The Maximum Number of Ways To Stab n Convex Nonintersecting Sets in the Plane Is $2n - 2$*

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Abstract. Let S be a collection of n convex, closed, and pairwise nonintersecting sets in the Euclidean plane labeled from 1 to n. A pair of permutations

$$\{(i_1, i_2, \ldots, i_{n-1}, i_n), (i_n, i_{n-1}, \ldots, i_2, i_1)\}$$

is called a *geometric permutation of S* if there is a line that intersects all sets of S in this order. We prove that S can realize at most $2n - 2$ geometric permutations. This upper bound is tight.

1. Introduction

Let S be a collection of n convex, closed, bounded, and pairwise nonintersecting sets in the Euclidean plane. We label the sets from 1 to n. A directed or undirected line that intersects all sets is called a *transversal of S*. Since no two sets intersect each other, a transversal intersects S in a well-defined order. In the case of an undirected line, such an order can be described by a pair of permutations, one being the reverse of the other. Such a pair is called a *geometric permutation of S*. For convenience we will represent a geometric permutation by any one of its two permutations.
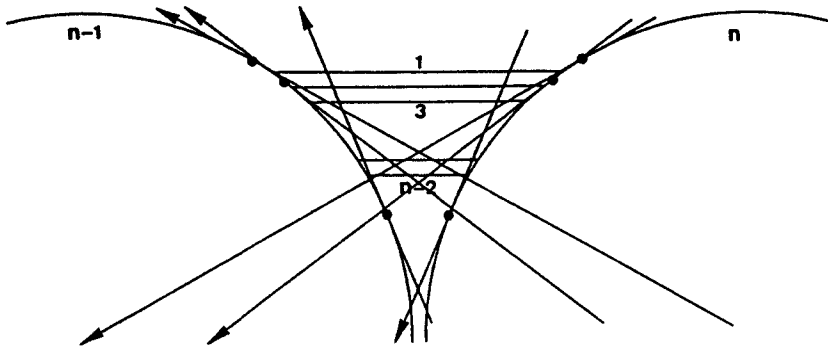
Fig. 1. A set with $2n-2$ geometric permutations.

For example, let S consist of two equally large disks, labeled $n-1$ and n, and $n-2$ horizontal line segments, labeled from 1 to $n-2$. The two disks are placed sufficiently close to each other such that their centers lie on a horizontal line h. The line segments are horizontal and lie above h such that their left endpoints lie sufficiently close to the boundary of the left disk and their right endpoints lie sufficiently close to the boundary of the right disk (see Fig. 1). The labels of the line segments increase from the top downward. In this example the geometric permutations are

$$(1, 2, \ldots, i, n, i+1, i+2, \ldots, n-2, n-1) \qquad \text{for} \quad i = n-2, n-3, \ldots, 0,$$

and

$$(n, n-2, n-3, \ldots, i+1, n-1, i, i-1, \ldots, 1) \qquad \text{for} \quad i = 0, 1, \ldots, n-2.$$

Thus, we see that $n \geq 4$ sets can realize as many as $2n-2$ geometric permutations. It is readily verified that for $n=2$ there is only one geometric permutation and that for $n=3$ the maximum number of geometric permutations is three.

The above example is taken from [KLZ] which also proves that $\binom{n}{2}$ is an upper bound on the maximum number of geometric permutations realized by n convex nonintersecting sets. A proof of the same upper bound can also be found in [KLL]. An upper bound of $6n+6$ on the number of geometric permutations has been shown recently in [W]. In this paper we prove that the lower bound of $2n-2$ is in fact tight.

2. The Upper Bound

We derive the tight upper bound on the maximum number of geometric permutations by proving a sequence of three lemmas. First we introduce a few definitions.

