COSMIC RAY KINETICS IN SPACE

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Abstract. This work gives a systematic description of the statistical theory of the propagation of cosmic ray charged particles through random electromagnetic fields in space. A kinetic equation is derived for the cosmic ray distribution function averaged over the statistical ensemble corresponding to a random field. Transition to the diffusion approximation is considered, and the problems of the scattering and acceleration of charged particles are analyzed. The theory of fluctuation effects in cosmic rays is briefly discussed.

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1. Introduction

The last twenty years in cosmic ray astrophysics have been marked by significant changes due to a crucial change in astronomy during the postwar years. The pronounced enlargement of the observation range has made it possible to include in the composition of the observational astronomy 'windows' superhigh-energy charged particles (cosmic rays) as well as photons from many intervals of the electromagnetic spectrum. It should be noted that the energy spectrum of the charged particles covered by the term 'cosmic rays' is extremely broad and is extended from the energies of 'escaping' particles ($\sim 10^3$ eV in the solar and interplanetary plasma) to energies of $\sim 10^{20}$ eV in magnetized plasma in the pulsar atmospheres. The range of problems associated with cosmic ray astrophysics is unusually broad. Cosmic ray astrophysics is of fundamental importance to studies of the physical nature of transient objects (the Universe, pulsars, quasars, galactic nuclei, supernovae) since it is the relativistic particle interactions with the medium (matter, fields, radiations) that yield the major portion of information about these

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objects. On the other hand, cosmic ray studies are of exceptional importance to the research into dynamical processes in the interstellar and interplanetary medium. A classical example is the propagation of galactic cosmic rays in the solar system. The charged particle propagation in space filled with magnetic fields (and such are all cosmic objects without exception) results in the deformation of the particle spatial–energy distribution, which is directly reflected in the experimentally observed spectra of cosmic ray intensity whose analysis makes it possible to derive valuable information on cosmic rays themselves and on the physical nature of the medium in which the charged particles propagate. Propagation of cosmic-ray charged particles in interstellar and interplanetary media is one of the most urgent problems of high-energy astrophysics in which the direct relation between the physical nature of various cosmic objects and the fine effects occurring during the relativistic particle interaction with magnetized cosmic plasma becomes naturally apparent.

The foundations of the theory of cosmic ray interaction with magnetized cosmic plasma were laid some thirty years ago. Fermi (1949) showed that, when 'colliding' with magnetized 'clouds' of cosmic plasma moving in the opposite direction, cosmic ray charged particles should accelerate. This effect, subsequently called the Fermi acceleration effect, is of fundamental importance in cosmic ray astrophysics and, in one modified form or another, is the basis of the theory of charged particle propagation in cosmic plasma. Serious studies of cosmic ray propagation processes began, however, about a decade ago. They were stimulated by an extensive development of experimental methods associated with the appearance of a worldwide network of neutron monitor stations and with launchings of recording equipment beyond the Earth's magnetosphere. The principles of the plasma theory were developed approximately at the same time in connection with thermonuclear studies. A certain aggregate of theoretical concepts that gives a consistent description of cosmic ray interaction with magnetized cosmic plasma has been formed as a result of the synthesis of achievements in the experimental studies of cosmic rays and plasma physics. Methods for deriving the basic equations of the cosmic ray transfer theory have been developed and research into various specific cases of particle propagation, which is valid for some extreme relations between the parameters characterizing the propagation, has been carried out. Unfortunately, the real observational data relate usually to the intermediate domain of the values of these parameters and, therefore, it is too early yet to expect a comprehensive quantitative agreement between theory and experiment. The analytical studies of the extreme cases, however, provide a deeper insight into the physical essence of the studied events and a qualitative idea about them that may be hoped to properly reflect real situations.

The range of questions discussed in the present work is associated mainly with those problems in propagation theory which must be studied on the basis of the kinetic equation. The fact is that the first attempts to construct a statistical theory of cosmic ray propagation were based on a simple model of isotropic diffusion

