Discrete Comput Geom 1:115–122 (1986)

Discrete & Computational
Geometry

Spherical Complexes and Nonprojective Toric Varieties

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Abstract. A combinatorial criterion for a toric variety to be projective is given which uses Gale-transforms. Furthermore, classes of nonprojective toric varieties are constructed.

1. Introduction

Let $\sigma := \mathbb{R}_+ a_1 + \cdots + \mathbb{R}_+ a_k$ be a cone in $\mathbb{R}^d$, where $a_1,\dots,a_k$ are primitive lattice points $\in \mathbb{Z}^d\setminus\{0\}$, and let σ have only 0 as an apex. If S is the unit sphere in $\mathbb{R}^d$, the intersection $\sigma_0 := \sigma \cap S$ is a spherical cell. Suppose Σ_0 is a spherical cell complex consisting of such cells. The corresponding cones form a system Σ called a *fan*. We can assume that every point $(\mathbb{R}_+ a_j) \cap S$ is a vertex of σ_0 for any $\sigma_0 \in \Sigma_0$. We may also consider Σ to be a cell complex, the vertices being one-dimensional cones $\mathbb{R}_+ a_j$.

If $\dim \sigma := \dim(\operatorname{aff} \sigma)$ (affine hull) equals k we call σ and σ_0 *simplicial*. We say Σ or Σ_0 is *simplicial* if every $\sigma \in \Sigma$ or $\sigma_0 \in \Sigma_0$ is simplicial, respectively. In the case of a simplicial fan we also look at Σ as being generated by projecting the simplexes $\sigma' := \operatorname{conv}\{a_1,\dots,a_k\}$, that is, $\sigma = \mathbb{R}_+\sigma'$ for all $\sigma \in \Sigma$. The simplicial complex $B_{\mathrm{st}}(Q)$ of all σ' thus defined bounds a star-shaped polyhedron Q with 0 in its kernel, provided Σ covers the whole space $\mathbb{R}^d$. Let $\check{\sigma} := \{x \mid \langle x, y\rangle \geq 0$ for all $y \in \sigma\}$ be the dual cone of $\sigma(\langle \cdot , \cdot \rangle =$ inner product), and let R_σ be the ring of all Laurent-polynomials $\Sigma a_j \mathbf{z}^j$, $a_j \in C$ (or any algebraically closed field), $\mathbf{z}^j := \mathbf{z}_1^{j_1} \cdots \mathbf{z}_d^{j_d}$, $j := (j_1,\dots,j_d) \in \check{\sigma} \cap \mathbb{Z}^d$, only finitely many a_j being $\neq 0$. $\operatorname{Spec} R_\sigma$ (the set of prime ideals of R_σ) is an affine variety. For any two $\sigma_1, \sigma_2 \in \Sigma$ we glue together $\operatorname{Spec} R_{\sigma_1}$ and $\operatorname{Spec} R_{\sigma_2}$ by the inclusion maps

$$R_{\sigma_1 \cap \sigma_2} \leftarrow R_{\sigma_1}, \qquad R_{\sigma_1 \cap \sigma_2} \leftarrow R_{\sigma_2}.$$

If this is done for all $\sigma_1, \sigma_2 \in \Sigma$ we obtain a variety X_Σ called *toric variety* (see

Kempf, Knudson, Mumford, and Saint-Donat [6], Oda [10], Danilov [2], and Teissier [14]; also [3]).

Any fan can easily be extended to a fan that covers all of $\mathbb{R}^d$. For X_Σ this means a compactification (completion). We assume in this article Σ to cover $\mathbb{R}^d$ and hence Σ_0 to have the sphere as its point set.

Our main goal is to extend some of the work of Oda and Miyake [10, 11] from three to higher dimensions. In particular, we study questions of projectiveness of X_Σ and construct classes of nonprojective toric varieties in all dimensions. We make use of the technique of the so-called Gale-transforms which proved to be very helpful in combinatorial convexity theory.

