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Double-Lattice Packings of Convex Bodies in the Plane

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Abstract. Mahler [7] and Fejes Tóth [2] proved that every centrally symmetric convex plane body K admits a packing in the plane by congruent copies of K with density at least $\sqrt{3}/2$. In this paper we extend this result to all, not necessarily symmetric, convex plane bodies. The methods of Mahler and Fejes Tóth are constructive and produce lattice packings consisting of translates of K. Our method is constructive as well, and it produces double-lattice packings consisting of translates of K and translates of $-K$. The lower bound of $\sqrt{3}/2$ for packing densities produced here is an improvement of the bounds obtained previously in [5] and [6].

1. Introduction and Preliminaries

A *convex body* is a compact convex set with an interior point. A convex body is *strictly convex* if its boundary contains no line segment. Throughout this paper, K will denote an arbitrary convex body in the Euclidean plane E, unless otherwise specifically assumed. If S is a subset of E, v is a vector in E, and r is a real number, then $v+S$ denotes the set $\{v+x : x \in S\}$, and rS denotes $\{rx : x \in S\}$. We write $-S$ instead of $(-1)S$. The set $v+S$ is called *the translate* of S *by* v. The area of S will be denoted by $|S|$, and, since each set considered here will be either a convex body or a polygon, the question of measurability need not be discussed.

For any pair of independent vectors u and v in E, the *lattice generated by* u and v is the set of vectors $L(u, v) = \{nu + kv : n$ and k are integers$\}$. The parallelogram spanned by u and v is called a *basic parallelogram* of the lattice.

A *packing of the plane with copies of* K is a family $\{K_i\}$ of convex bodies congruent to K whose interiors are mutually disjoint. A packing is a *lattice packing* if all of its members are translates of each other and the vectors of the translations

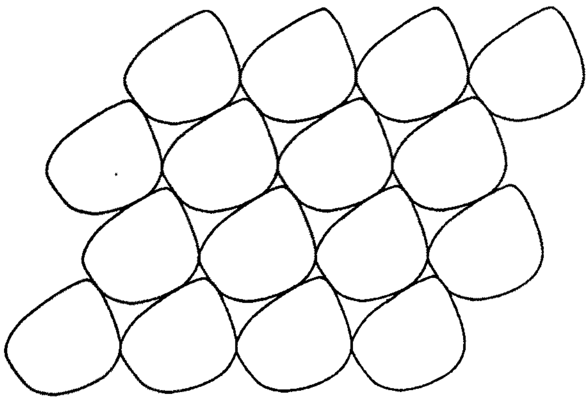

Fig. 1. Lattice packing.

form a lattice (see Fig. 1). A packing $\mathcal{P}$ is called a *double-lattice packing* if $\mathcal{P}$ is the union of two lattice packings $\mathcal{P}_0$ and $\mathcal{P}_1$ such that the 180° rotation about some point interchanges $\mathcal{P}_0$ and $\mathcal{P}_1$ (see Fig. 2). Obviously, the lattice of translations of $\mathcal{P}_0$ is the same as that of $\mathcal{P}_1$.

Each packing is assigned a real number in the interval [0, 1], called the density of the packing, which is, intuitively speaking, the percentage fraction of the plane occupied by the copies of K. A general definition of the density of a packing involves the notion of limit and requires a detailed investigation of its existence and uniqueness (see Chapter III, Section 1 of [4]). However, for lattice packings and for double-lattice packings, the density is easily defined using the area of a basic parallelogram of the underlying lattice of translations. If $\mathcal{P}$ is a lattice packing with copies of K, and if p is a basic parallelogram of the lattice of translations of $\mathcal{P}$, then the density of $\mathcal{P}$ equals $|K|/|p|$. Similarly, if $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$ is a double-lattice packing with copies of K, and if p is a basic parallelogram for the lattice of translations of each of $\mathcal{P}_0$ and $\mathcal{P}_1$, then the density of $\mathcal{P}$ is $2|K|/|p|$.