If a directed line $\vec{l}$ intersects all sets of S, then we say that $\vec{l}$ *induces* the permutation $(i_1, i_2, \ldots, i_n)$ that gives the order of sets along $\vec{l}$. Thus, the permutation induced by the line that coincides with $\vec{l}$ but has opposite direction is $(i_n, i_{n-1}, \ldots, i_1)$. Similarly, for an undirected transversal, t, we say that t *induces* the pair of permutations that represent the order in which it intersects S. A directed line $\vec{l}$ is said to be *tangent to* set i if $\vec{l}$ contains a point on the boundary of i and i is contained in one of the two closed half-planes defined by $\vec{l}$. Call $\vec{l}$ *left tangent to* i if this closed half-plane lies to the left of $\vec{l}$. This definition implies the important fact that two nonintersecting convex sets have at most two common left tangents.

For every angle $\alpha \in [0, 2\pi)$, we define $\vec{l}(\alpha)$ as the unique directed line that satisfies the following three conditions:

(i) α is the angle between the positive x-axis and $\vec{l}(\alpha)$.
(ii) No set of S is contained in the open half-plane to the left of $\vec{l}(\alpha)$.
(iii) At least one set of S is contained in the closed half-plane to the left of $\vec{l}(\alpha)$.

In Figs 1 and 2 lines $\vec{l}(\alpha)$ are shown for several angles α. It is clear from the definition of $\vec{l}(\alpha)$ that if there is a directed transversal of S with angle α, then $\vec{l}(\alpha)$ is the rightmost parallel transversal with this angle, that is, there is no transversal contained in the open half-plane to the right of $\vec{l}(\alpha)$. A line $\vec{l}(\alpha)$ of S is said to be *extreme* if it is tangent to at least two sets of S. For convenience, we also say that the undirected version of an extreme line is *extreme*. The significance of extreme lines derives from the following result.

Lemma 1. *Every (undirected) transversal t of S can be moved continuously to an extreme line without ever changing the induced geometric permutation.*

Proof. Let α_0 and $\alpha_1 = \alpha_0 + \pi$ be the angles of the two directed versions of t. Line t can be translated continuously to coincide with $\vec{l}_0 = \vec{l}(\alpha_0)$ or $\vec{l}_1 = \vec{l}(\alpha_1)$ without ever changing the induced geometric permutation. Let i_0 be the set contained in the closed half-plane to the left of $\vec{l}_0$ and let i_1 be the corresponding set for $\vec{l}_1$ (see Fig. 2). We can assume that i_0 and i_1 are unique, otherwise, $\vec{l}_0$ or $\vec{l}_1$ is an extreme line and we are done.
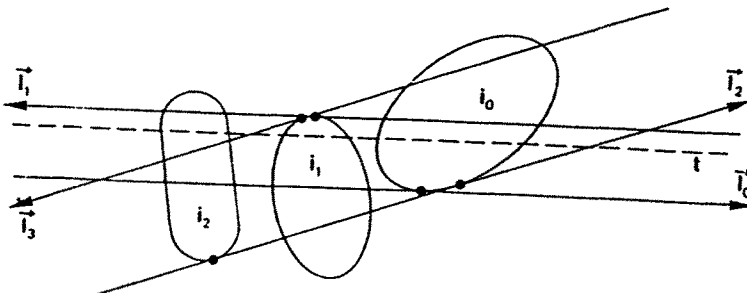

Fig. 2. Moving a transversal to an extreme position.

First we consider the case that $i_0 \neq i_1$ and $\vec{l}_0$ meets i_1 before i_0; thus, $\vec{l}_1$ meets i_0 before i_1 (see Fig. 2). Simultaneously, rotate $\vec{l}_0$ counterclockwise around i_0 and $\vec{l}_1$ counterclockwise around i_1 keeping $\vec{l}_0$ parallel to $\vec{l}_1$ until either $\vec{l}_0$ is tangent to some set other than i_0 or $\vec{l}_1$ is tangent to some set other than i_1. Let the lines which are rotations of $\vec{l}_0$ and $\vec{l}_1$ be labeled $\vec{l}_2$ and $\vec{l}_3$, respectively. Lines $\vec{l}_2$ and $\vec{l}_3$ are transversals of S and induce the same geometric permutation as t. There are four possible cases:

