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Integral Quadratic Forms, Kac–Moody Algebras, and Fractional Quantum Hall Effect. An *ADE*-$\mathcal{O}$ Classification

Jürg Fröhlich[1] and Emmanuel Thiran[1]

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The purpose of this paper is to present a rather comprehensive classification of incompressible quantum Hall states in the limit of large distance scales and low frequencies. In this limit, the description of low-energy excitations above the groundstate of an incompressible quantum Hall fluid is intimately connected to the theory of integral quadratic forms on certain lattices which we call quantum Hall lattices. This connection is understood with the help of the representation theory of algebras of gapless, chiral edge currents or, alternatively, from the point of view of the bulk effective Chern–Simons theory. Our main results concern the classification of quantum Hall lattices in terms of certain invariants and their enumeration in low dimensions and for a limited range of values of those invariants. Among physical consequences of our analysis we find explicit, discrete sets of plateau values of the Hall conductivity, as well as the quantum numbers of quasiparticles in fluids corresponding to any one among those quantum Hall lattices. Furthermore, we are able to predict transitions between structurally different quantum Hall fluids corresponding to the same filling factor. Our general results are illustrated by explicitly considering the following plateau values: $\sigma_H = N/(2N \pm 1)$, $N = 1, 2, 3, \ldots$, $\sigma_H = 5/13$, $8/5$, $5/3$, 1, and $\sigma_H = 1/2$.

KEY WORDS: Quantum Hall effect; Kac–Moody algebras; Abelian Chern–Simons theory; integral lattices; quadratic forms.

1. INTRODUCTION

The quantum Hall effect (QHE) is observed in electron gases confined to a planar region Ω and subject to a strong, uniform magnetic field $\mathbf{B}^{(0)} = (\mathbf{B}_{\parallel}^{(0)}, B_{\perp}^{(0)})$ transversal to Ω (i.e., with $B_{\perp}^{(0)} \neq 0$). Such systems

[1] Institut für Theoretishe Physik, ETH-Hönggerberg, CH-8093, Zurich, Switzerland. E-mail: JUERG@ITP.ETHZ.CH THIRAN@ITP.ETHZ.CH.

of electrons (and/or holes) are realized, experimentally, as inversion layers which are formed at the interface between a semiconductor and an insulator, e.g., in a MOSFET or in a heterostructure, such as one made of $GaAs/Al_xGa_{1-x}As$, when an electric field (gate voltage) is applied in the direction perpendicular to the interface. The quantum mechanical motion of the electrons in the direction perpendicular to the interface is then quantized—the electrons (or holes) are bound to the interface by a deep potential well. At very low temperatures, the gas of electrons (or holes) is therefore a very nearly two-dimensional system.

The domain Ω to which the electrons are confined is chosen to be a bounded subset of the (x, y) plane, typically a disk. When the magnetic field $\mathbf{B}^{(0)}$ has been turned on, one tunes the total electric current in the y direction to a value I_y and then measures the difference in the chemical potentials of the electrons (or holes) at the two edges of Ω transversal to the x direction, i.e., the voltage drop V_x in the x direction. The *Hall resistance* is defined as the ratio

$$R_H = \frac{V_x}{I_y} \tag{1.1}$$

Similarly, one can measure the longitudinal resistance

$$R_L = \frac{V_y}{I_y} \tag{1.2}$$

where V_y denotes the voltage drop in the y direction.

The surprising experimental discoveries made at the beginning of the 1980s by von Klitzing *et al.*[1] and Tsui *et al.*[2] can be summarized as follows: Let n denote the density of electrons (minus the density of holes), e the elementary electric charge, h Planck's constant, and c the velocity of light. One defines the *filling factor* ν, a dimensionless quantity, as

$$\nu = \frac{n}{B_\perp^{(0)}/(hc/e)} \tag{1.3}$$

where hc/e is the quantum of magnetic flux. If the electrons were free, spinless fermions ν would be the fraction of filled Landau levels. At very low temperatures, $T \approx 0$, the resistances R_H and R_L are functions of ν with the following remarkable properties:

(i) The dimensionless quantity

$$\sigma_H = \frac{h}{e^2} R_H^{-1} \tag{1.4}$$

where R_H^{-1} is the *Hall conductivity*, is constant on certain intervals, i.e., has plateaux, and the values of σ_H on all observed plateaux are *rational numbers*. The most pronounced plateaux have integer heights (integer QHE) which can be measured with extraordinary precision (one part in 10^8). They serve as new standards for the definition of e. Most plateaux have values $\sigma_H = n_H/d_H$, where n_H and d_H are relatively prime integers, and d_H is *odd* (odd denominator rule[(3)]), but a plateau at $\sigma_H = 5/2$ has been observed, too,[(4)] and in double-layer systems a plateau has been observed at $\sigma_H = 1/2$.[(5)]

(ii) Whenever (ν, σ_H) belongs to a plateau, R_L very nearly *vanishes*. Thus when σ_H has a plateau value the system is *free of dissipative processes*, and conversely. It is then called an "*incompressible quantum Hall (QH) fluid*."

(iii) The precision of plateau heights (but not their widths) is insensitive to sample preparation and geometry.

There is convincing evidence[(6,7,9)] that when σ_H is on a plateau of noninteger height the system exhibits fractionally charged excitations, the fractional charges being related to the denominator d_H in the value n_H/d_H of σ_H. Moreover, when the in-plane component $\mathbf{B}_\parallel^{(0)}$ of the magnetic field is varied, keeping ν fixed, it is found that certain plateaux disappear to reemerge, in some cases, at other values of $\mathbf{B}_\parallel^{(0)}$.[2] This strongly suggests that Zeeman energies, and thus electron spin, play an important role in a QH fluid at certain values of σ_H, such as $\sigma_H = 2/3$, $4/3$, $8/5$, $5/2$, etc.

For illustration, a table of observed plateau values for $0 < \sigma_H \leqslant 1$ is given in Table I. More details about observed plateaux and their special properties will be discussed in Section 7.

A variety of attempts at a theoretical explanation of these truly remarkable features of two-dimensional electron gases have been made, for both the integer QHE and the fractional QHE.[3] In both cases, ideas due to Laughlin have been seminal.[4]

In this paper, we further develop a line of thought initiated in refs. 18–22. In order to make this paper accessible to readers not familiar with the literature on the FQHE, we shall recall some of the key ideas proposed in the papers quoted above. The novel feature of this paper is that, starting from basic physical principles, it relates the observed plateau values of σ_H to the theory of *integral quadratic forms* on integral lattices

[2] See experimental results for tilted magnetic field transition at $\sigma_H = 8/5$ and $5/3$,[(8)] $\sigma_H = 4/3$,[(9)] $\sigma_H = 2/3$,[(10)] and $\sigma_H = 5/2$.[(4)] See also ref. 11.

[3] For reviews of the quantum Hall effect and comprehensive compilations of references see, e.g., refs. 12–15.

[4] See ref. 16 for IQH, ref. 17 for FQH; see also refs. 32 and 33.

Table I. Observed Plateau Values for $0<\sigma_H=n_H/d_H\leqslant 1$

d_H	σ_H
1	1
3	$\frac{1}{3}$, $\frac{2}{3}$
5	$\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$
7	$\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$
9	$\frac{2}{9}$, $\frac{4}{9}$, $\frac{5}{9}$
11	$\frac{2}{11}$, $\frac{3}{11}$, $\frac{4}{11}$, $\frac{5}{11}$, $\frac{6}{11}$, $\frac{7}{11}$
13	$\frac{3}{13}$, $\frac{4}{13}$, $\frac{6}{13}$, $\frac{7}{13}$, $\frac{9}{13}$
15	$\frac{4}{15}$

which the reader may have encountered in Lie group theory or number theory.[23,24] The logic of this relationship will involve a study of the algebras of chiral edge currents in QH fluids and their representation theory. In order to give the reader a rudimentary idea of what is involved, we shall summarize a few elementary facts about integral quadratic forms and describe some basic results.

Let V be an N-dimensional, real vector space with an inner product $(\cdot,\cdot)$. A basis $\{\mathbf{e}_1,..., \mathbf{e}_N\}$ of V is said to be *integral* iff its Gram matrix is integral, i.e.,

$$K_{IJ} := (\mathbf{e}_I, \mathbf{e}_J) \in \mathbb{Z} \qquad \text{for all} \quad I, J \tag{1.5}$$

Clearly $K_{IJ}=K_{JI}$, so $K=(K_{IJ})$ is a regular, symmetric $N\times N$ matrix with integer matrix elements. We define a lattice Γ by setting

$$\Gamma := \left\{ \mathbf{q} = \sum_{I=1}^{N} q^I \mathbf{e}_I : q^I \in \mathbb{Z} \text{ for all } I \right\} \tag{1.6}$$

Let $\{\boldsymbol{\varepsilon}^1,..., \boldsymbol{\varepsilon}^N\}$ be a basis of V *dual* to the basis $\{\mathbf{e}_1,..., \mathbf{e}_N\}$, i.e., satisfying $(\boldsymbol{\varepsilon}^I, \mathbf{e}_J)=\delta^I_J$, $I, J=1,..., N$. Here

$$\boldsymbol{\varepsilon}^I = \sum (K^{-1})^{IJ} \mathbf{e}_J$$

The basis $\{\boldsymbol{\varepsilon}^1,..., \boldsymbol{\varepsilon}^N\}$ generates the dual lattice

$$\Gamma^* := \left\{ \mathbf{n} = \sum_{I=1}^{N} n_I \boldsymbol{\varepsilon}^I : n_I \in \mathbb{Z} \text{ for all } I \right\} \tag{1.7}$$

Since $\mathbf{e}_J=\sum_{I=1}^N K_{JI}\boldsymbol{\varepsilon}^I$, with $K_{IJ}\in\mathbb{Z}$, we can view the lattice Γ as a sublattice of its dual Γ^*, $\Gamma\subseteq\Gamma^*$; and Γ is called self-dual if $\Gamma=\Gamma^*$.

A vector $\mathbf{v} \in V$ can be identified with a column vector $\vec{v} = (v^1, \ldots, v^N)^T$, called a *charge vector*, with $v^I = (\mathbf{v}, \boldsymbol{\varepsilon}^I)$, and with a row vector $\underset{\sim}{v} = (v_1, \ldots, v_N)$, called a *flux vector*, with $v_I = (\mathbf{v}, \mathbf{e}_I)$, $I = 1, \ldots, N$. Note that, in the product $(\cdot, \cdot)$ on V,

$$(\mathbf{q}, \mathbf{q}') = \vec{q}^T \cdot K\vec{q}' = \sum_{I,J} q^I K_{IJ} q'^J \qquad \text{for} \quad \mathbf{q}, \mathbf{q}' \in \Gamma \tag{1.8}$$

and

$$(\mathbf{n}, \mathbf{n}') = \underset{\sim}{n} K^{-1} \underset{\sim}{n}'^T = \sum_{I,J} n_I (K^{-1})^{IJ} n'_J \qquad \text{for} \quad \mathbf{n}, \mathbf{n}' \in \Gamma^* \tag{1.9}$$

By Kramer's rule,

$$(K^{-1})^{IJ} = \frac{1}{\Delta} \tilde{K}^{IJ} \tag{1.10}$$

where $\Delta = \det K \in \mathbb{Z}$, and $\tilde{K} = (\tilde{K}^{IJ})$, with $\tilde{K}^{IJ} = \tilde{K}^{JI} \in \mathbb{Z}$, is the cofactor matrix. Thus the matrix elements $(K^{-1})^{IJ}$ are rational numbers.

The set of vectors in Γ^* modulo vectors in Γ, Γ^*/Γ, is an Abelian group, and it is easy to see that its order is given by

$$|\Gamma^*/\Gamma| = \Delta \tag{1.11}$$

The lattice Γ is called *even* iff all scalar products $(\mathbf{q}, \mathbf{q}')$ are even integers, i.e., iff $K_{II} \in 2\mathbb{Z}$, for all $I = 1, \ldots, N$. Otherwise Γ is called *odd*.

We call Γ *Euclidean* iff the inner product $(\cdot, \cdot)$ is *positive-definite*, i.e., iff K is positive-definite.

Linear transformations of V mapping the lattice Γ onto itself form a group, denoted by $GL(N, \mathbb{Z})$, which is defined by

$$GL(N, \mathbb{Z}) := \{ S = (S_{IJ}) : S_{IJ} \in \mathbb{Z}, \forall I, J, \det S = \pm 1 \} \tag{1.12}$$

It contains the subgroup, $O(\Gamma)$, of all those invertible transformations which preserve the length of each lattice vector, i.e.,

$$O(\Gamma) := \{ S \in GL(N, \mathbb{Z}) : S^T K S = K \} \tag{1.13}$$

If $S \in O(\Gamma)$, then $S^{-1} \in O(\Gamma)$, and hence

$$S K^{-1} S^T = K^{-1} \tag{1.14}$$

i.e., $O(\Gamma) = O(\Gamma^*)$. Two integral quadratic forms, K_1 and K_2, are equivalent iff there is some matrix $S \in GL(N, \mathbb{Z})$ such that

$$K_2 = S^T K_1 S \tag{1.15}$$

The primary purpose of this paper is to derive the following basic connection between incompressible ($R_L = 0$) QH fluids and equivalence classes of integral quadratic forms on lattices.

Basic Result. An incompressible QH fluid is characterized by a pair of *integral, odd Euclidean lattices* Γ_e and Γ_h and two *linear forms* $\mathbf{Q}_e$ and $\mathbf{Q}_h$ on these lattices, i.e., vectors $\underset{\rightarrow}{Q}_x = ((Q_x)_1, \ldots, (Q_x)_{N_x})$ in Γ_x^* with

$$\mathbf{Q}_x(\mathbf{q}) = (\mathbf{Q}_x, \mathbf{q}) = \underset{\rightarrow}{Q}_x \cdot \vec{q} \qquad \forall \boldsymbol{q} \in \Gamma_x$$

satisfying the following two constraints:

(i) $\mathbf{Q}_x$ is a "*visible*" vector in Γ_x^*, i.e.,

$$\text{g.c.d.}((Q_x)_1, \ldots, (Q_x)_{N_x}) = 1$$

where g.c.d. denotes the greatest common divisor.

(ii) $\mathbf{Q}_x$ is an "*odd*" functional on Γ_x, i.e.,

$$\mathbf{Q}_x(\mathbf{q}) \equiv (\mathbf{q}, \mathbf{q})_x \mod 2 \tag{1.16}$$

[meaning that the parity of $\mathbf{Q}_x(\mathbf{q})$ is the same as that of $(\mathbf{q}, \mathbf{q})_x$] for $x = e, h$.

The Hall conductivity σ_H is given by

$$\sigma_H = \sigma_e - \sigma_h \tag{1.17}$$

where

$$\sigma_x = (\mathbf{Q}_x, \mathbf{Q}_x) = \underset{\rightarrow}{Q}_x \cdot K^{-1} \underset{\rightarrow}{Q}_x^T = \sum_{I,J}^{N_x} (Q_x)_I (K_x^{-1})^{IJ} (Q_x)_J \tag{1.18}$$

Clearly σ_e and σ_h, and hence σ_H, are *rational numbers*. Distinct vectors $\mathbf{Q}_x$ belonging to the same orbit denoted $[\mathbf{Q}_x]$, under the orthogonal group of the lattice $O(\Gamma_x)$, specify the same geometrical data. In all examples that we will have to deal with orbits have just two $\pm\mathbf{Q}_x$ or four elements. By (1.10),

$$\sigma_x = \frac{1}{\Delta_x} \sum (Q_x)_I \tilde{K}_x^{IJ} (Q_x)_J$$

where numerator, $\gamma_x = \sum (Q_x)_I \tilde{K}_x^{IJ} (Q_x)_J$, and denominator, Δ_x, are integers. Let l_x be their greatest common divisor, which we call the *level* of Γ_x:

$$l_x = \text{g.c.d.}(\gamma_x, \Delta_x)$$

Then

$$\sigma_x = \frac{n_x}{d_x}, \qquad \text{g.c.d.}\,(n_x, d_x) = 1$$

with

$$\begin{aligned} n_x &= \frac{1}{l_x} \sum (Q_x)_I \tilde{K}_x^{IJ} (Q_x)_J \\ d_x &= \frac{1}{l_x} \Delta_x \end{aligned} \tag{1.19}$$

$x = e, h$. It turns out that when d_x is *even*, then l_x must be even, too, in fact a multiple of 4, and the QH fluid will exhibit Laughlin vortices of electric charge $\pm e/2d_x$, where e is the elementary electric charge (see Theorem 6, Section 5).

Let us pause to explain some features of this result. The subscripts e and h stand for "electrons" and "holes," respectively. They indicate the nature of the basic charge carriers of the fluid. Fluids for which $\Gamma_e \neq \emptyset$ *and* $\Gamma_h \neq \emptyset$ are *composite fluids* containing both electrons and holes as basic charge carriers. The nature of the basic charge carriers can be inferred from the *chirality* (left or right) of the *edge currents* in the sample, given the direction of the external magnetic field. The chirality of edge currents is apparently experimentally measurable.[51]

Given the *Basic Result* described above, the task arises to classify incompressible QH fluids by classifying pairs of odd, integral, Euclidean lattices together with orbits of visible odd vectors in the dual lattices. Clearly, the classification problems for $x = e$ and h are identical, so that we may focus, e.g., on the classification of $(\Gamma_e, \mathbf{Q}_e)$ and henceforth drop the subscript e. We define a quantity $L_{\max}$ by setting

$$L_{\max} := \min_{\{\vec{q}_I\}}\left(\max_{M=1,\dots,N} (\mathbf{q}_M, \mathbf{q}_M)\right) \tag{1.20}$$

where the minimum is taken over all possible bases $\{\mathbf{q}_1, \dots, \mathbf{q}_N\}$ of Γ with the property that $\mathbf{Q}(\mathbf{q}_J) = 1$ for all $J = 1, \dots, N$ (such bases exist!). Physically, $L_{\max}$ has the following interpretation: If a state of the QH fluid is prepared which describes two electrons excited above the ground state of the system, one may consider the minimum of the modulus of their relative angular momentum in that state. For a proper choice of the quantum numbers of the state that minimum is at least $L_{\max}$. From the physics of Coulomb systems it is plausible that $L_{\max}$ satisfies an absolute upper bound, e.g.,

$$L_{\max} \leqq 9 \tag{1.21}$$

(in units where $\hbar = 1$).

The dimension N of the lattice Γ is the number of *independent* $U(1)$-edge currents of fixed chirality exhibited by the QH fluid. The discriminant Δ of the Gram matrix of a basis of Γ is related to the number of distinct fractionally charged Laughlin vortices of the fluid. It is plausible that for samples with a positive density of impurities the possible values of N and Δ are bounded by finite, positive numbers (which depend on the density of impurities).

The quantities N, Δ, and $L_{\max}$ are *invariants*. Thus the problem is to classify equivalence classes of odd, integral, Euclidean lattices Γ and $O(\Gamma)$ orbits $[\mathbf{Q}]$ of visible vectors $\mathbf{Q} \in \Gamma^*$ satisfying condition (1.16) with the property that the values of the invariants N, Δ, and $L_{\max}$ are bounded. Pairs of such data (for $x = e, h$) then classify incompressible QH fluids and determine the possible values of the Hall conductivity σ_H via Eqs. (1.17) and (1.18).

This classification problem is a very difficult, but *finite problem* in the "geometry of natural numbers."

We shall see that the structure of the lattices Γ_e, Γ_h and of the orbits $[\mathbf{Q}_e]$ and $[\mathbf{Q}_h]$ will determine much more than the value of σ_H. It will determine quantum numbers of quasiparticle excitations of the type of Laughlin vortices, certain properties of the spin wave functions of electrons or holes, and possible transitions as the values of components of the magnetic field or of the electron density are varied for a fixed value of the filling factor.

The reader may wonder why odd, integral, Euclidean lattices appear in the analysis of incompressible QH fluids. We shall see that such lattices describe the structure of all physically realizable representations of level $k = 1$ Kac–Moody algebras of chiral edge currents describing the boundary degrees of freedom of an incompressible QH fluid. The existence of such algebras of chiral edge currents can be derived from the electrodynamics of incompressible QH fluids by invoking a mechanism of gauge anomaly cancellation.[18–21]

In Section 2, we recall the basic facts concerning the electrodynamics of incompressible QH fluids and some features of their quantum mechanics (see refs. 19–21 for more details).

In Section 3, we show that the edge degrees of freedom of an incompressible QH fluid, which are related to the chiral boundary currents first described by Halperin,[26] are described by a quantum theory of chiral currents that exhibits an Abelian gauge anomaly exactly canceled by an Abelian gauge anomaly of the bulk degrees of freedom. This mechanism of gauge anomaly cancellation leads to a concept of *boundary–bulk duality* which is made precise by describing the theory of conserved bulk currents, in the limiting regime of large distance scales and low frequencies (scaling limit), in terms of an Abelian Chern–Simons gauge theory. The analysis of the space of physical states of the Chern–Simons theory, combined with natural assumptions on the spectrum of integrally charged quasiparticles of an incompressible QH fluid and their statistics, then leads to a proof of the "Basic Result" described above.

In Section 4, we present additional details concerning boundary–bulk duality and rederive the "Basic Result" by studying the algebras of chiral edge currents describing the boundary degrees of freedom of an incompressible QH fluid and their representation theory. We identify the physical states of the Chern–Simons theory describing the bulk currents with so-called conformal blocks of the algebras of chiral edge currents and derive some consequences for QH fluids on surfaces without boundary (of interest in the analysis of numerical experiments).

In Section 5, we begin with the main task set for this paper, the classification of integral, odd Euclidean lattices Γ and visible vectors $\mathbf{Q} \in \Gamma^*$ describing the physics in the scaling limit of incompressible QH fluids. A pair $(\Gamma, \mathbf{Q})$ of an integral, odd Euclidean lattice Γ and a visible vector $\mathbf{Q} \in \Gamma^*$ is called a *QH lattice*. We discuss some basic invariants of QH lattices and their physical meaning, arithmetic congruences between these invariants, and implications for the physical properties of incompressible QH fluids. Our analysis is organized in 12 short paragraphs, and the main results are summarized in seven theorems. Taking the "Basic Result" described above for granted, the material in Sections 5 and 6 can be read *without* being familiar with Sections 2–4. We say this to encourage theoreticians and experimentalists who are not familiar with current algebra and Chern–Simons gauge theory to proceed directly to Section 5, where they will find results which they may or should find relevant.

In Section 6, we present a constructive approach to finding QH lattices and deriving the value of the Hall conductivity σ_H and the spectrum of quasiparticles (Laughlin vortices) and their quantum numbers. Our methods are fairly effective in constructing the QH lattices corresponding to "elementary" QH fluids with $\sigma_H < 2$ which generalize the QH fluids with $\sigma_H = 1$, 1/3, 1/5,.... For a large class of such fluids, we present an *ADE-$\mathcal{O}$ classification*, where A, D, and E refer to the Lie algebras $su(n)$, $so(2n+4)$, $n = 2, 3, \ldots$, E_6, and E_7, respectively, and $\mathcal{O}$ stands for one- or two-dimensional, integral, odd Euclidean lattices which have been classified by Gauss.

These results enable us to associate QH lattices with all observed plateau values of σ_H and predict properties of the corresponding QH fluids, including phase transitions.

Section 7 summarizes our results on the construction of QH lattices in the form of explicit tables.

2. THE ELECTRODYNAMICS OF INCOMPRESSIBLE QH FLUIDS

We consider a two-dimensional gas of electrons (or holes) in a uniform, external magnetic field $\mathbf{B}^{(0)} = (\mathbf{B}_{\parallel}^{(0)}, B_{\perp}^{(0)})$, with $B_{\perp}^{(0)}$, the component of $\mathbf{B}^{(0)}$ perpendicular to the plane of the system, nonzero. In linear response theory, the connection between the electric field $\mathbf{E} = (E_x, E_y)$ in the plane of the system and the electric current density $\mathbf{i}_c$ is given by the *Ohm–Hall law*

$$\mathbf{E} = \rho \mathbf{i}_c \tag{2.1}$$

where

$$\rho = \begin{pmatrix} \rho_{xx} & -\rho_H \\ \rho_H & \rho_{yy} \end{pmatrix} \tag{2.2}$$

is the *resistivity tensor*. In two dimensions,

$$\rho_H = R_H$$

and, for a rectangular sample with edges of length l_x and l_y, $\rho_{xx} = R_L(l_y/l_x)$ and $\rho_{yy} = R_L(l_x/l_y)$. In particular,

$$R_L = 0 \Leftrightarrow \rho_{xx} = \rho_{yy} = 0 \tag{2.3}$$

in which case the *conductivity tensor* ρ^{-1} has the form

$$\rho^{-1} = \begin{pmatrix} 0 & \sigma_H \\ -\sigma_H & 0 \end{pmatrix} \qquad \text{with} \quad \sigma_H = \rho_H^{-1} = R_H^{-1} \tag{2.4}$$

in units where $e^2/h = 1$.

When $R_L = 0$, Eq. (2.1) thus takes the form

$$i_c^k = \sigma_H \varepsilon^{kl} E_l \qquad \text{(Hall law)} \tag{2.5}$$

where z_k is the kth component of the vector $\mathbf{z} = (z_x, z_y) \equiv (z_1, z_2)$, and

$$\varepsilon = (\varepsilon^{kl}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The conservation of electric charge is expressed, as usual, in the form of the continuity equation for charge and current density, i.e.,

$$\frac{\partial}{\partial t} \rho_c + \nabla \cdot \mathbf{i}_c = 0 \tag{2.6}$$

where ρ_c is the electric charge density, and $\mathbf{i}_c = (i_c^x, i_c^y)$ (with $i_c^z = 0$, as there is no current flowing in the direction perpendicular to the plane of the system).

Faraday's induction law for $\mathbf{E} = (E_x, E_y)$ and $B_\perp^{tot}$, the *total* z component of the magnetic field, is the equation

$$\frac{1}{c}\frac{\partial B_\perp^{tot}}{\partial t} + \nabla \wedge \mathbf{E} = 0 \tag{2.7}$$

Assuming temporarily that the spins of the electrons in the sample are frozen in the direction parallel or antiparallel to that of $\mathbf{B}^{(0)}$, we may ignore electron spin and treat the electrons as spinless fermions whose classical and quantum dynamics is insensitive to $\mathbf{B}_\parallel$ and $E_\perp$. In this situation, Eqs. (2.5)–(2.7) summarize the main features of the electrodynamics of QH fluids in the limit of large distance scales and small frequencies.