(Dorman, 1958; Ginsburg and Syrovatsky, 1963; Parker, 1963; Dorman, 1963), which was gradually further complicated and supplemented on the basis of phenomenological considerations and experimental data. This is the most rudimentary and most difficult part of the propagation theory associated with the solution of complex boundary-value problems in mathematical physics. Further progress in this field is possible on the basis of the development of numerical methods for solving partial differential equations. On the other hand, the subsequent development of the theory has shown that, even in the diffusion approximation, if the magnetic fields are transferred by moving plasma clouds, the corresponding equations are of somewhat unusual form, which has not been taken into account by many authors (Parker, 1963; Dorman, 1963). Such a situation takes place for interplanetary space where magnetic fields are transferred by solar wind at velocity $u_0 \approx 4 \times 10^7$ cms^{-1} and in the expanding shells of super-novae. Under these conditions, the particle motion is determinantly affected by the adiabatic deceleration mechanism associated with particle energy loss in 'collisions' with the radially recessed 'clouds' of magnetized cosmic plasma. The adiabatic deceleration mechanism was first examined by Ginsburg *et al.* (1955) who studied the problem of particle propagation in supernovae shells. It is the kinetic approach to the description of cosmic ray propagation, however, that has made it possible to establish the correct form of the diffusion approximation equations and the kinetic coefficients of these equations by establishing their relation to the parameters of the medium in which the particles propagate. Apart from these questions, there is a broad range of problems that cannot be solved at all if the diffusion approximation is to be used. Thus, the kinetic approach to the evolution of the distribution function of charged particles is necessary when studying the question of how the anisotropic flux of galactic cosmic rays transforms during the cosmic ray passage from the Galaxy to the solar system. The same approach is also necessary when studying the spatial–angular distribution of solar cosmic rays during the initial phase after their generation in chromospheric flares. The above examples do not exhaust the great number of problems that may be solved in terms of kinetic theory.

As was indicated above, we shall limit ourselves in the present review to works which treat the application of the kinetic equation to the description of cosmic ray propagation processes. The theory of cosmic ray transfer is set forth at great length and the various specific effects in terms of the isotropic diffusion model are analyzed in detail in a number of monographs (Ginsburg and Syrovatsky, 1963; Parker, 1963; Dorman, 1963).

The anisotropic diffusion model including the adiabatic deceleration of cosmic rays in radially expanding cosmic plasma was first developed by Parker (1965) and Dorman (1965). A consistent theory of cosmic ray propagation based on the kinetic equation was developed by Dolginov and Toptygin (1966). A great contribution to the development of this theory was made by Toptygin (1973, 1973a), Dolginov and Toptygin (1966, 1968), and Galperin *et al.* (1971). The collisionless equation was used by Dolginov and Toptygin (1966) to develop the theory of cosmic ray propagation in an inhomogeneous expanding medium which contains random magnetic field inhomogeneities transferred at a certain velocity by the cosmic plasma against the background of a regular magnetic field. The kinetic equation obtained was used by Dolginov ond Toptygin (1966) to make the transition to the diffusion approximation, to obtain the expression for the particle flux density including the adiabatic deceleration, and to examine some problems in the theory of cosmic ray transfer. Significant advances in the studies of propagation and acceleration of cosmic rays were made when studying the solar high-energy particles. Tverskoy (1967a; 1969) formulated the hypothesis of the transfer of fast particles behind the shock wave front where a developed magnetohydrodynamic turbulence occurs. It was assumed that the Larmor radius of particles is much smaller than the basic scale of turbulence. In this case, the kinetic equations describing the particle propagation are determined by a detailed form of the spectral function for a random magnetic field. It will be noted that the developed magnetohydrodynamic turbulence (Alfvén waves) was found as a result of direct measurement in interplanetary space (Belcher and Davis, 1969, 1971).

The rapid development of experimental techniques in recent years has made it possible to study intensively a new type of cosmic ray variations, namely, microvariations which are more or less regular changes in the cosmic ray intensity with a period of the ordrer of several tens of minutes and less. Comparison between theory and experimental measurements of the correlation function of the particle distribution function fluctuations is most informative to the study of microvariations. The theory of cosmic ray fluctuation effects was first developed by Shishov (1968).

We shall now outline briefly the contents of the review. In Section 2 the functional method is used to average the collisionless kinetic equation over a statistical ensemble corresponding to a random magnetic field and to obtain the kinetic equation for the averaged function of the cosmic ray distribution. In recent years the functional method has been intensively developed in quantum field theory (Schwinger, 1951; Fradkin, 1965), turbulence theory (Hopf, 1952; Monin and Yaglom, 1965, 1967), and especially in connection with problems of wave propagation in media with random inhomogeneities (Tatarsky, 1967; Klyatskin and Tatarsky, 1973). Transition to the diffusion approximation (Dolginov and Toptygin, 1966) is made in Section 3. Considered in Sections 4–6 are some problems of the theory of cosmic ray propagation in a strong regular magnetic field against the background of which a magnetohydrodynamic turbulence is excited (Tverskoy, 1967a; Toptygin, 1973a; Tsytovich, 1966; Dorman, 1972). The question of cosmic ray fluctuation effects is briefly considered in Section 7.

