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Random Projections of Regular Simplices*

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Abstract. Precise asymptotic formulae are obtained for the expected number of k-faces of the orthogonal projection of a regular n-simplex in n-space onto a randomly chosen isotropic subspace of fixed dimension or codimension, as the dimension n tends to infinity.

1. Introduction

Let T^n be an n-dimensional regular simplex in euclidean space $\mathbb{R}^n$. We project T^n orthogonally into a randomly chosen d-dimensional linear subspace with isotropic distribution and denote by $Ef_k(\Pi_d T^n)$ the expected value of the number f_k of k-faces of the projection. Here $d \in \{1, \ldots, n-1\}$ and $k \in \{0, 1, \ldots, d-1\}$ are given integers. We are interested in an asymptotic formula for $Ef_k(\Pi_d T^n)$ when the dimension n tends to infinity. This question, for the special case $k = 0$, was posed some years ago by J. E. Goodman and R. Pollack and was reported to one of us by P. Mani-Levitska. An answer is given by the following theorem.

Theorem 1. *For any given integers $0 \le k < d \le n - 1$,*

$$Ef_k(\Pi_d T^n) \sim \frac{2^d}{\sqrt{d}} \binom{d}{k+1} \beta(T^k, T^{d-1})(\pi \log n)^{(d-1)/2}$$

as n tends to infinity.

Here $\beta(T^k, T^{d-1})$ denotes the internal angle of the regular $(d-1)$-simplex T^{d-1}

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at one of its k-dimensional faces (see Section 3 for more information on this constant).

The question posed by Goodman and Pollack is part of a general program, which may be sketched as follows. Every configuration of $n + 1$ numbered points in general position in $\mathbb{R}^d$ is affinely equivalent to the orthogonal projection of the set of numbered vertices of a fixed regular simplex $T^n \subset \mathbb{R}^n$ onto a unique d-dimensional linear subspace of $\mathbb{R}^n$ (for a closely related assertion, see p. 121 of [7]). This fact induces a one-to-one correspondence between the (orientation-preserving) affine equivalence classes of such configurations and an open dense subset of the Grassmann manifold of oriented d-spaces in $\mathbb{R}^n$. Since the Grassmannian carries a unique rotation-invariant probability measure, Goodman and Pollack propose to use the correspondence in order to introduce a natural probability measure on the space of affine equivalence classes of configurations of the type above. This opens the way to treat various questions on probabilities of special configurations or on expectations of random variables associated with them. As one example, Goodman and Pollack mention a fresh approach to the classical Sylvester problem (see the Remark at the end of Section 3) and propose its generalization to probabilities of order types in higher dimensions. Independently, versions of the "Grassmann approach" were proposed by Vershik, see [13] and [12]. Vershik and Sporyshev [14], [15] used it to obtain an asymptotic upper estimate for the average number of steps required by a version of the simplex algorithm, when the number of variables tends to infinity while the number of constraints is fixed.

Theorem 1 has a (simpler) counterpart in which the projection is onto subspaces of fixed codimension:

Theorem 2. *For any given integers* $0 \leq k < n - d$,

$$Ef_k(\Pi_{n-d}T^n) \sim \binom{n+1}{k+1} = f_k(T^n)$$

as n tends to infinity.

It is not difficult, by means of some integral geometry, to derive a general expression for the expectation $Ef_k(\Pi_d P)$, where P is an arbitrary convex polytope in $\mathbb{R}^n$. We do this in Section 2, thus generalizing results of Miles [8]. This expression involves internal and external angles of the polytope P. As these are spherical volumes, an explicit computation will be impossible in almost any concrete higher-dimensional case (except for the cube); hence even for regular simplices we have to be satisfied with asymptotic expressions. The theorems are proved in Section 3.