In the "dictionary" that relates properties of Σ to properties of X_Σ we focus on three "words":

1. For $d = 2$, Σ can also be obtained by projecting the faces of a convex polyhedron P (see Fig. 1). For $d > 2$, this is, in general, not true. If it is true, we say Σ is *strongly polytopal.* X_Σ is called *projective* if it is globally the set of zeros of finitely many homogeneous polynomials in $d+1$ variables. The following equivalence is true (see for example, [2], page 118):

$$\Sigma \text{ strongly polytopal} \Leftrightarrow X_\Sigma \text{ projective.}$$

2. If σ is simplicial and if $\dim \sigma = d$, that is, $\sigma \in \Sigma^{(d)}$, we assign to the generating vectors $a_1, \ldots, a_d$ the determinant $\det \sigma := \det[a_1, \ldots, a_d]$. It can be shown ([10], page 12)

$$\det \sigma = \pm 1 \quad \text{for all } \sigma \in \Sigma^{(d)} \Leftrightarrow X_\Sigma \text{ is nonsingular.}$$

3. If in a cell complex $\mathscr{C}$ we choose a relative interior point p of a cell C ($p \in \operatorname{relint} C$), and if the star of C is replaced by the join of p to the boundary of this star, we say, a *stellar subdivision* $s(p, \mathscr{C})$ has been achieved (Fig. 2). We call a stellar subdivision $S(\mathbb{R}_+ a, \Sigma)$ *regular* if $a = a_1 + \cdots + a_k$ for $a_1, \ldots, a_k$ generating a cone of Σ. (The term "barycentric" used by Oda and Miyake is somewhat misleading.) There is a correspondence (see [10]):

$$\text{(regular) stellar subdivision of } \Sigma \to \text{blow-up of } X_\Sigma \text{ (along a nonsingular center).}$$

The inverse operation of a blow-up is called a *blow-down* (or σ-process).

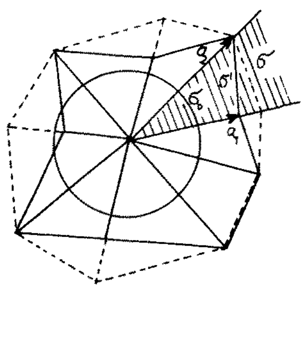

Fig. 1

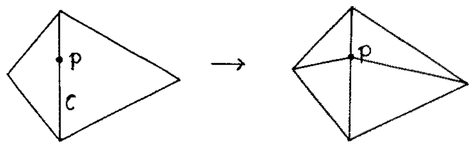

Fig. 2

2. Gale-Transforms and Facet-Splitting

Let $V := = \{a_1, \ldots, a_v\}$ be a finite set of points (vectors) in $\mathbb{R}^d$, and let $(\alpha_1, \ldots, \alpha_v)$ be an affine dependence of V, that is,

$$\alpha_1 a_1 + \cdots + \alpha_v a_v = 0, \qquad \alpha_1 + \cdots + \alpha_v = 0.$$

We choose a basis of the $(v-d-1)$-dimensional space of all affine dependences and write them as rows of a matrix

$$\begin{pmatrix} \alpha_{11} & \cdots & a_{1,v} \\ \vdots & & \vdots \\ \alpha_{v-d-1,1} & \cdots & \alpha_{v-d-1,v} \end{pmatrix} =: (\bar{a}_1, \ldots, \bar{a}_v).$$

The set of columns $\bar{V} := \{\bar{a}_1, \ldots, \bar{a}_v\}$ is called a *Gale-transform* of V (see, for example, Grünbaum [5] or McMullen and Shephard [9], Ewald and Voß [4], and, for a coordinate-free introduction, McMullen [8]).

Example. Consider in $\mathbb{R}^3$ the triangular prism with vertices $a_1 = (1,0,0)$, $a_2 = (0,1,0)$, $a_3 = (0,0,1)$, $a_4 = (0,-1,-1)$, $a_5 = (-1,0,-1)$, and $a_6 = (-1,-1,0)$. Let the rectangular faces be split as indicated in Fig. 3. Figure 4 presents a Gale-transform of $V = \{a_1, \ldots, a_6\}$. If $a_{i_1}, \ldots, a_{i_k}$ generate a cell ("face") σ of Σ we call $\bar{V} \setminus \{\bar{a}_{i_1}, \ldots, \bar{a}_{i_k}\}$ the *coface* $\bar{\sigma}$ of σ. We make use of a basic fact [13]:

Theorem. *Σ is strongly polytopal if and only if $\bigcap_{\sigma \in \Sigma} \operatorname{relint} \bar{\sigma} \neq \emptyset$.*

If, in particular, $0 \in \bigcap_{\sigma \in \Sigma} \operatorname{relint} \bar{\sigma}$, then $a_1, \ldots, a_v$ represent the vertices of a convex polytope.