Appendices I and II give the derivation of the equation for the correlation function of cosmic ray distribution function fluctuations and the solution of this equation for strong intensity fluctuations. Appendix III establishes the relationship between the various forms of the anisotropic transfer equation which are used in the theory of cosmic ray propagation.

2. Kinetic Equation

Consider a flux of non-interacting charged particles moving in a magnetic field with the regular, $\mathbf{H}_0(\mathbf{r}, t)$, and the random, $\mathbf{H}_1(\mathbf{r}, t)$, components

$$\begin{aligned} &\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0(\mathbf{r}, t) + \mathbf{H}_1(\mathbf{r}, t), \\ &\langle \mathbf{H}(\mathbf{r}, t) \rangle = \mathbf{H}_0(\mathbf{r}, t), \\ &\langle \mathbf{H}_1(\mathbf{r}, t) \rangle = 0. \end{aligned} \tag{2.1}$$

The angle brackets denote averaging over a statistical ensemble corresponding to a random magnetic field. As was indicated in Section 1, the fact that magnetic fields are transferred by the solar wind plasma at a certain velocity should be taken into account for interplanetary space. The velocity, which should be ascribed to the field, depends on the extent to which the magnetic field is frozen into the solar wind plasma. If the magnetic field is completely frozen into the plasma moving at velocity $|\mathbf{u}(\mathbf{r}, t)| \ll c$ (c is the speed of light) and having the regular, $\mathbf{u}_0(\mathbf{r}, t)$, and random, $\mathbf{u}_1(\mathbf{r}, t)$, components

$$\begin{aligned} &\mathbf{u}(\mathbf{r}, t) = \mathbf{u}_0(\mathbf{r}, t) + \mathbf{u}_1(\mathbf{r}, t), \\ &\langle \mathbf{u}(\mathbf{r}, t) \rangle = \mathbf{u}_0(\mathbf{r}, t), \\ &\langle \mathbf{u}_1(\mathbf{r}, t) \rangle = 0. \end{aligned} \tag{2.2}$$

then the effect of the induction electric field

$$\mathbf{E}(\mathbf{r}, t) = -\frac{1}{c}[\mathbf{u}(\mathbf{r}, t)\mathbf{H}(\mathbf{r}, t)] \tag{2.3}$$

on a particle should be included. Because of the transient nature of the processes occurring on the Sun and the turbulence development directly in interplanetary space, a broad spectrum of turbulent pulsations (Alfvén waves, magnetosound waves, etc.) is generated in the solar wind plasma apart from the plasma-frozen random inhomogeneities of the magnetic field. The stochastic electromagnetic fields of such pulsations significantly affect the charged particle motion. The distribution function $f(\mathbf{r}, \mathbf{p}, t)$ of non-interacting charged particles moving in the magnetic field determined by relations (2.1) satisfies the collisionless kinetic equation

$$\frac{\delta f}{\delta t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \mathbf{F}\frac{\partial f}{\partial \mathbf{p}} = 0, \tag{2.4}$$

where $\mathbf{v} = (c^2 p/\varepsilon)$ is the velocity and $\varepsilon = c(p^2 + m^2c^2)^{\frac{1}{2}}$ is the energy of a particle with momentum p and charge e. $\mathbf{F}$ is the force affecting the particle:

$$\mathbf{F} = e\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right). \tag{2.5}$$

The problem was formulated by Dolginov and Toptygin (1966) based on Equations (2.1)–(2.5).

Represent **F** in the form of the regular

$$\mathbf{F}_0 = e\left(\mathbf{E}_0 + \frac{1}{c}[\mathbf{v}\mathbf{H}_0]\right) \tag{2.6}$$

and the random

$$\mathbf{F}_1 = e\left(\mathbf{E}_1 + \frac{1}{c}[\mathbf{v}\mathbf{H}_1]\right) \tag{2.7}$$

parts. If the magnetic field is completely frozen into the plasma, the regular part of the electric field $\mathbf{E}_0$ is of the form

$$\mathbf{E}_0 = -\frac{1}{c}[\mathbf{u}_0\mathbf{H}_0] \tag{2.8}$$

and the random part is

$$\mathbf{E}_1 = -\frac{1}{c}([\mathbf{u}_0\mathbf{H}_1] + [\mathbf{u}_1\mathbf{H}]) \,. \tag{2.9}$$

