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A Limit Law for the Ground State of Hill's Equation

H. P. McKean[1]

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It is proved that the ground state $\Lambda(L)$ of $(-1)\times$ the Schrödinger operator with white noise potential, on an interval of length L, subject to Neumann, periodic, or Dirichlet conditions, satisfies the law

$$\lim_{L\uparrow\infty} P[(L/\pi)\,\Lambda^{1/2}\exp(-\tfrac{8}{3}\Lambda^{3/2})>x]=\begin{cases}1 & \text{for } x<0\\ e^{-x} & \text{for } x\geqslant 0\end{cases}$$

KEY WORDS: Schrödinger's operator; spectrum; diffusion.

1. INTRODUCTION

Let Q be Hill's operator[2] $-D^2+\mathrm{q}$, in which q is the standard white noise on a circle $0\leqslant x<L$ of large perimeter L. It is to be proved that, if $\Lambda(L)$ is the ground state of $-Q$ under (a) Neumann, (b) periodic, or (c) Dirichlet conditions, then

$$\lim_{L\uparrow\infty} P[(L/\pi)\,\Lambda^{1/2}(L)\exp(-\tfrac{8}{3}\Lambda^{3/2}(L)>x]=\begin{cases}1 & \text{for } x<0\\ e^{-x} & \text{for } x\geqslant 0\end{cases}$$

which is to say that $\Lambda(L)$ is well approximated by

$$(0.519+)(\lg L)^{2/3}+[(0.116+)\lg_2 L-(0.510-)-(0.346+)\lg \mathfrak{x}](\lg L)^{-1/3}$$

with an exponential variable $\mathfrak{x}$. $\Lambda(\text{Neumann})\leqslant\Lambda(\text{periodic})<\Lambda(\text{Dirichlet})$, so it suffices to deal with the first and the third. The proof employs a diffusion introduced and exploited by Halperin[5]; compare Section 4 below.

[1] Courant Institute of Mathematical Science, New York, New York.

[2] D signifies differentiation by x.

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2. DIRICHLET CASE

Fix $\lambda \in R$ and let $y_2(x, \lambda)$ be the sine-like solution of $Qf = \lambda f$ with $y_2(0) = 0$ and $y_2'(0) = 1$. The motion $\mathfrak{p}(x) = y_2'(x, \lambda)/y_2(x, \lambda)$ satisfies $\mathfrak{p}' = \mathfrak{q} - (\lambda + \mathfrak{p}^2)$, i.e., it is the diffusion with infinitesimal operator

$$G = (1/2)\, \partial^2/\partial p^2 - (\lambda + p^2)\, \partial/\partial p = \partial/\partial m(p)\, \partial/\partial s(p)$$

in which one sees the so-called scale $ds(p) = \exp[2(p^3/3 + \lambda p)]\, dp$ and speed measure $dm(p) = 2\exp[-2(p^3/3 + \lambda p)]\, dp$. The process starts at $\mathfrak{p}(0) = +\infty$, this being an entrance barrier[3]; it hits the exit barrier[4] $-\infty$ at the first root $\mathfrak{x}_1$ of $y_2(x, \lambda) = 0$, then reappears at $+\infty$, and so on; see Itô and McKean[(6)] for such matters.

We have $P[-\Lambda(L) > \lambda] = P[\mathfrak{x}_1 > L]$, the latter event being the same as: *no root of* $y_2(x, \lambda) = 0$ *for* $0 \leqslant x \leqslant L$. This probability is to be estimated for $L \uparrow \infty$ and $\lambda \downarrow -\infty$. To see what is happening, write $k\mathfrak{p}(kx)$ in place of $\mathfrak{p}(x)$ with $k = \sqrt{-\lambda}$. Then G is changed to $(-\lambda)^{-3/2}\, (1/2)\, \partial^2/\partial p^2 - (p^2 - 1)\, \partial/\partial p$, from which it appears that the diffusion acts like a chain of three states, $-\infty$, 0, $+\infty$, the motion being (almost) deterministic except for a pause (tunneling time) at 0 *alias* $-1 \leqslant p \leqslant +1$, whose mean must be appraised. This could be done by the methods of Friedlin and Vencel,[(1)] as reported in ref. 2, p. 326,[5] but I prefer another proof, without such scaling.