(i) Line $\vec{l}_2$ is left tangent to i_1 and therefore an extreme line.
(ii) Line $\vec{l}_3$ is left tangent to i_0 and therefore an extreme line.
(iii) Line $\vec{l}_2$ is tangent to some set i_2 other than i_0 and i_1. Since $\vec{l}_3$ is to the left of $\vec{l}_2$ and intersects all sets in S, $\vec{l}_2$ must be left tangent to i_2 (see Fig. 2). Thus, $\vec{l}_2$ is an extreme line.
(iv) Line $\vec{l}_3$ is tangent to some set i_2 other than i_0 and i_1. Since $\vec{l}_2$ is to the left of $\vec{l}_3$ and intersects all sets in S, $\vec{l}_3$ must be left tangent to i_2. Thus, $\vec{l}_2$ is an extreme line.

In the case where $i_0 \neq i_1$ and $\vec{l}_0$ meets i_1 after i_0 we do the rotation in clockwise order and arrive at the same four cases. If $i_0 = i_1$, then either order leads to cases (iii) or (iv). $\square$

Thus, an upper bound on the number of extreme lines also puts an upper bound on the number of geometric permutations. To derive such an upper bound, we associate S with a cyclic sequence of integers. Let $i(\alpha)$ be the set contained in the closed half-plane to the left of $\vec{l}(\alpha)$—if it is unique. Since $i(\alpha)$ is unique unless $\vec{l}(\alpha)$ is tangent to at least two sets, $i(\alpha)$ is defined except for at most a discrete number of angles α. We call the cyclic sequence

$$\mathscr{C}(S) = i_1 i_2 \cdots i_m$$

the *cycle of S* if:

(i) $i_j \neq i_{j+1}$, for $1 \leq j \leq m$ (where $i_{m+1} = i_1$), and
(ii) the circle of angles can be partitioned into m intervals $[\alpha_j, \alpha_{j+1})$, for $1 \leq j \leq m$ and $\alpha_{m+1} = \alpha_1$, such that $i(\alpha) = i_j$ if $\alpha \in (\alpha_j, \alpha_{j+1})$.

m is called the *length of* the cycle. The cycle of S can be constructed by rotating a directed line $\vec{l}$ through all lines $\vec{l}(\alpha)$ and by keeping track to which set $\vec{l}$ is left tangent. A new set is reached whenever $\vec{l}$ passes an extreme line. Thus, m is also the number of extreme lines of S.

Below, we argue about scattered subcycles of $\mathscr{C}(S)$ which are cyclic sequences that can be obtained from $\mathscr{C}(S)$ by removing some of its members. The remaining integers appear in the same order as in $\mathscr{C}(S)$.

Lemma 2.   *The cycle $\mathscr{C}(S)$ contains no scattered subcycle of the form abab, with $a \neq b$.*

Proof. Assume that there is a subcycle of the form *abab*. Then there are angles $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ such that

$$i(\alpha_1) = a, \qquad i(\alpha_2) = b, \qquad i(\alpha_3) = a, \quad \text{and} \quad i(\alpha_4) = b.$$

The definitions of lines $\bar{l}(\alpha)$ and of labels $i(\alpha)$ imply that for each one of the open intervals (α_1, α_2), (α_2, α_3), (α_3, α_4), and (α_4, α_1), there is a common left tangent of a and b whose angle is in this interval. However, since α and b do not intersect, there are only two directed lines that are left tangent to a and b, a contradiction.   □

Lemma 2 implies that if we remove all integers except for the a's and the b's from the cycle, then we get a cyclic sequence of the form

$$aa \cdots abb \cdots b.$$

We call a cyclic sequence that satisfies Lemma 2 and has no two equal integers in consecutive positions an $(n, 2)$-*Davenport–Schinzel cycle* or an $(n, 2)$-*cycle*, for short. Here, n refers to the largest number of different integers that can occur in the cycle. By construction of $\mathscr{C}(S)$, there is a one-to-one correspondence between the extreme lines of S and the pairs of consecutive integers in the cycle. Consequently, an upper bound on the maximum length of an $(n, 2)$-cycle is also an upper bound on the maximum number of extreme lines.