In accordance with relations (2.5)–(2.9), the kinetic Equation (2.4) will be written in the form

$$\left(\frac{\partial}{\partial t} + L_0\right)f + \mathbf{F}_1\mathbf{L}f = 0 \,, \tag{2.10}$$

where

$$L_0 = \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \mathbf{F}_0\mathbf{L}$$

is the operator associated with the regular part $\mathbf{F}_0$ of force **F** and $\mathbf{L} = (\partial/\partial\mathbf{p})$. The distribution function $f(\mathbf{r}, \mathbf{p}, t)$ is irregularly changed in space and time following the variations of the random field. The distribution function $\langle f(\mathbf{r}, \mathbf{p}, t)\rangle$ averaged over a statistical ensemble corresponding to a random field has real meaning. The method first developed in quantum field theory (Schwinger, 1951; Fradkin, 1965) and statistical hydromechanics (Hopf, 1952) (see also Monin and Yaglom, 1965, 1967), will be used to derive the equation satisfied with the function $\langle f(\mathbf{r}, \mathbf{p}, t)\rangle$. In recent years, this method has been intensively developed in problems of wave propagation in media with random inhomogeneities (Tatarsky, 1967; Klyatskin and Tatarsky, 1973). As is known (Hopf, 1952; Tatarsky, 1967) the statistical properties of the random function $\mathbf{F}_1(\mathbf{r}, \mathbf{p}, t) \equiv \mathbf{f}_1(\mathbf{x}, t)$ (the set of variables $\{\mathbf{r}, \mathbf{p}\}$ will henceforth be denoted with a single letter $\{\mathbf{x}\}$) are completely determined, if its characteristic functional is present:

$$\phi[\boldsymbol{\eta}(\mathbf{x}, t)] = \langle \exp i(\boldsymbol{\eta}\mathbf{F}_1)\rangle, \tag{2.11}$$

where

$$(\boldsymbol{\eta}\mathbf{F}_1) = \int \mathrm{d}\mathbf{x}\, \mathrm{d}t\, \eta_\alpha(\mathbf{x}, t) F_{1\alpha}(\mathbf{x}, t)$$

denotes the 'scalar product' in the functional space; the repeated indices here and below mean summation.

All the moments of the random field may be obtained from (2.11) as functional derivatives at the zero functional argument $\eta(\mathbf{x}, t)$.

$$\begin{aligned}
&\frac{1}{i}\frac{\delta\Phi[\boldsymbol{\eta}]}{\delta\eta_\alpha(\mathbf{x}, t)}\bigg|_{\boldsymbol{\eta}=0} = \langle F_{1\alpha}(\mathbf{x}, t)\rangle = 0 \\
&\frac{1}{i^2}\frac{\delta^2\Phi[\boldsymbol{\eta}]}{\delta\eta_\alpha(\mathbf{x}, t)\delta\eta_\lambda(\mathbf{x}', t')}\bigg|_{\boldsymbol{\eta}=0} = \langle F_{1\alpha}(\mathbf{x}, t)F_{1\lambda}(\mathbf{x}', t')\rangle = D_{\alpha\lambda}(\mathbf{x}, t; \mathbf{x}', t').
\end{aligned} \tag{2.12}$$

In the general case, the correlation tensor $D_{\alpha\lambda}$ is determined by the relation

$$D_{\alpha\lambda} = e^2\left\{T_{\alpha\lambda} + \frac{1}{c}(\varepsilon_{\alpha\beta\gamma}v_\beta\Pi_{\gamma\lambda} + \epsilon_{\lambda\mu\nu}v_\mu\Pi_{\nu\alpha}) + \varepsilon_{\alpha\beta\gamma}\varepsilon_{\lambda\mu\nu}v_\beta v_\mu B_{\gamma\nu}\right\}, \tag{2.12a}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the united antisymmetrical tensor of the third rank; $T_{\alpha\lambda} = \langle E_{1\alpha}E_{1\lambda}\rangle$ and $B_{\gamma\nu} = \langle H_{1\gamma}H_{1\nu}\rangle$ are the correlation tensors of the electric and magnetic fields, respectively; $\Pi_{\alpha\beta} = \langle H_{1\alpha}E_{1\beta}\rangle$ is the crossed correlation tensor of the electric and magnetic fields. If the magnetic field is completely frozen into the plasma, then $D_{\alpha\lambda}$ is of the form