Now $E(\exp(-\alpha \mathfrak{x}_1))$ is the reciprocal of $h_+(-\infty, \alpha)$, h_+ being the decreasing positive solution of $Gh = \alpha h$ with $h(\infty) = 1$ (see, e.g., ref. 6):

$$h_+(p, \alpha) = 1 + \alpha \int_p^\infty ds \int_{p_1}^\infty dm + \alpha^2 \int_p^\infty ds \int_{p_1}^\infty dm \int_{p_2}^\infty ds \int_{p_3}^\infty dm + \text{etc.}$$

Let $c(-\lambda)$ be the mean passage time

$$E(\mathfrak{x}_1) = \lim_{\alpha \downarrow 0} \frac{1}{\alpha} E(1 - \exp(-\alpha \mathfrak{x}_1)) = \int_{-\infty}^\infty ds \int_p^\infty dm$$

and note the appraisal[6]

$$c(-\lambda) = (2\pi)^{1/2} \int_0^\infty e^{-(q^3/6 + 2\lambda q)} \frac{dq}{\sqrt{q}} = \frac{\pi}{\sqrt{-\lambda}} \exp \frac{8}{3} (-\lambda)^{3/2} \times [1 + o(1)]$$

It is to be proved that $h_+(-\infty, \alpha/c)$ tends to $1 + \alpha$ as $\lambda \downarrow -\infty$; the limit law for $\Lambda(L)$ is an easy consequence of that: indeed, the limit law

[3] $\int_0^\infty dm \int_{-\infty}^p ds < \infty$, which means that paths come in from $+\infty$.
[4] $\int_{-\infty}^0 ds \int^\infty dm < \infty$, which means that paths actually arrive at $-\infty$ at a finite time.
[5] I owe these references to Varadhan (private communication).
[6] The integral concentrates at the saddle point $q = 2\sqrt{-\lambda}$ which makes the verification easy.

states that $L/c(-\Lambda)$ tends to an exponential variable, i.e., that e^{-x} is the limit, as $L\uparrow\infty$, of

$$P[L/c(-\Lambda)>x]=P[(L/x)<-\Lambda]=P[x<\mathfrak{x}_1/c]$$
$$\text{taken for}\quad \lambda=-c^{-1}(L/x)\downarrow -\infty$$

Proof. $1/h_+(-\infty,\alpha/c)$ is the Laplace transform of a probability measure on $[0,\infty)$, so it is enough to make the proof for $0<\alpha\leqslant 1/2$, say. But now

$$0\leqslant h_+\left(-\infty,\frac{\alpha}{c}\right)-(1+\alpha)$$
$$\leqslant\frac{1}{4c^2}\int_{-\infty}^{\infty}ds\int_{p_1}^{\infty}dm\int_{p_2}^{\infty}ds\int_{p_3}^{\infty}dm$$
$$+\frac{1}{8c^3}\int_{-\infty}^{\infty}ds\int_{p_1}^{\infty}dm\int_{p_2}^{\infty}ds\int_{p_3}^{\infty}dm\int_{p_4}^{\infty}ds\int_{p_5}^{\infty}dm\quad\text{etc.}$$
$$\leqslant\frac{1}{2c^2}\int_{-\infty}^{\infty}ds\int_{p_1}^{\infty}dm\int_{p_2}^{\infty}ds\int_{p_3}^{\infty}dm$$
$$=\frac{1}{2c^2}\int_0^{\infty}e^{-(q_1^3/6+2\lambda q_1)}\,dq_1\int_0^{\infty}e^{-(q_2^3/6+2\lambda q_2)}\,dq_2$$
$$\times\int_{-\infty}^{\infty}e^{-2q_1p_1^2}\,dp_1\int_{q_1/2+p_1+q_2/2}e^{-2q_2p_2^2}\,dp_2$$

to which the main contribution comes from a saddle point at $q_1=q_2=2\sqrt{-\lambda}$, as in the appraisal of c noted before, and the whole is exponentially small for $\lambda\downarrow-\infty$. It would be unprofitable to report the simple details.

3. NEUMANN CASE

The proof is similar. The diffusion is now $\mathfrak{p}(x)=y_1'(x,\lambda)/y_1(x,\lambda)$, y_1 being the cosine-like solution of $Qf=\lambda f$ with $y_1(0)=1$ and $y_1'(0)=0$; it is nothing but the old process starting not from $\mathfrak{p}(0)=+\infty$, but from $\mathfrak{p}(0)=0$, and one has $P[-\Lambda(L)>\lambda]=P[\mathfrak{x}_1>L\ \&\ \mathfrak{p}(L)>0]$, the latter event being the same as: *no root of* $y_1(x,\lambda)$ *for* $0\leqslant x\leqslant L$ *and* $y_1'(L,\lambda)>0$. Now $-\Lambda(L)$ is a decreasing function of L, so the same is true of $P[\mathfrak{x}_1>L$ etc.$]$, and it suffices, for the limit law, to check that

$$\int_0^{\infty}P[\mathfrak{x}_1>xc\ \&\ \mathfrak{p}(xc)>0]\,e^{-\alpha x}\,dx\qquad\text{tends to}\quad\frac{1}{1+\alpha}\quad\text{as}\quad\lambda\downarrow-\infty$$