Combining Eqs. (2.5)–(2.7), one easily finds that

$$\frac{\partial}{\partial t}\rho_c = \sigma_H \frac{1}{c}\frac{\partial B_\perp^{tot}}{\partial t} \tag{2.8}$$

We define $i_c^0 = c(\rho_c - en)$ as c times the difference between the charge density ρ_c and the uniform background density en of the system. By B we denote the difference between the actual z component of the magnetic field $B_\perp^{tot}$ and the z component $B_\perp^{(0)}$ of the uniform background magnetic field $\mathbf{B}^{(0)}$. Then Eq. (2.8) can be integrated in t to yield

$$i_c^0 = \sigma_H B \tag{2.9}$$

Faraday's induction law implies that the electromagnetic field ($\mathbf{E}$, B) can be derived from a vector potential $A = (A_0, \mathbf{A})$, $\mathbf{A} = (A_x, A_y)$:

$$\begin{aligned} \mathbf{E} &= -\nabla A_0 + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \wedge \mathbf{A} \end{aligned} \tag{2.10}$$

The vector potential A is determined by ($\mathbf{E}$, B) up to gauge transformations

$$A_a \to A_0 + \frac{1}{c}\frac{\partial \chi}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \nabla\chi$$

where χ is a scalar function. Setting $i_c = (i_c^0, \mathbf{i}_c)$, we can summarize the Hall law (2.5) and Eq. (2.9) in the equations

$$i_c^\mu = \sigma_H \varepsilon^{\mu\nu\lambda}\, \partial_\nu A_\lambda \tag{2.11}$$

or, using differential forms,

$$i_c = * \sigma_H \, dA \tag{2.12}$$

where $*$ denotes the Hodge $*$operation and d denotes exterior differentiation. Setting $J = *i_c$, i.e., $J_{\mu\nu} = \varepsilon_{\mu\nu\lambda} i_c^\lambda$ we find that (2.12) becomes

$$J = -\sigma_H \, dA \tag{2.13}$$

This equation can be derived from the action functional

$$\begin{aligned} S(A) &= \frac{\sigma_H}{4\pi} \int A \wedge dA - \frac{1}{2\pi} \int A \wedge J + \text{B.T.} \\ &= \frac{\sigma_H}{4\pi} \int \varepsilon^{\mu\nu\lambda} A_\mu \, \partial_\nu A_\lambda - \frac{1}{2\pi} \int A_\mu i_c^\mu \, d^3x + \text{B.T.} \end{aligned} \tag{2.14}$$

by setting the variation of $S(A)$ with respect to A to zero.[19] In (2.14), the integrations extend over the three-dimensional space-time, the cylinder $\Lambda = \Omega \times \mathbb{R}$, of the system, and B.T. stands for "boundary terms," i.e., terms only depending on the vector potential A restricted to the boundary, $\partial\Omega \times \mathbb{R}$, of the space-time of the system.

Treating electrons as noninteracting, classical particles of charge $-e$, one easily finds (by equating the electrostatic and the Lorentz force) that

$$\sigma_H = \frac{cen}{B_\perp} = \frac{e^2}{h} \nu \tag{2.15}$$

This equation is not far from what is actually observed in very pure samples, where the widths of the plateaux of σ_H are tiny, as long as ν is not too big. The important point is that when R_L is measured to *vanish* in some interval I of filling factors then σ_H remains *constant* over that interval, with a value that is some *rational multiple* of e^2/h. The classical law is followed only insofar as $\sigma_H = (e^2/h) \, n_H/d_H$ for all ν in I, where n_H/d_H is a rational number in the interval I. Steps toward a theoretical understanding of this remarkable "quantization" of the values of σ_H under the condition that R_L vanishes have been described in refs. 12–22 and references given there. Some of these steps will be recalled briefly below. But the main objective of this paper is to provide an understanding of *which rational multiples of* e^2/h *correspond to plateau values of* σ_H, and, given a plateau value of σ_H, to predict the spectrum of quasiparticles found in the system and to determine their electric charge, their statistics, and their spin. In trying to reach this objective we shall encounter the theory of integral quadratic forms on lattices.[23,24] But before we can understand how this happens, we must combine the electrodynamics of QH systems, as summarized in Eqs. (2.11) and (2.14), with *quantum mechanics*. In the remainder of this paper we shall employ units such that $e = \hbar = 1$ (unless mentioned otherwise).

There are different approaches toward quantizing a two-dimensional system of electrons coupled to an external vector potential $A = (A_0, \mathbf{A})$. One is to work with Feynman path integrals. In this approach one introduces a Grasssmann algebra with generators $\psi_s(x)$, $\psi_s(x)^*$, where $s = \pm 1/2$ denotes the z component of the electron spin and $x = (\mathbf{x}, t)$ is a space-time point belonging to $\Lambda = \Omega \times \mathbb{R}$. The action functional $S_\Lambda(\psi^*, \psi; A)$ is taken to be the usual action functional of nonrelativistic many-body theory where the fields ψ and ψ^* are coupled to A in the way familiar from the Pauli equation. All this is explained in much detail, e.g., in refs. 19–21; see also ref. 25.

Let $A^{(0)}$ denote the vector potential of the uniform background electromagnetic field $\mathbf{E}_\parallel^{(0)} = 0$, $B_\perp^{(0)}$, and let A be the vector potential of a small perturbing electromagnetic field $\mathbf{E}$, B, as in Eq. (2.10). [We set the components $E_\perp$ and $\mathbf{B}_\parallel$ to zero and work in a three-dimensional space-time, as before; see Eqs. (2.5)–(2.14).[5] If $E_\perp$ and $\mathbf{B}_\parallel$ do not vanish, one must interpret them as components of an $SU(2)$ gauge field coupling to the spin current, as explained in ref. 21. We shall not repeat these matters here.] A quantity of considerable interest is the *partition function*

$$Z_\Lambda(A) = N_\Lambda(A + A^{(0)})/N_\Lambda(A^{(0)}) \tag{2.16}$$

where

$$N_\Lambda(A) = \int \mathscr{D}\psi^* \, \mathscr{D}\psi \, \exp[iS_\Lambda(\psi^*, \psi; A)] \tag{2.17}$$

As long as space Ω is bounded and the density of electrons in Ω is finite, the path integral (2.17) is just a slick notation for an object that has a perfectly precise mathematical status for a large class of physically realistic model systems, including those considered in this paper (assuming that A is, e.g., uniformly bounded and smooth).

The important facts about the partition function $Z_\Lambda(A)$ are the following ones:

1. As long as $x_i \neq x_j$, for $i \neq j$ (noncoinciding arguments)

$$\begin{aligned} &(2\pi i)^n \frac{\delta^n}{\delta A_{\mu_1}(x_1) \cdots \delta A_{\mu_n}(x_n)} \ln Z_\Lambda(A) \\ &\quad = \langle T[j^{\mu_1}(x_1) \cdots j^{\mu_n}(x_n)] \rangle_\Lambda^c \end{aligned} \tag{2.18}$$

[5] Spin–orbit interactions are neglected; but see ref. 21.

where the right side denotes the connected, time-ordered Green function of n quantum mechanical current density operators $i^{\mu_1}(x_1),..., i^{\mu_n}(x_n)$ in a two-dimensional system of electrons coupled to an external vector potential $A + A^{(0)}$. [Thus $\ln Z_\Lambda(A)$ is the generating functional of the connected current Green functions.] In particular, defining the electric current density $i_c^\mu(x)$ at a space-time point x (as measured experimentally) as the expectation value of the quantum mechanical current density $i^\mu(x)$, we have that

$$i_c^\mu(x) = \langle i^\mu(x) \rangle_A = 2\pi i \frac{\delta}{\delta A_\mu(x)} \ln Z_\Lambda(A) \tag{2.19}$$

The functional

$$S_\Lambda^{\text{eff}}(A) = i \ln Z_\Lambda(A) \tag{2.20}$$

is customarily called the effective action of the system. Then Eq. (2.19) reads

$$i_c^\mu(x) = 2\pi \frac{\delta}{\delta A_\mu(x)} S_{\text{eff}}(A) \tag{2.21}$$

The second important fact about the partition function is its *gauge invariance.*

2. We have

$$Z_\Lambda(A + d\chi) = Z_\Lambda(A)$$

or

$$S_\Lambda^{\text{eff}}(A + d\chi) = S_\Lambda^{\text{eff}}(A) \tag{2.22}$$

for an arbitrary function χ on Λ. Equation (2.22) summarizes the *Ward identities* for a gas of electrons. It expresses the fact that all physical quantities of such a system are invariant under gauge transformations of A: $A \to A + d\chi$ [i.e., $A_0 \to A_0 + (1/c)\, \partial\chi/\partial t$, $\mathbf{A} \to \mathbf{A} + \nabla\chi$]. In other words, a system of nonrelativistic charged particles in a bounded region of space and at a finite density does not exhibit any gauge anomalies.

Next, we compare Eq. (2.21) to Eq. (2.11)—the Hall law—to find that Eq. (2.21) implies (2.11) if and only if

$$\begin{aligned} S_\Lambda^{\text{eff}}(A) &= \frac{\sigma_H}{4\pi} \int_\Lambda \varepsilon^{\mu\nu\lambda} A_\mu\, \partial_\nu A_\lambda + W(A\,|_{\partial\Lambda}) \\ &= \frac{\sigma_H}{4\pi} \int_\Lambda A \wedge dA + W(A\,|_{\partial\Lambda}) \end{aligned} \tag{2.23}$$

where $W(A|_{\partial\Lambda})$ stands for the boundary terms B.T. in Eq. (2.14), which will be discussed in the next section.

One should ask whether the form of $S_\Lambda^{\text{eff}}(A)$ given in (2.23) can be derived from the microscopic quantum mechanical dynamical laws of a two-dimensional electron gas under the condition that the longitudinal resistance R_L vanishes, and what an appropriate quantum mechanical reformulation of the equation $R_L = 0$ is. This question has been studied in ref. 21, where it has been proposed that the vanishing of R_L be interpreted as certain cluster decay properties of the connected current Green functions. Then one is able to show that the term $(\sigma_H/4\pi)\int_\Lambda A \wedge dA$, the so-called *Chern–Simons term*, is the leading contribution to the effective action $S_\Lambda^{\text{eff}}(A)$ in the regime of large distance scales and low frequencies. Moreover, it is the only contribution to the bulk effective action which *violates* gauge invariance, in the form of Eq. (2.22). This violation of gauge invariance essentially determines the boundary term $W(A|_{\partial\Lambda})$ which must cancel it exactly.

Thus, the hard analytical problem arising in the theory of the quantum Hall effect is to prove a certain kind of cluster decay properties of the connected current Green functions for certain values of the filling factor ν. Although this problem been studied analytically and numerically in much detail (see refs. 12 and 15 and references given there), it has not found a mathematically rigorous solution so far—not even for systems where the interactions between electrons can be ignored, but with disorder, which exhibit an integer QHE.

In this paper, we study an easier and yet quite nontrivial problem: Assuming that the analytical problem just described can be solved, i.e., that it is justified to use the effective action given in (2.23) in a description of the system in the limit of large distance scales and low frequencies —in accordance with the phenomenology of the QH effect, Eqs. (2.13), (2.14)—what can we say about the possible values of the coefficient σ_H; can we understand why it is quantized? To this question we find some surprising answers, which, incidentally, also shed some light on the analytical problem described above.

Our analysis is analogous to a group-theoretic analysis of symmetries of a quantum mechanical system, leaving the question open how one can solve its Schrödinger equation—except that in our problem we encounter Kac–Moody algebras of chiral currents, rather than ordinary groups and finite-dimensional Lie algebras. It is explained in the next section how Kac–Moody algebras arise in the study of quantum Hall systems. For details see refs. 26 and 18–22.

3. ANOMALY CANCELLATION AND $U(1)$-CURRENT ALGEBRA

In this section, we shall first determine the form of the all-important boundary term $W(A|_{\partial\Lambda})$ in the effective action $S_\Lambda^{\text{eff}}(A)$ given in Eq. (2.23). It turns out that this form is essentially determined by the gauge invariance (2.22) of the effective action. Let $\chi(x)$ be a gauge function not vanishing at the boundary $\partial\Lambda = \partial\Omega \times \mathbb{R}$ of the space-time region to which the electron gas is confined. Then

$$\begin{aligned} S_\Lambda^{\text{eff}}(A+d\chi) &= \frac{\sigma_H}{4\pi}\int_\Lambda (A+d\chi)\wedge dA + W((A+d\chi)|_{\partial\Lambda}) \\ &= S_\Lambda^{\text{eff}}(A) - \frac{\sigma_H}{4\pi}\int_{\partial\Lambda} d\chi \wedge A \\ &\quad + W((A+d\chi)|_{\partial\Lambda}) - W(A|_{\partial\Lambda}) \end{aligned} \tag{3.1}$$

where we have used that $d^2=0$, along with Stokes' theorem. Thus

$$W((A+d\chi)|_{\partial\Lambda}) - W(A|_{\partial\Lambda}) = \frac{\sigma_H}{4\pi}\int_{\partial\Lambda} d\chi\wedge A \tag{3.2}$$

This equation determines the general form of W, up to gauge-invariant terms. To see this, let us assume, for simplicity, that the system is confined to a region Ω of the (x, y) plane with the *topology of a disk*. Let L denote the length of the circumference of $\partial\Omega$. It is convenient to parametrize $\partial\Omega$ by an angle $\vartheta \in [0, 2\pi)$ and to introduce light-cone coordinates $u_\pm$ on $\partial\Lambda$:

$$u_\pm = \frac{1}{\sqrt{2}}\left(vt \pm \frac{L}{2\pi}\vartheta\right) \tag{3.3}$$

where v is some velocity. Interpreting the gauge field $A|_{\partial\Lambda}$ as a one-form α, we have that

$$\alpha(u) = \alpha_+(u)\,du_+ + \alpha_-(u)\,du_- \tag{3.4}$$

In light-cone coordinates the right side of (3.2) can be written as

$$\frac{\sigma_H}{4\pi}\int_{\partial\Lambda}[\alpha_+(u)\,\partial_-\chi(u) - \alpha_-(u)\,\partial_+\chi(u)]\,d^2u \tag{3.5}$$

With this, the general solution of Eq. (3.2) is found to be

$$W(\alpha) = \sigma_e W_L(\alpha) - \sigma_h W_R(\alpha) + G(\alpha) \tag{3.6}$$

where $G(\alpha)$ is a gauge-invariant functional of α,

$$W_{L/R}(\alpha)=\frac{1}{4\pi}\int_{\partial\Lambda}\left[\alpha_+(u)\,\alpha_-(u)-2\alpha_\mp(u)\frac{\partial_\pm^2}{\square}\alpha_\mp(u)\right]d^2u \tag{3.7}$$

and

$$\sigma_H=\sigma_e-\sigma_h \tag{3.8}$$

In (3.7), $\partial_\pm=\partial/\partial u_\pm$ and $\square=2\partial_+\partial_-$ is the two-dimensional d'Alembertian in light-cone coordinates. For further details see, e.g., ref. 21.

The problem we are now confronted with is to find out what Eqs. (3.6)–(3.8) tell us about the dynamics of boundary charge density waves in two-dimensional systems of electrons and holes in a transversal magnetic field. The answer is known from current algebra: $W_{L/R}(\alpha)$ is the *generating functional* of the connected Green functions of left-moving/right-moving chiral $U(1)$ currents localized on $\partial\Lambda$. These currents describe charged boundary density waves of the two-dimensional electron and hole gas. The study of charged excitations in the two-dimensional electron gas is closely related to the study of the representation theory of left- and right moving $U(1)$-current algebras. We have prejudices from physics concerning the charged low-energy excitations in an incompressible QH fluid, and these prejudices select a class of representations of the $U(1)$-current algebras which can be realized in such a fluid. Knowledge of this class of representations will imply knowledge of the possible values of σ_H.

The theory of chiral current algebras is perfectly symmetric under exchanging left-movers (L) with right-movers (R). We shall focus on left-movers, drop the subscript L and set $u:=u_+$ and $\sigma:=\sigma_e$. Let $J(u)$ be a left-moving current on $\partial\Lambda$. By (3.6) and (3.7), we have that the connected two-point current Green function in the ground state (vacuum), i.e., the second derivative of $W(\alpha)$ with respect to α_-, is given by

$$\frac{1}{4\pi^2}\langle J(u)\,J(u')\rangle^c=\frac{\sigma}{4\pi^2}\sum_{k\in\mathbb{Z}}\left(u-u'+k\frac{L}{\sqrt{2}}\right)^{-2} \tag{3.9}$$

All other connected Green functions vanish (at $\alpha=0$). We conclude that the commutator between two currents is given by

$$[J(u),J(u')]=i\sigma\delta'(u-u') \tag{3.10}$$

From these facts it follows that J is a derivative of a massless, chiral free field. The most general solution has the form

$$J(u)=(\mathbf{Q},\partial\boldsymbol{\phi}(u))=\underset{\sim}{Q}\cdot\partial\vec{\phi}(u)=\sum_{l=1}^{N}Q_l\partial\phi^l(u) \tag{3.11}$$

where

$$\underset{\sim}{Q} = (Q_1, \ldots, Q_N) \tag{3.12}$$

and

$$\vec{\phi}(u) = (\phi^1(u), \ldots, \phi^N(u))^T \tag{3.13}$$

is an N-tuple of massless, chiral free fields, for some $N = 1, 2, 3, \ldots$. The commutation relations of the fields $\phi^I(u)$, $I = 1, \ldots, N$, have the form

$$[\partial\phi^I(u), \partial\phi^J(u')] = i(C^{-1})^{IJ}\, \delta'(u - u') \tag{3.14}$$

for some positive-definite matrix $C = (C_{IJ})$. This matrix defines a scalar product $(\cdot, \cdot)$ on the space $\mathbb{R}^N$ of vectors $\boldsymbol{\phi}$ and $\mathbf{Q}$. Combining Eqs. (3.10), (3.11), and (3.14), we find the relation

$$\sigma = (\mathbf{Q}, \mathbf{Q}) = \underset{\sim}{Q} \cdot C^{-1} \underset{\sim}{Q}^T = \sum_{I,J=1}^{N} Q_I (C^{-1})^{IJ} Q_J \tag{3.15}$$

By choosing appropriate coordinates in field space $\mathbb{R}^N$ we can always transform C into the identity matrix.

The currents

$$J^I(u) := \partial\phi^I(u), \qquad I = 1, \ldots, N \tag{3.16}$$

generate a Kac–Moody algebra isomorphic to an N-fold tensor product of coupled chiral $\widehat{u(1)}$-current algebras.

In a QH fluid with negligible electron–electron interactions, every *filled Landau level* gives rise to a separate $\widehat{u(1)}$-current algebra at level 1 describing the edge currents first studied by Halperin[26]; see also ref. 21. The system is free of dissipative processes, with $R_L = 0$, precisely when the density of electrons is chosen such that the *extended states* of an integer number N of Landau levels are completely filled with electrons in the ground state of the system. In that case there are N independent $\widehat{u(1)}$-current algebras of edge currents. Choosing the sign of $B_\perp^{(0)}$ appropriately, these $\widehat{u(1)}$-current algebras are generated by left-moving currents for Landau levels filled with electrons, and right-moving currents for Landau levels filled with holes.

For N Landau levels filled with electrons, the vector $\mathbf{Q}$ is given by $\underset{\sim}{Q} = (1, \ldots, 1)$, the total electric edge current operator J is given by

$$J(u) = \sum_{I=1}^{N} \partial\phi^I(u) \tag{3.17}$$

and the matrix C is given by

$$C_{IJ} = \delta_{IJ}$$

so that

$$\sigma = N \tag{3.18}$$

Formulas (3.9)–(3.16) generalize what one knows from the integral quantum Hall effect[26] to general QH fluids of interacting electrons.[21]

The theory of chiral $U(1)$ currents described by formulas (3.9)–(3.16) has a Lagrangian description in terms of functional integrals. In this form, they can be coupled to external $U(1)$-vector potentials. The resulting theory exhibits a gauge anomaly given in terms of the actions W_L and W_R of Eq. (3.7). Since the theories of left- and right-movers are isomorphic, we shall focus on left-movers and omit the corresponding subscripts.

Let $\underset{\sim}{\alpha} = (\alpha_1, ..., \alpha_N)$ be an N-tuple of $U(1)$ gauge fields on the "cylinder" $\partial\Lambda = \partial\Omega \times \mathbb{R}$. We consider an action functional

$$\begin{aligned}
I_{\partial\Lambda}(\vec{\phi}, \underset{\sim}{\alpha}) := {} & \frac{1}{4\pi} \int_{\partial\Lambda} \partial_- \vec{\phi}^T(u) \cdot C \partial_+ \vec{\phi}(u) \, d^2u \\
& - \frac{1}{2\pi} \int_{\partial\Lambda} [\underset{\sim}{\alpha}_-(u) \cdot \partial_+ \vec{\phi}(u) - \underset{\sim}{\alpha}_+(u)(\partial_- \vec{\phi} - C^{-1} \underset{\sim}{\alpha}_-^T)(u)] \, d^2u \\
& + \frac{1}{4\pi} \int_{\partial\Lambda} \underset{\sim}{\alpha}_-(u) \cdot C^{-1} \underset{\sim}{\alpha}_+^T(u) \, d^2u
\end{aligned} \tag{3.19}$$

where $u := (u_+, u_-)$ and $C = (C_{IJ})$ is a positive-definite $N \times N$ matrix. Since the *nonchiral* fields $\phi^1, ..., \phi^N$ are coupled to external $U(1)$ gauge fields $\alpha_1, ..., \alpha_N$, the constraint that says that the physical degrees of freedom are described by left-moving components *cannot* be formulated by

$$\partial_- \vec{\phi}(u) = 0 \tag{3.20}$$

[i.e., $\vec{\phi}(u)$ independent of u_-], since (3.20) is *not* gauge-invariant. The correct gauge-invariant generalization of (3.20) is the equation

$$\partial_- \vec{\phi}(u) - C^{-1} \underset{\sim}{\alpha}_-^T(u) = 0 \tag{3.21}$$

For, under $U(1)$ gauge transformations, the fields $\vec{\phi}$ transform like *angles*,

$$\vec{\phi}(u) \mapsto {}^{\chi}\vec{\phi}(u) := \vec{\phi}(u) + C^{-1} \underset{\sim}{\chi}^T(u) \tag{3.22}$$

while

$$\underset{\sim}{\alpha}(u) \mapsto {}^{\chi}\underset{\sim}{\alpha}(u) := \underset{\sim}{\alpha}(u) + d\underset{\sim}{\chi}(u) \tag{3.23}$$

as usual, where $\underset{\rightarrow}{\chi} = (\chi_1, \ldots, \chi_N)$ is an N-tuple of scalar functions. Thus (3.21) is gauge-invariant.

Now, one checks by quadratic completion that

$$\mathcal{N}^{-1} \int \mathscr{D}\vec{\phi} \exp[iI_{\partial\Lambda}(\vec{\phi}, \underset{\rightarrow}{\alpha})]\, \delta(\partial_- \vec{\phi} - C^{-1}\underset{\rightarrow}{\alpha}_-^T)$$

$$= \exp\left\{\frac{i}{4\pi}\int_{\partial\Lambda}\left[\underset{\rightarrow}{\alpha}_+(u)\cdot C^{-1}\underset{\rightarrow}{\alpha}_-^T(u)\right.\right.$$

$$\left.\left. -2\underset{\rightarrow}{\alpha}_-(u)\cdot C^{-1}\frac{\partial_+^2}{\square}\underset{\rightarrow}{\alpha}_-^T(u)\right] d^2u\right\} \tag{3.24}$$

where $\mathcal{N}$ is a (divergent) normalization constant. The r.h.s. of (3.24) exhibits a $U(1)$ gauge anomaly canceled by that of a Chern–Simons action on Λ which depends on an N-tuple $\underset{\rightarrow}{A}$ of $U(1)$ vector potentials coupled through the matrix C^{-1}.

Next we set

$$\underset{\rightarrow}{\alpha}(u) := \underset{\rightarrow}{Q}\alpha(u) \qquad \text{with} \quad \alpha := A\,|_{\partial\Lambda} \tag{3.25}$$

where A is an external electromagnetic vector potential, and $\underset{\rightarrow}{Q}$ is an N-tuple of electric charges. Furthermore, we set

$$\sigma := \underset{\rightarrow}{Q}\cdot C^{-1}\underset{\rightarrow}{Q}^T \tag{3.26}$$

Comparing Eqs. (3.22) and (3.7), we find that

$$\mathcal{N}^{-1} \int \mathscr{D}\vec{\phi} \exp[iI_{\partial\Lambda}(\vec{\phi}, \underset{\rightarrow}{Q}\alpha)]\, \delta(\partial_- \vec{\phi} - C^{-1}\underset{\rightarrow}{Q}^T\alpha_-)$$

$$= \exp i\sigma W_L(\alpha) \tag{3.27}$$

The r.h.s. of (3.27) exhibits a $U(1)$ anomaly which is canceled by the $U(1)$ anomaly of

$$\exp i\frac{\sigma}{4\pi}\int_\Lambda A \wedge dA \tag{3.28}$$

Except for relative minus signs, the formulas for right-movers ($L \to R$) are identical. For a suitable choice of the direction of the uniform external magnetic field $\mathbf{B}^{(0)}$, left-moving edge currents are observed if the basic charge carriers are electrons, and right-moving ones are observed if the charge carriers are holes. Reversing the direction of $\mathbf{B}^{(0)}$ exchanges left with right.

These findings are quite important. We know from ref. 21 that the Chern–Simons term

$$i\frac{\sigma_H}{4\pi}\int_\Lambda A \wedge dA$$

is the only anomalous bulk term in the effective action $S^{\text{eff}}_\Lambda(A)$ of an incompressible QH fluid. Apparently, we learn from this that the degrees of freedom located near the boundary of such a fluid are described by N left-moving $U(1)$ currents with electric charges $(Q_{e1},..., Q_{eN})$ and coupled through a positive-definite matrix C_e, and by M right-moving $U(1)$ currents with electric charges $(Q_{h1},..., Q_{hM})$ and coupled through a positive-definite matrix C_h for certain as yet undetermined positive integers N and M. By (3.26)–(3.28), we have that

$$\sigma_H = \sigma_e - \sigma_h \tag{3.29}$$

with

$$\sigma_e = \underset{\to}{Q}_e \cdot C_e^{-1} \underset{\to}{Q}_e^T, \qquad \sigma_h = \underset{\to}{Q}_h \cdot C_h^{-1} \underset{\to}{Q}_h^T$$

The dynamics of the left-moving currents is described by a $(1+1)$-dimensional anomalous Lagrangian field theory with action $I_{\partial\Lambda}(\phi, \underset{\sim}{\alpha})$ given by (3.19) [a similar field theory $(L \to R)$ describes the right-movers].

So far, the only constraints on the still undetermined quantities N, M, $\underset{\to}{Q}_e$, $\underset{\to}{Q}_h$, C_e, and C_h are the ones described in Eq. (3.29). From the *physics* of incompressible QH fluids we shall derive further constraints on these quantities. Again, the arguments for left- and right-overs are similar, and we focus our attention on left-movers and drop subscripts.