Lemma 3. *If $i_1 i_2 \cdots i_m$ is an $(n, 2)$-cycle, then $m \leq 2n - 2$.*

Proof. The assertion is trivially true for $n = 2$. Assume inductively that Lemma 3 holds for $(n - 1, 2)$-cycles. We prove below that every $(n, 2)$-cycle has an integer a that occurs at most once. If this is true, then we can delete a and get an $(n - 1, 2)$-cycle after possibly removing also the predecessor of a—this has to be done if the predecessor and the successor of a are the same. Thus, every $(n, 2)$-cycle contains at most two integers more than a longest $(n\text{-}1, 2)$-cycle which implies the assertion.

To prove that there is always an integer a that occurs only once, we assume that such an integer does not exist. Let $i_j = i_k = b$ be two consecutive appearances of an integer, b, such that $k - j$ modulo m is a minimum. By Lemma 2, any integer c that occurs in the circular interval $i_{j+1}, i_{j+2}, \ldots, i_{k-1}$ cannot occur outside this interval. Thus, c must occur again within this interval which contradicts the minimality of $k - j$.   □

We have thus shown that $2n - 2$ is an upper bound on the number of extreme lines. By Lemma 1, this implies that $2n - 2$ is an upper bound on the number of geometric permutations of n convex, closed, bounded, and pairwise nonintersecting sets. If we reexamine the proofs of Lemmas 1-3, we realize that the only place where we use that the sets are bounded is where we assume that the lines $\bar{l}(\alpha)$ are well defined. They are well defined, however, even when S contains

unbounded sets, provided there is no direction such that a transversal normal to that direction can be translated into this direction arbitrarily far without ceasing to be a transversal. For example, this is already the case when only one of the sets in S is bounded. If this condition is not satisfied, then $\bar{l}(\alpha)$ is not defined in a few nondegenerate intervals of angles α. All steps of the above development go through if we treat such a nondegenerate interval in the same way as we treat a single angle where $\bar{l}(\alpha)$ is not defined. Thus, we can dispense with the assumption that the sets are bounded altogether. This implies the main result of this paper.

Theorem. *For $n \geq 4$, the maximum number of geometric permutations realized by n convex, closed, and pairwise nonintersecting sets in the plane is $2n - 2$. For $n = 1$, 2, 3 this maximum is equal to* 1, 1, 3, *respectively.*

3. Discussion

Lemma 3, the upper bound on the length of $(n, 2)$-cycles, can also be derived from the upper bound, $2n - 1$, on the length of a so-called $(n, 2)$-Davenport-Schinzel sequence, as proved in [A]. This is a sequence made up of at most n different integers that does not contain a scattered subsequence of the form $abab$ and does not contain a pair of adjacent integers that are equal. It is interesting to note that $(n, 2)$-cycles can be generalized to $(n, 2s)$-cycles, $s \geq 1$, which contain no scattered subcycles of the form $abab \cdots ab$ of length $2s + 2$. Upper and lower bounds on the maximum lengths of such cycles can be obtained by adapting the results in [HS], [S1], and [S2].

The number of geometric permutations of a collection S of n pairwise noninter-secting sets relates to the number of connected components of the interior of the so-called stabbing region as introduced in [EMPRWW]. The *stabbing region of* S is defined as the set of all points that are dual to transversals of S. For example, we can use the function that maps a point $p = (\pi_1, \pi_2)$ to the nonvertical line $y = \pi_1 x - \pi_2$, and vice versa, to realize the duality. Notice that this function is a one-to-one correspondence between points and lines and that it associates vertical lines with points at infinity. Figure 3 shows the stabbing region of three line segments. For each line segment the region of points dual to lines that intersect it is a double wedge which is the area swept out by a rotating line—this line corresponds to a point that moves along the line segment. Centers of rotation are indicated by dots in Fig. 3. Since the duality function associates vertical lines with points at infinity, the double wedge corresponding to a bounded line segment does not contain a vertical line. The stabbing region is then the intersection of the double wedges that correspond to the line segments. We can see that the three double wedges in Fig. 3 correspond to pairwise nonintersecting line segments since no two double wedges share a common line.

