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On the Contact Dimensions of Graphs

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Abstract. Every simple graph $G=(V,E)$ can be represented by a family of equal nonoverlapping spheres $\{S_v: v\in V\}$ in a Euclidean space R^n in such a way that $uv\in E$ if and only if S_u and S_v touch each other. The smallest dimension n needed to represent G in such a way is called the contact dimension of G and it is denoted by $\mathrm{cd}(G)$. We prove that (1) $\mathrm{cd}(T)<(7.3)\log|T|$ for every tree T, and (2)

$$m-1+\frac{n}{2}\left(1-\frac{n}{2\pi m}\left(\sqrt{\frac{n+4\pi m}{n}}-1\right)\right)<\mathrm{cd}(K_m+E_n)\le m-1+\left\lceil\frac{n}{2}\right\rceil,$$

where K_m+E_n is the join of the complete graph of order m and the empty graph of order n. For the complete bipartite graph $K_{n,n}$ this implies $(1.286)n-1<\mathrm{cd}(K_{n,n})<(1.5)n$.

1. Statement of the Results

For each simple graph $G=(V,E)$, there is an embedding f of V into a Euclidean space R^n such that

$$\|f(u)-f(v)\|=1 \text{ if } uv\in E \quad\text{and}\quad \|f(u)-f(v)\|>1 \text{ if } uv\notin E,$$

see [4]. The smallest dimension n for which such an embedding exists is called the *contact dimension* of G, and is denoted by $\mathrm{cd}(G)$. Here we present some bounds on the contact dimensions of trees and of the join of a complete graph and an empty graph. For a graph G, let $|G|$ denote the number of vertices of G.

Theorem 1. *For every tree T,* $\mathrm{cd}(T)<(7.3)\log|T|$.

Note that this estimate is sharp in the sense that there are trees T on n vertices with $\mathrm{cd}(T)>c\log n$, for some fixed $c>0$. In fact if G is a graph of n vertices and diameter d, then $\mathrm{cd}(G)>(\log n)/\log(d+1)$ [4, Theorem 2].

Let $K_m + E_n$ be the join of the complete graph K_m of order m and the empty graph E_n of order n, that is, $K_m + E_n$ is the complement of the disjoint union $E_m \cup K_n$. Define $d(m, n) = \mathrm{cd}(K_m + E_n)$. In [4] it was proved that $d(m, n) \leq m - 1 + \lceil n/2 \rceil$ and that for any n, there is an $m(n)$ such that if $m > m(n)$ then $d(m, n) = m - 1 + \lceil n/2 \rceil$. These results are improved in the following way.

Theorem 2.

$$m - 1 + \frac{n}{2}\left(1 - \frac{n}{2\pi m}\left(\sqrt{\frac{n + 4\pi m}{n}} - 1\right)\right) < d(m, n) \leq m - 1 + \left\lceil \frac{n}{2} \right\rceil.$$

Let us recall from [4] that, for $n \geq m$, $\mathrm{cd}(K_m + E_n) = \mathrm{cd}(K_{m,n})$ holds, where $K_{m,n}$ is the complete bipartite graph. This implies:

Corollary 1. *Suppose that $n \geq m$. Then*

$$m - 1 + \frac{n}{2}\left(1 - \frac{n}{2\pi m}\left(\sqrt{\frac{n + 4\pi m}{n}} - 1\right)\right) < \mathrm{cd}(K_{m,n}) \leq m - 1 + \left\lceil \frac{n}{2} \right\rceil.$$

Letting $m = n$ yields:

Corollary 2. $(1.286)n - 1 < d(n, n) = \mathrm{cd}(K_{n,n}) < (1.5)n.$

Since

$$\frac{n^2}{4\pi m}\left(\sqrt{\frac{n + 4\pi m}{n}} - 1\right) < \frac{1}{2}$$

is equivalent to $(n^3 - n^2)/\pi < m$, we have:

Corollary 3. *If $m > (n^3 - n^2)/\pi$, then $d(m, n) = m - 1 + \lceil n/2 \rceil$.*

Erdös and Füredi [2] used the probabilistic method to prove the existence of a set $X \subset R^n$ such that every angle spanned by three points of X is acute and $|X|$ grows exponentially in n.