The gauge-invariant $U(1)$-current operators

$$J^I_\mp(u) := \partial_\pm \phi^I(u) - (C^{-1}\underset{\to}{\alpha}^T_\pm)^I(u), \qquad I = 1,..., N \tag{3.30}$$

permit us to define N $U(1)$-charge operators: Let $s = (1/\sqrt{2})(u_+ - u_-)$, $t = (1/\sqrt{2}\,v)(u_+ + u_-)$; see (3.3). A gauge-invariant expression for the $U(1)$-charge operators $\vec{\mathcal{Q}} = (\mathcal{Q}^1,..., \mathcal{Q}^N)^T$ at time t is given by

$$\vec{\mathcal{Q}}_t := \oint_{\partial\Omega} \vec{J}_0(s,t)\, ds = \frac{1}{\sqrt{2}}\oint_{\partial\Omega} (\vec{J}_- - \vec{J}_+)(s,t)\, ds \tag{3.31}$$

In a Feynman path integral like the one appearing in Eq. (3.24), the fields $\vec{\phi}(s,t)$ can be chosen to be *periodic* in the space variable s with period L. We wish to consider a Feynman integral describing a transition of the boundary system from a state with $U(1)$ charges $\vec{q}_1$ at time t_1 to a state with $U(1)$ charges $\vec{q}_2$ at time t_2. By Eq. (3.31), and since the integration variables $\vec{\phi}$ are periodic in s, such a transition occurs if the external $U(1)$-gauge fields $\underset{\to}{\alpha}$ are chosen as follows: $\underset{\to}{\alpha} = \underset{\to}{\alpha}_+ \, du^+ + \underset{\to}{\alpha}_- \, du^- = \underset{\to}{\alpha}_0 \, dt + \underset{\to}{\alpha}_1 \, ds$, and the spatial components $\underset{\to}{\alpha}_1$ of $\underset{\to}{\alpha}$ are constrained to have the circulations

$$\oint_{\partial\Omega} \underset{\to}{\alpha}_1(s, t_l) \, ds = -\vec{q}_l^{\,T} \cdot C \tag{3.32}$$

If the boundary system consists of left-movers only, then we must impose the chiral constraint

$$\partial_- \vec{\phi}(u) - C^{-1} \underset{\to}{\alpha}_-^T(u) \equiv 0 \tag{3.33}$$

which, by periodicity of $\vec{\phi}$ in s, implies that $\underset{\to}{\alpha}_-$ can be gauged away. Then we have that

$$\underset{\to}{\alpha}_1 = -\frac{1}{\sqrt{2}} \underset{\to}{\alpha}_+, \qquad \underset{\to}{\alpha}_0 = \frac{1}{\sqrt{2}} \underset{\to}{\alpha}_+$$

with

$$\oint_{\partial\Omega} \underset{\to}{\alpha}_+(s, t_l) \, ds = \sqrt{2} \, \vec{q}_l^{\,T} \cdot C \tag{3.34}$$

for $l = 1, 2$. From (3.34), (3.33), (3.30), and (3.31) we finally obtain

$$\vec{\mathcal{Q}}_{t_l} = \vec{q}_l \qquad \text{for} \quad l = 1, 2 \tag{3.35}$$

Let us suppose that the gauge fields $\underset{\to}{\alpha}$ are the restrictions of N $U(1)$ gauge fields $\underset{\to}{A} = (A_1, ..., A_N)$ defined on the bulk space-time $\Lambda = \Omega \times \mathbb{R}$ of the QH fluid to the boundary $\partial\Lambda$. Then, by Stokes' theorem,

$$\oint_{\partial\Omega} \underset{\to}{\alpha} \, ds = \int_{\Omega} d\underset{\to}{A} \equiv \underset{\to}{n} \tag{3.36}$$

where the Ith component n_I of $\underset{\to}{n}$ is the *total flux* of the "magnetic field" $B_I = dA_I$ through space Ω, for $I = 1, ..., N$. By (3.32), $\underset{\to}{n}$ and the $U(1)$ charges $\vec{q}$ are related by the equation

$$\vec{q} = C^{-1} \underset{\to}{n}^T \tag{3.37}$$

We shall call the vectors $\underset{\to}{n}$ "*flux vectors*," while the $\vec{q}$'s are called "*charge vectors*."

The chiral theory described by the action (3.19) and the Feynman integral (3.24) can be equivalently described by a *topological Chern–Simons* *theory* on a space-time $\Lambda = \Omega \times \mathbb{R}$. This fact is called *boundary–bulk duality*.[21] To understand boundary–bulk duality, we introduce N $U(1)$ gauge fields

$$\vec{b} = (b^1, ..., b^N)^T \tag{3.38}$$

and N external vector potentials $\underset{\rightarrow}{A} = (A_1, ..., A_N)$ and define the Chern–Simons action

$$S_\Lambda(\vec{b}, \underset{\rightarrow}{A}) = \frac{1}{4\pi} \int_\Lambda \vec{b}^T \wedge C\, d\vec{b} + \frac{1}{2\pi} \int_\Lambda d\underset{\rightarrow}{A} \wedge \vec{b} + \text{B.T.} \tag{3.39}$$

where B.T. stands for (gauge-dependent) boundary terms. We note that $S_\Lambda(\vec{b}, \underset{\rightarrow}{A})$ is quadratic in $\vec{b}$. It is therefore not hard to show—modulo some subtle ties related to gauge fixing[30,31]—that

$$\begin{aligned} &\int \{\exp[-iS_\Lambda(\vec{b}, \underset{\rightarrow}{A})]\}\, \mathscr{D}\vec{b}\,|_{\text{g.f.}} \\ &\quad = \mathscr{N} \exp -i \left[\frac{1}{4\pi} \int_\Lambda \underset{\rightarrow}{A} \wedge C^{-1}\, d\underset{\rightarrow}{A}^T - W_L(C; \underset{\rightarrow}{A}\,|_{\partial\Lambda})\right] \end{aligned} \tag{3.40}$$

where $\mathscr{N}$ is a normalization factor, and

$$\begin{aligned} W_L(C; \underset{\rightarrow}{\alpha}) = \frac{1}{4\pi} \int_{\partial\Lambda} [&\underset{\rightarrow}{\alpha}_+(u) \cdot C^{-1} \underset{\rightarrow}{\alpha}_-^T(u) \\ &- 2\underset{\rightarrow}{\alpha}_-(u) \cdot C^{-1} \frac{\partial_+^2}{\Box} \underset{\rightarrow}{\alpha}_-^T(u)]\, d^2u \end{aligned} \tag{3.41}$$

and where "g.f." indicates some gauge fixing for the degrees of freedom located at $\partial\Lambda$.[31,19,20] Formally, Eq. (3.40) follows from (3.39) by quadratic completion. Thus the partition function of an incompressible QH fluid in the limit of large distance scales and low frequencies is obtained form a Chern–Simons theory for a certain number of Abelian gauge fields $\vec{b}$ coupled to external vector potentials $\underset{\rightarrow}{A} = \underset{\rightarrow}{Q}A$, where A is an electromagnetic vector potential. The Hall conductivity σ is given by $\sigma = \underset{\rightarrow}{Q} \cdot C^{-1} \underset{\rightarrow}{Q}^T$.

The gauge fields $\vec{b}$ can be interpreted as the *vector potentials of conserved currents*

$$\vec{\imath} = * d\vec{b} \tag{3.42}$$

with

$$i := \underset{\rightarrow}{Q} \cdot \vec{\imath} \tag{3.43}$$

the total electric density operator, in a description of the QH fluid valid, asymptotically, on large distance scales and at low frequencies. This has been discussed in detail in refs. 19 and 20.

The $U(1)$-charge operators associated with a current distribution $\vec{\imath}$ at time t are given by

$$\vec{\mathscr{Q}}_t = \int_\Omega \vec{\imath}^{\,0}(\vec{x}, t)\, d^2x$$

and the electric charge operator $\mathscr{Q}_t$ is given by

$$\mathscr{Q}_t = \underset{\rightarrow}{Q} \cdot \vec{\mathscr{Q}}_t$$

The quantum mechanical equations of motion obtained by varying the Chern–Simons action (3.39) with respect to $\vec{b}$ are given by

$$d\vec{b} = C^{-1} d\underset{\rightarrow}{A}^T \qquad \text{or} \qquad \vec{\imath} = C^{-1} * \underset{\rightarrow}{F}^T \tag{3.44}$$

where $\underset{\rightarrow}{F} = d\underset{\rightarrow}{A}$. Integrating these equations over all of space Ω, we obtain that

$$\vec{\mathscr{Q}}_t = C^{-1} \underset{\rightarrow}{n}_t^T \tag{3.45}$$

where

$$\underset{\rightarrow}{n}_t = \int_\Omega d\underset{\rightarrow}{A}(\vec{x}, t) \tag{3.46}$$

is the flux vector at time t.

Let us consider a state of the system with $U(1)$ charges given by a charge vector $\vec{q}$, corresponding to a vector of eigenvalues of $\vec{\mathscr{Q}}$. Then (3.45) implies that

$$\vec{q} = C^{-1}\, \underset{\rightarrow}{n}^T \tag{3.47}$$

This equation coincides with (3.37), as one would expect. The electric charge of the state (assuming that the electric charge of the ground state is set to zero) is then given by

$$q_{\mathrm{el}} = \underset{\rightarrow}{Q} \cdot \vec{q} = \underset{\rightarrow}{Q} \cdot C^{-1} \underset{\rightarrow}{n}^T \tag{3.48}$$

In particular, if $\underset{\rightarrow}{A} = \underset{\rightarrow}{Q} A$, where A is an external electromagnetic vector potential with magnetic flux $m = \int_\Omega dA$, then

$$q_{\mathrm{el}} = \sigma m, \qquad \sigma = \underset{\rightarrow}{Q} \cdot C^{-1} \underset{\rightarrow}{Q}^T \tag{3.49}$$

Let us consider a state of an incompressible QH fluid describing k electrons and l holes excited from the ground state by coupling the QH fluid to suitable external vector potentials $\underset{\rightarrow}{A}$. Suppose this state is described by a charge $\vec{q}$. Then we clearly have that

$$q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q} = l - k$$

We set $\underset{\rightarrow}{A}^{(0)} = \underset{\rightarrow}{Q} A^{(0)}$, where $A^{(0)}$ is a fixed background electromagnetic vector potential; see Eqs. (2.16), (2.17). Then $\underset{\rightarrow}{Q} A^{(0)} = \underset{\rightarrow}{A}^{(0)} + \underset{\rightarrow}{A}$, and the vector potentials $\underset{\rightarrow}{A}$ form an *additive group*. Hence the flux vectors $\underset{\rightarrow}{n}$ and, by Eq. (3.47), the charge vectors $\vec{q}$ of physical states of an incompressible QH fluid form an additive group. We denote the group of charge vectors of physical states by Γ_{phys}. This group contains a lattice, denoted by Γ, of $\vec{q}$-vectors with *integer* electric charge, i.e.,

$$\Gamma = \{\vec{q} \in \Gamma_{\text{phys}} : q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q} \in \mathbb{Z}\} \tag{3.50}$$

Now, the physics of incompressible QH fluids motivates us to require the following *Basic Hypotheses*:

A1. An arbitrary localized cluster of quasiparticle excitations of an incompressible QH fluid of electric charge $q \in \mathbb{Z}$ can be interpreted as a physical state of the system composed of $l + q$ holes and l electrons, for some $l = 0, 1, 2, \ldots$.

A2. Electrons and holes satisfy *Fermi statistics*. Thus, a cluster of quasiparticles of electric charge $q \in \mathbb{Z}$ is

$$\left.\begin{array}{r}\text{a fermion if } q \text{ is an odd integer}\\ \text{a boson if } q \text{ is an even integer}\end{array}\right\} \tag{3.51}$$

Wave functions of physical states of an incompressible QH fluid are *single-valued* in the positions of electrons or holes.

Let us explore the consequences of hypotheses A1 and A2. For this purpose, we consider histories of states of an incompressible QH fluid describing an excitation with charge vector $\vec{q}_1$ localized in a disk D_1 and an excitation with charge vector $\vec{q}_2$ localized in a disk D_2 disjoint from D_1. We suppose that D_1 and D_2 sweep out space-time tubes T_1 and T_2, as depicted in Fig. 1 (0), (1), and (2). According to Eqs. (3.44) and (3.47), such histories are described by coupling the gauge fields $\underset{\rightarrow}{b}$ to external vector potentials $\underset{\rightarrow}{A}^{(m)}$, with

$$\operatorname{supp} d\underset{\rightarrow}{A}^{(m)} \subseteq T_1^{(m)} \cup T_2^{(m)}$$

$$\operatorname{supp} d\underset{\rightarrow}{A}^{(m)}(t_0, \cdot) \subseteq D_1^{(m)} \cup D_2^{(m)}$$

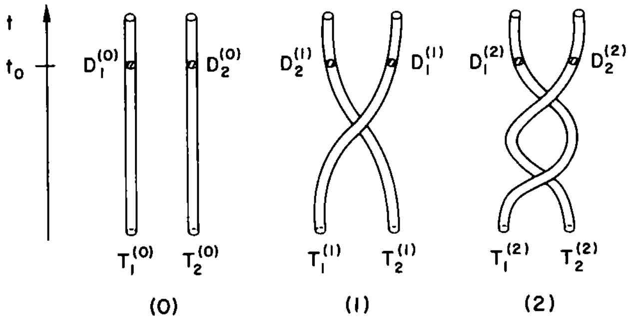

Fig. 1. Graphical representation of three possible histories ($m = 0$, 1, or 2) of a state of an incompressible quantum Hall fluid describing two excitations localized in two disjoint disks $D_1^{(m)}$ and $D_2^{(m)}$ sweeping out space-time tubes $T_1^{(m)}$ and $T_2^{(m)}$.

with

$$(d\underset{\rightarrow}{A}^{(m)})_{ij} = 0 \qquad \text{unless} \quad i = 1, j = 2, \quad \text{or } i = 2, j = 1 \tag{3.52}$$

and

$$\int_{D_\alpha^{(m)}} d\underset{\rightarrow}{A}^{(m)T}(\vec{x}, t_0) = C\vec{q}_\alpha, \qquad \alpha = 1, 2 \tag{3.53}$$

for $m = 0$, 1, 2, as follows from Eqs. (3.46), (3.47). We let

$$I^{(m)} := \int \{\exp[-iS_A(\vec{b}, \underset{\rightarrow}{A}^{(m)})]\} \mathscr{D}\vec{b}\,|_{\text{g.f.}} \tag{3.54}$$

be given by (3.40), with $\underset{\rightarrow}{A}^{(m)}$ as described above, for $m = 0, 1, 2$. We consider the two ratios $I^{(1)}/I^{(0)}$ and $I^{(2)}/I^{(0)}$. In these ratios we can pass to the limit $\Omega \nearrow \mathbb{R}^2$, $\Lambda \nearrow \mathbb{R}^3$. Using the explicit expression on the r.h.s. of Eq. (3.40), we can then calculate the limiting ratios explicitly. For $\vec{q}_1 = \vec{q}_2 = \vec{q}$, we find that

$$I^{(1)}/I^{(0)} = \exp i(\varphi_1 + \varphi_2) \exp \pi i(\vec{q}^T \cdot C\vec{q}) \tag{3.55}$$

where φ_ε is a phase only depending on

$$\underset{\rightarrow}{A}\,|_{T_\varepsilon^{(m)}}$$

for $\varepsilon = 1, 2$; $m = 0, 1$. For arbitrary $\vec{q}_1$ and $\vec{q}_2$, we find that

$$I^{(2)}/I^{(0)} = \exp i(\psi_1 + \psi_2) \exp 2\pi i(\vec{q}_1^T \cdot C\vec{q}_2) \tag{3.56}$$

where ψ_ε is a phase only depending on

$$\underset{\rightarrow}{A}\,|_{T_\varepsilon^{(m)}}$$

for $\varepsilon = 1, 2$; $m = 0, 2$. [For suitable choices of

$$\underset{\rightarrow}{A}\,|_{T_\varepsilon^{(m)}}, \qquad \varepsilon = 1, 2, \quad m = 0, 1, 2$$

the phases $\varphi_1, \varphi_2, \psi_1$, and ψ_2 actually vanish.]

The *Aharonov–Bohm phases*, $\exp \pi i(\vec{q}^T \cdot C\vec{q})$ and $\exp 2\pi i(\vec{q}_1^T \cdot C\vec{q}_2)$, describe the *statistics* of the quasiparticle excitations. Thus, if $\vec{q}_1 = \vec{q}_2 = \vec{q}$, with $q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q} = \pm 1$, the two excitations are holes or electrons and thus satisfy Fermi statistics. Hence, by Eq. (3.55), $\exp i\pi(\vec{q} \cdot C\vec{q}) = -1$, i.e., $\vec{q} \cdot C\vec{q}$ is an odd integer. More generally, by hypothesis A2, two excitations with charge vectors $\vec{q}_1 = \vec{q}_2 = \vec{q} \in \Gamma_{\text{phys}}$ are identical bosons if $q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q}$ is an even integer and identical fermions if $q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q}$ is an odd integer. Thus, using Eq. (3.55), we conclude that if $q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q}$ is an integer, then $\vec{q}^T \cdot C\vec{q}$ is an integer, and the parity of $\underset{\rightarrow}{Q} \cdot \vec{q}$ equals the parity of $\vec{q}^T \cdot C\vec{q}$, i.e.,

$$\underset{\rightarrow}{Q} \cdot \vec{q} \equiv \vec{q}^T \cdot C\vec{q} \mod 2 \tag{3.57}$$

Among the quasiparticles appearing in an incompressible QH fluid there is a single electron and hole. Thus

$$\underset{\rightarrow}{Q} \cdot \vec{q}_1 = 1 \qquad \text{for some vector } \vec{q}_1 \in \Gamma \tag{3.58}$$

We conclude that Γ is an integral lattice contained in $\mathbb{R}^N$, the matrix C determines an integral quadratic form $(\cdot, \cdot)$ on Γ, and $\mathbf{Q}$ is a vector with components Q in the dual, Γ^*, of Γ. By (3.40), (3.19), and (3.14), C is positive-definite and hence Γ is a Euclidean lattice. By (3.58) and (3.57), Γ contains a vector $\vec{q}_1$ such that

$$\underset{\rightarrow}{Q} \cdot \vec{q}_1 = 1 \qquad \text{and hence} \quad \vec{q}_1^T \cdot C\vec{q}_1 \quad \text{is odd} \tag{3.59}$$

Thus Γ is an *odd, integral Euclidean lattice*, and $\mathbf{Q}$ is a *visible vector* in the dual lattice Γ^* (i.e., the open line segment from the origin of Γ^* to $\mathbf{Q}$ does not contain any points of Γ^*):

$$\text{g.c.d.}(Q_1, \ldots, Q_N) = 1$$

Next, let us consider a state describing a cluster of quasiparticles with charge vector $\vec{q}_1 \in \Gamma$ localized in a disk D_1 and a cluster of quasiparticles with an arbitrary charge vector $\vec{q}_2 \in \Gamma_{\text{phys}}$ localized in a disk D_2. If the cluster with charge vector $\vec{q}_1$ makes a round trip around the cluster with charge vector $\vec{q}_2$, as depicted in Fig. 1(2), then the state is multiplied by an Aharonov–Bohm phase factor

$$\exp 2\pi i \vec{q}_1^T \cdot C\vec{q}_2$$

[see (3.56)]. Since $\vec{q}_1 \in \Gamma$, $\underset{\rightarrow}{Q} \cdot \vec{q}_1 \in \mathbb{Z}$, hence the cluster of quasiparticles with charge vector $\vec{q}_1$ corresponds to electrons and holes. Then hypothesis (A2) implies that

$$\exp 2\pi i \vec{q}_1^T \cdot C\vec{q}_2 = 1$$

i.e.,

$$\vec{q}_1^T \cdot C\vec{q}_2 \in \mathbb{Z} \qquad \text{for all} \quad \vec{q}_1 \in \Gamma, \quad \vec{q}_2 \in \Gamma_{\text{phys}} \tag{3.60}$$

It follows that

$$\Gamma_{\text{phys}} \subseteq \Gamma^* \tag{3.61}$$

By (3.49), and since the quadratic form $(\cdot, \cdot)$ is integral on Γ and $\mathbf{Q} \in \Gamma^*$, it follows that

$$\sigma = (\mathbf{Q}, \mathbf{Q}) = \underset{\rightarrow}{Q} \cdot C^{-1} \underset{\rightarrow}{Q}^T \quad \text{is a rational number} \tag{3.62}$$

If an incompressible QH fluid is composed of two such fluids with the property that the basic charge carriers of one fluid are electrons while the basic charge carriers of the other are holes, then the entire story told so far must be repeated with $(e, L, -, \cdots)$ replaced by $(h, R, +, \cdots)$. We then conclude that such a fluid is characterized, asymptotically on large distance scales and at low frequencies, by two integral, odd Euclidean lattices Γ_e and Γ_h, integral quadratic forms $(\cdot, \cdot)_e$ on Γ_e and $(\cdot, \cdot)_h$ on Γ_h, and visible vectors $\mathbf{Q}_e \in \Gamma_e^*$ and $\mathbf{Q}_h \in \Gamma_h^*$ such that

$$\sigma_H = \sigma_e - \sigma_h \tag{3.63}$$

with

$$\sigma_e = (\mathbf{Q}_e, \mathbf{Q}_e)_e, \qquad \sigma_h = (\mathbf{Q}_h, \mathbf{Q}_h)_h \tag{3.64}$$

It follows that σ_H is a rational number.

Comparing these conclusions with Eqs. (1.12)–(1.15), we find that we have established a first part of the "Basic Result" announced in Section 1.

4. A CRASH COURSE ON THE REPRESENTATION THEORY OF CHIRAL $\widehat{u(1)}$-CURRENT ALGEBRAS

In this section, we reconsider the main results of Section 3 from the point of view of the representation theory of chiral $\widehat{u(1)}$-current algebras.

In Eqs. (3.14) and (3.16) we found that an incompressible QH fluid in a uniform background magnetic field $\mathbf{B}^{(0)}$, with electrons as basic charge carriers, exhibits chiral edge currents,

$$J^I(u) = \partial\phi^I(u), \qquad I = 1,..., N \tag{4.1}$$

localized near the boundary $\partial\Omega$ of the system. For an appropriate choice of the direction of $\mathbf{B}^{(0)}$, these currents are left-movers, and $u = (1/\sqrt{2})[vt + (L/2\pi)\,\vartheta]$ is a light-cone coordinate on $\partial\Lambda = \partial\Omega \times \mathbb{R}$; see (3.3). The commutation relations of the currents J^I are given by

$$[J^I(u), J^L(u')] = i(C^{-1})^{IL}\,\delta'(u-u') \tag{4.2}$$

where C is a positive-definite $N \times N$ matrix; see (3.14). By choosing a suitable basis in field space $(\phi^1,..., \phi^N)$ we can always achieve that C is the identity matrix.

All unitary representations of these $\widehat{u(1)}$-current algebras can be constructed with the help of vertex operators[27,28]

$$V_L(u; \underset{\rightarrow}{n}) =: \exp i\sqrt{2}\,\underset{\rightarrow}{n}\cdot\vec{\phi}(u): \qquad \underset{\rightarrow}{n} \in \mathbb{R}^N \tag{4.3}$$

The vertex operators generate the operator (product) algebra

$$V_L(u; \underset{\rightarrow}{n})\, V_L(u'; \underset{\rightarrow}{n}')\, \underset{u\to u'}{\sim} (u-u')^{\Delta''-\Delta-\Delta'}\, V_L(u; \underset{\rightarrow}{n} + \underset{\rightarrow}{n}') \tag{4.4}$$

where

$$\begin{aligned} \Delta &= \tfrac{1}{2}(\mathbf{n}, \mathbf{n}) \equiv \tfrac{1}{2} \sum_{I,J=1}^{N} n_I (C^{-1})^{IJ} n_J \\ \Delta' &= \tfrac{1}{2}(\mathbf{n}', \mathbf{n}') \\ \Delta'' &= \tfrac{1}{2}(\mathbf{n}+\mathbf{n}', \mathbf{n}+\mathbf{n}') \end{aligned} \tag{4.5}$$

While the currents $J^I(u)$ are *periodic* operator-valued distributions of the light-cone variable u with period $L/\sqrt{2}$ [see Eqs. (3.3), (3.9)], the vertex operators $V_L(u; \underset{\rightarrow}{n})$ are in general *not* periodic in u, but should be viewed as operator-valued distributions on the covering space of the circle of circumference L, i.e., on the real line. By (4.2) and (4.3), they satisfy the quadratic Weyl algebra

$$V_L(u; \underset{\rightarrow}{n})\, V_L(u'; \underset{\rightarrow}{n}') = \{\exp[\pm\pi i\theta(\underset{\rightarrow}{n}, \underset{\rightarrow}{n})]\}\, V_L(u'; \underset{\rightarrow}{n}')\, V_L(u; \underset{\rightarrow}{n}) \tag{4.6}$$

for $u \gtrless u'$, where the phase $\theta(\underset{\rightarrow}{n}, \underset{\rightarrow}{n})$ is given by

$$\theta(\underset{\rightarrow}{n}, \underset{\rightarrow}{n}') = (\mathbf{n}, \mathbf{n}') = \sum_{I,J=1}^{N} n_I (C^{-1})^{IJ} n_J \tag{4.7}$$

The $U(1)$-charge operators $\vec{\mathcal{Q}}$ are given by

$$\vec{\mathcal{Q}} = (\mathcal{Q}^1, .., \mathcal{Q}^N)^T$$

with

$$\mathcal{Q}^I = \frac{1}{\sqrt{2}} \oint J^I(u)\, du \tag{4.8}$$

Equations (4.1)–(4.3) yield the commutation relations

$$[\mathcal{Q}^I, V_L(u; \underset{\sim}{n})] = q^I V_L(u; \underset{\sim}{n}) \tag{4.9}$$

where

$$q^I = \sum_{M=1}^{N} (C^{-1})^{IM} n_M \tag{4.10}$$

Let $\{J_k^I\}_{k \in \mathbb{Z}}$ be the Fourier coefficients of $J^I(u)$, i.e.,

$$J^I(u) = \sum_{k \in \mathbb{Z}} J_k^I \exp\left(2\pi i k \frac{\sqrt{2}\, u}{L}\right) \tag{4.11}$$

The vacuum state $|0\rangle$ of the $\widehat{u(1)}$-current algebras is characterized by the property that

$$J_k^I \,|0\rangle = 0 \qquad \text{for} \quad k = 0, 1, 2, ... \tag{4.12}$$

A dense set of states with vanishing $U(1)$ charges is obtained by acting with polynomials in the operators J_k^I, $k < 0$, on the vacuum $|0\rangle$. Let ψ be such a state. Then, formally,

$$V_L(u, \underset{\sim}{n})\, \psi \tag{4.13}$$

is a state[6] with $U(1)$ charges $\vec{q}(\underset{\sim}{n})$, where $q^I = q^I(\underset{\sim}{n})$ is given by Eq. (4.10). This follows from (4.9) and the fact that $\vec{\mathcal{Q}}\psi = 0$.

Every charge vector $\vec{q}$ of eigenvalues of $\vec{\mathcal{Q}}$ labels a distinct, unitary irreducible representation of the tensor product of N $\widehat{u(1)}$-current algebras. The representation space is spanned by the vectors (4.13), with $\vec{q}(\underset{\sim}{n}) = \vec{q}$. Thus the vertex operators $V_L(u; \underset{\sim}{n})$ play the role of Clebsch–Gordan operators in the representation theory of $\widehat{u(1)}$-current algebra.

Comparing Eqs. (4.9), (4.10), and (4.13) with Eqs. (3.24) and (3.31)–(3.35), we conclude that applying a vertex operator $V_L(u; \underset{\sim}{n})$ to

[6] Smearing (4.13) in u with a test function, one obtains a well-defined normalizable state.

some state ψ at time 0 corresponds, in a Feynman path integral formalism, to coupling the integration variables $\vec{\phi}$ in the path integral (3.24) to external gauge fields $\underset{\rightarrow}{\alpha}$ with the following properties: $\underset{\rightarrow}{\alpha}$ is the restriction to the boundary of N $U(1)$-gauge fields $\underset{\rightarrow}{A} = (A_1,..., A_N)$ on Λ that describe a vortex tube in Λ carrying fluxes $\underset{\rightarrow}{n}$ and contained in the half-space at positive time which ends, at time 0, in a magnetic monopole with magnetic charges $\underset{\rightarrow}{n}$ located in the point $(u, 0) \in \partial\Lambda$.

Repeating the discussion at the end of Section 3, after Eq. (3.49), we must ask which family of vertex operators $V_L(u; \underset{\rightarrow}{n})$ creates physical states of the algebras of chiral edge currents of an incompressible QH fluid when applied to states of charge 0. Clearly, we want these vertex operators to generate a closed operator algebra for the operator product specified in (4.4). Thus the charge vectors $\vec{q} = C^{-1}\underset{\rightarrow}{n}^T$ [see Eq. (4.10)] labeling physical representations of the algebras of chiral edge currents form an additive group Γ_{phys}. The electric charge of a state with $U(1)$ charges $\vec{q}$ is given by

$$q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q} \tag{4.14}$$

since, by Eq. (3.11), the electric edge current density J is given by

$$J = \underset{\rightarrow}{Q} \cdot \partial\vec{\phi} = \underset{\rightarrow}{Q} \cdot \vec{J} \tag{4.15}$$

See also eqs. (3.48), (4.9), and (4.10). The charge vectors $\vec{q}$ with integer electric charge $q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q}$ form a lattice $\Gamma \subset \Gamma_{\text{phys}}$. Hypotheses A1 and A2 of Section 3, after Eq. (3.50), can be reformulated as follows:

A1′. A vertex operator $V_L(u; \underset{\rightarrow}{n})$ with $q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q}(\underset{\rightarrow}{n}) = \underset{\rightarrow}{Q} \cdot C^{-1}\underset{\rightarrow}{n}^T = q \in \mathbb{Z}$ creates a boundary excitation of the system composed of $l + q$ holes and l electrons, $l = 0, 1, 2,...$.