In the course of the proof it turns out that the expected number of facets of the orthogonal projection of an n-dimensional regular simplex onto an isotropic d-dimensional random subspace of $\mathbb{R}^n$ coincides with the expected number of facets of the convex hull of $n + 1$ independent and normally distributed random points in $\mathbb{R}^d$. While we have no explanation for this fact, it allows us to deduce our asymptotic formula of Theorem 1 from a result by Raynaud [9] concerning convex hulls of normally distributed random points.

2. Expected Face Numbers of Random Projections

We begin with some preliminaries. By $\mathbb{R}^n$ we denote the n-dimensional euclidean vector space and by $G(n, d)$ the Grassmannian of d-dimensional linear subspaces of $\mathbb{R}^n$, endowed with the usual topology. The (unique) rotation-invariant probability measure on $G(n, d)$ is denoted by ν_d. For $L \in G(n, d)$, let Π_L be the orthogonal projection from $\mathbb{R}^n$ onto L.

A polytope in $\mathbb{R}^n$ is the convex hull of a nonempty finite set of points. The set of k-dimensional faces of a polytope P is denoted by $\mathscr{F}_k(P)$, and $f_k(P) = \text{card}\, \mathscr{F}_k(P)$ is the number of k-faces.

By an isotropic random d-subspace we understand a random variable, defined on some probability space, with values in $G(n, d)$ and with probability distribution ν_d. Let Λ be such a random d-subspace, and let P be a given n-polytope in $\mathbb{R}^n$. We denote the projection Π_Λ by Π_d, thus $\Pi_d P$ is a random polytope and $f_k(\Pi_d P)$ is an integer-valued random variable for any $k \in \{0, 1, \dots, d-1\}$. Its expectation is given by

$$Ef_k(\Pi_d P) = \int_{G(n,d)} f_k(\Pi_L P)\, d\nu_d(L). \tag{1}$$

By integrating relation (3.1) in [4] (with different notations) we obtain

$$Ef_k(\Pi_d P) = \sum_{F \in \mathscr{F}_k(P)} \gamma^{n-d,n}(P, F), \tag{2}$$

where $\gamma^{n-d,n}(P, F)$ is a Grassmann angle as defined by Grünbaum. Using formulae of spherical integral geometry, we can express this Grassmann angle in terms of internal and external angles. For (nonempty) faces F, G of P, let $\beta(F, G)$ be the internal and let $\gamma(F, G)$ be the external angle of G at its face F (see Chapter 14 of [3]). By definition, $\beta(F, F) = \gamma(F, F) = 1$ and $\beta(F, G) = \gamma(F, G) = 0$ if $F \not\subset G$. McMullen [6] defined

$$\Phi_r(F, G) := \sum_{J^r \in \mathscr{F}_r(G)} \beta(F, J^r)\gamma(J^r, G)$$

and deduced (p. 257) from results by Santaló [11] that

$$\gamma^{n-d,n}(P, F) = 2 \sum_{s \ge 0} \Phi_{d-1-2s}(F, P). \tag{3}$$

The formulation in [6] is in terms of polyhedral cones. In our case, these results have to be applied to cone$(0, P - z)$, where z is a point in the relative interior of the face F. From (2) and (3) we get

$$Ef_k(\Pi_d P) = 2 \sum_{s \geq 0} \sum_{F \in \mathscr{F}_k(P)} \sum_{G \in \mathscr{F}_{d-1-2s}(P)} \beta(F, G)\gamma(G, P). \tag{4}$$

Another representation can be obtained by using McMullen's bilinear angle sum relations. If Theorems 1 and 2 of [6] are applied to the image of cone$(0, P - z)$ under the orthogonal projection with kernel aff$(F - z)$ (compare Section 2 of [6]), we obtain

$$\sum_{r=k}^{n} \Phi_r(F, P) = 1$$

and

$$\sum_{r=k}^{n} (-1)^r \Phi_r(F, P) = 0.$$

Together with (3) this yields

$$\gamma^{n-d,n}(P, F) = 1 - 2 \sum_{s \geq 0} \Phi_{d+1+2s}(F, P)$$

and hence

$$Ef_k(\Pi_d P) = f_k(P) - 2 \sum_{s \geq 0} \sum_{F \in \mathscr{F}_k(P)} \sum_{G \in \mathscr{F}_{d+1+2s}(P)} \beta(F, G)\gamma(G, P). \tag{5}$$

Formula (4) for the case $k = d - 1$ (where no internal angles $\neq 1$ occur) and formula (5) for the case $d = n - 1$ (where no external angles $\neq 1$ occur) were already obtained by Miles [8, p. 234].