Two new proofs for this result are provided at the end of the paper, one semiconstructive and one constructive.

2. Proof of Theorem 1

Lemma 1. *Let O_k denote the surface area of the unit sphere in R^k, i.e., $O_k = 2\pi^{k/2}/\Gamma(k/2)$. Then*

$$((k-2)/(2\pi))^{1/2} < O_{k-1}/O_k < ((k-1)/(2\pi))^{1/2}.$$

Proof. Since $\log \Gamma(x)$ $(x>0)$ is a convex function, we have

$$\log \Gamma((k+1)/2)+\log \Gamma((k-1)/2)>2 \log \Gamma(k/2)$$

and hence $\Gamma((k+1)/2)/\Gamma(k/2)>\Gamma(k/2)/\Gamma((k-1)/2)$. Since

$$\frac{\Gamma((k+1)/2)}{\Gamma(k/2)} \frac{\Gamma(k/2)}{\Gamma((k-1)/2)}=\frac{k-1}{2},$$

we have

$$\frac{\Gamma(k/2)}{\Gamma((k-1)/2)}<\left(\frac{k-1}{2}\right)^{1/2}<\frac{\Gamma((k+1)/2}{\Gamma(k/2)}.$$

Since $O_{k-1}/O_k=\Gamma(k/2)/(\pi^{1/2}\Gamma((k-1)/2))$, we have the lemma. □

Lemma 2. *Let S be a unit sphere in R^{k+2} and $C(\alpha)$ be a spherical cap of angular radius α. Let $|S|$ and $|C(\alpha)|$ denote the surface areas of S and $C(\alpha)$. For $0<\alpha<\pi/2$ one has*

$$\frac{|S|}{|C(\alpha)|}\geq 2\left(\frac{k}{k+1}\right)^{1/2} \exp\left(\frac{k(\pi/2-\alpha)^2}{2}\right).$$

Proof. Since the function $f(t)=\exp(t^2/2)\cos t$ decreases on $[0, \pi/2]$, and since

$$|C(\alpha)|=O_{k+1}\int_{\pi/2-\alpha}^{\pi/2} \cos^k t\, dt,$$

we have

$$\begin{aligned}|C(\alpha)|&\leq O_{k+1}\int_{\pi/2-\alpha}^{\pi/2} \exp\left(\frac{-kt^2}{2}\right) dt\\ &< O_{k+1}\exp\left(\frac{-k(\pi/2-\alpha)^2}{2}\right)\int_0^{\infty} \exp\left(\frac{-kt^2}{2}\right) dt\\ &\leq O_{k+1}\left(\frac{\pi}{2k}\right)^{1/2} \exp\left(\frac{-k(\pi/2-\alpha)^2}{2}\right).\end{aligned}$$

Thus $|S|/|C(\alpha)|>(O_{k+2}/O_{k+1})(2k/\pi)^{1/2}\exp(k(\pi/2-\alpha)^2/2)$. Since $O_{k+2}/O_{k+1}>(2\pi/(k+1))^{1/2}$ by the above lemma, we have

$$\frac{|S|}{|C(\alpha)|}>2\left(\frac{k}{k+1}\right)^{1/2} \exp\left(\frac{k(\pi/2-\alpha)^2}{2}\right).$$ □

Corollary 4. *If* $k+2 \geq (7.3)\log n$, $n \geq 3$, *then* $|S|/|C(\pi/3)| > n$.