A2′. A vertex operator $V_L(u; \underset{\rightarrow}{n})$ must satisfy

$$\begin{aligned} &\textit{Fermi statistics} \text{ if } \quad q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q}(\underset{\rightarrow}{n}) \quad \text{is an } \textit{odd integer} \\ &\textit{Bose statistics} \text{ if } \quad q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q}(\underset{\rightarrow}{n}) \quad \text{is } \textit{even} \end{aligned} \tag{4.16}$$

If $q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q}(\underset{\rightarrow}{n})$ is an integer and $\vec{q}\,' \equiv \vec{q}(\underset{\rightarrow}{n}')$ belongs to Γ_{phys}, then

$$V_L(u; \underset{\rightarrow}{n})\, V_L(u'; \underset{\rightarrow}{n}') = \mp V_L(u'; \underset{\rightarrow}{n}')\, V_L(u; \underset{\rightarrow}{n}) \tag{4.17}$$

(independently of the sign of $u - u'$).

Combining A2′ with the Weyl relations (4.6), we conclude, in view of (4.17), that

$$\begin{aligned} &\text{if} \quad \underset{\rightarrow}{Q} \cdot \vec{q} \text{ is an odd integer, with } \vec{q} = C^{-1}\underset{\rightarrow}{n}^T, \text{ then} \\ &\vec{q}^{\,T} \cdot C\vec{q} = \underset{\rightarrow}{n} \cdot C^{-1}\underset{\rightarrow}{n}^T \text{ is an odd integer} \end{aligned} \tag{4.18}$$

$$\text{if} \quad \underset{\rightharpoondown}{Q} \cdot \vec{q} \text{ is an even integer, with } \vec{q} = C^{-1} \underset{\rightharpoondown}{n}^T, \text{ then}$$
$$\vec{q}^T \cdot C\vec{q} = \underset{\rightharpoondown}{n} \cdot C^{-1} \underset{\rightharpoondown}{n}^T \text{ is an even integer} \tag{4.19}$$

Furthermore,

$$\text{if} \quad \underset{\rightharpoondown}{Q} \cdot \vec{q} \text{ is an integer and } \vec{q}' \in \Gamma_{\text{phys}}, \text{ then}$$
$$\vec{q}^T \cdot C\vec{q}' = \underset{\rightharpoondown}{n} \cdot C^{-1} \underset{\rightharpoondown}{n}'^T \text{ is an integer} \tag{4.20}$$

as follows from (4.17), (4.6), and (4.7).

Thus, as in Section 3, we find that Γ is an integral, odd Euclidean lattice in $\mathbb{R}^N$, C defines a positive-definite, integral quadratic form on Γ, and Γ_{phys} is a lattice contained in or equal to the lattice Γ^* dual to Γ.

Next, we wish to make the connection between the two descriptions (boundary–bulk duality) of an incompressible QH fluid, (i) in terms of a topological Chern-Simons theory, and (ii) in terms of the representation theory of $\widehat{u(1)}$-current algebras, more precise. This connection has been described in the literature, starting with ref. 29; see also refs. 19, 20, 30, and 31. A key fact concerning this connection is the following (see ref. 30): In the Chern–Simons theory described in Section 3, (3.39)–(3.44), a physical state with $U(1)$ charges $\vec{q}_a$ concentrated at points (x_a, y_a) of Ω, $a = 1, \ldots, P$, is described by a *conformal block*[41]

$$\left\langle \prod_{a=1}^{P} V_L(z_a; \underset{\rightharpoondown}{n}_a) \right\rangle_{\underset{\rightharpoondown}{\alpha}^{(0)}} \tag{4.21}$$

of the chiral conformal field theory introduced in (3.19), (3.21)–(3.24) which describes the representation theory of $N\, \widehat{u(1)}$-current algebras. Here

$$z_a = x_a + iy_a \tag{4.22}$$

and the flux vectors $\underset{\rightharpoondown}{n}_a$ satisfy the equations

$$\underset{\rightharpoondown}{n}_a = \vec{q}_a^T C \tag{4.23}$$

The gauge fields $\underset{\rightharpoondown}{\alpha}^{(0)}$ are chosen to be the vector potentials of N uniform, neutralizing background "magnetic fields" with

$$\oint_{\partial\Omega} \underset{\rightharpoondown}{\alpha}^{(0)} = \sum_{a=1}^{P} \underset{\rightharpoondown}{n}_a \tag{4.24}$$

The conformal blocks in (4.21) are given by branches of the generally *multivalued* functions

$$\prod_{1 \leqslant a < b \leqslant P} (z_a - z_b)^{(\mathbf{q}_a, \mathbf{q}_b)} f_{\underset{\rightharpoondown}{\alpha}^{(0)}}(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) \tag{4.25}$$

where $f_{\alpha^{(0)}}$ are single-valued functions on $\Omega^{\times P}$, and $(\mathbf{q}_a, \mathbf{q}_b) = \vec{q}_a^{\,T} \cdot C\vec{q}_b$; see, e.g., ref. 36.

Note that the monodromy phases of the functions in (4.25) are precisely given by the phases

$$\exp 2\pi i\theta(\underset{\to}{n}_a, \underset{\to}{n}_b), \qquad \underset{\to}{n}_a = \vec{q}_a^{\,T} C \tag{4.26}$$

where θ is given in Eq. (4.7). This makes the connection between hypotheses A2 of Section 3 and A2′ above precise.

The function in (4.25) describes the asymptotic behavior at large distances of an amplitude describing a state of the QH fluid, where localized quasiparticle excitations of charge vectors $\vec{q}_a$ are present at the points (x_a, y_a) of Ω for $a = 1,\dots, P$. By (4.25) and (4.26), the quantities

$$\theta(\underset{\to}{n}_a, \underset{\to}{n}_b) = (\mathbf{n}_a, \mathbf{n}_b) = \underset{\to}{n}_a C^{-1} \underset{\to}{n}_b^T \tag{4.27}$$

are apparently the values of the *relative angular momentum* of the excitations at (x_a, y_a) and at (x_b, y_b); $1 \leqslant a, b \leqslant P$. If $Q \cdot \vec{q}_a = Q \cdot \vec{q}_b = -1$, the two excitations describe two single electrons. Then the relative angular momentum between these two electrons is given by

$$L_{ab} = (\mathbf{n}_a, \mathbf{n}_b) = (\mathbf{q}_a, \mathbf{q}_b) \tag{4.28}$$

The total, orbital angular momentum of the state described by the amplitude in (4.25) is then given by

$$L_{\text{tot}} = \sum_{1 \leqslant a < b \leqslant P} (\mathbf{q}_a, \mathbf{q}_b) \tag{4.29}$$

We shall see later that in a tensor product of N $\widehat{u(1)}$-current algebras one can imbed, in general in many different and inequivalent ways, and $\widehat{su}(2)$-current algebra.[42, 43],[7] This will enable us to describe *electron spin*, which has been neglected so far; see Section 6.

So far we have assumed that the QH fluid is confined to a domain Ω in the (x, y) plane. The connection between states of Chern–Simons theory and conformal blocks of massless, chiral free fields expressed in (4.21) enables us to study incompressible QH fluids on arbitrary surfaces Σ, e.g., surfaces without boundary and of arbitrary genus. Although such systems cannot be realized in the laboratory, their study is of some interest, e.g., for purposes of numerical simulations.

A ground state of Chern–Simons theory with an action given by (3.39) on a space-time $\Lambda = \Sigma \times \mathbb{R}$ is given by a conformal block of the conformal

[7] For the classification of conformal embeddings see ref. 48.

field theory corresponding to (3.19), (3.21) on the surface Σ without any punctures. These conformal blocks span a linear space of dimension Δ^g, where g is the genus of the surface Σ, and Δ is the order of the Abelian group Γ^*/Γ which is equal to the discriminant of the integral quadratic form on Γ given by $(\mathbf{q}, \mathbf{q}') = \vec{q}^T \cdot C\vec{q}'$, $\mathbf{q}, \mathbf{q}' \in \Gamma$.[34] Thus, if $\Gamma_{\text{phys}} = \Gamma^*$, then the QH fluid on a surface Σ of genus g, described by the Chern–Simons theory (3.39), has

$$\Delta^g \quad \text{degenerate groundstates,} \qquad \text{with} \quad \Delta = |\Gamma^*/\Gamma| \tag{4.30}$$

This result has previously been noticed in ref. 36.

It is a widely accepted heuristic idea that conformal blocks like those in (4.21) are likely to capture some of the main features of electronic ground-state wave functions of an incompressible QH fluid of N electrons, provided that $q_{\text{el}} = \underset{\rightarrow}{Q} \cdot \vec{q}_a = \underset{\rightarrow}{Q} C^{-1} \underset{\rightarrow}{n}_a^T = -1$ for $a = 1,..., P$; see, e.g., ref. 36. Of course, this idea does *not* logically follow from our analysis. However, for the Laughlin QH fluid at $\sigma_H = 1/3$ and other simple fluids, it has been quite successful,[32,33] for reasons that are not entirely understood. Taking the idea seriously and studying an incompressible QH fluid on a closed surface of genus g, one can make the following prediction of interest to people who do numerical simulations: We consider a gas of electrons on the surface Σ. Let Φ denote the total flux of the external magnetic field $\mathbf{B}^{(0)}$ through Σ. For simple topological reasons Φ is an integer, in units where $h/e = 1$. We imagine that there are N different species of electrons corresponding to charge vectors $\vec{q}^{(1)},..., \vec{q}^{(N)}$ which are a basis of the lattice Γ. Let l_i be the number of electrons of type $\vec{q}^{(i)}$ on Σ. Let us assume that the energies of eigenstates of the quantum mechanical Hamiltonian of the fluid which are orthogonal to all the ground states of the system are separated from the ground-state energies by a fairly large gap for a given value of Φ and for given $l_1,..., l_N$. It is then tempting to imagine that a ground-state wave function of this system is described by a conformal block of the conformal field theory with an action $I_\Sigma(\vec{\phi}, \underset{\rightarrow}{\alpha})$ as given in Eq. (3.19), where

$$\underset{\rightarrow}{\alpha} = \underset{\rightarrow}{Q}(A + \Omega) \tag{4.31}$$

where A is the vector potential whose field strength is the given external magnetic field $\mathbf{B}^{(0)}$ and Ω is the Levi-Civita spin connection on Σ in the representation with conformal spin 1/2, corresponding to the fact that electrons have spin 1/2. By the Gauss–Bonnet theorem, the integral of the curvature of the spin connection Ω over Σ is given by $1 - g$. Standard neutrality conditions for the conformal blocks of the field theory with action $I_\Sigma(\vec{\phi}, \underset{\rightarrow}{\alpha})$, with $\underset{\rightarrow}{\alpha}$ as in (4.31), imply that

$$\sum_{i=1}^{N} l_i \vec{q}^{(i)} = C^{-1} \underset{\rightarrow}{Q}^T (\Phi + (1 - g)) \tag{4.32}$$

and hence, by multiplying with $\underset{\rightarrow}{Q}$,

$$N_e = \sigma_H(\Phi + (1-g)) \tag{4.33}$$

where N_e is the total number of electrons in the system. Equations (4.32) and (4.33) are necessary conditions for the ground-state wave functions of an incompressible QH fluid on a surface Σ to be related to conformal blocks of an associated conformal field theory. Equation (4.33) reproduces the "shift formula" of ref. 37.

Whatever we have said about QH fluids composed of electrons applies also to QH fluids composed of holes after exchanging "e" and "h" ("left" and "right"). In our effective description, valid on large distance scales and at low frequencies, subsystems composed of electrons and subsystems composed of holes are independent of each other.

The main result established so far is the fact that the physics of an incompressible QH fluid in the scaling limit is coded into a pair of integral, odd Euclidean lattices Γ_e and Γ_h. The purpose of the next section is to summarize our results concerning a partial classification of such lattices and to apply these results to the analysis of incompressible QH fluids corresponding to experimentally observed plateaux.

5. GENERAL RESULTS ON THE CLASSIFICATION OF QH LATTICES

We start this section by recalling the notation introduced in Section 1 and summarizing the main results of Sections 3 and 4.

Let V be an N-dimensional, real vector space with inner product $(\cdot, \cdot)$. Let $\{\mathbf{x}_I\}_{I=1}^N$ be a basis of V and $\{\xi^I\}_{I=1}^N$ the basis dual to $\{\mathbf{x}_I\}_{I=1}^N$. A vector $\mathbf{v} \in V$ can be represented as a column vector $\vec{v} = (v^1, ..., v^N)^T$, with $v^I = (\mathbf{v}, \xi^I)$, or as a row vector $\underset{\rightarrow}{v} = (v_1, ..., v_N)$, with $v_I = (\mathbf{v}, \mathbf{x}_I)$, $I = 1, ..., N$. Then

$$\mathbf{v} = \sum_{I=1}^{N} v^I \mathbf{x}_I = \sum_{I=1}^{N} v_I \xi^I \tag{5.1}$$

and

$$(\mathbf{v}, \mathbf{v}') = \underset{\rightarrow}{v} \cdot \vec{v}' = \sum_{I=1}^{N} v_I v'^I = \sum_{I,J} v^I C_{IJ} v'^J = \sum_{I,J} v_I (C^{-1})^{IJ} v'_J \tag{5.2}$$

where

$$C_{IJ} = (\mathbf{x}_I, \mathbf{x}_J) \qquad \text{and} \qquad (C^{-1})^{IJ} = (\xi^I, \xi^J) \tag{5.3}$$

are the matrix elements of the Gram matrices corresponding to the bases $\{\mathbf{x}_I\}$ and $\{\xi^I\}$.

A basis $\{e_I\}_{I=1}^N$ of V is called *integral* if its Gram matrix, henceforth denoted K, with matrix elements

$$K_{IJ} = (\mathbf{e}_I, \mathbf{e}_J) \tag{5.4}$$

is integral. It determines an integral lattice $\Gamma \subset V$ given by

$$\Gamma = \left\{ \mathbf{q} = \sum_I q^I \mathbf{e}_I : q^I \in \mathbb{Z}, \text{ for all } I \right\} \tag{5.5}$$

Let $\{\boldsymbol{\varepsilon}^I\}_{I=1}^N$ be the basis of V dual to $\{\mathbf{e}_I\}_{I=1}^N$. Its Gram matrix is given by K^{-1}, with

$$(K^{-1})^{IJ} = (\boldsymbol{\varepsilon}^I, \boldsymbol{\varepsilon}^J)$$

It generates the dual lattice

$$\Gamma^* = \left\{ \mathbf{n} = \sum n_I \boldsymbol{\varepsilon}^I : n_I \in \mathbb{Z}, \text{ for all } I \right\}$$

A vector $\mathbf{v} \in V$ belongs to Γ iff the components $v^I = (\mathbf{v}, \boldsymbol{\varepsilon}^I)$ of $\vec{v}$ are integers; it belongs to Γ^* iff the components $v_I = (\mathbf{v}, \mathbf{e}_I)$ of $\underset{\rightarrow}{v}$ are integers.

The main results of Sections 3 and 4 can be summarized as follows [see Eqs. (3.62)–(3.64) and (4.14)–(4.20)]: Asympototically, on large distance scales and at low frequencies, an incompressible QH fluid is characterized by two integral, odd Euclidean lattices Γ_e and Γ_h with integral quadratic forms $(\cdot, \cdot)_e$ on Γ_e and $(\cdot, \cdot)_h$ on Γ_h and visible vectors $\mathbf{Q}_e \in \Gamma_e^*$ and $\mathbf{Q}_h \in \Gamma_h^*$ with the property that, for every vector $\mathbf{q} \in \Gamma_x$,

$$(\mathbf{Q}_x, \mathbf{q})_x \equiv (\mathbf{q}, \mathbf{q})_x \mod 2 \tag{5.6}$$

The Hall conductivity σ_H is given by

$$\sigma_H = \sigma_e - \sigma_h$$

with

$$\sigma_x = (\mathbf{Q}_x, \mathbf{Q}_x)_x \tag{5.7}$$

for $x = e, h$.

Vectors in Γ_x label multi-electron-hole configurations. Configurations of arbitrary quasiparticles are labeled by vectors in a lattice $(\Gamma_x)_{\text{phys}}$ with

$$\Gamma_x \subseteq (\Gamma_x)_{\text{phys}} \subseteq \Gamma_x^* \tag{5.8}$$

[see (3.50)]. With each vector $\mathbf{m} \in (\Gamma_x)_{\text{phys}}$ one can associate the electric charge of the corresponding state (normalized such that the charge of the ground state vanishes) which is given by

$$q_{\text{el}} = (\mathbf{Q}_x, \mathbf{m}) \tag{5.9}$$

and a statistical phase $\exp i\pi\theta_x(\mathbf{m}, \mathbf{m})$, with

$$\theta_x(\mathbf{m}, \mathbf{m}) \equiv (\mathbf{m}, \mathbf{m})_x \mod 2\mathbb{Z} \tag{5.10}$$

Our purpose is to summarize some of our main results concerning the classification of these data. Since our entire analysis is symmetric under interchange of $x = e$ with $x = h$, we shall drop the subscript x whenever possible.

Thus, let $(\Gamma, \mathbf{Q} \in \Gamma^*)$ be an N-dimensional "QH lattice," with $\Gamma \subseteq \Gamma^* \subset V \simeq \mathbb{R}^N$. Linear transformations of $\mathbb{R}^N$ mapping Γ onto itself form a group, denoted by $GL(N, \mathbb{Z})$, which consists of all integral $N \times N$ matrices $S = (S_{IJ})$ of determinant $\det S = \pm 1$. Hence two pairs $(K_1, \underset{\rightarrow}{Q}_1)$ and $(K_2, \underset{\rightarrow}{Q}_2)$ of positive-definite, integral $N \times N$ matrices and visible vectors in Γ^* describe the same QH fluid iff

$$K_1 = S^T K_2 S, \qquad \underset{\rightarrow}{Q}_1 = \underset{\rightarrow}{Q}_2 S, \qquad \text{for some} \quad S \in GL(N, \mathbb{Z}) \tag{5.11}$$

This will be abbreviated by writing $(K_1, \underset{\rightarrow}{Q}_1) \sim (K_2, \underset{\rightarrow}{Q}_2)$. The group $GL(N, \mathbb{Z})$ contains the subgroup $O(\Gamma)$ of all those transformations S that preserve the quadratic form on Γ, i.e.,

$$S^T K S = K \tag{5.12}$$

in a given basis.

Every integral, odd lattice Γ has a basis $\{\mathbf{q}_I\}_{I=1}^N$, called "*symmetric*," such that

$$(\mathbf{Q}, \mathbf{q}_I) = 1 \qquad \text{for all} \quad I = 1, ..., n \tag{5.13}$$

and hence $K_{II} = (\vec{q}_I, \vec{q}_I)$ is odd for all I. In the dual basis, $\mathbf{Q}$ has components $\underset{\rightarrow}{Q} = (1, ..., 1)$. Furthermore, there always exists a basis $\{\mathbf{q}, \mathbf{e}_1, ..., \mathbf{e}_{N-1}\}$, called "*normal*," such that

$$(\mathbf{Q}, \mathbf{q}) = 1, \qquad (\mathbf{Q}, \mathbf{e}_I) = 0, \qquad I = 1, ..., N-1 \tag{5.14}$$

and hence

$$K_{00} = (\mathbf{q}, \mathbf{q}) \quad \text{is odd}, \qquad K_{II} = (\mathbf{e}_I, \mathbf{e}_I) \quad \text{is even} \tag{5.15}$$

for $I = 1, ..., N-1$.

By a *QH lattice* we henceforth mean a pair of an *integral, odd, Euclidean lattice* Γ and a *visible vector* $\mathbf{Q} \in \Gamma^*$ satisfying the parity constraint (5.6). Given a basis in Γ, a QH fluid is characterized by a pair (K, Q) of a positive-definite, integral matrix K and a row vector Q, with g.c.d. $(Q_1,..., Q_N) = 1$, where Q_I are the components of $\mathbf{Q}$ in the dual basis. Our aim is now to find invariants for pairs (K, Q) that enable us to distinguish certain inequivalent QH lattices and are useful for a partial classification of QH lattices. Details of our results will appear in a separate article.[45]

Among the most elementary *invariants of* QH *lattices* are the following:

1. The *dimension* N of the lattice Γ.

2. The *oddness* of Γ (i.e., Γ is of type I, in the nomenclature of ref. 23; even lattices are said to be of type II and can apparently not describe QH fluids).

3. The *discriminant* Δ of the quadratic form $(\cdot, \cdot)$ on Γ. It can be defined as the determinant of the Gram matrix K associated to a given basis of Γ. By (5.11), det K is an invariant.

Note that the space Γ^*/Γ of cosets of Γ^* modulo Γ is an Abelian group. Its order is denoted by $|\Gamma^*/\Gamma|$. It is easy to derive that

$$\Delta = \det K = |\Gamma^*/\Gamma| \tag{5.16}$$

As pointed out in (4.30), Δ is the ground-state degeneracy of the QH fluid described by $(\Gamma, \mathbf{Q})$ on a torus.

Lattices with $\Delta = +1$ are called unimodular, or self-dual, and appear only in the description of QH fluids with *integer* Hall conductivity (IQHE), while QH fluids exhibiting a *fractional* quantum Hall effect (FQHE) are always described by *non-self-dual* lattices.

4. An *invariant* $L_{\max}$ is defined by setting

$$L_{\max} = \min_{\{\mathbf{q}_I\}_{I=1}^N} \left(\max_{J=1,...,N} (\mathbf{q}_J, \mathbf{q}_J) \right) \tag{5.17}$$

where the minimum is taken over all *symmetric* bases of Γ. Let $\mathbf{q}^\#$ be a basis vector for which $(\mathbf{q}^\#, \mathbf{q}^\#) = L_{\max}$. Since $\mathbf{q}^\#$ is an element of a symmetric basis, it follows that $q_{el} = (\mathbf{Q}, \mathbf{q}^\#) = 1$ corresponds to a state of the QH fluid, where one electron with quantum numbers $-\mathbf{q}^\#$ has been created from the ground state. By (4.28), $L_{\max}$ is the minimum of the modulus of the angular momentum of a state describing two electrons with quantum numbers $-\mathbf{q}^\#$ created from the ground state.

Since the matrix $K=(K_{IJ})$ defined by $K_{IJ}=(\mathbf{q}_I, \mathbf{q}_J)$ for a basis $\{\mathbf{q}_I\}$ minimizing $\max_{J=1,\ldots,N}(\mathbf{q}_J, \mathbf{q}_J)$, is positive-definite, Hadamard's inequality implies that

$$\Delta = \det K \leqslant L_{\max}^N \tag{5.18}$$

In a real, incompressible QH fluid, $L_{\max}$ satisfies a universal upper bound

$$L_{\max} \leqslant L_* < \infty \tag{5.19}$$

with $L_* \approx 9$. To understand this, we recall that the suppression of relative angular momenta l between pairs of electrons with $|l| < L_{\max}$ is due to the Coulomb repulsion between the electrons, which has a finite strength. Furthermore, if $L_{\max}$ were very large, the electron density of the system would be so small that the formation of a Wigner lattice would lower the energy of the system. However, the formation of a Wigner lattice destroys the incompressibility of the system.

It is easy to see[45] that the bound (5.19) on $L_{\max}$ and a bound on the dimension N of the lattice Γ yield upper bounds on the numerator and denominator of the Hall conductivities σ_H of incompressible QH fluids. Thus, if $L_{\max}$ and N are bounded above, the possible values of the Hall conductivity of incompressible QH fluids form a *finite* set of rational numbers.

We should ask whether one can find a universal upper bound on the dimension N of QH lattices. Unfortunately, we do not know any method of determining an *explicit* bound on N. However, heuristically, it is clear that N cannot be arbitrarily large in a real QH fluid. There are two reasons for that: A real QH fluid has a finite density of impurities. These impurities tend to cause mixing between different chiral edge currents, so that the number of *independently conserved* edge currents—which is the dimension of the QH lattice—is limited by the strength and density of impurities. Furthermore, the specific heat of the edge degrees of freedom of an incompressible QH fluid is proportional to the dimension N of the lattice Γ (which is equal to the central charge of the conformal field theory describing the edge currents). Thus, finiteness of the specific heat of a QH fluid implies an upper bound on the dimension N. (These issues deserve, however, a more careful analysis.)

We may now state our first general result concerning the classification of incompressible QH fluids.

Theorem 1. Consider an incompressible QH fluid described by two integral, odd Euclidean lattices Γ_e and Γ_h of dimensions $N^{(e)}$ and $N^{(h)}$, respectively. Assume that

$$N^{(e)}, N^{(h)} \leqslant N_* < \infty \tag{5.20}$$

and that the value of the invariant $L_{\max}$ satisfies the bound (5.19) for both lattices Γ_e and Γ_h.

Then the number of inequivalent pairs of lattices Γ_e and Γ_h satisfying (5.19) and (5.20) is *finite*. (It is bounded by a number depending on L_* and N_*.) Moreover, the set of values of the Hall conductivity

$$\sigma_H = \sigma_e - \sigma_h$$

is a *finite* set of rational numbers.

Remarks. Details of the proof of this theorem will be presented in ref. 45; see also refs. 23 and 24. Unfortunately, as N_* and L_* grow somewhat large, the number of inequivalent pairs of lattices becomes unmanageably large. As long as $N \leqslant 8$ and $\Delta \leqslant 13$, a complete list of QH lattices is known for $0 < \sigma_H < 2$. Fairly exhaustive tables will be given in Section 7 (see also ref. 42).

Our bounds on the number of possible values of σ_H grows exponentially in N_*.

From now on, we focus our attention on the classification of pairs $(\Gamma_e, \mathbf{Q} \in \Gamma_e^*)$ of incompressible QH fluids composed of electrons, and we drop the subscript e.

5. A lattice Γ is called *decomposable* iff

$$\Gamma = \Gamma_1 \oplus \Gamma_2 \tag{5.21}$$

for two sublattices Γ_1 and Γ_2 with the property that

$$(\mathbf{q}_1, \mathbf{q}_2) = 0 \qquad \text{for all} \quad \mathbf{q}_1 \in \Gamma_1 \quad \text{and all} \quad \mathbf{q}_2 \in \Gamma_2 \tag{5.22}$$

Otherwise, it is called *indecomposable*.

If Γ is decomposable, then Γ^* is decomposable. A QH fluid is called *composite* iff the associated lattice Γ is decomposable. Otherwise, it is called *elementary*. Let $\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_k$ be the decomposition of a lattice Γ into indecomposable sublattices, and let $\Gamma^* = \Gamma_1^* \oplus \cdots \oplus \Gamma_k^*$ and $\mathbf{Q} = \mathbf{Q}_k$, with $\mathbf{Q}_i \in \Gamma_i^*$, be the corresponding decompositions of Γ^* and $\mathbf{Q}$. If σ_H denotes the Hall conductivity of a composite QH fluid with lattice Γ, then

$$\sigma_H = (\mathbf{Q}, \mathbf{Q}) = \sum_{i=1}^{k} (\mathbf{Q}_i, \mathbf{Q}_i) = \sum_{i=1}^{k} \sigma_H^i \tag{5.23}$$

where σ_H^i is the Hall conductivity of the elementary QH fluid with lattice Γ_i.