Note that the concepts of double-lattice packing and its density are affine-invariant, just as for lattice packings (see p. 26 of [10]). Specifically, let $\mathcal{P}$ be a

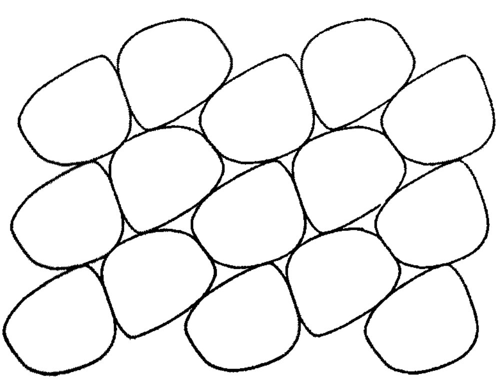

Fig. 2. Double-lattice packing.

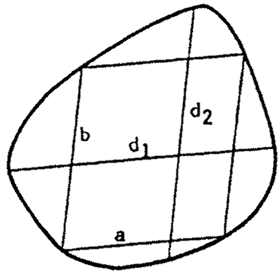

Fig. 3. Extensive parallelogram; $a \geq d_1/2$, $b \geq d_2/2$.

double-lattice packing of the plane with copies of K and let A be an affine transformation of the plane. Then $A(\mathcal{P})$ is a double-lattice packing of the plane with copies of $A(K)$ and the density of $A(\mathcal{P})$ is the same as that of $\mathcal{P}$.

A *chord* of K is a segment whose endpoints lie on the boundary of K. A chord of K is a *diameter* of K if it is at least as long as any chord parallel to it. If v is a unit vector, the *length* of K *in the direction* of v is the length of the diameter of K parallel to v. A polygon is *inscribed* in K if each of its vertices lies on the boundary of K. A parallelogram inscribed in K is *extensive* if the length of each of its sides is at least one-half of the length of K in the direction of the side (see Fig. 3). Notice that extensive parallelograms are affine-invariant, i.e., if q is an extensive parallelogram inscribed in K and if A is an affine transformation of the plane, then $A(q)$ is an extensive parallelogram inscribed in $A(K)$.

2. Double-Lattice Packings Generated by Extensive Parallelograms

Assume that q is an extensive parallelogram inscribed in K, place the origin at one of the vertices of q, and let u and v be the vectors spanning q. For every pair of integers i, j, and for $\varepsilon = 0, 1$, let $K^{\varepsilon}(i,j) = 2(iu + jv) + (-1)^{\varepsilon}K$. Observe that the family $\mathcal{P} = \{K^{\varepsilon}(i,j)\}$ is a double-lattice packing of the plane with copies of K, namely $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$, where $\mathcal{P}_0 = \{K^0(i,j)\}$ and $\mathcal{P}_1 = \{K^1(i,j)\}$ (see Fig. 4). The lattice of translations is generated by $2u$ and $2v$, its basic parallelogram is $p = 2q$, and the density of $\mathcal{P}$ equals $2|K|/|p| = |K|/2|q|$. We say that the double-lattice packing is *generated* by the extensive parallelogram q.

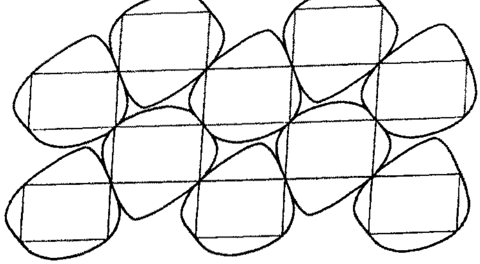

Fig. 4. Double-lattice packing generated by an extensive parallelogram.

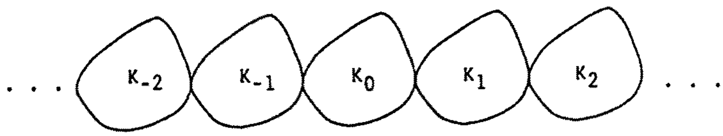

Fig. 5. A row of translates of K.

Among all double-lattice packings of the plane with copies of K there exists one of maximum density. This can be easily shown using a compactness-type argument. In this section we show how to construct such a packing.

Theorem 1. *If K is strictly convex, and if $\mathcal{P}$ is a maximum density double-lattice packing of the plane with copies of K, then $\mathcal{P}$ is generated by a minimum area extensive parallelogram inscribed in K.*

Proof. Let $K_0 \in \mathcal{P}$. Obviously, K_0 touches another member of $\mathcal{P}$.