Proof. It is easy to check that if $k+2 \geq (7.3)\log n > (72/\pi^2)\log n$ then $|S|/|C(\pi/3)| > n$. □

Proof of Theorem 1. Let d be the smallest integer such that the ratio of the surface area of the unit sphere in R^d and the area of the spherical cap of angular radius 60° is greater than $n-2$. Then $d \leq (7.3)\log n$ by Lemma 2. We show that T is embeddable in R^d in such a way that $\|u-v\| = 1$ if uv is an edge of T, and >1 otherwise. To prove this we use induction on $|T|$. It is trivial if $|T| = 1$. Suppose it is true for $n-1$, and let $|T| = n$. Let x be a vertex of T of degree 1, $T_0 = T - \{x\}$, and let y be the unique neighbor of x in T. By induction, we can embed T_0 in R^d. Let S be the unit sphere in R^d around y and $C_1, C_2, \ldots, C_{n-2}$ the intersection of the other $n-2$ unit spheres (drawn around the remaining $n-2$ vertices of T_0) with S. Clearly, C_i is a spherical cap of angular radius $\leq 60°$. By the choice of d, surf. area$(S) > \sum_i$ surf. area(C_i), hence we can place x on the surface of S so that the distances from x to all other ponts in T_0 are greater than 1. Thus T is also embeddable in R^d.

3. Proof of Theorem 2

A point set in Euclidean space is said to be *dispersed* if any two points of the set are at distance more than 1. The following theorem was proved in [4, Theorem 6].

Theorem [4]. $d(m,n) \leq k+m-1$ *if and only if a sphere of radius* $s(m) := ((m+1)/(2m))^{1/2}$ *in* R^k *contains n dispersed points.*

Proof of Theorem 2. Since the upper bound was proved in [4], we only show the lower bound. Suppose $k+m-1 = d(m,n)$. Then, by the above theorem, a sphere S of radius $s(m)$ in R^k contains n dispersed points. Let $\alpha = \arcsin((\frac{1}{2})^{1/2}/s(m))$. Since S contains n dispersed points, there is a spherical cap $C = C(\alpha)$ of angular radius α such that C contains $\lceil n|C|/|S|\rceil$ dispersed points, where $|\ |$ denotes the surface area. This can be seen in the following way. Let x_i, $i = 1, \ldots, n$, be n dispersed point on S. Consider a "random spherical cap" $\mathbf{C}$ of angular radius α, and define random variables $\mathbf{v}_i$, $i = 1, \ldots, n$, by $\mathbf{v}_i = 1$ if $x_i \in \mathbf{C}$, $\mathbf{v}_i = 0$ otherwise. Then the expected value of the sum $\mathbf{v}_1 + \cdots + \mathbf{v}_n$ equals $n|C|/|S|$. Hence there must be a spherical cap $C = C(\alpha)$ which contains at least $\lceil n|C|/|S|\rceil$ dispersed points.

Next we show that if C contains k dispersed points then the boundary bC of C also contains k dispersed points. Let $z \in C$ be the "center" of C, and let y be any point of the boundary bC of C. Then, for $m \geq 2$,

$$\|y-z\|^2 = 2s(m)^2(1-\cos\alpha) = ((m+1)/m)(1-(1-\sin^2\alpha)^{1/2})$$
$$= (1+1/m)(1-(1/(m+1))^{1/2}) < 1.$$

Let x_i, $i=1,\dots,k$, be k dispersed points on C. Then $x_i \neq z$ for all i. For each i, let y_i be the point of bC such that the geodesic path on S connecting y_i and z passes through x_i. (The point y_i is the point where the great circle passing through z and x_i intersects with bC.)

Claim. The points y_i, $i=1,\dots,k$, are dispersed.