A pair $(\Gamma, \mathbf{Q})$ of a decomposable lattice Γ and a vector $\mathbf{Q} \in \Gamma^*$ is called *improper* iff, in the decomposition of $\mathbf{Q} = \mathbf{Q}_1 + \cdots + \mathbf{Q}_k$ associated with the decomposition of $\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k$,

$$\mathbf{Q}_i = 0 \qquad \text{for at least one } i \tag{5.24}$$

Obviously, an elementary QH fluid with lattice Γ_i and vector $\mathbf{Q}_i = 0$ has a vanishing Hall conductivity. Moreover, it does not mix with any other components of a given QH fluid. We may therefore discard improper QH lattices $(\Gamma, \mathbf{Q})$ throughout our analysis and focus on the classification of *indecomposable* QH lattices.

Next, we discuss some further invariants of QH lattices $(\Gamma, \mathbf{Q})$.

6. In the basis of Γ^* dual to a given basis of Γ, the vector $\mathbf{Q}$ has integer components $\underset{\to}{Q} = (Q_1, ..., Q_N)$. The only $GL(N, \mathbb{Z})$-invariant associated with an integral vector $\underset{\to}{Q}$ is the greatest common division (g.c.d.) of its components,

$$q = \text{g.c.d.}(Q_1, ..., Q_N) \tag{5.25}$$

Geometrically, $q - 1$ is the number of points in Γ^* on the open, straight-line segment joining the origin of Γ^* to $\mathbf{Q}$. Physically, $\pm q$ is the *electric charge* of the particles of which the QH fluid is composed (in units where $e = 1$). For a QH fluid composed of electrons or holes, we have that $q = 1$, which is equivalent to requiring that $\mathbf{Q}$ be a *visible* vector in Γ^*. Since q is the only $GL(N, \mathbb{Z})$-invariant associated with $\underset{\to}{Q}$, visibility of $\mathbf{Q}$ implies that one can always choose *symmetric* bases of Γ for which $\underset{\to}{Q} = (1, ..., 1)$ and *normal* bases of Γ for which $\underset{\to}{Q} = (1, 0, ..., 0)$; see Eqs. (5.13)–(5.15). Given a *fixed* integral matrix K, the ambiguity in choosing a basis in Γ with Gram matrix K is described by the group $O(\Gamma)$; see (5.12). An $O(\Gamma)$-invariant associated with $\mathbf{Q}$ is its *orbit* $[\mathbf{Q}]$ under $O(\Gamma)$. It is an "experimental fact" about lattices of not too large dimension and not too large discriminant that the orbit $[\mathbf{Q}]$ of the *shortest* odd visible vector $\mathbf{Q}$ is *unique*, and in most cases the orbit $[\mathbf{Q}]$ contains only $\pm\mathbf{Q}$, i.e., $\mathbf{Q}$ is a "face vector" in the terminology of ref. 49.

7. Let $K = (K_{IJ})$ be the Gram matrix of a basis $\{\mathbf{e}_I\}_{I=1}^N$ of Γ, and K^{-1} the Gram matrix of the basis $\{\boldsymbol{\varepsilon}^I\}_{I=1}^N$ of Γ^* dual to the basis $\{\mathbf{e}_I\}_{I=1}^N$. By Kramer's rule,

$$K^{-1} = \Delta^{-1} \tilde{K} \tag{5.26}$$

where $\Delta = \det K$ is the discriminant of the quadratic form $(\cdot, \cdot)$ on Γ and $\tilde{K}$ is the matrix of cofactors obtained from K; clearly $\tilde{K}$ is a positive-definite integral matrix and Δ is an integer, so that the matrix elements of K^{-1} are rational numbers. By Eqs. (3.15) or (5.7),

$$\sigma_H = (\mathbf{Q}, \mathbf{Q}) = \sum_{I,J} Q_I (K^{-1})^{IJ} Q_J = \Delta^{-1} \sum_{I,J} Q_I \tilde{K}^{IJ} Q_J \tag{5.27}$$

Clearly, the *length squared* $(\mathbf{Q}, \mathbf{Q})$ of the vector $\mathbf{Q}$ is an *invariant* of a QH lattice $(\Gamma, \mathbf{Q})$ [generally coarser than the orbit $[\mathbf{Q}]$ of $\mathbf{Q}$ under $O(\Gamma)$ discussed in paragraph **6**]. Since Δ is an invariant of Γ,

$$\gamma := \Delta(\mathbf{Q}, \mathbf{Q}) = \sum_{I,J} Q_I \tilde{K}^{IJ} Q_J \tag{5.28}$$

is a numerical invariant of $(\Gamma, \mathbf{Q})$. It is a positive integer. Although γ is, *a priori*, an invariant of the *pair* $(\Gamma, \mathbf{Q})$, it is actually often related to a numerical invariant of the lattice Γ alone.

Theorem 2. Let Γ be an integral, odd lattice with an *odd* discriminant Δ, and let $\mathbf{Q}$ be an arbitrary *odd* vector of Γ^* [i.e., $(\mathbf{Q}, \mathbf{q}) \equiv (\mathbf{q}, \mathbf{q})$ mod 2, for arbitrary $\mathbf{q} \in \Gamma$]. Let

$$\gamma = \Delta(\mathbf{Q}, \mathbf{Q})$$

Then γ modulo 8 is an invariant of Γ.

The proof is an easy exercise; but see ref. 45.

In general $\gamma = \Delta(\mathbf{Q}, \mathbf{Q})$ need not be coprime to Δ. We define

$$l = \text{g.c.d.}(\gamma, \Delta) \tag{5.29}$$

The integer l is called the *level* of a QH lattice $(\Gamma, \mathbf{Q})$. Writing σ_H as a fraction of two coprime integers n_H and d_H, we have that

$$\gamma = l n_H, \qquad \Delta = l d_H \tag{5.30}$$

An indecomposable QH lattice $(\Gamma, \mathbf{Q})$ of level $l = 1$ is called a *minimal* QH lattice.

8. Next, we attempt to characterize the lattice $\Gamma_{\text{phys}} \subseteq \Gamma^*$ of vectors in Γ^* which are quantum numbers of configurations of quasiparticles created from the ground state of an incompressible QH fluid described by the QH lattice $(\Gamma, \mathbf{Q})$; see Eq. (3.61) or the remark after (4.20). Obviously

$$\Gamma \subseteq \Gamma_{\text{phys}} \subseteq \Gamma^*_{\text{phys}} \subseteq \Gamma^* \tag{5.31}$$

or

$$\Gamma \subseteq \Gamma^*_{\text{phys}} \subseteq \Gamma_{\text{phys}} \subseteq \Gamma^* \tag{5.32}$$

The discussion of the inclusions (5.31) and (5.32) determines four Abelian groups, $\Gamma^*_{\text{phys}}/\Gamma$, which is isomorphic to $\Gamma^*/\Gamma_{\text{phys}}$, $\Gamma_{\text{phys}}/\Gamma^*_{\text{phys}}$, and Γ^*/Γ. The orders of these groups are denoted by p, p, r, and Δ, respectively. Then (5.32) implies that

$$\Delta = p^2 \cdot r \tag{5.33}$$

This simple equation limits the possible choices of Γ_{phys}

(a) If Δ is *square-free*, then

$$\text{either} \quad \Gamma_{\text{phys}} = \Gamma \quad \text{or} \quad \Gamma_{\text{phys}} = \Gamma^* \tag{5.34}$$

In a system of noninteracting electrons, one obviously has that $\Gamma_{\text{phys}} = \Gamma$. However, in this case, $\Gamma = I_N \equiv \mathbb{Z}^N$ is *self-dual*. But if electrons interact with each other and σ_H is fractional (FQHE), then $\Gamma \neq \Gamma^*$, and one expects that $\Gamma_{\text{phys}} = \Gamma^*$, as suggested by the analysis of the simplest fractional QH fluids, such as the Laughlin fluids.[16,17,32]

(b) If $\Delta = p^2$, then

$$\text{either} \quad \Gamma_{\text{phys}} = \Gamma, \quad \text{or} \quad \Gamma \subset \Gamma_{\text{phys}} = \Gamma^*_{\text{phys}} \subset \Gamma^*, \quad \text{or} \quad \Gamma_{\text{phys}} = \Gamma^* \tag{5.35}$$

The alternative, $\Gamma \subset \Gamma_{\text{phys}} = \Gamma^*_{\text{phys}} \subset \Gamma^*$, can sometimes be excluded by showing that there are no self-dual lattices between Γ and Γ^*. In this case, $\Gamma_{\text{phys}} = \Gamma^*$ is the alternative which is most likely realized in an actual QH fluid of interacting electrons.

(c) If $d_H = \Delta/l = p^2 r$ for some $p = 2, 3,\ldots$, then if Γ_{phys} properly contains Γ, one can prove that there are quasiparticles of *fractional* electric charge satisfying *Bose* or *Fermi statistics*.[45] If $\Gamma_{\text{phys}}/\Gamma$ has order p, then these quasiparticles are in fact local relative to all other quasiparticles of the system. A QH fluid with such a spectrum of quasiparticles would be somewhat exotic, and we expect that, again, $\Gamma_{\text{phys}} = \Gamma^*$.

It should be emphasized, however, that the issue of whether Γ_{phys} properly contains Γ or whether $\Gamma_{\text{phys}} = \Gamma^*$ lies beyond the scope of the present analysis and can only be decided on the basis of a detailed understanding of the quantum mechanics of incompressible QH fluids. Moreover, it is worth remembering that the Basic Hypothesis A2 of Section 3 or A2′ of Section 4 puts nontrivial constraints on the charge–statistics relation for arbitrary quasiparticle excitation in Γ_{phys}. As a consequence, it is not always possible to set $\Gamma_{\text{phys}} = \Gamma^*$ in the general case of nonminimal QH lattices. We will not pursue here those issues, referring the interested reader to ref. 45 for further details.

9. Let us now assume that we consider a QH fluid described by a QH lattice $(\Gamma, \mathbf{Q})$ and with $\Gamma_{\text{phys}} = \Gamma^*$. A quantity of considerable theoretical and experimental interest is the *smallest, nonzero, fractional electric charge* of a quasiparticle appearing in this system. We define

$$q^* = \min_{\mathbf{n} \in \Gamma^*, (\mathbf{Q}, \mathbf{n}) \neq 0} |(\mathbf{Q}, \mathbf{n})| \tag{5.36}$$

Let $\underset{\rightarrow}{n} = \vec{q}^T K$ be the flux vector corresponding to a charge vector $\vec{q}$ of a quasiparticle $\mathbf{n} \in \Gamma^*$ ($\underset{\rightarrow}{n}$ gives the components of $\mathbf{n}$ in the basis of Γ^*). Since

$$q_{\text{el}} = (Q, n) = \underset{\rightarrow}{Q} \cdot \vec{q} = \underset{\rightarrow}{Q} K^{-1} \underset{\rightarrow}{n}^T = \Delta^{-1} \underset{\rightarrow}{Q} \tilde{K} \underset{\rightarrow}{n}^T \tag{5.37}$$

and $\underset{\rightarrow}{Q} \tilde{K} \underset{\rightarrow}{n}^T$ is an integer for arbitrary $\underset{\rightarrow}{n}$, and q^* is an integer multiple of Δ^{-1}. In general q^* is not equal to Δ^{-1}, and we may define an *invariant* g of $(\Gamma, \mathbf{Q})$ by setting

$$q^* = g\Delta^{-1} \tag{5.38}$$

By (5.37),

$$g = \text{g.c.d.}((\underset{\rightarrow}{Q}\tilde{K})_1, ..., (\underset{\rightarrow}{Q}\tilde{K})_N) \tag{5.39}$$

where $(\underset{\rightarrow}{Q}\tilde{K})_i$ is the ith component of $\underset{\rightarrow}{Q}\tilde{K}$. Since $\underset{\rightarrow}{Q}\tilde{K}K = \Delta \underset{\rightarrow}{Q}$, where $\underset{\rightarrow}{Q}$ is visible, and since K is an integral matrix, it follows that g divides Δ. The invariant γ is given by $\gamma = \underset{\rightarrow}{Q}\tilde{K}\underset{\rightarrow}{Q}^T$; see (5.28). By (5.39), g thus divides γ. Hence g divides the g.c.d. of γ and Δ, which is the level l of the QH lattice. This allows us to define an integer λ, the "*charge parameter*" of the QH lattice, by setting

$$\lambda = \frac{l}{g} = \frac{\text{g.c.d.}(\gamma, \Delta)}{\text{g.c.d.}(\underset{\rightarrow}{Q}\tilde{K})} \tag{5.40}$$

The invariant λ determines the value of the smallest, nonzero fractional electric charge q^* in terms of the denominator d_H of the Hall conductivity σ_H:

$$q^* = \frac{1}{\lambda d_H} \tag{5.41}$$

since $q^* = g\Delta^{-1} = gl^{-1}d_H^{-1} = \lambda^{-1}d_H^{-1}$, by (5.38) and (5.40).

The numerical invariants of QH lattices found so far can be arranged in the form of a symbol

$$_N\left(\frac{n_H}{d_H}\right)^g_\lambda \tag{5.42}$$

with $l = g \cdot \lambda$ (the level), $\Delta = l d_H = |\Gamma^*/\Gamma|$, $\gamma = l n_H$, and $\sigma_H = n_H/d_H$. For minimal QH lattices, $l = 1$ and hence $\lambda = g = 1$ and $\Delta = d_H$.

10. Next, we define a more subtle invariant of QH lattices $(\Gamma, \mathbf{Q})$ related to their *spectrum of Laughlin vortices* and their *statistical phases*: the *genus* of a lattice Γ (see e.g., refs. 23 and 24. The Abelian group Γ^*/Γ is determined by N positive integers $d_1, \ldots, d_N$, some of which may be equal to 1. They have the propeties that

$$\begin{aligned} &d_i \text{ divides } d_{i+1} \text{ and} \\ &\Delta = \det K = d_1 d_2 \cdots d_N \end{aligned} \tag{5.43}$$

Geometrically, there is a basis $\{\mathbf{r}_l\}_{l=1}^N$ of Γ such that $\{d_l^{-1}\mathbf{r}_l\}_{l=1}^N$ is a basis of Γ^*. Thus the group Γ^*/Γ has the following factorization into cyclic subgroups $\mathbb{Z}_{d_l}$:

$$\Gamma^*/\Gamma \simeq \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_N} \tag{5.44}$$

The physical interpretation of (5.44) is that if $\Gamma_{\text{phys}} = \Gamma^*$, then the number of factors in (5.44) for which $d_l > 1$ is the *number of different elementary quasiparticles*, or *Laughlin vortices*, of the QH fluid.

QH lattices always involve an *odd* lattice Γ. Thus all vectors $\mathbf{n} + \Gamma$ of Γ^*/Γ have the same length $(\mathbf{n}, \mathbf{n})$ mod $\mathbb{Z}$:

$$\begin{aligned} (\mathbf{n}+\mathbf{q}, \mathbf{n}+\mathbf{q}) &= (\mathbf{n}, \mathbf{n}) + 2(\mathbf{n}, \mathbf{q}) + (\mathbf{q}, \mathbf{q}) \\ &\equiv (\mathbf{n}, \mathbf{n}) \mod \mathbb{Z} \end{aligned}$$

for all $\mathbf{q} \in \Gamma$. Hence $(\cdot, \cdot)$ defines a quadratic form ϑ on Γ^*/Γ with values in $\mathbb{Q}$ mod $\mathbb{Z}$. By (5.6), the *squares* of the statistical phases $\exp i\pi\theta(\mathbf{n}, \mathbf{n})$, i.e., the *monodromies*, associated with vectors $\mathbf{n} \in \Gamma^*$ uniquely fix the quadratic form ϑ on Γ^*/Γ and conversely.

We now define the genus of a lattice. Two lattices Γ_1 and Γ_2 have the same genus iff they have the same dimension, the same parity (or type), and there is an isomorphism between Γ_1^*/Γ_1 and Γ_2^*/Γ_2 which preserves the quadratic form ϑ (i.e., the monodromy phases of the vectors in Γ^*).

Transcribed in physical jargon, two odd lattices Γ_1 and Γ_2 with equal dimension have the same genus iff they have isomorphic families of Laughlin vortices with identical monodromy phases $\exp i2\pi\theta$.

It should be emphasized that, in general, there can be several inequivalent lattices in the same genus. In fact, the number of equivalence classes in a given genus tends to ∞ as $N \to \infty$.[24] For fairly small values of N (e.g., $N \leqslant 7$) and of Δ (e.g., $\Delta \leqslant 25$), the situation is, however, much less discouraging than suggested by this general result, so that the genus is a very useful invariant for the classification of QH lattices which has a fairly direct physical interpretation.

11. In the following, we summarize some interesting *congruences* between the various invariants of QH lattices discussed so far. Proofs of our results will appear in ref. 45. A first example of such a congruence is the one stated in Theorem 2, i.e., if $(\Gamma, \mathbf{Q}_1)$ and $(\Gamma, \mathbf{Q}_2)$ are two QH lattices with the same Γ, then

$$\gamma_1 = \Delta(\mathbf{Q}_1, \mathbf{Q}_1) \equiv \Delta(\mathbf{Q}_2, \mathbf{Q}_2) = \gamma_2 \mod 8 \tag{5.45}$$

A second example is the following result:

Theorem 3. Let $(\Gamma, \mathbf{Q})$ be a QH lattice, and assume that Δ and $\gamma = \Delta(\mathbf{Q}, \mathbf{Q})$ are *odd* integers.

Then the dimension N of Γ is odd, and

$$\gamma \equiv N \mod 4 \tag{5.46}$$

Equation (5.46) generalizes the equation $\sigma_H = \gamma = N$ valid in QH fluids of noninteracting electrons. For minimal QH lattices (i.e., $l = 1$), Theorem 3 can be sharpened.

Theorem 4. Let $(\Gamma, \mathbf{Q})$ be a minimal QH lattice (so that $\gamma = n_H$, $\Delta = d_H$). Then:

(a) d_H is odd, and $\Gamma^*/\Gamma \simeq \mathbb{Z}_{d_H}$.

(b) If n_H is even, then the dimension N is even.

(c) If n_H is odd, then N is odd, and $n_H \equiv N \mod 4$.

Theorem 5. If $(\Gamma, \mathbf{Q})$ is a QH lattice with an *even* charge parameter λ, then the invariant $g = l/\lambda$ is even, too, if either d_H is even and n_H is odd, or d_H is odd and n_H is even. If $\lambda = g = 2$, then

$$\Gamma^*/\Gamma \simeq \mathbb{Z}_{4d_H} \tag{5.47}$$

Proofs of these results will appear in ref. 45.

As indicated in part (a) of Theorem 4, there are apparently no minimal QH lattices corresponding to a Hall conductivity $\sigma_H = n_H/d_H$ with an *even* denominator. Since this is a fundamental result for the analysis of plateaux at $\sigma_H = 1/2$, observed in double-layer systems,[5] we state it in a separate theorem.

Theorem 6. The charge parameter λ of a QH lattice $(\Gamma, \mathbf{Q})$ of arbitrary dimension N corresponding to a Hall conductivity $\sigma_H = (\mathbf{Q}, \mathbf{Q}) = n_H/d_H$ with an *even* denominator d_H is *even.*

By Theorem 5, we have that the invariant g defined in (5.39) is then *even* as well, and hence, by (5.40), the level l of such a QH lattice is a multiple of 4. Thus there are no minimal $(l = 1)$ QH lattices with even denominator d_H. However, in view of Theorems 5 and 6, there are still some distinguished QH lattices $(\Gamma, \mathbf{Q})$ corresponding to a Hall conductivity σ_H with an *even* denominator, namely those whose level $l = \lambda g$ is 4, i.e.,

$$\lambda = g = 2 \tag{5.48}$$

In this case, Eq. (5.47) implies that $\Gamma^*/\Gamma \simeq \mathbb{Z}_{4d_H}$.

These results have the following interesting consequences: If $(\Gamma, \mathbf{Q})$ is a QH lattice corresponding to a Hall conductivity $\sigma_H = n_H/d_H$ with even denominator d_H, and if $\Gamma_{\text{phys}} = \Gamma^*$, then there are quasiparticles of electric charge $q^* = 1/\lambda d_H$ [see (5.4)], where λ is even. In particular, for $\sigma_H = 1/2$ or $\sigma_H = 5/2$, one predicts the existence of quasiparticles of charge $\pm e/4$, where e is the elementary electric charge. This theoretical prediction could be tested experimentally.

We recall that one of the basic physical hypotheses on which our analysis of incompressible QH fluids is based is that a configuration of quasiparticles described by $\mathbf{q} \in \Gamma$ with odd electric charge is a fermion, while if the electric charge is even it is a boson. This expresses a relation between electric charge and statistics. It is natural to ask whether there is such a relation between charge and statistics for configurations corresponding to arbitrary vectors $\mathbf{n} \in \Gamma^*$. In the following, we answer this question in the affirmative for minimal QH lattices with an odd denominator d_H.

12. *A charge–statistic theorem.* The purpose of this paragraph is to show that, for any *minimal* QH lattice $(\Gamma, \mathbf{Q})$ corresponding to a Hall conductivity $\sigma_H = n_H/d_H$ with an *odd* denominator d_H, the statistical phase $\theta(\mathbf{n}, \mathbf{n})$—see Eq. (5.6)—of an arbitrary vector $\mathbf{n} \in \Gamma^*$ is fixed by its electric charge $q_{el} = (\mathbf{Q}, \mathbf{n})$. This is a theorem on the *connection between charge and statistics.* We shall see that this connection is fixed by σ_H alone. This is due to the fact that the *genus* (see paragraph **10**) of a minimal QH lattice with odd d_H is fixed by σ_H.

Special cases of our general theorem have been noticed before. It is well known that there is a charge–statistics connection for the Laughlin fluids corresponding to $\sigma_H = 1/d_H$, with d_H odd. For certain hierarchy QH fluids, a charge–statistics connection has been found by Block and Wen.[46]

Our first observation is that, for a minimal QH lattice $(\Gamma, \mathbf{Q})$ with an odd d_H,

$$\Gamma^*/\Gamma \simeq \mathbb{Z}_{d_H} \tag{5.49}$$

[i.e., $d_1 = \cdots = d_{N-1} = 1$, $d_N = d_H$, in Eqs. (5.43), (5.44)]. A generator of Γ^*/Γ is the vector $\mathbf{Q}$. For, the multiples $r\mathbf{Q}$, $r = 0, 1, 2, \ldots$, of $\mathbf{Q}$ form a subgroup of Γ^*/Γ, whose order we denote by r_*. Hence $r_*\mathbf{Q} \in \Gamma$, and therefore

$$(r_*\mathbf{Q}, \mathbf{n}) = r_* \underset{\to}{Q} K^{-1} \underset{\to}{n}^T = \frac{r_*}{d_H} \underset{\to}{Q} \tilde{K} \underset{\to}{n}^T \in \mathbb{Z}$$

for arbitrary $\mathbf{n} \in \Gamma^*$. Since the invariant $g = 1$, for a minimal QH lattice, r_*/d_H must be an integer. Hence $r_* = d_H$, and

$$\Gamma^*/\Gamma = \{r\underset{\to}{Q}\}_{r=0}^{d_H-1} \simeq \mathbb{Z}_{d_H}$$

(this proves part of Theorem 4a). In this situation, $n_H = d_H(\mathbf{Q}, \mathbf{Q})$ fixes the genus of the lattice Γ, i.e., it fixes the quadratic form $\vartheta(\mathbf{n}) \equiv (\mathbf{n}, \mathbf{n}) \bmod \mathbb{Z}$, $\mathbf{n} \in \Gamma^*$, introduced in paragraph **10**. For, thanks to (5.49), every $\underset{\to}{n} \in \Gamma^*$ can be written as $\mathbf{n} = r\mathbf{Q} + \mathbf{q}$, $\mathbf{q} \in \Gamma$. Then

$$\vartheta(r\mathbf{Q} + \Gamma) = r^2(\mathbf{Q}, \mathbf{Q}) = r^2 \frac{n_H}{d_H} \bmod \mathbb{Z} \tag{5.50}$$

This shows that ϑ is fixed by the *quadratic class of* n_H *modulo* d_H, which thus fixes the genus of Γ. In particular, the monodromy phases $\exp 2i\pi\theta(\mathbf{n}, \mathbf{n})$ of all vectors $\mathbf{n} \in \Gamma^*$ are fixed by r, which is fixed by the electric charge of $\mathbf{n} \bmod \mathbb{Z}$, and by n_H and d_H. We would like to show that, not only the monodromy phases, but the statistical phases, or half-monodromies $\exp i\pi\theta$ are fixed by the electric charges. The key idea here is to use the parity constraint

$$(\mathbf{Q}, \mathbf{q}) \equiv (\mathbf{q}, \mathbf{q}) \mod 2 \qquad \text{for all} \quad \mathbf{q} \in \Gamma \tag{5.51}$$

[see (5.6)]. Thus for $\mathbf{n} = r\mathbf{Q} + \mathbf{q}$, $\mathbf{q} \in \Gamma$, the statistical phase is

$$\begin{aligned} \frac{\mu(\mathbf{n})}{d_H} &:= \theta(\mathbf{n}, \mathbf{n}) = (\mathbf{n}, \mathbf{n}) \equiv r^2(\mathbf{Q}, \mathbf{Q}) + (\mathbf{q}, \mathbf{q}) \bmod 2 \\ &\equiv r^2(\mathbf{Q}, \mathbf{Q}) + (\mathbf{Q}, \mathbf{q}) \mod 2 \end{aligned} \tag{5.52}$$

while the electric charge is

$$\frac{\varepsilon(\mathbf{n})}{d_H} = (\mathbf{Q}, \mathbf{n}) = r(\mathbf{Q}, \mathbf{Q}) + (\mathbf{Q}, \mathbf{q}) \tag{5.53}$$

and hence

$$\begin{aligned}\mu(\mathbf{n}) &\equiv r^2 n_H + d_H(\mathbf{Q}, \mathbf{q}) \bmod 2d_H \\ \varepsilon(\mathbf{n}) &= r n_H + d_H(\mathbf{Q}, \mathbf{q})\end{aligned} \tag{5.54}$$

Now, if n_H is odd,

$$n_H \mu(\mathbf{n}) = \varepsilon(\mathbf{n})^2 \quad \bmod 2d_H \tag{5.55}$$

and since g.c.d. $(n_H, 2d_H) = 1$, n_H is invertible modulo $2d_H$. Its inverse mod $2d_H$ is denoted by $(n_H)^{-1}$. Then Eq. (5.55) implies the following result:

Theorem 7 (Charge–statistics connection):

$$q_{\text{el.}}(\mathbf{n}) = (\mathbf{Q}, \mathbf{n}) = \frac{\varepsilon(\mathbf{n})}{d_H} \Rightarrow \theta(\mathbf{n}, \mathbf{n}) \equiv (n_H)^{-1} \frac{\varepsilon(\mathbf{n})^2}{d_H} \quad \bmod 2 \tag{5.56}$$

This is the desired connection between charge and statistics, provided n_H is *odd.* If n_H is *even*, then it is no longer invertible mod $2d_H$. Defining

$$(n_H)^{-1} := 2(2n_H)^{-1} + d_H \tag{5.57}$$

where $(2n_H)^{-1}$ is the inverse of $2n_H$ modulo d_H, we find that the charge–statistics connection (5.56) still holds.[45]

If $(\Gamma, \mathbf{Q})$ is a minimal QH lattice with an *even* denominator d_H, and hence $\lambda = g = 2$, it is tempting to think that charge and spin of a vector $\mathbf{n} \in \Gamma^*$ determine its statistical phase. (We realize that we have not defined the "spin" of vectors in Γ^*; but see ref. 45). For nonminimal QH lattices, the electric charge in general does not determine the statistical phase (except for vectors in Γ). For more details, see ref. 45.