Case I. K_0 touches a translate of K_0, call it K_1. Let u be the vector of the translation from K_0 to K_1. Since u belongs to the lattice of translations of $\mathcal{P}$, there is in $\mathcal{P}$ a "row" $R_0 = \cdots K_{-2} \cup K_{-1} \cup K_0 \cup K_1 \cup K_2 \cdots$ of translates of K_0 in which $K_{i+1} = u + K_i$ (see Fig. 5). It follows that $\mathcal{P}$ is the union of a sequence of nonoverlapping rows $\{R_j\}$ such that every other row is a translate of R_0 and each of the remaining rows is a translate of $-R_0$. Since $\mathcal{P}$ is of maximum density, two neighboring rows are as close to each other as is possible. This implies that K_0 touches two copies of K in each of the two rows neighboring R_0. Each of these four copies of K is a translate of $-K_0$, and it is easy to notice that the points at which they touch K_0 are vertices of a parallelogram q inscribed in K_0. This parallelogram is extensive, and it generates the packing $\mathcal{P}$.

Case II. K_0 does not touch any translate of itself in $\mathcal{P}$. In this case, K_0 touches at least two translates of $-K_0$, say K_{-1} and K_1. The vector of the translation from K_{-1} to K_1 belongs to the lattice of $\mathcal{P}$ and it produces an "alternating row" $\cdots K_{-2} \cup K_{-1} \cup K_0 \cup K_1 \cup K_2 \cdots$ in which each K_{2i} is a translate of K_0 and each K_{2i+1} is a translate of $-K_0$ (see Fig. 6). Now the entire packing $\mathcal{P}$ consists of translates of that one alternating row. In a similar fashion as in Case I we conclude that K_0 touches four translates of $-K_0$, and, again, we find the extensive parallelogram q inscribed in K_0 which generates $\mathcal{P}$.

Since the density of $\mathcal{P}$ depends only on the area of q, and since that density is maximum, the area of q is minimum among all extensive parallelograms

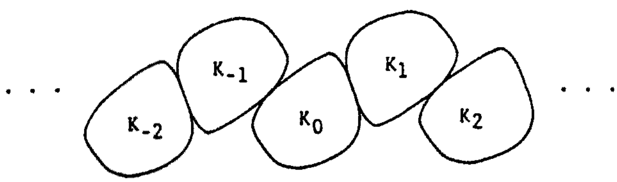

Fig. 6. An alternating row.

inscribed in K_0, and the proof is complete. However, it can be noticed now that the minimality of the area of q implies that the length of one of the sides of q actually equals one-half of the length of K_0 in the direction of that side. Therefore K_0 actually touches a translate of itself, and Case II is not possible at all. □

Remark 1. If K is not strictly convex, the conclusion of the above theorem does not necessarily hold. However, in this case there exists a double-lattice packing with maximum density which is generated by a minimum-area extensive parallelogram inscribed in K. This can be obtained by approximating K with a sequence of strictly convex bodies K_n and then selecting a convergent subsequence of double-lattice packings.

Remark 2. Theorem 1 and the above remark yield an algorithm for finding a maximum density double-lattice packing with copies of K which goes as follows. For any diameter d of K, find a pair of chords parallel to d, each of length equal to one-half of the length of d. These two chords define a parallelogram $q(d)$ inscribed in K, which turns out to be extensive (see Lemma 1 of the following section). Now vary d and find a critical position of $d = d_0$ such that $q(d_0)$ is of minimum area. This minimum-area extensive parallelogram generates a maximum density double-lattice packing with copies of K. In general, locating the critical diameter d_0 may be a problem, but in many special cases, as in the following examples, the diameter d_0 is easy to find.

Examples. An application of the algorithm described in Remark 2 to the case when K is a regular pentagon results in a double-lattice packing of density $(5-\sqrt{5})/3 = 0.92131\ldots$, shown in Fig. 7. This packing may have the maximum

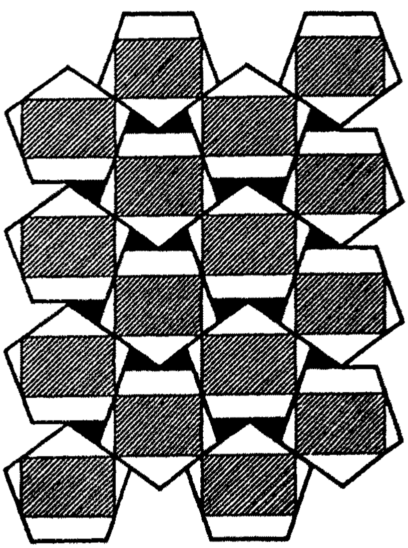

Fig. 7. Maximum density double-lattice packing with regular pentagons.