Proof. Let x_{ji} be the orthogonal projection of x_j on the plane determined by x_i, z, and the center o of sphere S. Then one of the angles $\measuredangle y_i o x_{ji}$, $\measuredangle z o x_{ji}$ is not less than the angle $\measuredangle x_i o x_{ji}$. Hence $\max\{\|y_i - x_{ji}\|^2, \|z - x_{ji}\|^2\} \geq \|x_i - x_{ji}\|^2$, and hence $\max\{\|y_i - x_j\|^2, \|z - x_j\|^2\} \geq \|x_i - x_j\|^2$. However, since $\|z - x_j\| < 1$, we have $\|y_i - x_j\| > 1$ for $i \neq j$. Similarly, we can conclude $\|y_i - y_j\| > 1$ for $i \neq j$. Thus y_i $i=1,\dots,k$, are dispersed.

Now, by Rankin's theorem [5], a sphere of radius $\leq (\frac{1}{2})^{1/2}$ in R^{k-1} contains at most k dispersed points. Since the radius of bC is $(\frac{1}{2})^{1/2}$, bC contains at most k dispersed points, and so does the cap $C = C(\alpha)$ by the above argument. Hence $n|C|/|S| \leq k$.

Let us evaluate $|C|/|S|$. Since

$$|C| = O_{k-1} s(m)^{k-1} \int_0^{\alpha} (\sin\theta)^{k-2}\, d\theta$$
$$= O_{k-1} s(m)^{k-1} \left(\int_0^{\pi/2} (\sin\theta)^{k-2}\, d\theta - \int_{\alpha}^{\pi/2} (\sin\theta)^{k-2}\, d\theta \right)$$

and

$$|S| = 2O_{k-1} s(m)^{k-1} \int_0^{\pi/2} (\sin\theta)^{k-2}\, d\theta,$$

we have

$$|C|/|S| = \tfrac{1}{2} - (O_{k-1}/O_k) \int_{\alpha}^{\pi/2} (\sin\theta)^{k-2}\, d\theta$$
$$> \tfrac{1}{2} - (O_{k-1}/O_k)(\pi/2 - \alpha)$$
$$> \tfrac{1}{2} - (O_{k-1}/O_k)(1/m)^{1/2}$$

(because $\pi/2 - \alpha < \tan(\pi/2 - \alpha) = (1/m)^{1/2}$). Hence, by Lemma 1, $|C|/|S| > \frac{1}{2} - ((k-1)/(2\pi m))^{1/2}$. So

$$k > n/2 - n((k-1)/(2\pi m))^{1/2} > n/2 - n(k/(2\pi m))^{1/2},$$

and from this we have

$$k > \frac{n}{2}\left(1 - \frac{n}{2\pi m}\left(\sqrt{\frac{n+4\pi m}{n}} - 1\right)\right).$$

Thus we have

$$d(m,n) > \frac{n}{2}\left(1 - \frac{n}{2\pi m}\left(\sqrt{\frac{n+4\pi m}{n}} - 1\right)\right) + m - 1. \qquad \square$$

4. Points Without Obtuse Angle

Erdös and Füredi [2] proved that for every ε there exists a $\delta = \delta(\varepsilon)$ and points $P_1, \ldots, P_m$ on the unit sphere in R^d so that $m > (1+\delta)^d$ and for all $1 \le h < i < j \le m$ all angles of the triangle $P_hP_iP_j$ lie between $\pi/3 - \varepsilon$ and $\pi/3 + \varepsilon$.

Their proof is probabilistic.

Here we derive a bound using Lemma 2.

Theorem 3. *For every* $0 < \beta < \pi/2$ *there exist points* $P_1, \ldots, P_m$ *on the unit sphere* S *in* R^{k+2} *so that* $m > (k/(k+1))^{1/2} \exp(k\beta^2/2)$ *and all distances* P_iP_j, $1 \le i \le j \le m$, *satisfy*