This completes our survey of general results on the classification of QH lattices. It should be remarked that a complete classification of one- and two-dimensional QH lattices, based on results of Gauss, is known, and that the classification of three-dimensional QH lattices with small $L_{\max}$ is possible.[42]

6. THE ADE-$\mathcal{O}$ CLASSIFICATION OF QH LATTICES

In this section we complement the general results discussed in Section 5 by presenting a more constructive analyses of QH lattices.

An N-dimensional, integral Euclidean lattice Γ contains a distinguished $(N-k)$-dimensional sublattice Γ_W, the so called *Witt sublattice* $(k = 0, 1, \ldots, N)$,

$$\Gamma_W \oplus \mathcal{O}_k \subseteq \Gamma \tag{6.1}$$

Γ_W is defined to be the sublattice of Γ generated by all vectors of length squared 1 and 2. One shows that

$$\Gamma_W = \Gamma_{\text{root}} \oplus I_l \tag{6.2}$$

where I_l is an l-dimensional, simple hypercubic generated by l orthonormal basis vectors of length 1, and Γ_{root} is a direct sum of root lattices of the Lie algebras $A_{m-1} = su(m)$, $D_{m+2} = so(2m+4)$, $m = 2, 3,\ldots, E_6, E_7$, and E_8.

Since Γ is integral, it also contains a maximal sublattice $\mathcal{O}_k$ of dimension

$$k = N - \dim \Gamma_W \tag{6.3}$$

generated by vectors of length squared 3, 4,... and orthogonal to Γ_W. Thus we have the inclusions

$$\Gamma_W \oplus \mathcal{O}_k \subseteq \Gamma \subseteq \Gamma^* \subseteq \Gamma_W^* \oplus \mathcal{O}_k^* \tag{6.4}$$

The sublattice $\Gamma_W \oplus \mathcal{O}_k$ is called the *Kneser shape* of Γ.[23] The Witt sublattice Γ_W can be further decomposed:

$$\Gamma_W = \Gamma_{sd} \oplus \mathring{\Gamma}_W \tag{6.5}$$

where Γ_{sd} is the direct sum of I_l and of all the E_8 root lattices contained in Γ_W. Clearly $\Gamma_{sd} = \Gamma_{sd}^*$ is *self-dual.*

Thus (6.4) can be sharpened by writing

$$\Gamma_{sd} \oplus \mathring{\Gamma}_W \oplus \mathcal{O}_k \subseteq \Gamma \subseteq \Gamma^* \subseteq \Gamma_{sd} \oplus \mathring{\Gamma}_W^* \oplus \mathcal{O}_k^* \tag{6.6}$$

and hence

$$\Gamma = \Gamma_{sd} \oplus \mathring{\Gamma} \subseteq \Gamma_{sd} \oplus \mathring{\Gamma}^* = \Gamma^* \tag{6.7}$$

If $(\Gamma, \mathbf{Q})$ is a QH lattice then, by (6.7), the corresponding QH fluid is *composite*: In accordance with (6.7),

$$\mathbf{Q} = \mathbf{Q}_{sd} + \mathbf{Q}'$$

and

$$\sigma_H = (\mathbf{Q}, \mathbf{Q}) = (\mathbf{Q}_{sd}, \mathbf{Q}_{sd}) + (\mathbf{Q}', \mathbf{Q}') = \sigma_{Hsd} + \sigma'_H \tag{6.8}$$

By construction,

$$\Gamma_{sd} = I_l \oplus \Gamma_{E_8} \tag{6.9}$$

where Γ_{E_8} is a direct sum of E_8 root lattices. Accordingly, $\bar{\mathbf{Q}}_{sd} = \mathbf{Q}_l + \mathbf{Q}_{E_8}$, and $(\Gamma_{E_8}, \mathbf{Q}_{E_8})$ would have to be a QH sublattice. But Γ_{E_8} is an *even* lattice and therefore does *not* correspond to an incompressible QH fluid of electrons or holes. (It might, however, appear in the study of a QH effect for surface layers of superfluid He_3, as studied in ref. 21.) Thus, for QH lattices $(\Gamma, \mathbf{Q})$, Γ does *not* contain a Γ_{E_8} sublattice. The QH sublattice $(I_l, \mathbf{Q}_l)$ describes, of course, a QH fluid exhibiting an integer quantum Hall effect.

In the following we may always assume that Γ is indecomposable, and then $I_l = \emptyset$, i.e., $l = 0$. Thus,

$$\mathring{\Gamma}_W = \Gamma_W \tag{6.10}$$

does not contain any Γ_{E_8} or I_l sublattices.

Returning to the decomposition (6.4), we note that $\Gamma/(\Gamma_W \oplus \mathcal{O}_k) \simeq (\Gamma_W^* \oplus \mathcal{O}_k^*)/\Gamma^*$ is a finite Abelian group, henceforth denoted by $\mathcal{G}$ and called the *glue group.* Clearly

$$\mathcal{G} \simeq \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r} \tag{6.11}$$

where $p_1,..., p_r$ are numbers > 1, with $p_i | p_{i+1}$, $i = 1,..., r-1$; and $r \leqslant N$. The generator of $\mathbb{Z}_{p_i}$ can be interpreted, geometrically, as a coset $\mathbf{g}_i + (\Gamma_W + \mathcal{O}_k)$, for some vector $\mathbf{g}_i \in \Gamma$. If we like to work with a unique $\mathbf{g}_i$, we may choose $\mathbf{g}_i$ to be the *shortest* vector in its coset. This vector is called a *gluing vector*. It then follows that

$$\Gamma = \langle \mathbf{g}_1,..., \mathbf{g}_r, \Gamma_W \oplus \mathcal{O}_k \rangle \tag{6.12}$$

Returning to (6.4), and recalling that $|\Gamma^*/\Gamma| = \Delta$, we find that

$$|\Gamma_W^*/\Gamma_W| \cdot |\mathcal{O}_k^*/\mathcal{O}_k| = \Delta\, |\mathcal{G}|^2 = \Delta \prod_{i=1}^{r} p_i^2 \tag{6.13}$$

The order of $|\Gamma_W^*/\Gamma_W|$ is easy to calculate if we know which root lattices appear in Γ_W. It is well known (see, e.g., ref. 42) that

$$|\Gamma_W^*/\Gamma_W| = \det C \tag{6.14}$$

where C is the Cartan matrix (i.e., the Gram matrix of a basis of simple roots) of the root lattices appearing in Γ_W, with Γ_W^* the direct sum of the corresponding weight lattices. Then

$$\det C_{A_{m-1}} = m, \qquad \det C_{D_{m+2}} = 4, \qquad m = 2, 3,... \tag{6.15}$$

and

$$\det C_{E_6} = 3, \qquad \det C_{E_7} = 2 \tag{6.16}$$

In accordance with (6.4), every glue vector g can be written as $\mathbf{\Omega}+\mathbf{v}$, where $\mathbf{\Omega}\in\Gamma_W^*$ and $\mathbf{v}\in\mathcal{O}_k{}^*$. Following ref. 23, we introduce the following notations:

(a) If Γ_W is the root lattice of A_{m-1}, we choose a basis in Γ_W^* consisting of elementary weights dual to a basis of simple roots of Γ_W. The elements $\{\mathbf{\Omega}^a\}_{a=1}^{m-1}$ of such a basis can be labeled by their m-ality a, and we abreviate $\mathbf{\Omega}^a$ by $[a]$. The 0-vector is the weight vector of the trivial representation and is denoted by $[0]$. We have that

$$([a],[a])\equiv(\mathbf{\Omega}^a,\mathbf{\Omega}^a)=\frac{a(m-a)}{m} \tag{6.17}$$

(b) If Γ_W is the root lattice of D_m, $m\geqslant 4$, then $[0]$ stands for the 0-vector (weight of trivial representation), $[1]$ stands for the weight vector of the spinor representation, $[2]$ for the weight vector of the vector representation, and $[3]$ for the weight vector of the conjugate spinor representation. It is known that

$$([1],[1])=([3],[3])=\frac{m}{4},\qquad ([2],[2])=1 \tag{6.18}$$

(c) For Γ_W the root lattice of E_7, there is only one weight vector to be specified, the one corresponding to the 56-dimensional fundamental representation, which is denoted by $[1]$ and has length squared

$$([1],[1])=\tfrac{3}{2} \tag{6.19}$$

(d) If Γ_W is the root lattice of E_6, Γ_W^* is generated by Γ_W and the weight vectors of the 27-dimensional fundamental representation and of its contragredient representation which are denoted by $[1]$ and $[2]$, respectively, and have length squared

$$([1],[1])=([2],[2])=\tfrac{4}{3} \tag{6.20}$$

(e) If the $\mathcal{O}_k$-sublattice is one-dimensional, $k=1$, it is generated by a single vector $\mathbf{x}$ which is determined by its length squared $\mathbf{s}=(\mathbf{x},\mathbf{x})$. The vector $\boldsymbol{\xi}$ dual to $\mathbf{x}$ then has length squared $1/s$. The $\mathcal{O}_1^*$ component $\mathbf{v}$ of a glue vector $\mathbf{g}_i$ is then a multiple of $\boldsymbol{\xi}$, i.e., $\mathbf{v}=r\boldsymbol{\xi}$, or $\mathbf{v}=(r/s)\,\mathbf{x}$. We then abbreviate $\mathbf{v}$ by $[r/s]$.

If the $\mathcal{O}_k$ sublattice is two-dimensional, $k=2$, we choose a basis $\{\mathbf{x}_1,\mathbf{x}_2\}$ in $\mathcal{O}_2$ and describe $\mathcal{O}_2$ by three integers a, b, c, where $a=(\mathbf{x}_1,\mathbf{x}_1)$, $b=(\mathbf{x}_1,\mathbf{x}_2)$, and $c=(\mathbf{x}_2,\mathbf{x}_2)$, so that $\left(\begin{smallmatrix} a & b\\ b & c\end{smallmatrix}\right)$ is the Gram matrix of the basis

$\{\mathbf{x}_1, \mathbf{x}_2\}$. Then $|\mathcal{O}_2^*/\mathcal{O}_2| = ac - b^2$. The $\mathcal{O}_2^*$ component $\mathbf{v}$ of a glue vector $\mathbf{g}$ can be expanded in the basis $\mathbf{x}_1, \mathbf{x}_2$,

$$\mathbf{v} = \frac{r_1}{s_1}\mathbf{x}_1 + \frac{r_2}{s_2}x_2 \tag{6.21}$$

and we abreviate $\mathbf{v}$ by the symbol

$$\left[\frac{r_1}{s_1}, \frac{r_2}{s_2}\right]$$

Thanks to Gauss' work in number theory, the classification of all two-dimensional $\mathcal{O}$-lattices is known.

Fortunately, in our search for QH lattices $(\Gamma, \mathbf{Q})$ for QH fluids corresponding to experimentally observed plateaux of σ_H, the $\mathcal{O}$ sublattices of the lattices Γ that arise are essentially all one- or two-dimensional. This enables us to come up with precise predictions which we shall present in Section 7.

After this digression we continue our general analysis of QH lattices $(\Gamma, \mathbf{Q})$. Let us return to the inclusions of Eq. (6.4),

$$\Gamma_W \oplus \mathcal{O}_k \subseteq \Gamma \subseteq \Gamma^* \subseteq \Gamma_W^* \oplus \mathcal{O}_k^* \tag{6.22}$$

By (6.5) and (6.10), Γ_W is an *even* sublattice of Γ. If $\Gamma_W \neq \varnothing$, the inclusion of $\Gamma_W \oplus \mathcal{O}_k$ in Γ must be proper, and hence the order of the glue group $\mathcal{G}$ is at least 2. Let

$$\mathbf{Q} = \mathbf{Q}_W + \mathbf{Q}' \tag{6.23}$$

with $\mathbf{Q}_W \in \Gamma_W^*$, $\mathbf{Q}' \in \mathcal{O}_k^*$, be the decomposition of the $\mathbf{Q}$ vector corresponding to (6.22). Since Γ_W is even, condition (5.1) implies that

$$(\mathbf{Q}_W, \mathbf{q}) \equiv 0 \mod 2 \tag{6.24}$$

for all $\mathbf{q} \in \Gamma_W$, i.e., $\mathbf{Q}_W \in 2\Gamma_W^*$. Thus, by Eqs. (6.17)–(6.20),

$$(\mathbf{Q}_W, \mathbf{Q}_W) \geqslant 2 \tag{6.25}$$

unless $\mathbf{Q}_W = 0$. Now

$$\begin{aligned}\sigma_H &= (\mathbf{Q}', \mathbf{Q}') + (\mathbf{Q}_W, \mathbf{Q}_W) \\ &\geqslant (\mathbf{Q}', \mathbf{Q}') + 2\end{aligned} \tag{6.26}$$

unless $\mathbf{Q}_W = 0$.

Our discussion is summarized in the following theorem.

Theorem 8. Let $(\Gamma, \mathbf{Q})$ be an indecomposable QH lattice corresponding to a Hall conductivity $\sigma_H = (\mathbf{Q}, \mathbf{Q}) < 2$. Then

$$\Gamma_W \oplus \mathcal{O}_k \subseteq \Gamma \subset \Gamma^* \subseteq \Gamma_W^* \oplus \mathcal{O}_k^* \tag{6.27}$$

all inclusions being *proper* if $\Gamma_W \neq \emptyset$, Γ_W is a direct sum of root lattices corresponding to A_{m-1}, D_{m+2}, $m = 2, 3, \ldots$, E_6, and E_7 and $\mathcal{O}_k$ is a k-dimensional lattice, with $k \geqslant 1$, generated by vectors of length squared 3, 4,..., and $\mathbf{Q} \in \Gamma^*$ is *orthogonal* to Γ_W (i.e., $\mathbf{Q} \in \mathcal{O}_k^*$).

This result has a rather remarkable corollary concerning symmetries of the edge currents of the QH fluid described by $(\Gamma, \mathbf{Q})$: Let $\mathcal{G}$ denote the Lie algebra—in general a direct sum of A_m, D_l, E_6, E_7—whose root lattice is given by Γ_W. Then the algebra generated by the *chiral currents* $\partial\phi^I(u)$, $I = 1, \ldots, N$ defined in (4.1) [see also (3.11) and (3.14)] and the *vertex operators* introduced in (4.3),

$$\{\gamma(\mathbf{n})\, V_L(u; \mathbf{n}) \colon \mathbf{n} \in \Gamma_W\}$$

where the $\gamma(\mathbf{n})$'s are certain "cocycles" which can be found, e.g., in ref. 27, contains the non-Abelian *Kac–Moody algebra* $\hat{\mathcal{G}}_1$ (at level 1). It is generated by the operators

$$\{\underset{\sim}{e}_I \cdot \partial\vec{\phi}(u)^T, \gamma(\mathbf{n}_\alpha)\, V_L(u; \mathbf{n}_\alpha)\} \tag{6.28}$$

where $\{\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_{N-k}\}$ is a basis of orthonormal [with respect to the metric given by the matrix C^{-1}, see (4.2)] row vectors of the $(N-k)$-dimensional subspace of $\mathbb{R}^N$ containing Γ_W, and the vectors $\mathbf{n}_\alpha$ are *simple roots* in Γ_W, i.e., $(\mathbf{n}_\alpha, \mathbf{n}_\alpha) = 2$, for all α.

The operators in (6.28) are *neutral*, i.e., do not transfer any electric charge, since, by Theorem 8, $(\mathbf{Q}, \mathbf{n}) = 0$, for any $\mathbf{n} \in \Gamma_W$ (provided $\sigma_H < 2$).

The Kac–Moody algebra $\hat{\mathcal{G}}_1$ has only finitely many inequivalent, irreducible, unitary representations labeled by the cosets Γ_W^*/Γ_W. Every such coset is represented by an *elementary weight* $\mathbf{\Omega} \in \Gamma_W^*$.

Let $\mathbf{m} \in \Gamma_{\text{phys}} \subseteq \Gamma^*$ correspond to a physical state of the algebra of edge currents of an incompressible QH fluid described by the QH lattice $(\Gamma, \mathbf{Q})$. Let

$$\mathbf{m} = \mathbf{m}_W + \mathbf{m}', \qquad \text{with} \quad \mathbf{m}_W \in \Gamma_W^*, \quad \mathbf{m}' \in \mathcal{O}_k^*$$

be its decomposition corresponding to (6.27). Then $\mathbf{m}_W = \mathbf{\Omega} + \mathbf{l}$, with $\mathbf{l} \in \Gamma_W$, where $\mathbf{\Omega}$ is an elementary weight corresponding to an irreducible, unitary representation $\pi_{\mathbf{\Omega}}$ of $\hat{\mathcal{G}}_1$. The physical state labeled by $\mathbf{m}$ then transforms according to the representation $\pi_{\mathbf{\Omega}}$ under elements of $\hat{\mathcal{G}}_1$.

The Kac–Moody algebra $\hat{\mathscr{G}}_1$ contains a subalgebra of *global symmetry generators*, the zero modes of the Kac–Moody currents, which generate the Lie algebra $\mathscr{G}$. The corresponding Lie group $G = \exp \mathscr{G}$ is the group of global symmetries of the edge degrees of freedom of the QH fluid. If a state of the algebra of edge currents corresponding to the element $\mathbf{m} \in \Gamma_{\text{phys}}$ transforms under a highest-weight representation $\pi_{\mathbf{\Omega}}$ of G, then $\mathbf{\Omega} \equiv \mathbf{m}_W$ mod Γ_W.

Given a Lie algebra $\mathscr{G}$ of rank $N-k \geqslant 1$, there are, in general, various *conformal embeddings* of Kac–Moody algebras $\widehat{su}(2)_r$ at level $r \geqslant 1$ into the Kac–Moody algebra $\hat{\mathscr{G}}_1$ (see refs. 43 and 48 for reviews of conformal embeddings). Depending on the quantum mechanical properties of the QH fluid described by $(\Gamma, \mathbf{Q})$, it is sometimes possible to interpret an algebra $\widehat{su}(2)_r$ conformally embedded in $\hat{\mathscr{G}}_1$ as an algebra of *chiral edge spin currents* describing the spin degrees of freedom of the edge states of the QH fluid. In this case the group G of global symmetries of the edge states contains $SU(2)_{\text{spin}}$ as a subgroup. The possible values of the spin s labeling irreducible, unitary representations of $\widehat{su}(2)_r$ are given by $s = 0,\ 1/2,\dots,\ r/2$.

Conformal embeddings of current algebras $\widehat{su}(p)_q$ into $\hat{\mathscr{G}}_1$, $p = 2, 3,\dots,$ $q = 1, 2,\dots,$ may describe *internal symmetries* encountered in a description of the QH fluid valid, asymptotically, at large distance scales and low frequencies. For example, p may be related to the number of layers (or valleys) of the QH fluid.

Thus, remarkably, our theory of QH lattices is clever enough to discover that *electrons have spin* and thus, in spite of the external magnetic field applied to the system, the edge states of a QH fluid may carry chiral spin currents. This remark is important for the analysis of *spin-singlet QH fluids.* Physical states of the edge current algebra of electric charge ± 1 transforming *trivially* under $SU(2)_{\text{spin}}$ then describe *spin-polarized electrons.* Our general theory predicts that there are QH fluids composed of spin-singlet "bands" of electrons and of "bands" of fully spin-polarized electrons. Similar remarks apply to internal symmetries.

It may be clear at this point that our theory of QH lattices enables us in many cases to understand *transitions* observed in QH fluids, as, for example, the external magnetic field is tilted, keeping the filling factor ν of the QH fluid fixed[8–11]: If, for a given value n_H/d_H of the Hall conductivity σ_H one can find several distinct QH lattices $(\Gamma_1, \mathbf{Q}_1),\dots, (\Gamma_r, \mathbf{Q}_r)$, with $(\mathbf{Q}_1, \mathbf{Q}_1) = \cdots = (\mathbf{Q}_r, \mathbf{Q}_r)$, which, however, differ in that they have distinct global symmetry groups and different degrees of spin polarization, then a QH fluid with Hall conductivity $\sigma_H = n_H/d_H$ is predicted to exhibit transitions when external control parameters, such as the in-plane component of the external magnetic field, are changed.

All this will be discussed in more detail in refs. 42 and 45.

To conclude this section, we present a classification of QH lattices $(\Gamma, \mathbf{Q})$ with a one-dimensional $\mathcal{O}_1$ sublattice and with Q orthogonal to Γ_W. We also describe a series of QH lattices with "maximal symmetry" for which many quantities of practical interest can be calculated explicitly.

Theorem 8 and the results in a paragraph **6** of Section 5 [see also Eqs. (5.14) and (5.15)] yield the following general result of considerable practical interest.

Theorem 9. Let $(\Gamma, \mathbf{Q})$ be an N-dimensional QH lattice with $\sigma_H < 2$ and

$$\Gamma_W \oplus \mathcal{O} \subset \Gamma \subset \Gamma^* \subset \Gamma_W^* \oplus \mathcal{O}^*$$

where $\mathcal{O} = \mathcal{O}_1$ is one-dimensional.

Then $\mathbf{Q}$ is orthogonal to Γ_W, and there exists a normal basis $\{\mathbf{q}, \mathbf{e}_1, \ldots, \mathbf{e}_{N-1}\}$ for Γ [see Eqs. (5.14) and (5.15), i.e., $(\mathbf{Q}, \mathbf{q}) = 1$, $(\mathbf{Q}, \mathbf{e}_I) = 0$, $I = 1, \ldots, N-1$], with the following properties:

(a) $\{\mathbf{e}_1, \ldots, \mathbf{e}_{N-1}\}$ generate an *even* lattice Γ_0, with

$$\Gamma_W \subseteq \Gamma_0 \subset \Gamma_0^* \subseteq \Gamma_W^* \tag{6.29}$$

(b) $\mathbf{q} = (d_H/n_H)\,\mathbf{Q} + \boldsymbol{\omega}$, where $\boldsymbol{\omega} \in \Gamma_W^*$.

(c) The Gram matrix of the basis $\{\mathbf{q}, \mathbf{e}_1, \ldots, \mathbf{e}_{N-1}\}$ has the form

$$K = \left(\begin{array}{c|c} 2p+1 & \underset{\rightharpoondown}{\omega} \\ \hline \underset{\rightharpoondown}{\omega}^T & K_0 \end{array}\right) \tag{6.30}$$

where p is a positive integer, K_0 is the Gram matrix of the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_{N-1}\}$ of Γ_0, and the dual components $\underset{\rightharpoondown}{\omega}$ of the vector $\boldsymbol{\omega}$ are given by $\omega_I = (\boldsymbol{\omega}, \mathbf{e}_I)$, $I = 1, \ldots, N-1$. Furthermore,

$$\begin{aligned} \Delta = \det K &= (2p+1)\det K_0 - \underset{\rightharpoondown}{\omega}\tilde{K}_0\underset{\rightharpoondown}{\omega}^T \\ &= \det K_0(2p+1-(\boldsymbol{\omega}, \boldsymbol{\omega})) \end{aligned} \tag{6.31}$$

and

$$\sigma_H = (\mathbf{Q}, \mathbf{Q}) = (K^{-1})_{00} = \Delta^{-1}\det K_0 > \frac{1}{2p+1} \tag{6.32}$$

Remarks. 1. It can often be ruled out that there is an even lattice Γ_0 which is *not* self-dual [see (6.7)–(6.10)], and which contains Γ_W properly:

(a) If Γ_W^*/Γ_W does not contain any nontrivial subgroup, then $\Gamma_0 = \Gamma_W$. As an application, if Γ_W is the root lattice of E_6 or E_7, then $\Gamma_0 = \Gamma_W$. Similarly, since Γ_0 cannot be self-dual, $\Gamma_0 = \Gamma_W$ if Γ_W is the root lattice of D_m, $m \geqslant 4$.

(b) If Γ_W is the root lattice of A_m, then $\Gamma_0 = \Gamma_W$, for $m \leqslant 14$. The first exception appears for A_{15}, corresponding to a symmetry group $SU(16)$.

(c) If Γ_W is a direct sum of root lattices, the situation is more complicated. For example, if Γ_W is a direct sum of four one-dimensional root lattices of A_1 there is an even lattice properly containing Γ_W and contained in Γ_W^*. However, the corresponding QH lattice is equivalent to one where Γ_W is the root lattice of D_4. It can describe a QH fluid with $\sigma_H = 1/2$.

Thus, for not too large values of the dimension N, $\Gamma_0 = \Gamma_W$ is the typical case.

See ref. 45 for more details.

2. *Definition.* A *maximally symmetric, elementary QH fluid* is one corresponding to a QH lattice $(\Gamma, \mathbf{Q})$, where Γ is indecomposable, $\mathcal{O} = \mathcal{O}_1$ is one-dimensional, $\Gamma_0 = \Gamma_W$, and $\mathbf{Q} \in \mathcal{O}_1^*$ is orthogonal to Γ_W. For such a QH fluid, the matrix element $K_{00} = 2p + 1$ of the Gram matrix K given in (6.30) is the invariant $L_{\max}$ defined in Eq. (5.17) of paragraph **4** of Section 5, provided $\boldsymbol{\omega} \in \Gamma_W^*$ is chosen to be of minimal length in its coset modulo Γ_W [see Eqs. (6.17)–(6.20)]. In that paragraph, we described reasons (e.g., the Wigner lattice instability) for expecting that

$$2p + 1 = L_{\max} \leqslant L_* \approx 9 \tag{6.33}$$

[see (5.19)]. By (6.31) and (6.32),

$$\sigma_H^{-1} = 2p + 1 - (\boldsymbol{\omega}, \boldsymbol{\omega}) \tag{6.34}$$

Hence, σ_H has an absolute lower bound in this class of maximally symmetric QH fluids $\sigma_H \geqslant 1/L_{\max} \geqslant 1/L_* \approx 1/9$. Let

$$\Gamma_W = \Gamma^{(1)} \oplus \cdots \oplus \Gamma^{(s)} \tag{6.35}$$

be the decomposition of Γ_W into indecomposable root lattices of A_{m-1}, D_{m+2}, $m = 2, 3, \ldots, E_6$, and E_7. Let $\boldsymbol{\omega} = \boldsymbol{\omega}^{(1)} + \cdots + \boldsymbol{\omega}^{(s)}$, with $\boldsymbol{\omega}^{(i)} \in \Gamma^{(i)*}$, be the corresponding decomposition of the vector $\boldsymbol{\omega}$. Since, for an elementary QH fluid, Γ is indecomposable, $\boldsymbol{\omega}^{(i)} \neq 0$, for all $i = 1, \ldots, s$. Let N_A be the number of A_{m-1} root lattices, N_D the number of D_{m+2} root lattices, and N_6 and N_7 the numbers of E_6 and E_7 root lattices appearing in (6.35), with $N_A + N_D + N_6 + N_7 = s$. Then, by Eqs. (6.17)–(6.20),

$$(\omega, \omega) \geqslant \tfrac{1}{2} N_A + N_D + \tfrac{4}{3} N_6 + \tfrac{3}{2} N_7 \tag{6.36}$$

Since $\sigma_H > 0$, Eqs. (6.34) and (6.36) yield the following inequality.

Theorem 10:

$$\tfrac{1}{2} N_A + N_D + \tfrac{4}{3} N_6 + \tfrac{3}{2} N_7 < 2p + 1 \leqslant L_* \tag{6.37}$$

with $L_* \approx 9$.

Thus, the number s of sublattices of Γ_W appearing in the decomposition (6.35) satisfies a *universal upper bound*, $s < 2L_* \approx 18$.

