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Covering the Plane with Convex Polygons

János Pach

Department of Mathematics, State University of New York at Stony Brook, Stony Brook, NY 11794
and Mathematical Institute of the Hungarian Academy of Sciences, Budapest 1364, Pf. 127

Abstract. It is proved that for any centrally symmetric convex polygonal domain P and for any natural number r, there exists a constant $k = k(P, r)$ such that any k-fold covering of the plane with translates of P can be split into r simple coverings.

1. Introduction

A system of sets $\mathscr{S} = \{S_i | i \in I\}$ is said to form a *k-fold covering of X* if every element of X is contained in at least k members of $\mathscr{S}$. A 1-fold covering is called a *simple covering* or, briefly, a *covering.*

In 1980 at a meeting on discrete geometry in Salzburg I proposed the following conjecture (See [4].)

Conjecture 1.1. *There exists a sufficiently large integer k such that any k-fold covering of the plane with open unit discs can be decomposed into two simple coverings.*

Though many promising attempts have been made to attack this problem, Conjecture 1.1 is still unsettled.

The main result of this paper is the following.

Theorem 1. *Let P be an open domain bounded by a centrosymmetric convex closed polygon. Then there exists a natural number* $k = k(P)$ *such that any k-fold covering of* $\mathbb{R}^2$ *with translates of P can be decomposed into two simple coverings.*

As a matter of fact, in Section 2 we shall prove this result in a slightly stronger form (see Theorem 3).

Given any $\varepsilon > 0$ and a system $\mathscr{S} = \{S_i | i \in I\}$ of centrosymmetric sets, let $(1+\varepsilon)\mathscr{S}$ denote the set-system obtained from $\mathscr{S}$ by replacing each S_i by its $1+\varepsilon$ times larger homothetic copy centered at the same point.

The following assertion is an immediate consequence of Theorem 1.

Corollary 1.2. *Let D be an open domain bounded by a centrosymmetric convex closed curve (e.g., a circle), and let $\varepsilon > 0$. Then there exists a natural number $k = k(D, \varepsilon)$ with the property that any k-fold plane covering $\mathscr{D}$ with translates of D can be decomposed into two parts $\mathscr{D}_1 \cup \mathscr{D}_2$ such that $(1+\varepsilon)\mathscr{D}_1$ and $(1+\varepsilon)\mathscr{D}_2$ are simple coverings.*

However, a little better result can be established without using Theorem 1.

Theorem 2. *Let $D \subset \mathbb{R}^n$ be an open domain bounded by a centrosymmetric convex closed surface, and let $\varepsilon > 0$. Then there exists a natural number $k' = k'_n(\varepsilon)$ having the property that any k'-fold covering of $\mathbb{R}^n$ with translates of D can be decomposed into two set-systems $\mathscr{D}_1 \cup \mathscr{D}_2$ such that $\mathscr{D}_1$ and $(1+\varepsilon)\mathscr{D}_2$ are simple coverings.*

Proof. An easy compactness argument shows that it is sufficient to prove our statement for k'-fold coverings $\mathscr{D} = \{D_i | i \in I\}$ of an arbitrarily large cube $C \subseteq \mathbb{R}^n$, where $|\mathscr{D}| = |I|$ is finite.

Suppose that the center of D is at 0, and let $0 < \varepsilon < 1$. Then there exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that $D = \{x \in \mathbb{R}^n | \|x\| < 1\}$. The n-dimensional space equipped with this norm is usually called a *Minkowski space* whose *gauge body* is D (cf. [8]). Let c_i denote the center of D_i, i.e., $D_i = c_i + D$ $(i \in I)$. Let us select a *maximal* subset $J \subset I$ with the property that $\|c_i - c_j\| \geq \varepsilon$ for any $i, j \in J$.

Given any $x \in C$, put

$$J(x) := \{ j \in J | x \in D_j \}.$$

Since $c_j + \frac{1}{2}\varepsilon D$ $(j \in J(x))$ are pairwise disjoint subsets of $x + \frac{3}{2}D$, we obtain

$$|J(x)| \leq \frac{\mathrm{Vol}_n(\frac{3}{2}D)}{\mathrm{Vol}_n(\frac{1}{2}\varepsilon D)} = \left(\frac{3}{\varepsilon}\right)^n.$$

Thus, if k' exceeds this value then $\mathscr{D}_1 := \{D_i | i \in I \setminus J\}$ forms a covering of the cube C.