$$2(1 - \sin\beta) \le P_iP_j^2 \le 2(1 + \sin\beta). \tag{4.1}$$

Proof. Let $P_1, \ldots, P_m$ be a system of points on the unit sphere S in R^{k+2} satisfying (4.1) and such that the addition of any further point would violate (4.1). Let $D_i(\pi/2 - \beta)$ be the spherical double cap of angular radius $\pi/2 - \beta$ centered at P_i (a double cap is the union of two diametrically opposite caps). Then the union of $D_i(\pi/2 - \beta)$ for $i = 1, \ldots, m$ has to cover all points on the sphere. (In fact, if $Q \notin D_i(\pi/2 - \beta)$ then by elementary computation

$$2(1 - \sin\beta) < P_iQ^2 < 2(1 + \sin\beta)$$

holds.) However, $|D_i(\pi/2 - \beta)| = 2|C(\pi/2 - \beta)|$ and Lemma 2 yields

$$m \ge |S|/(2|C(\pi/2 - \beta)|) \ge (k/(k+1))^{1/2} \exp(k\beta^2/2),$$

as desired. $\square$

Remark. Note that the proof of Theorem 3 shows that every maximal (nonextendable) set satisfying (4.1) is exponentially large. Thus one can construct such a set by adding the points one by one.

Corollary 5. *There exist points* $P_1, \ldots, P_m \in S \subset R^{k+2}$ *so that all triangles* $P_hP_iP_j$, $1 \le h < i < j \le m$, *are acute and*

$$m \ge (1.0594)^k (k/(k+1))^{1/2}$$

holds.

Proof. It is sufficient to choose β so that $\sin\beta = \frac{1}{3}$ and apply Theorem 3. □

Corollary 6. *There are points* $P_1, \ldots, P_m \in S \subset R^{k+2}$ *so that all triangles* $P_h P_i P_j$, $1 \le h < i < j \le m$, *have all angles between* 59° *and* 61° *and* $m \ge (1.00011)^k (k/(k+1))^{1/2}$ *holds.*

Proof. This time one chooses β so that $\sin 30.5° = \frac{1}{2}((1+\sin\beta)/(1-\sin\beta))^{1/2}$ (for this value of β one checks that $\sin 29.5° < \frac{1}{2}((1-\sin\beta)/(1+\sin\beta))^{1/2}$ holds) and applies Theorem 3. □

In [2] somewhat better bounds are obtained both in Corollaries 5 and 6. The reason is that Erdös and Füredi choose the points from the 2^n vertices of the cube which span no obtuse angles. Thus in the case of Corollary 5 one has to get rid of right angles only. Let us recall the standard correspondence between the vertices of the n-cube and the subsets of an n-element set X. By Pythagoras' theorem (see [2]) three vertices corresponding to subsets $A, B, C \subset X$ span a right angle at C if and only if $A \cap B \subset C \subset A \cup B$ holds.

One can use a recent result of Friedman [3] to obtain an explicit construction for such a family of exponential size (and, consequently, of exponentially many vertices of the n-cube with all angles acute). In fact, a special case of Friedman's [3] Theorem 5.7 gives an explicit construction for more than 59^d sequences of 59 symbols and of length $10^{32}d$ so that for any three sequences there is at least one place where all three are different. (This result of Friedman was used by Alon [1] to obtain explicit construction for other related families of exponential size.)

Now let b be the smallest integer so that there exists a family $\{F_1, \ldots, F_{59}\}$ of subsets of $\{1, 2, \ldots, b\}$ without three sets F_h, F_i, F_j satisfying $F_h \cap F_i \subset F_j \subset F_h \cup F_i$ (to be more explicit one can also take $b = 59$ and $F_i = \{i\}$). Then in each sequence replace each appearance of the ith symbol by a $(0, 1)$-sequence of length b corresponding to F_i. With $n = 10^{32}bd$ this gives $59^d > 2^{5n/c}$ (where $c = 10^{32}b$), i.e., exponentially many vertices of the n-cube without right angles.

Finally, let us call the reader's attention to a forthcoming interesting paper of Reiterman *et al.* [6] on sphericity and another related dimension of graphs.

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