3. Regular Simplices

Now we turn to the special case where P is an n-dimensional regular simplex T^n. Let $F \in \mathscr{F}_k(T^n)$ be a k-dimensional face of T^n, where $k \in \{0, \ldots, n-1\}$. The set of exterior unit normal vectors to T^n at some relatively interior point of F is an $(n-k-1)$-dimensional regular spherical simplex lying in an $(n-k-1)$-dimensional great subsphere S^{n-k-1} of the unit sphere of $\mathbb{R}^n$. Its spherical edge-length is equal to the angle between the exterior normal vectors of two distinct facets of T^n, which is given by $\arccos(-1/n)$. By $v(m, \alpha)$ we denote the m-dimensional spherical measure of a regular spherical simplex in S^m of edge-length α, and $\omega_m = 2\pi^{(m+1)/2}/\Gamma((m+1)/2)$ is the total measure of S^m. Thus the external angle of T^n at its k-face F is given by

$$\gamma(F, T^n) = \frac{v(n-k-1, \arccos(-1/n))}{\omega_{n-k-1}}. \tag{6}$$

For the internal angle of T^n at the k-face F we obtain

$$\beta(F, T^n) = \frac{v(n-k-1, \arccos(1/(k+2)))}{\omega_{n-k-1}}. \tag{7}$$

In the following we consider T^k, for $k < n$, as a face of T^n.
First we treat the case $k = d-1$ of Theorem 1. In this case, (4) reduces to

$$Ef_{d-1}(\Pi_d T^n) = 2\binom{n+1}{d}\gamma(T^{d-1}, T^n). \tag{8}$$

Let $m \geq 2$. If

$$-\frac{1}{m-1} < \cos\alpha \leq 0, \tag{9}$$

then a formula going back to Ruben [10] (see Section 3.3 of [5] and p. 283 of [2] for short proofs) says that

$$\frac{v(m-1, \alpha)}{\omega_{m-1}} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{A_m(\alpha)t} e^{-s^2}\, ds\right)^m dt,$$

where

$$A_m(\alpha) = \left(\frac{-\cos\alpha}{1+(m-1)\cos\alpha}\right)^{1/2}.$$

For $m = n-d+1$ and $\cos\alpha = -1/n$, condition (9) is satisfied, hence

$$\gamma(T^{d-1}, T^n) = \sqrt{\frac{d}{\pi}}\int_{-\infty}^{\infty} e^{-dt^2}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{t} e^{-s^2}\, ds\right)^{n-d+1} dt,$$

which gives

$$\begin{aligned} Ef_{d-1}(\Pi_d T^n) &= 2\binom{n+1}{d}\sqrt{\frac{d}{\pi}}\int_{-\infty}^{\infty} e^{-dt^2}\varphi(t)^{n-d+1}\, dt \\ &= 2\binom{n+1}{d}\sqrt{\frac{d}{\pi}}\int_{0}^{\infty} e^{-dt^2}\{\varphi(t)^{n-d+1} + [1-\varphi(t)]^{n-d+1}\}\, dt \end{aligned} \tag{10}$$

with

$$\varphi(t) := \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t} e^{-s^2}\, ds.$$