$$D_{\alpha\lambda} = \left(\frac{e}{c}\right)^2 \varepsilon_{\alpha\beta\gamma}\varepsilon_{\lambda\mu\nu}\{w_\beta w_\mu B_{\gamma\nu} - w_\beta H_{0\nu}S_{\mu\gamma} - w_\mu H_{0\gamma}S_{\beta\nu} + H_{0\gamma}H_{0\nu}Q_{\beta\mu}\}, \tag{2.14}$$

where $\mathbf{w} = \mathbf{v} - \mathbf{u}_0$, and $S_{\mu\gamma} = \langle u_{1\mu}H_{1\gamma}\rangle$ is the crossed correlation tensor of the magnetic and velocity fields; $Q_{\beta\mu} = \langle u_{1\beta}u_{1\mu}\rangle$ is the correlation tensor of the velocity field. When writing (2.14) we neglected the term $(e/c)^2\langle[\mathbf{u}_1\mathbf{H}_1]_\alpha[\mathbf{u}_1\mathbf{H}_1]_\lambda\rangle$ assuming it to be small.

After multiplying Equation (2.10) by $\exp i(\boldsymbol{\eta}\mathbf{F}_1)$ and averaging the obtained expression over the statistical ensemble corresponding to the random field we shall get the equation in functional derivatives

$$\left(\frac{\partial}{\partial t} + L_0\right)\mathcal{F}[\boldsymbol{\eta}; \mathbf{x}, t] = i\mathbf{L}_\alpha\left\{\frac{\delta \ln \Phi[\boldsymbol{\eta}]}{\delta\eta_\alpha(\mathbf{x}, t)} + \frac{\delta}{\delta\eta_\alpha(\mathbf{x}, t)}\right\}\mathcal{F}[\boldsymbol{\eta}; \mathbf{x}, t] \tag{2.15}$$

relative to the functional

$$\mathcal{F}[\boldsymbol{\eta}; \mathbf{x}, t] = \frac{\langle f(\mathbf{x}, t)\exp i(\boldsymbol{\eta}\mathbf{F}_1)\rangle}{\Phi[\boldsymbol{\eta}]} \tag{2.16}$$

whose value coincides, at the zero functional argument $\boldsymbol{\eta} = 0$, with the distribution function averaged over the statistical ensemble corresponding to the random field:

$$\mathcal{F}[0; \mathbf{x}, t] = \langle f(\mathbf{x}, t)\rangle. \tag{2.17}$$

The following procedure is usually used (Monin and Yaglom, 1965, 1967; Tatarsky, 1967) to obtain the equation for the averaged distribution function (2.17) from Equation (2.15).

Represent the functional $\mathcal{F}[\boldsymbol{\eta}(\mathbf{x}, t); \mathbf{x}, t]$ in the form of the functional power series

$$\mathscr{F}[\boldsymbol{\eta}; \mathbf{x}, t] = \mathscr{F}_0(\mathbf{x}, t) + \int d\mathbf{x}_1 \, dt_1 \eta_\alpha(\mathbf{x}_1, t_1) \mathscr{F}_{1\alpha}(\mathbf{x}, t; \mathbf{x}_1, t_1)$$

$$+ \int d\mathbf{x}_1 \, dt_1 \, d\mathbf{x}_2 \, dt_2 \eta_\alpha(\mathbf{x}_1, t_1) \eta_\beta(\mathbf{x}_2, t_2) \mathscr{F}_{2\alpha\beta}(\mathbf{x}, t; \mathbf{x}_1, t_1; \mathbf{x}_2, t_2)$$

$$+ \int d\mathbf{x}_1 \, dt_1 \, d\mathbf{x}_2 \, dt_2 \, d\mathbf{x}_3 \, dt_3 \eta_\alpha(\mathbf{x}_1, t_1) \eta_\beta(\mathbf{x}_2, t_2) \eta_\gamma(\mathbf{x}_3, t_3)$$

$$\times \mathscr{F}_{3\alpha\beta\gamma}(\mathbf{x}, t; \mathbf{x}_1, t_1; \mathbf{x}_2, t_2; \mathbf{x}_3, t_3) + \ldots, \tag{2.18}$$

where $\mathscr{F}_0$, $\mathscr{F}_{1\alpha}$, $\mathscr{F}_{2\alpha\beta}, \ldots$ are the power functionals of the zero, first, second, etc. powers, respectively. The expansion $\delta \ln \Phi[\boldsymbol{\eta}]/\delta\eta_\alpha(\mathbf{x}, t)$ is the functional power series, the n-th term of which is determined by the form of the correlation tensor of the $(n+1)$-th rank