For any $x \in C$, choose a $D_i \in \mathscr{D}_1$ $(i \notin J)$ which contains x. By the maximal property of J, now there exists a $j \in J$ such that $\|c_i - c_j\| < \varepsilon$. Then $\|x - c_j\| \leq \|x - c_i\| + \|c_i - c_j\| < 1 + \varepsilon$, i.e., x is covered by $c_j + (1+\varepsilon)D$. In other words, $(1+\varepsilon)(\mathscr{D} \setminus \mathscr{D}_1)$ is a covering of C, as desired. □

Essentially the same argument yields the following slight generalization of Theorem 2: Let $D \subset \mathbb{R}^n$ be an open domain bounded by a centrosymmetric convex closed surface, let $\varepsilon > 0$ and suppose that $r \geq 2$ is an integer. Then there exists a $k' = k'_{n,r}(\varepsilon)$ (independent of D) such that any k'-fold covering $\mathscr{D}$ of $\mathbb{R}^n$ with translates of D can be decomposed into r parts $\mathscr{D}_1 \cup \mathscr{D}_2 \cup \cdots \cup \mathscr{D}_r$ such that $\mathscr{D}_1, (1+\varepsilon)\mathscr{D}_2, \ldots, (1+\varepsilon)\mathscr{D}_r$ are simple coverings.

For more problems and results on multiple coverings consult [3], [4], [5].

2. Proof of Theorem 1

We shall reformulate our problem in a little more convenient dual form.

Let $v_1, v_2, \ldots, v_n$ and 0 be the vertices of P (in cyclic order) and the center of P, respectively. For any $x, y \in \mathbb{R}^2$, let $P(xy)$ denote a congruent copy of P translated by $\overrightarrow{xy}$.

Consider now a k-fold covering $\{P_j | j \in J\}$ of the plane with translates of P, where k will be specified later. Let c_j denote the center of P_j, i.e., $P_j = P(0c_j)$. Using the fact that P is centrosymmetric, we obtain that, for any $x \in \mathbb{R}^2$ and $j \in J$, x is covered by P_j if and only if $c_j \in P(0x)$. Thus, the number of c_js contained in $P(0x)$ is at least k.

Let us divide the plane by straight lines into disjoint congruent squares (cells) of sides

$$\delta := \min_r \min_{s,t \neq r} \frac{d(v_r, v_s v_t)}{\sqrt{2}}, \tag{1}$$

where $d(v_r, r_s v_t)$ denotes the distance between v_r and the line $v_s v_t$. Using standard compactness arguments, we can assume without loss of generality that

(a) no straight line $c_i c_j$ $(i \neq j)$ is parallel to any edge of P;
(b) every c_j is contained in the interior of some square of the above cell decomposition;
(c) every cell contains only finitely many c_js.

Since any translate of P has nonempty intersection with at most $((\max d(v_r, v_s))/\delta + 2)^2$ cells, we obtain that for every $x \in \mathbb{R}^2$ there is a cell S such that the number of c_js contained in $P(0x) \cap S$ is at least

$$k' := k\delta^2 \Big/ \Big(\max_{r,s} d(v_r, v_s) + 2\delta \Big)^2. \tag{2}$$

Hence it is enough to prove the following.

Theorem 1′. *There exists a sufficiently large natural number $k = k(P)$ with the property that any finite system of points $\mathscr{C} = \{c_i | i \in I\}$ arranged in a square S of side δ and satisfying* (a) *can be coloured by two colours* (*red and green*) *so that every translate of P covering at least k' members of $\mathscr{C}$ contains points of both colours* (cf. (1), (2)).

The set of all points $c_i \in \mathscr{C}$, for which there exists a vertex v_r of P $(1 \leq r \leq n)$ such that $P(v_r c_i) \cap \mathscr{C} = \varnothing$, is said to be the *boundary* of $\mathscr{C}$ and is denoted by $Bd\mathscr{C}$. (Note that $P(v_r c_i)$ is an open set.) For any $c_i \in Bd\mathscr{C}$ let

$$\operatorname{type}(c_i) := \{ r | 1 \leq r \leq n, P(v_r c_i) \cap \mathscr{C} = \varnothing \}.$$