Proof. The left-hand side may be evaluated as

$$\int_0^\infty P[\mathfrak{x}_1 > x \,\&\, \mathfrak{p}(x) > 0]\, e^{-(\alpha/c)x} \frac{dx}{c} = \frac{h_-(0)}{c} \int_0^\infty h_+ \, dm$$
$$= -\frac{h_-(0)\, h'_+(0)}{c}$$

in which $h_+ = h_+(p, \alpha/c)$ is as before and h_- is the allied increasing solution of $Gh = (\alpha/c)h$ with $h(-\infty) = 0$ and $h'_- h_+ - h_- h'_+ = 1$.[7] Now

$$h_-\left(p, \frac{\alpha}{c}\right) = s(p) + \frac{\alpha}{c} \int_{-\infty}^{p} ds \int_{-\infty}^{p_1} s\, dm$$
$$+ \frac{\alpha^2}{c^2} \int_{-\infty}^{p} ds \int_{-\infty}^{p_1} dm \int_{-\infty}^{p_2} ds \int_{-\infty}^{p_3} s\, dm \quad \text{etc.}$$

if the normalization is ignored, and, from the symmetry $p \to -p$ which exchanges s and $m/2$, one sees that $h'_-(0) = h_+(0)$ and $h'_+(0) = -2(\alpha/c)\, h_-(0)$, whence the quantity to be studied is the reciprocal of $\alpha + (c/2)\, h_+^2(0)/h_-^2(0)$ with the *un*normalized function h_-. The proof is finished by checking three small items confirming that $(c/2)\, h_+^2(0)/h_-^2(0) = 1 + o(1)$.

Item 1: Note that, for $\lambda \downarrow -\infty$, $c = 2\int_{-\infty}^{\infty} ds \int_p^\infty dm$ may be identified with $2s^2(0)$; this is done by saddle point, as before.

Item 2:

$$h_+(0) = 1 + (\alpha/c) \int_0^\infty ds \int_p^\infty h_+ \, dm$$
$$= 1 + c^{-1} \times O\left[h_+(0) \int_0^\infty ds \int_p^\infty dm \right]$$

and

$$c^{-1} \int_0^\infty ds \int_p^\infty dm = \int_0^\infty dq \int_0^\infty e^{-2pq(p+q)}\, dp/2s^2(0)$$
$$= o(1)$$

This shows that $h_+(0) = 1 + o(1)$.

[7] The prime signifies differentiation with regard to the scale $s(p) = \int_{-\infty}^{p} \exp[2(q^3/3 + \lambda q)]\, dq$.

Item 3:

$$h_-(0) = -(c/2\alpha)\, h'_+(0)$$
$$= (1/2)\int_0^\infty h_+\, dm \sim (1/2)\int_0^\infty dm = s(0)$$

by symmetry, so $(2/c)\, h_-^2(0)/s^2(0) = 1 + o(1)$.

4. INTEGRATED DENSITY OF STATES

The present method of diffusion affords a pretty complete picture of the statistics of the spectrum of Q with simple proofs. The integrated density of states $N(\lambda) = \lim_{L\uparrow\infty} L^{-1} \times$ (the number of eigenvalues $\leqslant \lambda$) provides a nice additional example: roughly, it is half the winding number about the origin of the path $[y_2(x, \lambda), y_2'(x, \lambda)]$: $0 \leqslant x < L$, *alias* the number of passages of $\mathfrak{p}(x)$: $0 \leqslant x < L$ from $+\infty$ to $-\infty$ and, if this number is n, then L approximates the sum $\mathfrak{x}_n$ of the passage times, so that $N(L) = \lim_{n\uparrow\infty} n/\mathfrak{x}_n = 1/E(\mathfrak{x}_1)$, by the law of large numbers—in short,

$$\frac{1}{N(\lambda)} = (2\pi)^{1/2} \int_0^\infty e^{-(q^3/6 + \lambda q)} \frac{dq}{\sqrt{q}}$$

The fact is due to Frisch and Lloyd,[3] and the pretty proof to Halperin[5]; compare Lifshits *et al.* (ref. 7, pp. 172–174) and also Fukushima and Nakao.[4]

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