$$\frac{\delta \ln \Phi[\boldsymbol{\eta}]}{\delta\eta_\alpha(\mathbf{x}, t)} = -\int d\mathbf{x}_1 \, dt_1 \eta_\beta(\mathbf{x}_1, t_1) D_{\alpha\beta}(\mathbf{x}, t; \mathbf{x}_1, t_1)$$

$$+ \int d\mathbf{x}_1 \, dt_1 \, d\mathbf{x}_2 \, dt_2 \eta_\beta(\mathbf{x}_1, t_1) \eta_\gamma(\mathbf{x}_2, t_2)$$

$$\times D_{\alpha\beta\gamma}(\mathbf{x}, t; \mathbf{x}_1, t_1; \mathbf{x}_2, t_2) + \ldots. \tag{2.19}$$

Substituting (2.18) and (2.19) in (2.15), we shall equate the functionals of the same power in the left and right parts of (2.15) to each other. The resultant infinite chain of connected equations is

$$\left(\frac{\partial}{\partial t} + L_0\right) \mathscr{F}_0(\mathbf{x}, t) = i\mathscr{L}_\alpha \mathscr{F}_\alpha(\mathbf{x}, t; \mathbf{x}, t)$$

$$\left(\frac{\partial}{\partial t} + L_0\right) \mathscr{F}_{1\alpha}(\mathbf{x}, t; \mathbf{x}_1, t_1) = -iD_{\alpha\beta}(\mathbf{x}, t; \mathbf{x}_1, t_1) \mathscr{L}_\beta \mathscr{F}_0(\mathbf{x}, t)$$

$$+ 2i\mathscr{L}_\beta \mathscr{F}_{2\alpha\beta}(\mathbf{x}, t; \mathbf{x}, t; \mathbf{x}_1, t_1) \tag{2.20}$$

$$\left(\frac{\partial}{\partial t} + L_0\right) \mathscr{F}_{2\alpha\beta}(\mathbf{x}, t; \mathbf{x}_1, t_1; \mathbf{x}_2, t_2) = -D_{\alpha\beta\gamma}(\mathbf{x}, t; x_1, t_1; \mathbf{x}_2, t_2) \mathscr{L}_\gamma \mathscr{F}_0(\mathbf{x}, t)$$
$$- iD_{\alpha\gamma}(\mathbf{x}, t; \mathbf{x}_1, t_1) \mathscr{L}_\gamma \mathscr{F}_{1\beta}(\mathbf{x}, t; \mathbf{x}_2, t_2)$$
$$- iD_{\beta\gamma}(\mathbf{x}, t; \mathbf{x}_2, t_2) \mathscr{L}_\gamma \mathscr{F}_{1\alpha}(\mathbf{x}, t; \mathbf{x}_1, t_1)$$
$$+ 3i\mathscr{L}_\gamma \mathscr{F}_{3\alpha\beta\gamma}(\mathbf{x}, t; \mathbf{x}, t; \mathbf{x}_1, t_1; \mathbf{x}_2, t_2).$$

It was taken into account when writing (2.20) that a symmetrization over the arguments and indices of the factors $\eta_\alpha(\mathbf{x}, t)$ should be made in the higher terms of expansions (2.18) and (2.19).

Assuming one of the functionals $\mathscr{F}_n$ to be zero, we shall obtain a closed set of equations. Thus, assuming $\mathscr{F}_2 = 0$, we obtain the following set of equations from (2.20);

$$\left(\frac{\partial}{\partial t} + L_0\right)\mathscr{F}_0(\mathbf{x}, t) = i\mathscr{L}_\alpha \mathscr{F}_{1\alpha}(\mathbf{x}, t; \mathbf{x}, t)$$
$$\left(\frac{\partial}{\partial t} + L_0\right)\mathscr{F}_{1\alpha}(\mathbf{x}, t; \mathbf{x}_1, t_1) = -iD_{\alpha\lambda}(\mathbf{x}, t; \mathbf{x}_1, t_1)\mathscr{L}_\lambda \mathscr{F}_0(\mathbf{x}, t). \tag{2.21}$$