Let us define on $Bd\mathscr{C}$ a directed graph $\vec{G}$ in the following way. Two boundary points $c_i, c_j \in Bd\mathscr{C}$ are connected by a *directed edge* (directed straight line segment) $\overrightarrow{(c_i, c_j)} \in E(\vec{G})$ if and only if there exists a translate P' of P with vertices $v'_1, v'_2, \ldots, v'_n$ such that $P' \cap \mathscr{C} = \varnothing$ and c_j and c_i are lying on two consecutive sides of P', i.e.,

$$c_j \in [v'_{r-1}, v'_r], \qquad c_i \in [v'_r, v'_{r+1}] \quad \text{for some } r \ (1 \leq r \leq n), \tag{3}$$

where the indices of v are taken mod n. Because of the choice of δ (see (1)), all vertices of P', except perhaps v'_r, are outside S. It is also clear by property (a) that

$\overline{P}'$ (the closure of P') cannot contain any element of $\mathscr{C}$ distinct from c_i and c_j. Taking into account that the vector $\overrightarrow{c_i c_j}$ is in the interior of the convex cone induced by the vectors $\overrightarrow{v'_{r+1}v'_r} = \overrightarrow{v_{r+1}v_r}$ and $\overrightarrow{v'_r v'_{r-1}} = \overrightarrow{v_r v_{r-1}}$, and these cones are openly disjoint for different rs, we obtain that the natural number r satisfying (3) is uniquely determined. Let type$(\overrightarrow{c_i, c_j}) := r$. Obviously,

$$\text{type}\left(\overrightarrow{c_i, c_j}\right) \in \text{type}(c_i) \cap \text{type}(c_j) \quad \text{for any } \left(\overrightarrow{c_i, c_j}\right) \in E(\vec{G}). \tag{4}$$

Further, let

$$E^r := \left\{ \left(\overrightarrow{c_i, c_j}\right) \in E\left(\vec{G}\right) | \text{type}\left(\overrightarrow{c_i, c_j}\right) = r \right\}.$$

Proposition 2.1. *For any r $(1 \le r \le n)$, the points belonging to E^r form a simple directed chain*, i.e., *a sequence $(c_0^r, c_1^r, \dots, c_{j(r)}^r)$ such that*

(i) *$(c_i^r, c_{i+1}^r) \in E^r$ $(0 \le i < j(r))$ and E^r has no more elements;*

(ii) *$\overrightarrow{c_i^r c_{i+1}^r}$ is in the interior of the convex cone of the vectors $\overrightarrow{v_{r+1}v_r}$ and $\overrightarrow{v_r v_{r-1}}$ $(0 \le i < j(r))$.*

Proof. Let r $(1 \le r \le n)$ be fixed and let $(c_0^r, c_1^r, \dots, c_j^r)$ be a *maximal* sequence with the property that $(\overrightarrow{c_i^r, c_{i+1}^r}) \in E^r$ for every $(0 \le i < j)$. It follows now from the definitions that there exist $x_1, x_2, \dots, x_j \in \mathbb{R}^2$ such that

$$T^r := P\left(v_{r-1}c_0^r\right) \cup \left(\bigcup_{0 < i \le j} P\left(v_r x_i\right) \right) \cup P\left(v_{r+1}c_j^r\right) \tag{5}$$

is disjoint from $\mathscr{C}$, but $c_{i-1}^r, c_i^r \in \overline{P}(v_r x_i)$, hence $\overline{T}^r$ (the closure of T^r) contains $c_0^r, c_1^r, \dots, c_j^r$.

Suppose, in order to obtain a contradiction, that there is an edge $(\overrightarrow{c, c'}) \in E^r \setminus \{(\overrightarrow{c_i^r, c_{i+1}^r}) | 0 \le i < j\}$. Then one can find an $x \in \mathbb{R}^2$ satisfying $P(v_r x) \cap \mathscr{C} = \varnothing$ and $c, c' \in \overline{P}(v_r x)$. In view of assumption (a) and the fact that $T^r \cap \mathscr{C} = \varnothing$, we have $x \notin \overline{T}^r$. However, in this case $P(v_r x) \cap \{c_0^r, c_1^r, \dots, c_j^r\} \ne \varnothing$. This contradiction establishes (i).

The second part of the statement is evident. □

Proposition 2.2. *Let $|\mathscr{C}| \ge 2$ and $c \in Bd\mathscr{C}$. Suppose that $\{r, r+1, \dots, s-1, s\}$ is a maximal interval* (mod n) *all of whose elements belong to* type(c). *Then*

(i) *There exist $c_i, c_j \in Bd\mathscr{C}$ such that $(\overrightarrow{c_i, c}) \in E^r$, $(\overrightarrow{c, c_j}) \in E^s$.*

(ii) *If $s \ne r$ then c is the endpoint of E^r and the initial point of E^s*, i.e., $c = c_{j(r)}^r = c_0^s$.