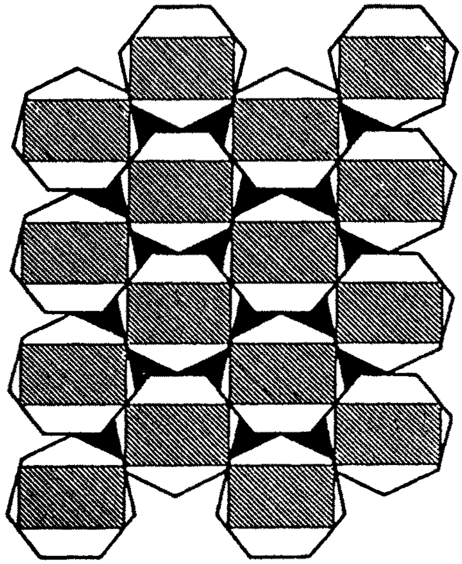

Fig. 8. Maximum density double-lattice packing with regular heptagons.

density among all, not necessarily double-lattice, packings with congruent regular pentagons. A similarly obtained double-lattice packing with congruent regular heptagons is shown in Fig. 8. This packing's density equals 0.8926..., and, again, this is possibly the maximum density of any packing with congruent regular heptagons. This conjecture implies the conjecture of Blind [1] which asserts that there exists a noncentrally symmetric convex body K whose maximum packing density is smaller than 0.9024.... The constant 0.9024... is conjectured by Reinhardt [8] to be the greatest lower bound for all maximum packing densities of the centrally symmetric convex plane bodies. This motivates the following.

Problem. Let $n \geq 5$ be an odd integer. Is the maximum density among all packings with congruent regular n-gons attained through a double-lattice packing?

Note that if K is a regular polygon with an even number of sides, then, due to its central symmetry, the maximum density among all packings with copies of K is attained through a lattice packing, according to a well-known result of Fejes Tóth [3] and Rogers [9].

3. Two Certain Types of Inscribed Extensive Parallelograms and a High-Density Packing Generated by One of Them

We are now concerned with the extensive parallelograms inscribed in K with one pair of sides parallel to a given direction. In particular, we focus on the two extreme cases: when the two sides are as short as possible, and when they are as long as possible.

Lemma 1. *Let v be an arbitrary direction and let d be the length of K in the direction of v. If c_1 and c_2 are two chords parallel to v and of length $d/2$ each, then the parallelogram whose opposite sides are these two chords is extensive.*

Lemma 2. *Let v be an arbitrary direction, let k_1 and k_2 be the lines of support for K, parallel to v, and let w be the distance between these two lines. If e_1 and e_2 are two equal length chords parallel to v and such that the distance between the two lines defined by these chords equals $w/2$, then the parallelogram whose opposite sides are these two chords is extensive.*

We omit the very simple proofs of the above lemmas.

For an arbitrary direction v, the parallelogram described by Lemma 1 is called the *half-length* parallelogram in the direction of v, and the one described by Lemma 2 is called the *half-width* parallelogram in the direction of v (see Fig. 9).

Theorem 2. *For every direction v, the area of either the half-length or the half-width parallelogram in the direction of v, inscribed in K, is less than or equal to $|K|/\sqrt{3}$.*

Proof. Let k_1 and k_2 be the two lines of support for K, parallel to v, and let A_1 and A_2 be respective points at which k_1 and k_2 touch K. Let d be the diameter of K parallel to v. Denote the endpoints of d by B_1 and B_2 and let m_1 and m_2 be parallel lines of support of K, touching K at B_1 and B_2, respectively. The lines k_1, k_2, m_1, and m_2 bound a parallelogram containing K which we will assume is a square of side 4; this can be arranged through a suitable affine transformation of the plane. Denote by L_1, L_2, L_3, and L_4 the vertices of the half-length parallelogram in K in the direction of v, and let W_1, W_2, W_3, and W_4 be the vertices of the half-width parallelogram in K in the direction of v, as shown in Fig. 10. Let x be the length of the chord W_1W_2 and let y be the distance between the lines of the chords L_1L_2 and L_3L_4. Obviously, each of the numbers x and y is between 2 and 4. The areas of the two extensive parallelograms $L_1L_2L_3L_4$ and $W_1W_2W_3W_4$ are equal to $2y$ and $2x$, respectively. Let D denote