The region of points dual to lines crossing a closed, compact set is given by a convex and a concave function (it is the set of points between those two functions including the points on the functions), and it is necessarily connected.
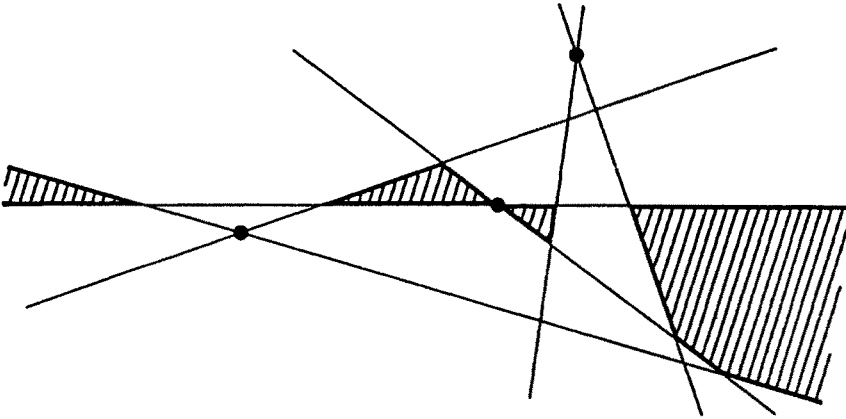
Fig. 3. Stabbing region of three nonintersecting line segments.

The so-called stabbing region is now the intersection of n such regions. Thus, it is the set of points below or on the lower envelope of n convex functions and above or on the upper envelope of n concave functions. This intersection is not necessarily connected. An extreme line is dual to a "breakpoint" on either envelope. Each connected component of the interior of the stabbing region has at least one such breakpoint in its boundary, unless it is unbounded. The upper bound on the number of extreme lines given in Lemmas 1–3 thus implies that the interior of the stabbing region of n convex and pairwise nonintersecting sets consists of at most $2n - 1$ connected components—one more than the maximum number of extreme lines since it is possible that the two unbounded components, if they exist, contain only one point dual to an extreme line. In [EMPRWW] it is proved that the interior of the stabbing region of n line segments, intersecting or nonintersecting, consists of at most $n + 1$ connected components. This implies that n nonintersecting line segments realize at most n geometric permutations—a result independently shown in [KLZ].

From the computational point of view it is interesting to ask how fast the geometric permutations of S can be computed. To answer this question, we assume that the sets in S are *computationally simple*, that is, we assume that:

(i) The common tangents of two sets can be computed in constant time (this includes the two inner tangents of the sets).

(ii) Given an angle α and a set i, we can compute the directed line with angle α that is left tangent to i in constant time.

(iii) Given a directed line, we can determine which one of two sets it intersects first in constant time.

With these assumptions, we can use the divide-and-conquer approach that was exploited in [EMPRWW] to compute transversals for line segments and generalized in [AB] to arbitrary sets meeting some computational restrictions. Details about this approach can be found in those two papers. It leads to an algorithm that takes $O(n \log n)$ time to construct the stabbing region as defined above.

Each connected component represents an equivalence class of lines that define the same geometric permutation—except for the two unbounded components which represent the same equivalence class, if they exist. To compute all geometric permutations of S, we recursively store the permutation with its associated component. A component of the entire set is obtained as the intersection of two components computed recursively, and its permutation is obtained by merging the two permutations associated with the components. Using condition (iii), the merge step can be performed in linear time. Since the stabbing region consists of at most $2n - 1$ connected components, this yields an $O(n^2)$-time algorithm. The algorithm is optimal in the worst case since it produces up to quadratic output.

Finally, there is the problem of generalizing the results to three and higher dimensions. It is shown in [W] that the number of geometric permutation defined by lines that intersect n convex sets in $d \geq 3$ dimensions is $O(n^{2d-2})$. A lower bound of $\Omega(n^{d-1})$ is claimed in [KLZ].

Acknowledgments

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