3. The family of QH lattices described in Theorems 9 and 10 can be extended as follows: Suppose the Gram matrix of a normal basis of Γ has the form

$$K = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\overset{(0)}{K}} & \begin{array}{c} \underset{\to}{\omega} \\ \hline 0 \end{array} \\ \hline \underset{\to}{\omega}^T & 0 & K_0 \end{array} \right) \tag{6.38}$$

with

$$\underset{\to}{Q} = (1, 0, ..., 0) \tag{6.39}$$

where K_0 is the Gram matrix of an even lattice Γ_0 with $\Gamma_W \subseteq \Gamma_0 \subset \Gamma_0^* \subseteq \Gamma_W^*$.

Then

$$\Delta = \det K_0 (\overset{(0)}{\Delta} - (\omega, \omega) \overset{(0)}{\gamma}) \qquad \text{and} \qquad \gamma = \det K_0 \overset{(0)}{\gamma} \tag{6.40}$$

where

$$\overset{(0)}{\Delta} = \det \overset{(0)}{K}, \qquad \overset{(0)}{\gamma} = \overset{(0)}{\Delta} \cdot (\overset{(0)}{K}{}^{-1})_{00}$$

Hence

$$\sigma_H = \frac{\gamma}{\Delta} = \frac{\overset{(0)}{\gamma}}{\overset{(0)}{\Delta} - (\omega, \omega) \overset{(0)}{\gamma}} = (\overset{(0)}{\sigma}{}_H^{-1} - (\omega, \omega))^{-1} \tag{6.41}$$

This completes our survey of general results on the classification of QH lattices. More details on these results and their proofs can be found in refs. 42 and 45. The task that remains is to apply these results to the analysis of experimentally observed incompressible QH fluids corresponding to specific plateau values of σ_H. In carrying out this task, the tables of Conway and Sloane[23] of low-dimensional lattices are extremely useful. Section 7 presents a survey. Further details will appear elsewhere.

7. TABLES OF QH LATTICES AND OBSERVED PLATEAUX OF σ_H

In this section we complete the discussion of Section 6 on the ADE-$\mathcal{O}$ classification of QH lattices by providing explicit tables of such lattices and correlating them with corresponding values of σ_H. In carrying out this task, we use the tables of integral Euclidean lattices compiled by Conway and Sloane,[23] whose notations we follow.

Let Γ be an integral, Euclidean lattice with Kneser shape $\Gamma_W \oplus \mathcal{O}_k \subset \Gamma$, where Γ_W is the Witt sublattice of Γ and $\mathcal{O}_k$ is a k-dimensional sublattice of Γ generated by vectors of length squared 3, 4,.... Then

$$\Gamma = \langle \mathbf{g}_1,..., \mathbf{g}_r, \Gamma_W \oplus \mathcal{O}_k \rangle \tag{7.1}$$

where $\mathbf{g}_1,..., \mathbf{g}_r$ are the gluing vectors (usually chosen as the shortest vectors in their $\Gamma_W \oplus \mathcal{O}_k$ coset).

(a) We describe Γ_W by giving a list of A_{m-1}, D_{m+2}, $m = 2, 3,..., E_6$, and E_7 whose root lattices are contained in Γ_W.

(b) We describe $\mathcal{O}_k$ by specifying the Gram matrix of a basis $\{\mathbf{x}_1,..., \mathbf{x}_k\}$ of $\mathcal{O}_k$.

For $k = 1$, this Gram matrix is denoted by p_1 for some $p = 1, 2, 3,...$.

For $k = 2$, it is denoted by $(a^b\, c)_2$, with $a = (\mathbf{x}_1, \mathbf{x}_1)$, $b = (\mathbf{x}_1, \mathbf{x}_2)$, and $c = (\mathbf{x}_2, \mathbf{x}_2)$.

We shall not introduce special notations for $k \geqslant 3$, since our tables only contain lattices with $k = 1$ or 2.

(c) Every glue vector $\mathbf{g}$ is decomposed as

$$\mathbf{g} = \mathbf{\Omega} + \mathbf{v}$$

where $\mathbf{\Omega} \in \Gamma_W^*$ is an elementary weight vector which we denote by $[a_1, a_2,...] \equiv [a_1] + [a_2] + \cdots$, where the notation $[a]$ is the one introduced in (6.17)–(6.20), and $\mathbf{v} \in \mathcal{O}_l^*$ is denoted by $[r/s]$ if $k = 1$ and by $[r_1/s_1, r_2/s_2]$ if $k = 2$; see point (e) of Section 6, in particular Eq. (6.21).

In our table of QH lattices, we shall also indicate the discriminant Δ of the lattice and the maximal dimension $N_* = N_*(\Delta)$ up to which we have scanned all lattices, using the notation $\Delta(N_*)$, the order $|\mathscr{G}|$ of the glue group $\mathscr{G}$, the components of Q of vector $\mathbf{Q}$, written in the basis $\{\xi_1,..., \xi_k\}$ of $\mathcal{O}_k^*$ dual to the basis $\{\mathbf{x}_1,..., \mathbf{x}_k\}$ of $\mathcal{O}_k$. To be precise, we list explicitly all the equivalent charge vectors [i.e., $\mathbf{Q}$'s belonging to a fixed $O(\Gamma)$ orbit; see eq. (5.12)] associated to the QH lattice. In all cases considered, there are just two ($\pm\mathbf{Q}$) or four vectors in each orbit. We only list QH lattices with $\sigma_H < 2$, so that by Theorem 8 (Section 6), $\mathbf{Q}$ belongs to $\mathcal{O}_k^*$.

Finally, we shall display the symbol for the QH lattice, ${}_N(n_H/d_H)_\lambda^g$, introduced in Section 5, Eq. (5.42). We repeat here, for convenience, how the basic invariants can be read from the symbol. N is the lattice dimension and $\Delta = \lambda g d_H$ its discriminant; λ is the charge parameter and $l = \lambda \cdot g$ is the level. For minimal QH lattices, i.e., when $\lambda = g = 1$, we shall omit the superscript g and the subscript λ from the symbol.

It is rather striking that in all but three cases of Table II this symbol suffices to fully characterize the QH lattice.

In the three exceptional cases, we add a sign to specify the genus of the lattice, [see ${}_9^{\pm}(1)_2^2$], following conventions of Convay and Sloane,[23] or if the genus is the same for both lattices, we merely distinguish the two QH lattices by a prime [see ${}_{12}(10/7)$, ${}_{12}(10/7)'$ and ${}_9(6/5)_1^2$ $({}_9(6/5)_1^2)'$].

A remarkable feature of Table II emerges if we focus on low-dimensional, minimal (level 1) QH lattices: To a fixed value of the Hall conductance σ_H, there corresponds just *one* or *no* elementary quantum Hall fluid! The "missing fractions" will be reviewed explicitly in Section 7.2. For more details see refs. 42 and 45.

7.1. Indecomposable QH Lattices with $\sigma_H < 2$ and $\Delta \leqslant 19$

Our main interest being in minimal ($\Delta = d_H$) QH lattices, which, by Theorem 4, necessarily have odd discriminant Δ, we have omitted from our list the even-discriminant lattices with $\Delta = 12$, 14, 16, and 18. This reduces the size of Table II quite substantially. We list separately, in Section 7.3, the QH lattices with $\Delta = 8$ and $d_H = 2$.

Finally, we note that there are no QH lattices with discriminant $\Delta = 2$ in the selected range of values for σ_H.

7.2. Values of $\sigma_H < 2$, $d_H \leqslant 19$ Not Corresponding to an Indecomposable QH Lattice at Level 1

One of the interesting features of Table II of QH lattices is the nonexistence of minimal indecomposable QH lattices with dimension $N \leqslant N_*(\Delta)$ for certain specific values of $\sigma_H < 2$. Furthermore, using the constructive method described in Section 6, one can readily check whether

Table II. All Indecomposable Quantum Hall Lattices for $\sigma_H < 2$ and Discriminant $\Delta \leqslant 19$ ($\Delta \neq 8, 12, 14, 16, 18$) in Low Dimension

$\Delta(N_*)$	$\lvert\mathscr{G}\rvert$	$\Gamma_W \mathcal{O}_k[\mathbf{g}_1; \mathbf{g}_2; \ldots]$	$\pm\underline{Q}$	${}_N(\sigma_H)^g_\lambda$
1 (18)	1	1_1	$\underline{\xi}$	${}_1(1)$
2 (16)	—	—	—	—
3 (14)	1	3_1	$\underline{\xi}$	${}_1(\frac{1}{3})$
	2	$E_7\, 6_1\, [1, \frac{1}{2}]$	$2\underline{\xi}$	${}_8(\frac{2}{3})$
	4	$D_9\, 12_1\, [1, \frac{1}{4}]$	$4\underline{\xi}$	${}_{10}(\frac{4}{3})$
4 (12)	2	$D_8\, 4_1\, [1, \frac{1}{2}]$	$2\underline{\xi}$	${}_9^+(1)^2_2$
	2	$E_7\, A_1\, 4_1\, [1, 1, \frac{1}{2}]$	$2\underline{\xi}$	${}_9^-(1)^2_2$
5 (12)	1	5_1	$\underline{\xi}$	${}_1(\frac{1}{5})$
	2	$A_1\, 10_1\, [1, \frac{1}{2}]$	$2\underline{\xi}$	${}_2(\frac{2}{5})$
	3	$E_6\, 15_1\, [1, \frac{1}{3}]$	$3\underline{\xi}$	${}_7(\frac{3}{5})$
	4	$D_7\, 20_1\, [1, \frac{1}{4}]$	$4\underline{\xi}$	${}_8(\frac{4}{5})$
	6	$E_7\, A_2\, 30_1\, [1, 1, \frac{1}{6}]$	$6\underline{\xi}$	${}_{10}(\frac{6}{5})$
	4	$D_9\, (3\,{}^1\,7)_2\, [1, \frac{1}{4}, \frac{1}{4}]$	$\underline{\xi}_1 + 3\underline{\xi}_2$	${}_{11}(\frac{7}{5})$
6 (11)	2	$D_6\, 6_1\, [1, \frac{1}{2}]$	$2\underline{\xi}$	${}_7(\frac{2}{3})^2_1$
	4	$D_7\, A_1\, 12_1\, [1, 1, \frac{1}{4}]$	$4\underline{\xi}$	${}_9(\frac{4}{3})^2_1$
	5	$A_9\, 15_1\, [4, \frac{1}{5}]$	$5\underline{\xi}$	${}_{10}(\frac{5}{3})^2_1$
	4	$E_7\, A_3\, 12_1\, [1, 1, \frac{1}{4}]$	$4\underline{\xi}$	${}_{11}(\frac{4}{3})^2_1$
7 (12)	1	7_1	$\underline{\xi}$	${}_1(\frac{1}{7})$
	3	$A_2\, 21_1\, [1, \frac{1}{3}]$	$3\underline{\xi}$	${}_3(\frac{3}{7})$
	4	$D_5\, 28_1\, [1, \frac{1}{4}]$	$4\underline{\xi}$	${}_6(\frac{4}{7})$
	2	$E_7\, 14_1\, [1, \frac{1}{2}]$	$2\underline{\xi}$	${}_8(\frac{2}{7})$
	6	$E_6\, A_1\, 42_1\, [1, 1, \frac{1}{6}]$	$6\underline{\xi}$	${}_8(\frac{6}{7})$
	9	$A_8\, 63_1\, [4, \frac{1}{9}]$	$9\underline{\xi}$	${}_9(\frac{9}{7})$
	12	$D_7\, A_2\, 84_1\, [1, 1, \frac{1}{12}]$	$12\underline{\xi}$	${}_{10}(\frac{12}{7})$
	2	$E_7\, ({}_3\,{}^1\,{}_5)_2\, [1, \frac{1}{2}, \frac{1}{2}]$	$\underline{\xi}_1 - \underline{\xi}_2$	${}_9(\frac{5}{7})$
			$\underline{\xi}_1 + 3\underline{\xi}_2$	${}_9(\frac{13}{7})$
	$2 \cdot 2$	$D_8\, ({}_4\,{}^2\,{}_8)_2\, [1, \frac{1}{2}, 0; 2, 0, \frac{1}{2}]$	$2\underline{\xi}_1 + 2\underline{\xi}_2$	${}_{10}(\frac{8}{7})$
	$2 \cdot 4$	$D_9\, A_1\, ({}_6\,{}^2\,{}_{10})_2\, [0, 1, 0, \frac{1}{2}; 1, 1, \frac{1}{4}, \frac{1}{4}]$	$2\underline{\xi}_1 - 2\underline{\xi}_2$	${}_{12}(\frac{10}{7})$
	10	$E_7\, A_4\, 70_1\, [1, 1, \frac{1}{10}]$	$10\underline{\xi}$	${}_{12}(\frac{10}{7})'$
8 (10)		See Section 7.3		
9 (8)	1	9_1	$\underline{\xi}$	${}_1(\frac{1}{9})$
	2	$A_1\, 18_1\, [1, \frac{1}{2}]$	$2\underline{\xi}$	${}_2(\frac{2}{9})$
	4	$A_3\, 36_1\, [1, \frac{1}{4}]$	$4\underline{\xi}$	${}_4(\frac{4}{9})$
	5	$A_4\, 45_1\, [2, \frac{1}{5}]$	$5\underline{\xi}$	${}_5(\frac{5}{9})$
	7	$A_6\, 63_1\, [3, \frac{1}{7}]$	$7\underline{\xi}$	${}_7(\frac{7}{9})$
	8	$A_7\, 72_1\, [3, \frac{1}{8}]$	$8\underline{\xi}$	${}_8(\frac{8}{9})$
	2	$A_5\, 6_1\, [3, \frac{1}{2}]$	$2\underline{\xi}$	${}_6(\frac{2}{3})^3_1$
	$2 \cdot 2$	$D_6\, 6_1\, 6_1\, [1, \frac{1}{2}, 0; 3, 0, \frac{1}{2}]$	$2\underline{\xi}_1 \pm 2\underline{\xi}_2$	${}_8(\frac{4}{3})^3_1$

Table II. *(Continued)*

$\Delta(N_*)$	$\lvert\mathscr{G}\rvert$	$\Gamma_W \mathcal{O}_k[\mathbf{g}_1; \mathbf{g}_2; \ldots]$	$\pm\underrightarrow{Q}$	${}_N(\sigma_H)^g_\lambda$
10 (10)	3	$A_5\, 15_1\, [2, \frac{1}{3}]$	$3\underrightarrow{\xi}$	${}_6(\frac{3}{5})^2_1$
	4	$D_5 A_1\, 20_1\, [1, 1, \frac{1}{4}]$	$4\underrightarrow{\xi}$	${}_7(\frac{4}{5})^2_1$
	6	$D_6 A_2\, 30_1\, [1, 1, \frac{1}{6}]$	$6\underrightarrow{\xi}$	${}_9(\frac{6}{5})^2_1$
	8	$A_7 A_1\, 40_1\, [3, 1, \frac{1}{8}]$	$8\underrightarrow{\xi}$	${}_9(\frac{8}{5})^2_1$
	2	$E_7\, ({}_4{}^2{}_6)_2\, [1, 0, \frac{1}{2}]$	$2\underrightarrow{\xi}_2$	${}_9(\frac{4}{5})^2_1$
			$2\underrightarrow{\xi}_1 + 2\underrightarrow{\xi}_2$	$({}_9(\frac{6}{5})^2_1)'$
11 (8)	1	11_1	$\underrightarrow{\xi}$	${}_1(\frac{1}{11})$
	1	$({}_3{}^1{}_4)_2$	$\underrightarrow{\xi}_1$	${}_2(\frac{4}{11})$
			$\underrightarrow{\xi}_1 + 2\underrightarrow{\xi}_2$	${}_2(\frac{12}{11})$
			$\underrightarrow{\xi}_1 - 2\underrightarrow{\xi}_2$	${}_2(\frac{20}{11})$
	6	$A_2 A_1\, 66_1\, [1, 1, \frac{1}{6}]$	$6\underrightarrow{\xi}$	${}_4(\frac{6}{11})$
	5	$A_4\, 55_1\, [1, \frac{1}{5}]$	$5\underrightarrow{\xi}$	${}_5(\frac{5}{11})$
	3	$E_6\, 33_1\, [1, \frac{1}{3}]$	$3\underrightarrow{\xi}$	${}_7(\frac{3}{11})$
	7	$A_6\, 77_1\, [2, \frac{1}{7}]$	$7\underrightarrow{\xi}$	${}_7(\frac{7}{11})$
	2	$E_7\, 22_1\, [1, \frac{1}{2}]$	$2\underrightarrow{\xi}$	${}_8(\frac{2}{11})$
	14	$A_6 A_1\, 154_1\, [3, 1, \frac{1}{14}]$	$14\underrightarrow{\xi}$	${}_8(\frac{14}{11})$
	2.2	$D_6\, ({}_6{}^2{}_8)_2\, [1, \frac{1}{2}, 0; 2, 0, \frac{1}{2}]$	$2\underrightarrow{\xi}_1 + 2\underrightarrow{\xi}_2$	${}_8(\frac{10}{11})$
			$2\underrightarrow{\xi}_1 - 2\underrightarrow{\xi}_2$	${}_8(\frac{18}{11})$
13 (8)	1	13_1	$\underrightarrow{\xi}$	${}_1(\frac{1}{13})$
	2	$A_1\, 26_1\, [1, \frac{1}{2}]$	$2\underrightarrow{\xi}$	${}_2(\frac{2}{13})$
	3	$A_2\, 39_1\, [1, \frac{1}{3}]$	$3\underrightarrow{\xi}$	${}_3(\frac{3}{13})$
	2	$A_1\, ({}_3{}^1{}_9)_2\, [1, \frac{1}{2}, \frac{1}{2}]$	$\underrightarrow{\xi}_1 - \underrightarrow{\xi}_2$	${}_3(\frac{7}{13})$
			$\underrightarrow{\xi}_1 + 3\underrightarrow{\xi}_2$	${}_3(\frac{15}{13})$
	10	$A_4 A_1\, 130_1\, [2, 1, \frac{1}{10}]$	$10\underrightarrow{\xi}$	${}_6(\frac{10}{13})$
	6	$A_5\, 78_1\, [1, \frac{1}{6}]$	$6\underrightarrow{\xi}$	${}_6(\frac{6}{13})$
	4	$D_5\, ({}_7{}^2{}_8)_2\, [1, \frac{1}{2}, \frac{1}{4}]$	$\underrightarrow{\xi}_1 - 2\underrightarrow{\xi}_2$	${}_7(\frac{11}{13})$
	4	$D_7\, 52_1\, [1, \frac{1}{4}]$	$4\underrightarrow{\xi}$	${}_8(\frac{4}{13})$
	12	$D_5 A_2\, 156_1\, [1, 1, \frac{1}{12}]$	$12\underrightarrow{\xi}$	${}_8(\frac{12}{13})$
	3	$E_6\, ({}_5{}^1{}_8)_2\, [1, \frac{1}{3}, \frac{1}{3}]$	$\underrightarrow{\xi}_1 + 2\underrightarrow{\xi}_2$	${}_8(\frac{8}{13})$
			$3\underrightarrow{\xi}_1$	${}_8(\frac{24}{13})$
15 (8)	1	15_1	$\underrightarrow{\xi}$	${}_1(\frac{1}{15})$
	4	$D_5\, 60_1\, [1, \frac{1}{4}]$	$4\underrightarrow{\xi}$	${}_6(\frac{4}{15})$
	7	$A_6\, 105_1\, [1, \frac{1}{7}]$	$7\underrightarrow{\xi}$	${}_7(\frac{7}{15})$
	6	$A_5\, 6_1\, 15_1\, [1, \frac{1}{2}, \frac{1}{3}]$	$2\underrightarrow{\xi}_1 \pm 3\underrightarrow{\xi}_2$	${}_7(\frac{19}{15})$
	2	$E_7\, 30_1\, [1, \frac{1}{2}]$	$2\underrightarrow{\xi}$	${}_8(\frac{2}{15})$
	14	$A_6 A_1\, 210_1\, [2, 1, \frac{1}{14}]$	$14\underrightarrow{\xi}$	${}_8(\frac{14}{15})$
	$4 \cdot 2$	$D_5 A_1\, 6_1\, 20_1\, [1, 1, 0, \frac{1}{4}; 2, 1, \frac{1}{2}, 0]$	$2\underrightarrow{\xi}_1 \pm 2\underrightarrow{\xi}_2$	${}_8(\frac{22}{15})$
	3	$A_2\, ({}_6{}^3{}_9)_2\, [1, \frac{1}{3}, \frac{1}{3}]$	$3\underrightarrow{\xi}_2$	${}_4(\frac{6}{5})^3_1$
			$2\underrightarrow{\xi}_1 + \underrightarrow{\xi}_2$	${}_4(\frac{2}{3})^5_1$

Table II. *(Continued)*

$\Delta(N_*)$	$\lvert\mathscr{G}\rvert$	$\Gamma_W \mathcal{O}_k[\mathbf{g}_1;\mathbf{g}_2;\ldots]$	$\pm\underline{Q}$	${}_N(\sigma_H)^g_\lambda$
	$2\cdot 2$	$D_4\,({}_8\,{}^2\,{}_8)_2\,[1, \frac{1}{2}, 0; 3, 0, \frac{1}{2}]$	$2\underline{\xi}_1+2\underline{\xi}_2$	${}_6(\frac{4}{5})^3_1$
			$2\underline{\xi}_1-2\underline{\xi}_2$	${}_6(\frac{4}{3})^5_1$
	3	$A_2\,A_2\,15_1\,[1, 1, \frac{1}{3}]$	$3\underline{\xi}$	${}_5(\frac{3}{5})^3_1$
	6	$A_5\,A_2\,30_1\,[3, 1, \frac{1}{6}]$	$6\underline{\xi}$	${}_8(\frac{6}{5})^3_1$
	$2\cdot 2$	$D_6\,6_1\,10_1\,[1, \frac{1}{2}, 0; 3, 0, \frac{1}{2}]$	$2\underline{\xi}$	${}_8(\frac{2}{3})^5_1$
17 (7)	1	17_1	$\underline{\xi}$	${}_1(\frac{1}{17})$
	2	$A_1\,34_1\,[1, \frac{1}{2}]$	$2\underline{\xi}$	${}_2(\frac{2}{17})$
	1	$({}_3\,{}^1\,{}_6)_2$	$\underline{\xi}_1$	${}_2(\frac{6}{17})$
			$\underline{\xi}_1+2\underline{\xi}_2$	${}_2(\frac{14}{17})$
			$-\underline{\xi}_1+2\underline{\xi}_2$	${}_2(\frac{22}{17})$
	2	$A_1\,({}_5\,{}^1\,{}_7)_2\,[1, \frac{1}{2}, \frac{1}{2}]$	$\underline{\xi}_1-\underline{\xi}_2$	${}_3(\frac{7}{17})$
			$\underline{\xi}_1+3\underline{\xi}_2$	${}_3(\frac{23}{17})$
			$3\underline{\xi}_1+\underline{\xi}_2$	${}_3(\frac{31}{17})$
	4	$A_3\,68_1\,[1, \frac{1}{4}]$	$4\underline{\xi}$	${}_4(\frac{4}{17})$
	4	$A_3\,({}_8\,{}^2\,{}_9)_2\,[1, \frac{1}{4}, \frac{1}{2}]$	$-2\underline{\xi}_1+\underline{\xi}_2$	${}_5(\frac{13}{17})$
			$2\underline{\xi}_1+3\underline{\xi}_2$	${}_5(\frac{21}{17})$
	10	$A_4\,A_1\,170_1\,[1, 1, \frac{1}{10}]$	$10\underline{\xi}$	${}_6(\frac{10}{17})$
	3	$E_6\,51_1\,[1, \frac{1}{3}]$	$3\underline{\xi}$	${}_7(\frac{3}{17})$
	15	$A_4\,A_2\,255_1\,[2, 1, \frac{1}{15}]$	$15\underline{\xi}$	${}_7(\frac{15}{17})$
	4	$D_5\,({}_3\,{}^1\,{}_{23})_2\,[1, \frac{1}{4}, \frac{1}{4}]$	$\underline{\xi}_1+3\underline{\xi}_2$	${}_7(\frac{11}{17})$
			$-\underline{\xi}_1+5\underline{\xi}_2$	${}_7(\frac{27}{17})$
19 (7)	1	19_1	$\underline{\xi}$	${}_1(\frac{1}{19})$
	1	$({}_4\,{}^1\,{}_5)_2$	$\underline{\xi}_2$	${}_2(\frac{4}{19})$
			$3\underline{\xi}_2$	${}_2(\frac{36}{19})$
			$2\underline{\xi}_1+\underline{\xi}_2$	${}_2(\frac{20}{19})$
			$2\underline{\xi}_1-\underline{\xi}_2$	${}_2(\frac{28}{19})$
	2	$A_1\,({}_3\,{}^1\,{}_{13})_2\,[1, \frac{1}{2}, \frac{1}{2}]$	$\underline{\xi}_1+\underline{\xi}_2$	${}_3(\frac{7}{19})$
			$\underline{\xi}_1-3\underline{\xi}_2$	${}_2(\frac{23}{19})$
	3	$A_2\,57_1\,[1, \frac{1}{3}]$	$3\underline{\xi}$	${}_3(\frac{3}{19})$
	$2\cdot 2$	$A_1\,A_1\,({}_8\,{}^2\,{}_{10})_2\,[1, 1, \frac{1}{2}, 0; 1, 0, 0, \frac{1}{2}]$	$2\underline{\xi}_1+2\underline{\xi}_2$	${}_4(\frac{14}{19})$
			$2\underline{\xi}_1-2\underline{\xi}_2$	${}_4(\frac{22}{19})$
	5	$A_4\,95_1\,[2, \frac{1}{5}]$	$5\underline{\xi}$	${}_5(\frac{5}{19})$
	12	$\dot{A}_3\,A_2\,228_1\,[1, 1, \frac{1}{2}]$	$12\underline{\xi}$	${}_6(\frac{12}{19})$
	5	$A_4\,({}_9\,{}^1\,{}_{11})_2\,[1, \frac{2}{5}, \frac{1}{5}]$	$\underline{\xi}_1+3\underline{\xi}_2$	${}_6(\frac{16}{19})$
			$3\underline{\xi}_1-\underline{\xi}_2$	${}_6(\frac{24}{19})$
	4	$D_5\,({}_5\,{}^1\,{}_{16})_2\,[1, \frac{1}{2}, \frac{1}{4}]$	$\underline{\xi}_1-2\underline{\xi}_2$	${}_7(\frac{11}{19})$
			$3\underline{\xi}_1+2\underline{\xi}_2$	${}_7(\frac{35}{19})$
	6	$A_5\,({}_{10}\,{}^4\,{}_{13})_2\,[1, \frac{1}{6}, \frac{1}{3}]$	$4\underline{\xi}_1+\underline{\xi}_2$	${}_7(\frac{31}{19})$
			$3\underline{\xi}_2$	${}_7(\frac{15}{19})$

a minimal, maximally symmetric lattice of arbitrary dimension exists for these fractions. In this subsection, we list the "missing fractions," i.e., those values of $\sigma_H < 2$ and $d_H \leqslant 19$ that do not correspond to any minimal, maximally symmetric QH lattice and for which no minimal indecomposable QH lattice with dimension strictly below some dimension $\mathcal{N} \geqq N_*(\Delta)$ exists. (We have used Theorem 4 to optimize our estimate on $\mathcal{N}$.) The value of σ_H and the optimal dimension $\mathcal{N}$ are indicated in the symbol $\sigma_H(\mathcal{N})$:

$d_H = 3$ $\frac{5}{3}(17)$
$d_H = 5$ $\frac{8}{5}(14)$; $\frac{9}{5}(13)$
$d_H = 7$ $\frac{11}{7}(15)$
$d_H = 9$ —
$d_H = 11$ $\frac{8}{11}(10)$; $\frac{9}{11}(9)$; $\frac{13}{11}(9)$; $\frac{15}{11}(11)$; $\frac{16}{11}(10)$; $\frac{17}{11}(9)$;
$d_H = 13$ $\frac{5}{13}(9)$; $\frac{14}{13}(10)$; $\frac{16}{13}(10)$; $\frac{19}{13}(11)$; $\frac{20}{13}(10)$; $\frac{22}{13}(10)$; $\frac{25}{13}(9)$
$d_H = 15$ $\frac{8}{15}(10)$; $\frac{13}{15}(9)$; $\frac{16}{15}(10)$; $\frac{23}{15}(11)$; $\frac{29}{15}(9)$
$d_H = 17$ $\frac{5}{17}(9)$; $\frac{9}{17}(9)$; $\frac{20}{17}(8)$; $\frac{24}{17}(8)$; $\frac{25}{17}(9)$; $\frac{28}{17}(8)$; $\frac{29}{17}(9)$; $\frac{30}{17}(8)$; $\frac{33}{17}(9)$
$d_H = 19$ $\frac{8}{19}(8)$; $\frac{10}{19}(8)$; $\frac{13}{19}(9)$; $\frac{18}{19}(8)$; $\frac{29}{19}(9)$; $\frac{32}{19}(8)$; $\frac{33}{19}(9)$; $\frac{37}{19}(9)$

Of course, many among these values of σ_H have been observed in FQH experiments. For some fractions, e.g., 5/3, 8/5, 9/5, and 11/7, the bound on the dimension $\mathcal{N}$ is high enough to practically rule out the possibility of a reasonable, elementary, minimal QH lattice. In those cases we have but two options to understand the observed incompressible state: either we must consider *nonminimal* (i.e., higher-level) QH lattices, at the price of encountering more complicated quasiparticle spectra (no charge $\rightarrow$ statistics connection and/or exotic fractional charges for elementary vortices); or we must account for the observed σ_H by means of a *composite fluid*, using, for example, electron–hole conjugation.