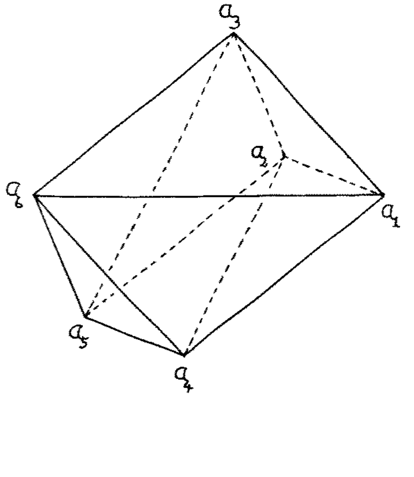

Fig. 3

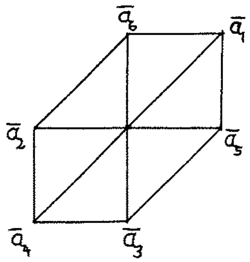

Fig. 4

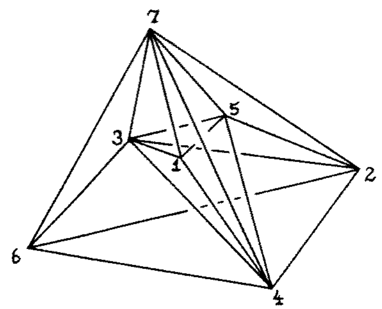

Fig. 5

In the above example, the prism without face-splitting has a face-structure that satisfies the latter condition. If the splittings are carried out, however, among the cofaces there are $\bar{a}_1\bar{a}_4\bar{a}_6$, $\bar{a}_2\bar{a}_5\bar{a}_4$, $\bar{a}_3\bar{a}_6\bar{a}_5$ which have no relative interior point in common. So we obtain a nonstrongly polytopal fan Σ.

All determinants of Σ except $\det[a_4, a_5, a_6]$ are ± 1. Applying the regular stellar subdivision, $S(\mathbb{R}_+ a, \Sigma)$ where $a = a_4 + a_5 + a_6$ provides a nonsingular, nonprojective variety $X_{S(\mathbb{R}_+ a, \Sigma)\Sigma}$.

An analogous construction for $d = 4$ can be obtained as follows. Consider the subdivision of a three-simplex as indicated in Fig. 5. It consists of double-simplexes $\Delta_1 := 12457$, $\Delta_2 := 23567$, $\Delta_3 := 31647$, and four simplexes 1435, 2346, 1235, 1357. A Gale-transform $\bar{1}, \ldots, \bar{7}$ of $1, \ldots, 7$ is shown in Fig. 6. This decomposition of the simplex can be looked at as the Schlegel-diagram of a four-polytope P, that is, a central projection of P into one of its facets. A direct construction of P can be obtained by finding a Gale-transform of the points in Fig. 6 and taking their convex hull. It is known that Fig. 6 represents again a

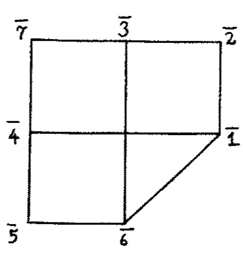

Fig. 6

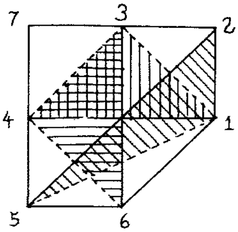

Fig. 7

Gale-transform of P. The double-simplexes $\Delta_1, \Delta_2, \Delta_3$ can be looked at as analogues of the rectangular faces of the prism in Fig. 3 which are two-dimensional double-simplexes.

Now we may split each of the facets $\Delta_1, \Delta_2, \Delta_3$ by a one-dimensional or a two-dimensional diagonal into three or two simplexes. There are eight typical combinations of such facet-splittings, two of which turn out to provide non-strongly polytopal fans:

I. Split Δ_1 at 12, Δ_2 at 56, and Δ_3 and 347;

II. split Δ_1 at 457, Δ_2 at 237, and Δ_3 at 16.

Figures 7 and 8 provide for cases I and II, respectively, three cofaces that have no relative interior point in common.

For any $d \geq 3$ we obtain the following statement. By *facet-splitting* we mean generally the straight subdivision of the facets of a convex polytope into convex polytopes whose vertices are all vertices of the original polytope.

Theorem 1. *Let P be a convex d-polytope, $d \geq 3$, $0 \in \operatorname{int} P$, with v rational vertices, and let P have at least $v - d$ facets which are simplicial but not simplexes. Then by appropriate facet-splittings we obtain at least one complex $B(P)$ on the boundary of P such that $\Sigma = \Sigma(B(P))$ is not strongly polytopal.*

Proof. A Gale-transform of the vertex set $\operatorname{vert} P$ of P spans a space of dimension $v - d - 1$. Let $\Delta'_1, \ldots, \Delta'_{v-d}$ be simplicial facets that are not simplexes. If Δ'_j has more than $d+1$ vertices, we apply facet-splittings until we obtain a piece Δ_j of Δ'_j that has precisely $d+1$ vertices. So let $\Delta_1, \ldots, \Delta_{v-d}$ be $(d-1)$-cells of a cell-complex $B_0(P)$ realized on the boundary of P.