(iii) *If $s \ne r, r+1$ then $E^t = \varnothing$ for all $t \in \{r+1, \dots, s+1\}$.*

Proof. Since $|\mathscr{C}| \ge 2$, type$(c) \ne \{1, 2, \dots, n\}$.

Part (i) is an immediate consequence of the maximality of $\{r, r+1, \dots, s\}$.

To prove (ii), suppose indirectly that there exists a $c' \in bd\mathscr{C}$ such that, e.g., $(\overrightarrow{c, c'}) \in E^r$. Then $c' \in P(v_{r+1}c)$, contradicting $r+1 \in$ type(c).

Assume finally that $t-1, t, t+1 \in \text{type}(c)$, but $E^t \neq \varnothing$. That is, one can choose $c_g, c_h \in Bd\mathscr{C}$, $x \in \mathbb{R}^2$ satisfying $P(v_t x) \cap \mathscr{C} = \varnothing$ and $c_g, c_h \in \bar{P}(v_t x)$. Obviously, $c_g, c_h \notin \bar{P}(v_{t-1}c) \cup \bar{P}(v_t c) \cup \bar{P}(v_{t+1}c)$, which implies that $c \in P(v_t x)$. This contradiction proves (iii). □

Lemma 2.3

$$\Delta := \left(c_0^1, c_1^1, \ldots, c_{j(1)}^1 = c_0^2, c_1^2, \ldots, c_{j(2)}^2 = c_0^3, \ldots, c_{j(n-1)}^{n-1} = c_0^n, c_1^n, \ldots, c_{j(n)}^n\right)$$

is a cyclically ordered sequence of the elements of $Bd\mathscr{C}$ $(c_{j(n)}^n = c_0^1)$ *having the following properties.*

(i) *Every* $c \in Bd\mathscr{C}$ *occurs in* Δ *at least one and at most twice.*

(ii) *If some* $c \in Bd\mathscr{C}$ *occurs in* Δ *twice, then* c *is called a singular point and* $\text{type}(c) = \{r, r + \frac{1}{2}n\}$ *for some* $1 \le r \le n \pmod n$. *Moreover,* $\text{type}(c) = \text{type}(c^*)$ *for any two singular points* $c, c^* \in Bd\mathscr{C}$.

(iii) *Connecting each pair of consecutive elements of* Δ *by a straight line segment, we obtain a closed polygon which does not intersect itself. (For the sake of simplicity, this polygon will also be denoted by* Δ.)

Proof. The first part of (i) is obvious by Proposition 2.2(i).

Let $c \in Bd\mathscr{C}$ and suppose without loss of generality that $c = c_i^1$ for some i $(0 \le i < j(1))$. If c' and c'' are any two consecutive members of

$$\Delta' = \left(c_{i+1}^1, \ldots, c_{j(1)}^1 = c_0^2, c_1^2, \ldots, c_{j(2)}^2 = c_0^3, \ldots, c_{j(n/2)}^{n/2} = c_0^{n/2+1}\right),$$

and e is a straight line through c parallel to $\overrightarrow{v_1 v_n} = \overrightarrow{v_{n/2} v_{n/2+1}}$, then, by Proposition 2.1(ii), $d(c'', e) > d(c', e)$. Consequently, the elements of Δ' are different from each other and from c. Exactly the same can be said about the sequence

$$\Delta'' = \left(c_{j(n/2+1)}^{n/2+1} = c_0^{n/2+2}, c_1^{n/2+2}, \ldots, c_{j(n/2+2)}^{n/2+2} = c_0^{n/2+3}, \ldots, c_{j(n)}^n = c_0^1, \ldots, c_{i-1}^1\right).$$

Since c can be identical with at most one point of $(c_1^{n/2+1}, c_2^{n/2+1}, \ldots, c_{j(n/2+1)-1}^{n/2+1})$, the second part of (i) is also true.