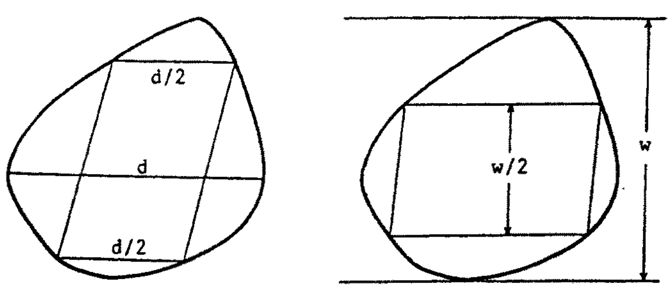

Fig. 9. The half-length and the half-width parallelograms.

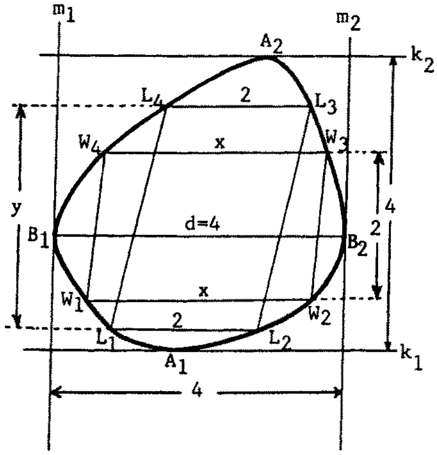

Fig. 10. Constructing the dodecagon D inscribed in K.

the dodecagon $B_1W_1L_1A_1L_2W_2B_2W_3L_3A_2L_4W_4$ which is inscribed in K. The area of D is readily computed as follows:

(1) The combined area of the two triangles $L_1A_1L_2$ and $A_2L_3L_4$ equals $4-y$.
(2) The combined area of the two trapezoids $L_1L_2W_2W_1$ and $L_3L_4W_4W_3$ equals $(x+2)(y-2)/2$.
(3) The combined area of the remaining two trapezoids $B_1B_2W_2W_1$ and $B_1B_2W_3W_4$ equals $4+x$.

Thus the total area of D equals $6+xy/2$.

Let $t=\min(x,y)$. On one hand, the area of one of the two parallelograms in question equals $2t$. On the other hand, the area of K is greater than or equal to that of D, since K contains D. Therefore $|K|\geq 6+xy/2\geq 6+t^2/2$. It follows that $|K|/2t\geq 3/t+t/4$, and, since $3/t+t/4\geq\sqrt{3}$ for all $t>0$, we conclude that $|K|/2t\geq\sqrt{3}$, which is equivalent to $2t\leq|K|/\sqrt{3}$. □

Corollary. *Every convex plane body K admits a double-lattice packing of the plane with density of at least $\sqrt{3}/2$.*

Proof. By Theorem 2 there exists an extensive parallelogram q inscribed in K and with $|q|\leq(\sqrt{3}/3)|K|$. This parallelogram generates a double-lattice packing of the plane with copies of K, as described in Section 2. The density of this packing equals $|K|/2|q|\geq\sqrt{3}/2$. □

Remark 3. In the proof of Theorem 2, the direction v was chosen arbitrarily, and each of the two extensive parallelograms constructed there had a side parallel to v. Therefore the above corollary can be strengthened somewhat by adding that one of the basic vectors of the translation lattice of the double-lattice packing is parallel to the prescribed direction. In this phrasing, the inequality obtained for

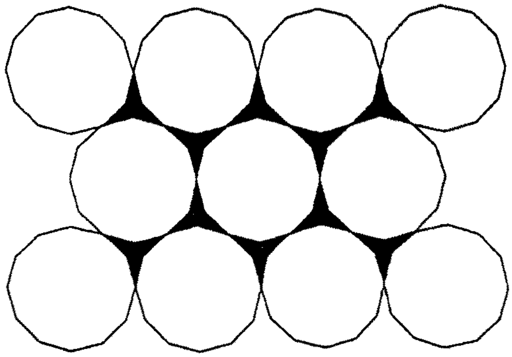

Fig. 11. Maximum density (double-) lattice packing with regular dodecagons under the constraint that one basic translation is parallel to the central diagonal. Density: $\sqrt{3}/2$.

the density is the best possible, as the example of the regular dodecagon shows, in which the prescribed direction is parallel to its central diagonal (see Fig. 11).

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