To solve the equation set (2.21), we introduce the functions $\varphi_\alpha(\mathbf{x}, t; \mathbf{x}_1, t_1)$ according to the relation

$$\mathscr{F}_{1\alpha}(\mathbf{x}, t; \mathbf{x}_1, t_1) = \exp(-L_0 t)\varphi_\alpha(\mathbf{x}, t; \mathbf{x}_1, t_1). \tag{2.22}$$

The effect of the operator $\exp(-L_0 t)$ on the arbitrary function of coordinates and momenta is known to consist in replacing $\mathbf{r}$ by $\mathbf{r} - \Delta\mathbf{r}(t)$ and $\mathbf{p}$ by $\mathbf{p} - \Delta\mathbf{p}(t)$, where $\Delta\mathbf{r}(t)$ and $\Delta\mathbf{p}(t)$ are the changes of the radius-vector and momentum of the particle in a regular field during time t. Substituting (2.22) in the second equation of the set (2.21), we shall get

$$\mathscr{F}_{1\alpha}(\mathbf{x}, t; \mathbf{x}_1, t_1) = -i\int_0^t \mathrm{d}t' \exp[-L_0(t-t')]D_{\alpha\lambda}(\mathbf{x}, t'; \mathbf{x}_1, t_1)\mathscr{L}_\lambda \mathscr{F}_0(\mathbf{x}, t). \tag{2.23}$$

The Equation for the function $\mathscr{F}_0(\mathbf{x}, t)$ will be found from the first equation of the set (2.21) using (2.23):

$$\left(\frac{\partial}{\partial t} + L_0\right)\mathscr{F}(\mathbf{r}, \mathbf{p}, t)$$
$$= \mathscr{L}_\alpha \int_0^t \mathrm{d}t' \exp[-L_0(t-t')]D_{\alpha\lambda}(\mathbf{r}, \mathbf{p}, t'; \mathbf{r}_1, \mathbf{p}_1, t)\mathscr{L}_\lambda \mathscr{F}(\mathbf{r}, \mathbf{p}, t'). \tag{2.24}$$

We returned here to the previous denominations $\{\mathbf{x}\} \rightarrow \{\mathbf{r}, \mathbf{p}\}$ and omitted index 0 for the function $\mathscr{F}_0(\mathbf{r}, \mathbf{p}, t)$. In accordance with (2.21), we set that in the right part of (2.24) $t_1 = t$ and it should be borne in mind that, after the action of the operator $\exp[-L_0(t-t')]$, we should set that $\mathbf{r}_1 = \mathbf{r}$ and $\mathbf{p}_1 = \mathbf{p}$.

For further analysis of equation (2.24), the dependence of the correlation tensor of the coordinate axes and time should be specified. If the magnetic field is completely frozen into the plasma, the most general form of the correlation tensor $D_{\alpha\lambda}$ compatible with the experimental data and satisfying Maxwell's equations is

$$D_{\alpha\lambda}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = D_{\alpha\lambda}(\boldsymbol{\rho}, \mathbf{r} - \mathbf{u}_0 T), \tag{2.25}$$

where

$$\boldsymbol{\rho} = \tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2), \quad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \quad T = t_2 - t_1.$$

The random magnetic field described by the correlation tensor (2.25) corresponds to the case where the turbulence is a certain aggregate of regions whose scale is of the order of the random field correlation radius. The turbulence is homogeneous within each of such regions, but the total intensity of the magnetic field turbulent pulsations varies slowly from one turbulent region to another. In accordance with this, the first argument in the right hand side of (2.25) describes the gradual variations in the turbulent pulsation intensity from one turbulent region to another and reflects the fact that the pulsation intensity varies appreciably only when $\boldsymbol{\rho}$ changes by a value of the order of the random field correlation radius l_c. The second argument describes the local structure of turbulence, which is universal inside the region with a characteristic size of the order of l_c. The writing of the second argument in the form of $\mathbf{r}-\mathbf{u}_0T$ presupposes that the natural movement of magnetic field inhomogeneities may be neglected and that it may be assumed that all the spatial-time variations of the random magnetic field are associated with the transfer of random inhomogeneities at velocity $\mathbf{u}_0$ (Monin and Yaglom, 1965; 1967). If the turbulence is not only homogeneous but also statistically isotropic, the correlation second-rank tensor of the random magnetic field is of the following form (Monin and Yaglom, 1965; 1967; Dolginov and Toptygin, 1968);

$$D_{\alpha\lambda}(\boldsymbol{\rho},\mathbf{r}) = \tfrac{1}{3}\langle H_1^2(\boldsymbol{\rho})\rangle \left\{ \psi\left(\frac{r}{l_c}\right)\delta_{\alpha\lambda} - \psi_1\left(\frac{r}{l_c}\right)\frac{r_\alpha r_\lambda}{r^2}\right\}$$

$$\psi_1\left(\frac{r}{l_c}\right) = \psi\left(\frac{r}{l_c}\right) - \frac{2l_c^2}{r^2}\int_0^{r/l_c} \mathrm{d}y\, y\psi(y), \tag{2.26}$$

where $\psi(r/l_c)$ is some scalar function which is assumed to be known from observations; $\langle H_1^2(\boldsymbol{\rho})\rangle$ is the mean square of the random magnetic field.