Furthermore, if $c = c_i^1$ occurs in Δ twice then $c = c_j^{n/2+1}$ for some $0 < j < j(n/2+1)$; hence, by (4), $\text{type}(c) \supseteq \{1, \frac{1}{2}n+1\}$. It is easily seen that $\text{type}(c)$ cannot have any other element, i.e., $\text{type}(c) = \{1, \frac{1}{2}n+1\}$. To prove the second part of (ii), suppose indirectly that there is another singular point $c^* \in Bd\mathscr{C}$ with $\text{type}(c^*) = \{r, \frac{1}{2}n + r\}$, $r \neq 1, \frac{1}{2}n+1$. Then c^* is an element of

$$\Delta''' = \left(c_{i+1}^1, \ldots, c_{j(1)}^1 = c_0^2, c_1^2, \ldots, c_{j(2)}^2 = c_0^3, \ldots, c_{j(n/2)}^{n/2} = c_0^{n/2+1}, \ldots, c_{j-1}^{n/2+1}\right)$$

and all points of this sequence are contained in the convex cone determined by the vectors $\overrightarrow{v_1 v_n}$ and $\overrightarrow{v_2 v_1}$, whose apex is at c. Thus, either $P(v_r c^*)$ or $P(v_{r+n/2} c^*)$ contains c, the desired contradiction.

Finally, let c and c' be any two consecutive elements of Δ, e.g., $c = c_i^1$ and $c' = c_{i+1}^1$ $(0 \le i < j(1))$. Then there exists an $x \in \mathbb{R}^2$ satisfying $c, c' \in \bar{P}(v_1 x)$ and

$P(v_1x)\cap C=\varnothing$. The same argument as the one used in the proof of (i) shows that Δ' and Δ'' cannot cross the edge $\overrightarrow{(c,c')}$. On the other hand, both c and c' are situated outside the region $T^{n/2+1}$ (defined by (5)), and

$$x\notin P\left(v_{n/2}c_0^{n/2+1}\right)\cup\left(\bigcup_{0\le h\le j(n/2+1)}P\left(v_{n/2+1}c_h^{n/2+1}\right)\right)\cup P\left(v_{n/2+2}c_{j(n/2+1)}^{n/2+1}\right).$$

From this, one can easily infer that the missing piece $(c_0^{n/2+1},c_1^{n/2+1},\dots,c_{j(n/2+1)}^{n/2+1})$ of Δ cannot cross $\overrightarrow{(c,c')}$ either, which completes the proof of (iii).

Note, however, that Δ can "touch" itself. For example, it is possible that $c'=c_h^{n/2+1}$ and $c=c_{h+1}^{n/2+1}$ for some $0\le h<j(\frac{1}{2}n+1)$, i.e., $\overrightarrow{(c,c')}\in E^1$ and $\overrightarrow{(c',c)}\in E^{n/2+1}$. □

The following assertion is a simple corollary to Lemma 2.3(iii).

Corollary 2.4. *There exists a 2-colouring f of the boundary of $\mathscr{C}$ with black and white (f: $Bd\mathscr{C}\to\{B,W\}$) such that there are no two consecutive black points and no three consecutive white points on Δ.*

Lemma 2.5. *Let P' be any translate of P. Then $P'\cap Bd\mathscr{C}$ is the union of at most two intervals of consecutive elements of Δ.*

Proof. By the choice of δ (see (1)), the square $S\supset\mathscr{C}$ is so small that it can intersect at most two sides of P' ($[v_n',v_1']$ and $[v_1',v_2']$, say), and these two sides are necessarily consecutive. For a contradiction, assume without loss of generality that there are two edges $\overrightarrow{(c,c')},\overrightarrow{(d,d')}\in E(\vec{G})$ crossing $[v_n',v_1']$ such that $c,d\in P'$ and $c',d'\notin P'$. By Proposition 2.1(ii) it is obvious that $\overrightarrow{(c,c')},\overrightarrow{(d,d')}\in E^1\cup E^2\cup\cdots\cup E^{n/2}$, i.e., all of c, c', d, and d' are elements of the sequence

$$\Delta'_0=\left(c_0^1,c_1^1,\dots,c_{j(1)}^1=c_0^2,c_1^2,\dots,c_{j(2)}^2=c_0^3,\dots,c_{j(n/2)}^{n/2}=c_0^{n/2+1}\right).$$

Let e denote a straight line through c_0^1 parallel to $[v_n',v_1']$. Similarly, as in the proof of Lemma 2.3(i), we can see that all elements of Δ'_0 are on the same side of e. Moreover, if b and b' are any two consecutive elements of Δ'_0 (and b comes first), then their distances from e satisfy $d(b',e)>d(b,e)$. Hence Δ'_0 can intersect $[v_n',v_1']$ only once, contradiction. □

Lemma 2.6. *Let P' be any translate of P containing exactly two boundary points of $\mathscr{C}$, i.e., $P'\cap Bd\mathscr{C}=\{d_0,d_1\}$. Then, either d_0 and d_1 are two consecutive elements of Δ, or there exist another translate P'' of P and $\lambda\in\{0,1\}$ such that*

(i) $P''\cap\mathscr{C}\subseteq P'\cap\mathscr{C}$, $|P''\cap\mathscr{C}|\ge\frac{1}{2}|P'\cap\mathscr{C}|$;

(ii) $P''\cap Bd\mathscr{C}=\{d_\lambda\}$.