To each Δ_j let $\bar{\Delta}_j$ be a coface which is $(v-d-2)$-dimensional and hence spans a hyperplane H_j in $\mathbb{R}^{v-d-1}$. Now Δ_j can be split into simplexes using a Radon partition of $\operatorname{vert} \Delta_j$ into subsets D_j, D'_j such that $D_j \cup D'_j = \operatorname{vert} \Delta_j$,

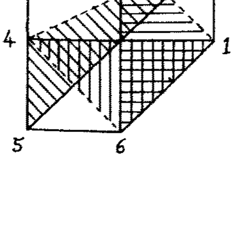

Fig. 8

$D_j \cap D_j' = \varnothing$, $(\operatorname{conv} D_j) \cap (\operatorname{conv} D_j') \neq \varnothing$. We obtain $(v-d-1)$-dimensional cofaces $\operatorname{conv}(\bar{\Delta}_j \cup \{\bar{a}_{i_j}\})$, $a_{i_j} \in \operatorname{vert} \Delta_j$, which can lie on either side of H_j depending on whether $a_{i_j} \in D_j$ or $a_{i_j} \in D_j'$. Since $\Delta_1', \ldots, \Delta_{v-d}'$ are simplicial, all splittings of the Δ_j are independent. Hence they can be chosen in such a way that

$$\bigcap_{j=1}^{v-d} \operatorname{relint} \operatorname{conv}\left(\bar{\Delta}_j \cup \{\bar{a}_{i_j}\}\right) = \varnothing .$$

By Shephard's theorem, this proves our assertion. □

Remark 1. Polytopes P as assumed in Theorem 1 do exist for any $d \geq 3$. Let, for example, C be a d-dimensional cube with 0 as its center, and consider in any one-dimensional face pq of C the supporting hyperplane H such that $H \cap C = pq$ and such that H is perpendicular to the plane spanned by 0, p, and q. Then the half-spaces bounded by such H and containing 0 intersect in a polytope P that has $v = 2^d + 2d$ vertices and $d \cdot 2^{d-1}$ simplicial facets that are not simplexes. Since $v - d = 2^d + d < d \cdot 2^{d-1}$ for $d \geq 3$ there are sufficiently many such faces available. Further examples for $d = 4$ can be found in Altshuler and Steinberg [1].

Remark 2. In many cases there will be more than one nonstrongly polytopal fan that can be constructed from P. If, for example, d is even and $v - d$ is odd, then the two possible facet-splittings of Δ_j are nonisomorphic. Replacing $\operatorname{conv}(\bar{\Delta}_j \cup \{a_{i_j}\})$ by $\operatorname{conv}(\bar{\Delta}_j \cup \{a_{k_j}\})$ where a_{i_j}, a_{k_j} are in different sets D_j, D_j', $j = 1, \ldots, v-d$, provides a fan that is nonisomorphic to the first one. This example generalizes cases I and II in the above four-dimensional example.

Remark 3. If the hyperplanes H_j are linearly dependent, then, in general, less than $v - d$ facet splittings will do to obtain nonstrongly polytopal fans. The same is true in many cases where the Δ_j have more than $d+1$ vertices.

3. Canonical Extensions

We present now a further method of constructing nonprojective toric varieties from given ones. If the variety X_Σ we start with has no singularities the same is true for the new ones. Also the possibility of turning the variety into a projective space by blow-ups and -downs is preserved.

Let Σ be a simplicial fan in $\mathbb{R}^d$, and let $V := \{a_1, \ldots, a_v\}$ be the set of its generating primitive lattice vectors. We embed $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$, replace a vertex $(a_j, 0)$, say $(a_1, 0)$, by $(a_1, 1)$, and join $(a_1, 1)$ to the complement of the star of $(a_1, 0)$ in the complex $B_{\text{st}}(Q)$ on the boundary of the star-shaped polytope such that Σ projects the faces of $B_{\text{st}}(Q)$ (see Section 1). Then we join $(0, -1)$ to the boundary of the complex thus constructed. We obtain a complex $\tilde{B}_{\text{st}}(\tilde{Q})$ which bounds a star-shaped polytope $\tilde{Q}$ in $\mathbb{R}^{d+1}$. We call $\tilde{B}_{\text{st}}(\tilde{Q})$ or its associated fan $\tilde{\Sigma}$ a *canonical extension* of $B_{\text{st}}(Q)$ or Σ, respectively. Also $X_{\tilde{\Sigma}}$ is then called a *canonical extension* of X_Σ (Fig. 9). (According to Provan and Billera [12] $\tilde{B}_{\text{st}}(\tilde{Q})$ is the *simplicial wedge* of $B_{\text{st}}(Q)$ on a_1; according to Klee and Kleinschmidt [7] the *dual wedge*.)