The second option, compositeness, generally appears to be simpler and more natural. We shall discuss the examples $\sigma_H = 5/3$ and 8/5 explicitly in Section 7.5.

We also note that there is no elementary QH fluid with $\sigma_H = 5/13$. Moreover, this value of σ_H has no natural composite explanation, since $\sigma_H = 1/13$ and $\sigma_H = 2/13$ fluids have not been observed, and, as $5/13 < 1/2$, electron–hole conjugation $(5/13 = 1 - 8/13)$ can presumably be excluded. This result should be contrasted with predictions of standard hierarchy schemes, Haldane–Halperin or Jain–Goldman,[50] which predict an incompressible state corresponding to $\sigma_H = 5/13$. Up to now, *no* $\sigma_H = 5/13$ plateau has been observed experimentally. Its persistent absence would represent an interesting, partial experimental justification of the additional hypotheses on which the classification presented in this work is based, namely low dimension and minimality of the QH lattices describing chiral edge currents.

We also emphasize the absence of indecomposable QH lattices at $\sigma_H = 8/15$, $9/17$, and $10/19$, which are three successive *unobserved* plateaux in the "second main sequence," $\sigma_H = m/(2m-1)$, $m = 1, 2, \ldots$, of fractions converging to 1/2. In Section 7.4 we explain why an observation (or not) of these fractions would be an interesting experimental input for our theoretical understanding of quantum Hall fluids.

7.3. QH Lattices with $d_H = 2$, $\Delta = 8$

If $\sigma_H = n_H/2$, then the invariants g and λ [see Eqs. (5.39), (5.40), paragraph **9**, Section 5) are at least 2 (Theorems 5 and 6, paragraph **11**, Section 5). Thus Δ is divisible by 8. We present in Table III all *indecomposable* QH lattices with $l = \lambda \cdot g = 4$ and $\Delta = 8$ in dimension $N \leqslant 10$. Since Δ and N_* are fixed, they are not displayed. Otherwise notations are identical to those used in Table II. Only $\sigma_H = 1/2$ or $3/2$ can appear since we have limited ourselves to $\sigma_H < 2$ as in Table II.

7.4. The Maximally Symmetric A and E Series of Indecomposable QH Lattices at Level 1

As examples of natural families of QH lattices we present the maximally symmetric, indecomposable QH lattices with a Witt sublattice given by a root lattice of A or E. [The $\mathcal{O}$ sublattice is one-dimensional, and $\Gamma_e = \Gamma_W$; see (6.29), (6.31), and (6.32).] The Dynkin diagrams describing these Witt sublattices are given in Fig. 2a.

The Gram matrix K of a normal basis $\{\mathbf{q}, \mathbf{e}_1, \ldots, \mathbf{e}_{N-1}\}$ of Γ, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_{N-1}\}$ is a basis of simple roots for Γ_W, has the form

$$K = \left(\begin{array}{c|c} 2p+1 & \underset{\rightarrow}{\omega} \\ \hline \underset{\rightarrow}{\omega}^T & C \end{array} \right) \tag{7.2}$$

Table III. All Indecomposable Quantum Hall Lattices for $\sigma_H = 1/2$ or $3/2$ and Discriminant $\Delta = 8$ in Low Dimension[a].

$\lvert\mathcal{G}\rvert$	$\Gamma_W\,\mathcal{O}_k[\mathbf{g}]$	$\pm \underset{\rightarrow}{Q}$	${}_N(\sigma_H)^g_\lambda$
1	$({}_3\,{}^1\,{}_3)_2$	$\underset{\rightarrow}{\xi}_1 + \underset{\rightarrow}{\xi}_2$	${}_2(\frac{1}{2})^2_2$
2	$A_1 A_1 8_1\ [1, 1, \frac{1}{2}]$	$2\underset{\rightarrow}{\xi}$	${}_3(\frac{1}{2})^2_2$
2	$A_3 8_1\ [2, \frac{1}{2}]$	$2\underset{\rightarrow}{\xi}$	${}_4(\frac{1}{2})^2_2$
2	$D_m 8_1\ [2, \frac{1}{2}]$, $4 \leqslant m \leqslant 9$	$2\underset{\rightarrow}{\xi}$	${}_{m+1}(\frac{1}{2})^2_2$
3	$E_6 ({}_4\,{}^2\,{}_7)_2\ [1, \frac{1}{3}, \frac{1}{3}]$	$3\underset{\rightarrow}{\xi}_2$	${}_8(\frac{3}{2})^2_2$
6	$E_6 A_1 A_1 24_1\ [1, 1, 1, \frac{1}{6}]$	$6\underset{\rightarrow}{\xi}$	${}_9(\frac{3}{2})^2_2$
6	$E_6 A_3 24_1\ [1, 2, \frac{1}{6}]$	$6\underset{\rightarrow}{\xi}$	${}_{10}(\frac{3}{2})^2_2$

[a] $N \leqslant 10$.

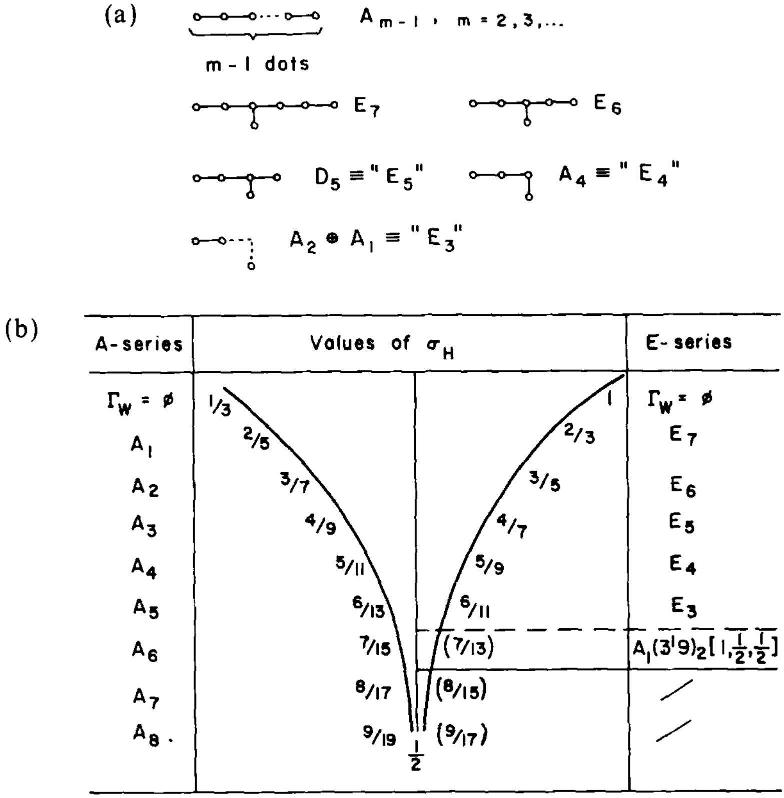

Fig. 2. (a) Dynkin diagrams describing the Witt sublattices for the A and E series of maximally symmetric quantum Hall lattices. (b) Values of σ_H for the principal (i.e., $L_{max} = 3$) A and E series of maximally symmetric quantum Hall lattices.

where C is the Cartan matrix, i.e., the Gram matrix of $\{e_1,..., e_{N-1}\}$, and

$$\omega = [1] \quad \text{for} \quad A_{m-1}, m = 2, 3,..., E_7, E_6, E_5$$

$$\omega = [2] \quad \text{for} \quad E_4$$

$$\omega = [1][1] \quad \text{for} \quad E_3$$

By formulas (6.31) and (6.32), and using Eqs. (6.15)–(6.20), we have that

$$\Delta = \det K = 2pm + 1, \quad \sigma_H = \frac{m}{2pm+1}, \quad \text{for } A_{m-1}, \; m = 2, 3,... \tag{7.3}$$

$$\Delta = \det K = 2pm - 1, \quad \sigma_H = \frac{m}{2pm-1}, \quad \text{for } E_{8-(m-1)}, \; m = 2, 3, 4, 5, 6 \tag{7.4}$$

These functions also appear as Hall conductivities of "hierarchy states"; see, e.g., refs. 38, 50,[8] and, for the A series refs. 19 and 20. For $p = 1$, i.e., for $L_{\max} = 2p + 1 = 3$ [see Eq. (6.33)], we get the fractions shown in Fig. 2b.

Besides being singled out by their high symmetry, the QH lattices of the E series and those of the A series up to dimension 7 can be shown to be the *unique*, *smallest dimensional*, minimal indecomposable lattices for the corresponding fraction. This can be inferred from Table II. For the A series the strength of our results is limited by the available tables of lattices given in ref. 23.

Figure 2b shows very clearly the *asymmetry* between the two series: The E series is finite, describing five nontrivial lattices of decreasing dimension; the A series is infinite, with increasing dimension. The fraction that immediately follows the smallest fraction of the E series is 7/13, experimentally observed, to which one can associate a unique three-dimensional, indecomposable, minimal QH lattice [see Table II, ${}_3(7/13)$]. It is not a maximally symmetric lattice, however, since the $\mathcal{O}$ part in the Kneser shape is *two-dimensional*. But it is interesting to note that its Witt part is an $SU(2)$ sublattice.

We stress that, for the next smaller fractions 8/15, 9/17, and 10/19, no indecomposble minimal solutions are available (see Section 7.2). If feasible, this would be a fruitful range for an experimental search of *composite fluids*. Typical solutions could be $8/15 = (1/3) + (1/5)$, or $8/15 = (4/15) + (4/15)$. (Note that 4/15 is an observed fraction.) These two composite fluids could be distinguished by their elementary fractional charges, $e_1^* = e/3$ and $e_2^* = e/5$, for the first composite fluid, while $e^* = (1/\lambda)(e/15)$, for the second fluid.

7.5. Examples: The Plateaux at $\sigma_H = 8/5,\ 5/3,\ 1,\ 1/2$

If the external magnetic field $\mathbf{B}^{(0)}$ acting on a QH sample is not very large, and the effective g-factor of the electrons in the two-dimensional fluid is small, then if the fluid is incompressible, it can be in a "spin-singlet state." In this case, there will be *chiral edge spin currents* generating an $\widehat{su}(2)_1$ Kac–Moody algebra.

Suppose now that the component of $\mathbf{B}^{(0)}$ in the plane of the sample is increased (tilting of $\mathbf{B}^{(0)}$), while the filling factor ν is kept constant. Then one must expect that, at a critical value of the tilting angle, a transition from the spin-singlet state to a state of the system where the spins of all electrons are polarized will occur.

[8] For a review see ref. 33.

We shall argue that the QH fluid corresponding to the plateau at $\sigma_H = 8/5$ is an example of a QH fluid exhibiting such a transition.

Consulting the table in Section 7.2, we observe that there is no indecomposable, minimal QH lattice corresponding to $\sigma_H = 8/5$. We are therefore forced to look for QH lattices which are either decomposable or nonminimal. Decomposable QH lattices describe composite QH fluids. For small values of the in-plane component of $\mathbf{B}^{(0)}$ the following particle–hole composite QH fluid is a natural candidate for an incompressible state with $\sigma_H = 8/5$.

(a) One pair of QH lattices corresponding to $\sigma_H = 8/5$ is $(\Gamma_e, \mathbf{Q}_e) \oplus (\Gamma_h, \mathbf{Q}_h)$, with

$$\Gamma_e = I_2, \ \underset{\rightarrow}{Q}_e = (1, 1)$$
$$\Gamma_h = A_1 \ 10_1 \ [1, \tfrac{1}{2}], \qquad \underset{\rightarrow}{Q}_h = 2\underset{\rightarrow}{\xi}$$

corresponding to a Gram matrix $K_h = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$, with $\underset{\rightarrow}{Q}_h = (1, 0)$, in a normal basis. Clearly $(\Gamma_h, \mathbf{Q}_h)$ is the *electron–hole conjugate* of the QH fluid with $\sigma_H = 2/5$ discussed in Section 7.4. In a normal basis, the Gram matrix of Γ_e is given by $K_e = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, with $\underset{\rightarrow}{Q}_e = (1, 0)$. The Witt sublattices of Γ_e and Γ_h are the root lattice of A_1, so, as explained in Section 6, (6.28), there is an $\widehat{su}(2)_1$ Kac–Moody algebra of chiral edge currents which may be interpreted as *spin currents*. This explanation of $\sigma_H = 8/5$ thus corresponds to a "*spin-singlet*" *state*,[8] which we expect to be realized at small values of $\mathbf{B}_\parallel^{(0)}$.

As we increase $\mathbf{B}_\parallel^{(0)}$, keeping the filling factor constant, the Zeeman energy of electrons increases. We thus expect that, at some value of $\mathbf{B}_\parallel^{(0)}$, the QH fluid described above becomes unstable and a transition to a new incompressible state occurs as $\mathbf{B}_\parallel^{(0)}$ is increased further.[8-11] This new state is likely to contain a fully polarized, completely occupied lowest Landau level. The following is a plausible lattice.

(b) A pair of QH lattices associated to $\sigma_H = 8/5 = 1 + 3/5$ is $(\Gamma_e^1, \mathbf{Q}_e^1) \oplus (\Gamma_e^2, \mathbf{Q}_e^2)$, with $\Gamma_e^1 = I_1 \equiv 1_1$, $\underset{\rightarrow}{Q}_e^1 = (1)$, and $(\Gamma_e^2, \mathbf{Q}_e^2)$ is the E_6 solution corresponding to $\sigma_H = 3/5$ which we have discussed in Section 7.4. Obviously, the QH fluid corresponding to this particular decomposable QH lattice is partially *spin-polarized*.

Note that the QH fluid described in (a) exhibits edge currents of *both* chiralities, while the edge currents of the one described in (b) have all the *same* chirality. Experiments testing the chirality of edge currents are reported in ref. 51.

It illustrates an aspect of our general analysis that if we give up the condition of minimality we can find further QH lattices with $\sigma_H = 8/5$, in particular there are *indecomposable*, *nonminimal* QH lattices corresponding to $\sigma_H = 8/5$. An example of such a lattice is $(\Gamma_e, \mathbf{Q}_e)$ with $\Gamma_e = A_7 A_1 40_1[3, 1, 1/8]$, $\underset{\rightarrow}{Q}_e = 8\xi$, and symbol ${}_9(8/5)_1^2$; see Table II. It is tempting to interpret the $\vec{SU}(2)$ corresponding to the A_1 sublattice of Γ_e as describing electron spin, while the $SU(8)$ corresponding to A_7 describes asymptotic (approximate) internal symmetries. Thus an elementary, incompressible QH fluid described by $(\Gamma_e, \mathbf{Q}_e)$ would have a "spin-singlet" ground state. Because of the large internal symmetry it is perhaps unlikely that such a fluid can be realized in a monolayer.

In contrast to the plateau at $\sigma_H = 8/5$ which exhibits a transition, as the external magnetic field $\mathbf{B}^{(0)}$ is tilted (keeping the filling factor constant), no such transition has been observed for the plateau at $\sigma_H = 5/3$. It is natural to ask whether our analysis permits us to understand this.

As emphasized in Section 7.2, no indecomposable minimal QH lattice exists for $\sigma_H = 5/3$ in dimension below 17, and, moreover, no minimal, maximally symmetric QH lattice can be constructed in any dimension! Apparently, the most natural explanations for $\sigma_H = 5/3$ will thus be found among *composite fluids*. (Nonminimal indecomposable QH lattices with $\sigma_H = 5/3$ have high dimensions and a fairly wierd structure.) Indeed, a widely accepted picture of the $\sigma_H = 5/3$ state is to view it as an electron–hole conjugate form of the state at $\sigma_H = 1/3$, i.e., to interpret it as a composite fluid $5/3 = 2 - 1/3$. In QH lattice language, this corresponds to a decomposition:

$$(\Gamma, \mathbf{Q}) = (\Gamma_e, \mathbf{Q}_e) \oplus (\Gamma_h, \mathbf{Q}_h) \tag{7.5}$$

with $\sigma_e = (\mathbf{Q}_e, \mathbf{Q}_e) = 2$ and $\sigma_h = (\mathbf{Q}_h, \mathbf{Q}_h) = 1/3$.

The electron part $(\Gamma_e, \mathbf{Q}_e)$ has integral Hall conductance $\sigma_e = 2$; a very natural choice is therefore a composite of two elementary fluids with $\sigma_H = 1$:

$$(\Gamma_e, \mathbf{Q}_e); \qquad \Gamma_e = 1_1 \oplus 1_1, \qquad \underset{\rightarrow}{Q}_e = (1, 1); \qquad \sigma_e = 1 + 1$$

Our classification results strongly restrict the QH lattices that can be associated to the hole fluid with $\sigma_h = 1/3$. There is a unique minimal QH lattice, and all nonminimal solutions have necessarily a nontrivial value for the charge parameter λ.

This is the content of the following simple lemma.

Lemma 11. Let $(\Gamma, \mathbf{Q})$ be an indecomposable QH lattice with $(\mathbf{Q}, \mathbf{Q}) = \sigma_H = 1/d_H$, $d_H = 1, 3, 5, \ldots$. Then:

1. There is a unique minimal QH lattice $(\Gamma = (d_H), \underset{\rightarrow}{Q} = (1))$.

2. If dim $\Gamma \geqslant 2$, then the charge parameter λ is at least 2 (hence $l \geqslant 2$).

Proof. Recall that $g = \text{g.c.d.}(Q\tilde{K})$ and $\gamma = \tilde{K}_{11}$, in a normal basis; see (5.39) and (5.28). Thus g divides γ. If $g = \gamma$, then there is a normal basis in which $\tilde{K}$ has the form

$$\tilde{K} = \begin{pmatrix} \gamma & \gamma & 0 & \cdots & 0 \\ \gamma & & & & \\ 0 & & * & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$$

Thus, by a further basis transformation, $\tilde{K}$ can be brought to the form

$$\tilde{K} = \begin{pmatrix} \gamma & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}$$

and hence Γ is decomposable, which contradicts our hypothesis. By (5.40) it then follows that $\lambda \geqslant 2$. QED

As a consequence, if dim $\Gamma_h > 1$, then we predict that the smallest fractional electric charge is $e^* = e/3\lambda$, with $\lambda \geqslant 2$. But experiments reported in refs. 6 and 7 suggest that $e^* = e/3$ for the $\sigma_H = 1/3$ state. This favors the idea that Γ_h is the one-dimensional lattice (3), and $\underline{Q}_h = (1)$. Obviously, this lattice does not contain an A_1 sublattice, and hence it describes a state of fully spin-polarized holes.

The lattice $(\Gamma_e, \mathbf{Q}_e)$ appearing in the decomposition (7.5) describes a composite QH fluid with two bands of oppositely spin-polarized electrons.

These results nicely fit experimental data[8-11] indicating that when the external magnetic field $\mathbf{B}^{(0)}$ is tilted the incompressible state at $\sigma_H = 5/3$ remains stable, no matter how large $\mathbf{B}_{\parallel}^{(0)}$ is. Our results suggest that the $\sigma_H = 5/3$ state will exhibit edge currents of both chiralities. This might be tested in edge magnetoplasmon experiments.[51]

Next, we wish to analyze QH lattices with $\sigma_H = 1$. According to Lemma 11, the *only* elementary QH fluid *without* excitations of fractional electric charge corresponds to the QH lattice $(\Gamma_e = (1), \underline{Q}_e = (1))$. It is tempting to ask whether there in a natural QH lattice with charge parameter $\lambda \geqslant 2$ corresponding to $\sigma_H = 1$. The corresponding QH fluid would then exhibit fractional electric charges which might arise as a consequence of electron–electron interactions. Furthermore, it might happen that in such a fluid there is no preferred direction for spin polarization, i.e., one would observe spin waves. The corresponding QH lattice would then have to contain an A_1 root lattice.

Our general analysis shows that QH lattices corresponding to $\sigma_H = 1$ with these properties exist. We encounteres two examples in Table II:

$$\Gamma = D_8\, 4_1\, [1, \tfrac{1}{2}], \qquad \underset{\to}{Q} = 2\underset{\to}{\xi} \tag{7.6}$$

with symbol ${}_9^+(1)_2^2$, and

$$\Gamma = E_7\, A_1\, 4_1\, [1, 1, \tfrac{1}{2}], \qquad \underset{\to}{Q} = 2\underset{\to}{\xi} \tag{7.7}$$

with symbol ${}_9^-(1)_2^2$.

These examples have discriminant $\Delta = 4$ and charge parameter $\lambda = 2$. They also offer natural possibilities to imbed $SU(2)_{\text{spin}}$ in their symmetry groups. However, their symmetry groups and the dimension of Γ are frighteningly large.

There are, however, fairly natural two- and three-dimensional QH lattices with $\sigma_H = 1$. If the invariant $L_{\max}$ [whose physical significance is well understood; see Eq. (5.17)] is constrained to take the value 3, then we find the following lattices:

The unique two-dimensional QH lattice is

$$\Gamma = (3\ {}^1\ 3)_2, \qquad \underset{\to}{Q} = \underset{\to}{\xi}_1 - \underset{\to}{\xi}_2 \tag{7.8}$$

corresponding to the symbol ${}_2(1)_2^4$.

Note that the lattice $\Gamma = (3\ {}^1\ 3)_2$ is also encountered in the analysis of $\sigma_H = 1/2$, but in combination with a $\underset{\to}{Q}$-vector $\underset{\to}{Q} = \underset{\to}{\xi}_1 + \underset{\to}{\xi}_2$.

Clearly, the QH lattice displayed in (7.8) describes a QH fluid of spin-polarized electrons, since Γ does not contain an A_1 sublattice. However, in three dimensions, we find a QH lattice containing an A_1 sublattice:

$$\begin{aligned} &\Gamma = A_1\, (4_1)\, (6_1)\, [1, \tfrac{1}{2}, \tfrac{1}{2}] \\ &\underset{\to}{Q} = 2\underset{\to}{\xi}_1 \end{aligned} \tag{7.9}$$

with symbol ${}_3(1)_2^6$.

A second three-dimensional QH lattice $(\Gamma, \mathbf{Q})$ with symbol ${}_3(1)_2^8$ is described by its $\underset{\sim}{K}$-matrix

$$K = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}, \qquad \underset{\to}{Q} = (1, 1, 1) \tag{7.10}$$

in a symmetric basis. These two QH lattices can be shown to exhaust the list of QH lattices with $L_{\max} = 3$, $\sigma_H = 1$ in three dimensions.[42]

We notice that only the QH lattice displayed in (7.9) can account for $SU(2)_{\text{spin}}$ in an elementary QH fluid of unpolarized electrons, has $L_{\max} = 3$, and has small dimension. It predicts the existence of excitations with half-integer electric charge, just as in the case of the lattices described in (7.6) and (7.7). However, it has a rather large discriminant $\Delta = 12$, while the lattices in (7.6) and (7.7) have a $\Delta = 4$. This may cast some doubt on the stability of a QH fluid described by (7.9).

In conclusion, one might argue that the QH lattices given in (7.7) and in (7.9) are rather natural candidates for the description of an unpolarized, elementary QH fluid with $\sigma_H = 1$ and with an $SU(2)_{\text{spin}}$ symmetry.

Finally we note that the lattices in (7.7) and (7.9) could also describe spin-polarized, elementary QH fluids in *double-layer systems*, with the $SU(2)$ symmetry acting on the layer index.

Next, we study QH lattices describing elementary QH fluids with $\sigma_H = 1/2$, a plateau that is observed in double-layer systems. Besides the "boring" solution,

$$\Gamma = (3\ {}^1\ 3)_2, \qquad \underset{\rightarrow}{Q} = \underset{\rightarrow}{\xi}_1 + \underset{\rightarrow}{\xi}_2 \tag{7.11}$$

there are "more imaginative" solutions that are listed as the second, third, and fourth lattices in Table III. Among these three, the most natural one is

$$\Gamma = A_1\ A_1\ 8_1\ [1, 1, \tfrac{1}{2}], \qquad \underset{\rightarrow}{Q} = 2\underset{\rightarrow}{\xi} \tag{7.12}$$

It describes a QH fluid composed of two species of spin-unpolarized electrons (corresponding to the two layers) exhibiting an $SU(2)_{\text{spin}} \times SU(2)_{\text{layer}}$ symmetry. Electrons transform according to the spin-1/2 representations of both $SU(2)$ symmetries. There are excitations of fractional electric charge $e/4$, spin 1/2, and "isospin" 0, or spin 0 and isospin 1/2. Three excitations of one kind and one of the other kind reconstitute an electron.

A look at Table III suggests that, in certain double-layer systems, one should be able to realize an elementary QH fluid with $\sigma_H = 3/2$ (second but last lattice in Table III), although such fluids would exhibit large internal symmetries.

Should we expect to find QH fluids with $\sigma_H = 1/4$ or 1/6? What one can show is that, for a two- or three-dimensional QH lattice with $\sigma_H = 1/4$, $L_{\max} \geqslant 5$, and this is also true for maximally symmetric QH lattices. Moreover, since g, $\lambda \geqslant 2$, the discriminant Δ of all QH lattices with $\sigma_H = 1/4$ is always $\geqslant 16$, while QH lattices expected to correspond to experimentally observed plateau values have discriminants $\Delta \leqslant 15$. For

$\sigma_H = 1/6$, the corresponding bounds are $L_{\max} \geqslant 7$ and $\Delta \geqslant 24$. These large values of $L_{\max}$ and Δ hint at an explanation of why plateaux at $\sigma_H = 1/4$, 1/6 are not observed: Tentative QH fluids with $\sigma_H = 1/4$, 1/6 would presumably have a very small gap and/or be threatened by the Wigner crystal instability.

In a separate paper[42] we analyze fairly systematically QH lattices corresponding to many observed plateaux of σ_H and make predictions concerning those plateaux that exhibit transitions between different incompressible QH fluids when external parameters such as the density or the in-plane magnetic field are varied. Considering an example such as $\sigma_H = 2/3$ somewhat systematically shows that this is a rather complicated task, because when there are many QH lattices corresponding to the same value of σ_H one must appeal to physical principles to find out which lattices have a chance to describe experimentally realizable QH fluids.

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