Proof. Let $v_1',v_2',\dots,v_n'$ denote the vertices of P', and suppose again without loss of generality that the square S intersects the sides $[v_n',v_1']$ and $[v_1',v_2']$ only.

Assume first that $d_{1-\lambda}\notin P(v_1d_\lambda)$ for $\lambda=0,1$. Then $P(v_1d_\lambda)\cap\mathscr{C}=\varnothing$ ($\lambda=0,1$), otherwise P' ($\supseteq P(v_1d_\lambda)\cap\mathscr{C}$) would contain some $d\in Bd\mathscr{C}$ ($d\ne d_0,d_1$), contradicting $|P' \cap Bd\mathscr{C}| = 2$. Thus, both d_0 and d_1 belong to E^1 and, by Proposition 2.1, they can be joined by a directed polygon

$$\left(d_\lambda = c_i^1, c_{i+1}^1, \ldots, c_j^1 = d_{1-\lambda}\right) \quad \text{where } 0 \le i < j \le j(1),\ \lambda \in \{0,1\}.$$

Since all points of this polygon are in $P' \cap Bd\mathscr{C}$, we have $j = i+1$, i.e., d_0 and d_1 are two consecutive elements of Δ.

Suppose next $d_1 \in P(v_1 d_0)$, and let $\{r, r+1, \ldots, s\}$ be a maximal interval (mod n) all of whose elements belong to type(d_0). Note that in this case $1 \notin$ type(d_0). By Proposition 2.2, d_0 has two neighbors (on Δ), d_0^- and d_0^+, such that type($\overrightarrow{d_0^-, d_0}$) $= r$ and type($\overrightarrow{d_0, d_0^+}$) $= s$.

If $r \in \{2, 3, \ldots, \frac{1}{2}n\}$ (or $s \in \frac{1}{2}n+2, \frac{1}{2}n+3, \ldots, n\}$), then d_0^- (d_0^+, resp.) is in $P' \supseteq P(v_1 d_0) \cap S$, hence $d_0^- = d_1$ ($d_0^+ = d_1$, resp.) and the lemma holds.

Consider now the only remaining case $r = s = \frac{1}{2}n + 1$. Let e (and e^*) be a straight line through d_0 parallel to $[v_n', v_1']$ (and $[v_1', v_2']$, resp.), and let x (x^*) denote the intersection point of e and $[v_1', v_2']$ (e^* and $[v_n', v_1']$, resp.). Then

$$|P' \cap \mathscr{C}| \le |P(v_1 x) \cap \mathscr{C}| + |P(v_1 x^*) \cap \mathscr{C}| + \left|P(v_{n/2+1} d_0) \cap \mathscr{C}\right|,$$

where the last term is zero. Thus, either $P'' := P(v_1 x)$ or $P'' := P(v_1 x^*)$ meets the requirements of the lemma. □

This motivates the following.

Definition 2.7. Let r be a natural number. A point $c \in Bd\mathscr{C}$ is called *r-rich* if there exists a translate P'' of P such that $P'' \cap Bd\mathscr{C} = \{c\}$ and $|P'' \cap C| \ge r$.

Lemma 2.8. *Let P' be a translate of P, $r \ge 2$ a natural number, and suppose that $c^-, c, c^+ \in P'$ are three consecutive elements of Δ (in this order). If c is r-rich, then $|P' \cap (\mathscr{C} \setminus Bd\mathscr{C})| \ge r - 1$.*

Proof. Suppose without loss of generality that $c^- = c_i^1$, $c = c_{i+1}^1$ for some i ($0 \le i < j(1)$). Then, by Proposition 2.2 and Lemma 2.3, type($\overrightarrow{c^-, c}$) $= 1$, type($\overrightarrow{c, c^+}$) $= s$ for some s ($1 \le s \le \frac{1}{2}n + 1$) and type($c$) $\supseteq \{1, 2, \ldots, s\}$.