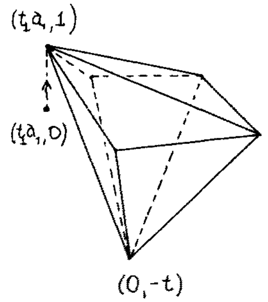

Fig. 9

$\tilde{B}_{st}(\tilde{Q})$ can also be obtained by doubling $\bar{a}_1$ in a Gale-transform $\bar{V}$ of V: $\bar{a}_1 = \bar{a}_{v+1}$. The additional affine dependence $\bar{a}_1 - \bar{a}_{v+1} = 0$ provides us the new vertices $(a_1, 1), (a_2, 0), \ldots, (a_v, 0), (0, -1)$ in the extended original space. This interpretation of the canonical extension should be kept in mind but is not necessary for what follows.

Theorem 2. *Let $X_{\tilde{\Sigma}}$ be a canonical extension of X_{Σ}.*

(1) *If X_{Σ} has dimension d, $X_{\tilde{\Sigma}}$ has dimension $d+1$.*
(2) *If X_{Σ} is projective, so is $X_{\tilde{\Sigma}}$.*
(3) If X_{Σ} is nonprojective, so is $X_{\tilde{\Sigma}}$.
(4) If X_{Σ} is nonsingular and can, by blow-ups and -downs, be transformed into a projective space, the same is true for $X_{\tilde{\Sigma}}$.

Proof. (1) True, by definition.

(2) Let $t_1 a_1, \ldots, t_v a_v$, $t_j > 0$, $j = 1, \ldots, v$, be vertices of a convex polytope. Then $(t, a_1, 0)$ is outside $P_0 := \text{conv}\{(t_1 a_1, 1), (t_2 a_2, 0), \ldots, (t_v a_v, 0)\}$. Hence, if $t > 0$ is sufficiently large, the line segment joining $(t_1 a_1, 1)$ and $(0, -t)$ is also outside P_0. Therefore, $\tilde{\Sigma}$ is also strongly polytopal, the realizing polytope being $\text{conv}(P_0 \cup \{0, -t\})$.

(3) Suppose $\tilde{\Sigma}$ were strongly polytopal, being realized by a polytope $\tilde{P}$. Then $P := \tilde{P} \cap \{X_{d+1} = 0\}$ is a realization for Σ, a contradiction.

(4) The determinants of $d+1$ rows associated with facets of P_0 evidently reduce, up to a factor ± 1, to determinants of d rows associated with the facets of $B_{st}(Q)$, hence are ± 1.

We apply first a stellar subdivision $S(\mathbb{R}_+ p, \tilde{\Sigma})$ where $p = (a_1, 0) + (0, -1)$. The complex $\mathscr{C}' := [B_{st}(Q) \setminus \text{star}(a_1, B(Q))] \cup [p \cdot \text{link star}(a_1, B(Q))]$ ($p \cdot \mathscr{C} := \{\text{conv}(\{p\} \cup \sigma) | \sigma \in \mathscr{C}\}$ the join of p and $\mathscr{C}$) is isomorphic to $B(Q)$. Hence regular stellar subdivisions and inverses applied successively to $B_{st}(Q)$ correspond to analogous operations for $\mathscr{C}'$ and can naturally be extended to operations for $\tilde{B}_{st}(\tilde{Q})$. If $B_{st}(Q)$ is thus transformed into a d-simplex, $\tilde{B}_{st}(\tilde{Q})$ is being transformed into a double-simplex which, in turn, is readily transformed into a simplex. (Compare Provan and Billera [12] and Klee and Kleinschmidt [7].) □

Theorem 2 provides a construction method for nonprojective toric varieties in all dimensions $d > 3$. In particular, we have from the examples presented in

Section 2:

Theorem 3. (1) *For any $d \geq 3$ there exist nonprojective toric varieties with $v = d + 3$ exceptional divisors.*

(2) *For any $d \geq 3$ there exist nonsingular, nonprojective toric varieties having $v = d + 4$ exceptional divisors.*

Remark. If X_Σ can be blown down, this only carries over to $X_{\tilde{\Sigma}}$ if $\mathbb{R}_+ a_1 \neq \mathbb{R}_+ p$ in $S(\mathbb{R}_+ p, \Sigma)$.

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Received July 3, 1985.