The right-hand side of (10) is precisely expression (1.13) in [9], with N replaced by $n + 1$ and n replaced by d. There it represents the expected number of facets of the convex hull of $n + 1$ random points in $\mathbb{R}^d$, chosen independently according to a normal distribution. The asymptotic behavior of (10) for $n \to \infty$ was established by Raynaud. (More details of the necessary estimations can be found in Appendix II of [1], where an extension is treated. For a different approach, see [15].) From his result, we conclude that

$$Ef_{d-1}(\Pi_d T^n) \sim \frac{2^d}{\sqrt{d}} (\pi \log n)^{(d-1)/2}. \tag{11}$$

For arbitrary $k \in \{0, \ldots, d-1\}$, we see from (4) that

$$Ef_k(\Pi_d T^n) = 2 \sum_{s \geq 0} \binom{n+1}{d-2s} \binom{d-2s}{k+1} \beta(T^k, T^{d-2s-1}) \gamma(T^{d-2s-1}, T^n).$$

Now we observe that here the number of nonzero summands does not depend on n and that

$$2\binom{n+1}{d-2s} \gamma(T^{d-2s-1}, T^n) \sim \frac{2^{d-2s}}{\sqrt{d-2s}} (\pi \log n)^{(d-2s-1)/2}$$

for $n \to \infty$, by (8) and (11) with d replaced by $d - 2s$. It follows that the term with $s = 0$ is dominating, which completes the proof of Theorem 1.

For the proof of Theorem 2 we use (5), where we replace d by $n - d$. Thus we obtain

$$Ef_k(\Pi_{n-d} T^n) - \binom{n+1}{k+1}$$
$$= -2 \sum_{s \geq 0} \binom{n+1}{d-2s-1} \binom{n-d+2s+2}{k+1} \beta(T^k, T^{n-d+2s+1}) \gamma(T^{n-d+2s+1}, T^n).$$

Here the number of nonzero summands does not depend on n. Further, $\gamma(T^m, T^n) \leq 1$ and $\beta(T^k, T^m) \leq 2^{k-m}$, hence $\beta(T^k, T^{n-k+2s+1}) \leq c \cdot 2^{-n}$ with a constant c independent of n. Since the binomial coefficients in the last sum are bounded by fixed powers of n, Theorem 2 follows immediately.

Remark. If four random points are chosen in the plane, what is the probability that they form a convex quadrilateral, that is, that none of the points lies inside the triangle formed by the other three? If the points are independently and uniformly distributed inside a given convex domain K, then this question is the classical problem of Sylvester. The probability $p(K)$ of obtaining a quadrilateral in this case satisfies

$$\tfrac{2}{3} \leq p(K) \leq 1 - \frac{35}{12\pi^2} = 0.7048,$$

with equality on the left if K is a triangle and on the right if K is an ellipse, and only in these cases.

Goodman and Pollack suggested a reformulation of Sylvester's original question which does not require the points to lie inside a preassigned region, but instead uses the probability measure for 4-point configurations described in the introduction. In other words, and extended to higher dimensions, we ask (for example) for the probability p_{n+1} that the orthogonal projection $\Pi_{n-1}T^n$ of the n-dimensional regular simplex onto an isotropic random hyperplane has $n+1$ vertices. Since $\Pi_{n-1}T^n$ has either n or $n+1$ vertices,

$$\begin{aligned} n(1 - p_{n+1}) + (n+1)p_{n+1} &= Ef_0(\Pi_{n-1}T^n) \\ &= f_0(T^n) - 2 \sum_{F \in \mathscr{F}_0(T^n)} \beta(F, T^n) \\ &= (n+1) - \frac{2(n+1)v(n-1, \pi/3)}{\omega_{n-1}} \end{aligned}$$

by (5) (or an easy direct argument) and (7). This yields

$$p_{n+1} = 1 - \frac{2(n+1)v(n-1, \pi/3)}{\omega_{n-1}}.$$

Especially,

$$p_4 = 3 - \frac{6}{\pi} \arccos \tfrac{1}{3} = 0.6490,$$

as already computed by Goodman and Pollack.

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