Using the fact that c is r-rich, we can choose a translate P'' of P satisfying the conditions described in Definition 2.7. Let $v_1'', v_2'', \ldots, v_n''$ denote the vertices of P'', and assume as above that S intersects the sides $[v_{t-1}'', v_t''], [v_t'', v_{t+1}'']$. It is easily seen that $t \ne \frac{1}{2}n+2, \frac{1}{2}n+3, \ldots, n$ and $t \ne s+1, s+2, \ldots, s+\frac{1}{2}n-1$ (mod n), otherwise P'' would cover either c^- or c^+. If $t = \frac{1}{2}n+1$, then $|P'' \cap \mathscr{C}| \ge r \ge 2$ readily implies that P'' contains another boundary point of C distinct from c, contradicting the assumptions. Hence

$$t \in \{1, 2, \ldots, s\}. \tag{6}$$

The boundary of P'' intersects both $[c^-, c]$ and $[c, c^+]$. Let the corresponding intersection points be denoted by d^- and d^+.

If d^- and d^+ are on the same edge of P'' ($d^-, d^+ \in [v_{t-1}'', v_t'']$, say), then by (6) all points of $P'' \cap \mathscr{C}$ are lying in the triangle $d^- c d^+$. However, this triangle is completely covered by any convex set containing c^-, c and c^+, thus

$$|P'\cap(\mathscr{C}\setminus Bd\mathscr{C})| \geq |P''\cap(\mathscr{C}\setminus Bd\mathscr{C})| \geq r-1.$$

If d^- and d^+ are on different edges of P'', then $d^+\in[v''_{t-1}, v''_t]$, $d^-\in[v''_t, v''_{t+1}]$, and all points of $P''\cap\mathscr{C}$ are in the quadrangle $Q=(d^-, c, d^+, v''_t)$. Let $v'_1, v'_2, \dots, v'_n$ denote the vertices of P', and suppose that S intersects the sides $[v'_{r-1}, v'_r]$ and $[v'_r, v'_{r+1}]$ only.

We claim that $P'\supseteq Q$. For if not then $[v'_{r-1}, v'_r]\cup[v'_r, v'_{r+1}]$ would cross the boundary of Q at least twice. Since $c^-, c, c^+\in P'$ and P' is convex, none of these intersection points can be on $[d^-, c]\cup[c, d^+]$. Further, no side of P' can intersect both $[v''_{t-1}, v''_t]\supseteq[d^+, v''_t]$ and $[v''_t, v''_{t+1}]\supseteq[v''_t, d^-]$. This implies $[v'_{r-1}, v'_r]\cap[v''_{t-1}, v''_t]\neq\varnothing$, $[v'_r, v'_{r+1}]\cap[v''_t, v''_{t+1}]\neq\varnothing$, which is impossible. Hence, $P'\supseteq Q\supseteq P''\cap(\mathscr{C}\setminus Bd\mathscr{C})$ and the lemma follows. □

We are now in the position to prove Theorem 1′.

Let f: $Bd\mathscr{C}\to\{B, W\}$ be a 2-colouring having the properties stated in Corollary 2.4. Let us define a 2-colouring of $\mathscr{C}$ with red and green (g: $\mathscr{C}\to\{R, G\}$), as follows. For any $x\in\mathscr{C}$, let

$$g(x) := \begin{cases} G & \text{if } x\in Bd\mathscr{C} \text{ and } x \text{ is } \tfrac{1}{2}k'\text{-rich or } f(x)=W, \\ R & \text{otherwise.} \end{cases}$$

Consider now any translate P' of P covering at least k' elements of $\mathscr{C}$. We distinguish two cases.

Case A. $P'\cap(\mathscr{C}\setminus Bd\mathscr{C})\neq\varnothing$. Then $f(c)=R$ for any $c\in P'\cap(\mathscr{C}\setminus Bd\mathscr{C})$. If $|P'\cap Bd\mathscr{C}|\geq 3$ then, by Lemma 2.5, P' contains two consecutive elements of Δ. According to Corollary 2.4, at least one of these two points should be green.

Thus we can assume that $|P'\cap Bd\mathscr{C}|\leq 2$ and P' contains no two consecutive elements of Δ. By Lemma 2.6 there is a $\frac{1}{2}k'$-rich point $d\in P'\cap Bd\mathscr{C}$ which is green by definition. (Note that $P'\cap Bd\mathscr{C}\neq\varnothing$.)

