lines l_1 and l_2 are followers (since there is no horizontal line above l_1 and below l_2) and yet both have Hansen-type (2, 0, 0). The configuration

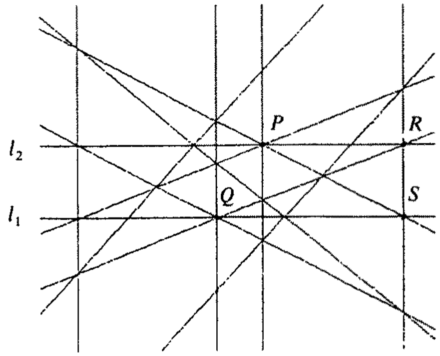

Fig. 21. E_{14}—the smallest counterexample to subtheorem 17. The 14th line is at infinity.

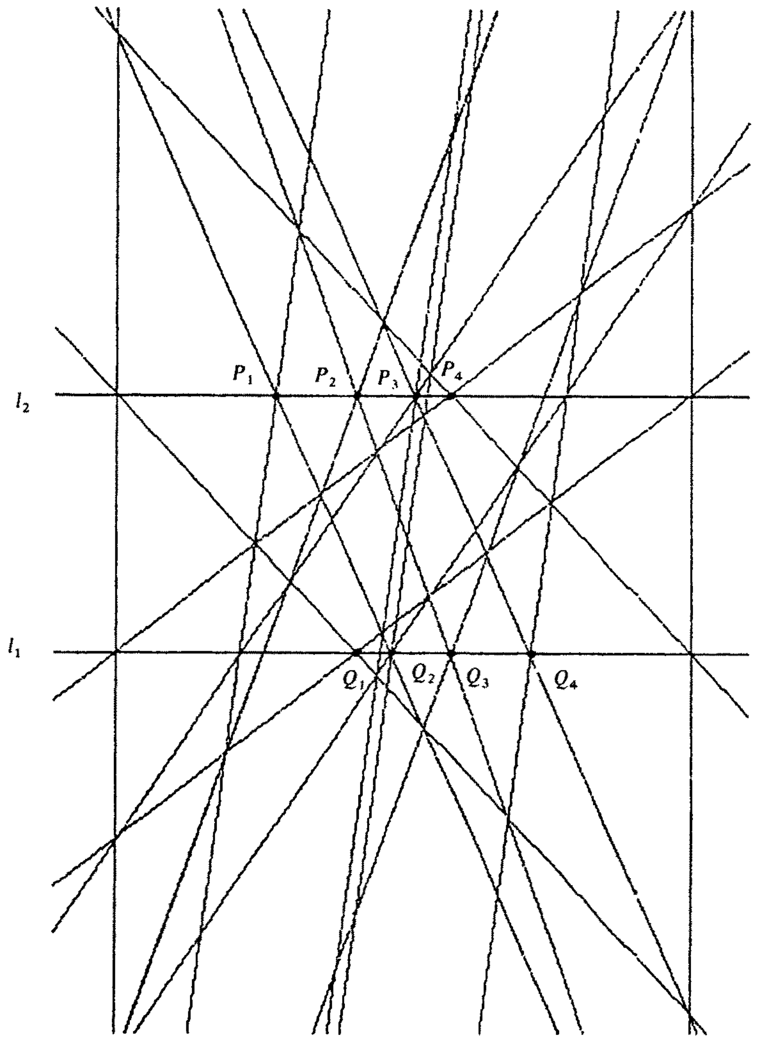

Fig. 22. E_{20}. The 20th line is at infinity.

in Fig. 21 is easily seen to exist by requiring that the ratio PR/QS be α where $\alpha^3 + \alpha^2 = 1$, i.e., $\alpha \approx 0.754877666$. For every n of the form $n = 8 + 6k$, where k is a positive integer, there is a configuration E_n of n lines having a pair of followers each with Hansen-type (2, 0, 0). The configurations E_{20} and E_{26} are given in Figs. 22 and 23.

Finally, we remark that up to $k(2k + 1)$ additional lines can be added to the configuration E_n, $n = 8 + 6k$, to obtain even more counterexamples. For instance, in Fig. 22, any line joining P_i and Q_j with $4 \geq i \geq j \geq 1$ can be added without altering the Hansen-type of l_1 and l_2. It can be shown that $s \geq n/2$ in all such

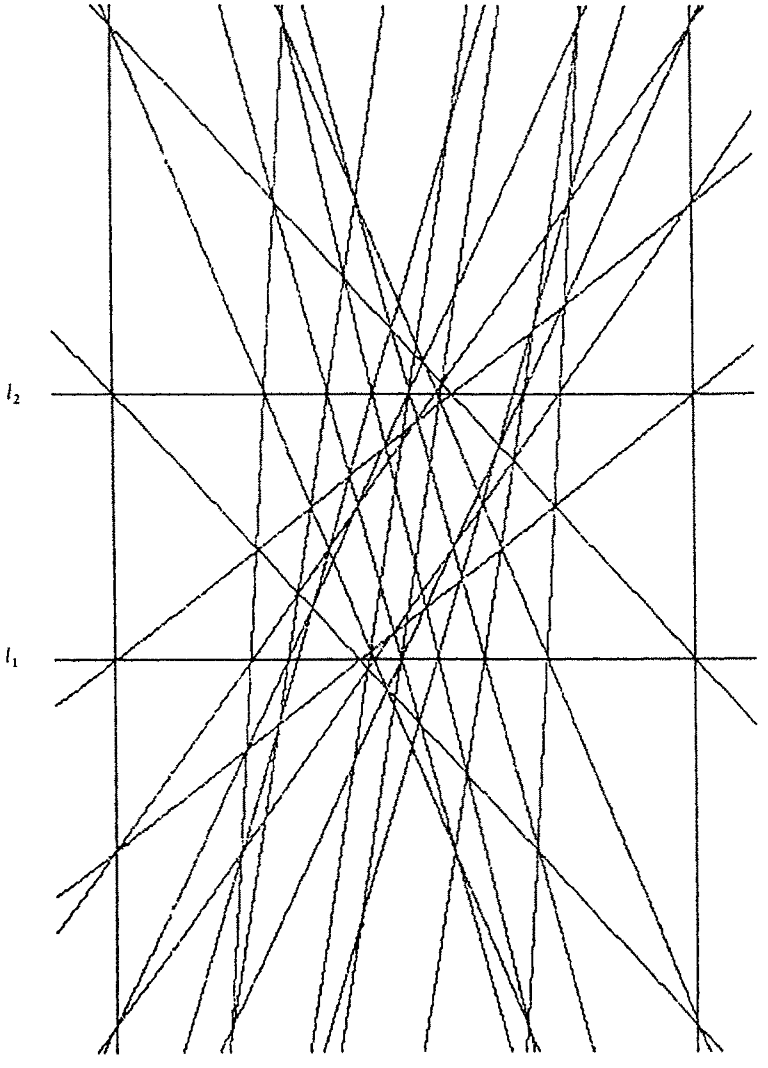

Fig. 23. E_{26}. The 26th line is at infinity.

examples. There is some evidence to suggest that these are the only counter-examples.

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Note added in proof. Theorem 2.15 can be proved more simply as follows. By Theorem 2.12, $s \geq 2B$, where B denotes the number of type (2, 0) lines. Since an ordinary point is associated to at most six lines, Theorem 2.13 yields $6s \geq 2B + 3(n - B)$. Thus $s \geq \max\{2B, (3n - B)/6\} \geq 6n/13$.