Case B. $P'\cap(\mathscr{C}\setminus Bd\mathscr{C})=\varnothing$. By Lemma 2.5, P' contains at least $\frac{1}{2}k'$ consecutive elements of Δ. Let them be denoted by $c_1, c_2, \dots, c_m$ ($m\geq\frac{1}{2}k'$). Suppose that $k'\geq 10$. Since no two consecutive elements of Δ are red, there are at least two c_is ($1\leq i\leq m$) which are coloured green.

Assume now, in order to obtain a contradiction that $g(c_i)=G$ for all i ($1\leq i\leq m$). In view of Corollary 2.4, there are no three consecutive white points on Δ; hence at least one of $c_2, c_3, \dots, c_{m-1}$ is $\frac{1}{2}k'$-rich. However, in this case it follows immediately from Lemma 2.8 that $|P'\cap(\mathscr{C}\setminus Bd\mathscr{C})|\geq\frac{1}{2}k'-1>0$, the desired contradiction.

Therefore, taking (1) and (2) into account, Theorems 1 and 1′ are true for $k'\geq 10$, i.e., if

$$k \geq 20\left(\frac{\max_{r,s} d(v_r, v_s)}{\min_r \min_{s,t\neq r} d(v_r, v_s v_t)}+\sqrt{2}\right)^2.$$

Note that our colouring g: $C \to \{R, G\}$ has the following interesting additional property.

Proposition 2.9. *Let P' be any translate of P covering at least k' elements of $\mathscr{C}$. Then* $|\{c \in P' \cap \mathscr{C} | g(c) = R\}| \geq \frac{1}{6}(k'-8)$.

If $k'' := \frac{1}{6}(k'-8) \geq 10$, then repeating the above argument for $\mathscr{C}' := \{c \in \mathscr{C} | g(c) = R\}$ and k'', we obtain that the points of $\mathscr{C}'$ can be recoloured by two colours (pink and violet) so that, leaving the points of $\mathscr{C} \backslash \mathscr{C}'$ unchanged (green), any translate of P covering at least k' elements of $\mathscr{C}$ will contain at least one point of each colour.

Hence, by induction we can establish the following generalization of Theorem 1.

Theorem 3. *Let P be an open domain bounded by a centrosymmetric convex closed polygon in the plane, and let r be a natural number. Then there exists a constant $k = k(P, r)$ such that any k-fold covering of $\mathbb{R}^2$ with translates of P can be decomposed into r simple coverings.*

Note that using a beautiful lemma of Beck and Fiala [2], one can easily prove the following slight generalization of a result of Beck [1], related to our Theorem 1′.

Theorem 4. *Let P be an open domain bounded by a centrosymmetric convex closed polygon having n vertices. Then any finite system of points $\mathscr{C} \subseteq \mathbb{R}^2$ can be partitioned into two parts $\mathscr{C}_R \cup \mathscr{C}_G$ (red and green) so that $||P' \cap \mathscr{C}_R| - |P' \cap \mathscr{C}_G|| \leq \gamma n^2(\log|\mathscr{C}|)^4$ for every translate P' of P. (γ is an absolute constant.)*

However, if $|\mathscr{C}|$ is large, then this result does not give any nontrivial information about the discrepancy of the above partition on small sets, and the methods of the classical theory of irregularities in point-distributions seem to break down as well (cf. [1], [6], [7]).

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References

1. J. Beck, Balanced two-colorings of finite sets in the square *I*, Combinatorica 1 (1981), 327–335.
2. J. Beck and T. Fiala, "Integer-making" theorems, Discrete Applied Math. 3 (1981), 1–8.
3. G. Fejes Tóth, New results in the theory of packing and covering, in Convexity and its Applications (P. M. Gruber and J. M. Wills, eds.), Birkhäuser Verlag, Basel, 318–359 1983.
4. J. Pach, Decomposition of multiple packing and covering, Diskrete Geometrie, 2. Kolloq. Math. Inst. Univ. Salzburg, 169–178 1980.
5. C. A. Rogers, Packing and covering, Cambridge Tracts 54, Cambridge University Press, Cambridge, 1964.
6. K. F. Roth, On irregularities of distribution, Mathematika 7 (1954), 73–79.
7. W. M. Schmidt, Irregularities of distribution VII, Acta Arithmetica 21 (1972), 45–50.
8. F. A. Valentine, Convex Sets, McGraw-Hill, New York, 1964.

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