Taking account of (2.25) and setting $t-t'=\tau$, Equation (2.24) can be written in the form

$$\left(\frac{\partial}{\partial t}+L_0\right)\mathcal{F}(\mathbf{r},\mathbf{p},t) = \mathcal{L}_\alpha \int_0^t \mathrm{d}\tau \exp(-L_0\tau) \times D_{\alpha\lambda}\left(\frac{\mathbf{r}+\mathbf{r}_1}{2},\mathbf{p}_1;\mathbf{r}_1-\mathbf{r}-\mathbf{u}_0\tau,\mathbf{p}\right)\mathcal{L}_\lambda\mathcal{F}(\mathbf{r},p,t-\tau). \tag{2.27}$$

The right part of Equation (2.27) is different from zero for time intervals of the order of the time of the random field correlation. Assuming that the correlation time is small as compared with the characteristic time of changes of the mean distribution function $\mathscr{F}$, we may write

$$\left(\frac{\partial}{\partial t}+L_0\right)\mathscr{F}(\mathbf{r},\mathbf{p},t)=\mathscr{L}_\alpha\int_0^\infty \mathrm{d}\tau\exp(-L_0\tau)$$

$$\times D_{\alpha\lambda}\left(\frac{\mathbf{r}+\mathbf{r}_1}{2},\mathbf{p}_1;\mathbf{r}_1-\mathbf{r}-\mathbf{u}_0\tau,\mathbf{p}\right)\mathscr{L}_\lambda\mathscr{F}(\mathbf{r},\mathbf{p},t). \tag{2.28}$$

If the particle momentum in the regular magnetic field H_0 varies little at distances of the order of the random field correlation radius, it may be assumed that $\Delta\mathbf{r}(\tau)=\mathbf{v}\tau$ and $\Delta\mathbf{p}(\tau)=0$ in the case of the action of the operator $\exp(-L_0\tau)$.

Then

$$\left(\frac{\partial}{\partial t}+L_0\right)\mathscr{F}(\mathbf{r},\mathbf{p},t)=\mathscr{L}_\alpha D_{\alpha\lambda}(\mathbf{r},\mathbf{p})\mathscr{L}_\lambda\mathscr{F}(\mathbf{r},\mathbf{p},t)$$

$$D_{\alpha\lambda}(\mathbf{r},\mathbf{p})=\int_0^\infty \mathrm{d}\tau D_{\alpha\lambda}(\mathbf{r},\mathbf{v}\tau-\mathbf{u}_0\tau;\mathbf{p}). \tag{2.29}$$

It is included in (2.29) that the first argument in $D_{\alpha\lambda}$ describes the slow change of the mean square of the random field with distance (see (2.25)) and, therefore, the effect of operator $\exp(-L_0\tau)$ on it may be disregarded.

At $u_1=0$ the Equations (2.27)–(2.29) transform into equations first obtained by Dolginov and Toptygin (1966) using the diagram techniques.

Let us elucidate the character of the approximations used in deriving the kinetic equation. When obtaining Equations (2.27)–(2.29), we closed the equation chain by setting $F_2=0$. This assumption is valid if the corrections to the distribution function $\mathscr{F}(\mathbf{r},\mathbf{p},t)$, associated with the inclusion of the next terms of the functional series (2.18) are small. In the considered case, however, the next non-vanishing approximation need not be calculated and the known quantum analogy (see, for example, Bonch–Bruevich and Tyablikov, 1961) may be used according to which the approximation based on the approximation $F_2=0$ corresponds to the Born approximation of the disturbance theory. If the magnetic field is stationary in time, the condition of applicability of the Born approximation is that the change of the particle momentum δp in the random field $\mathbf{H}_1$ is small as compared with the particle momentum p.

The ratio of these values

$$\frac{\delta p}{p}\approx\frac{e\sqrt{\langle H_1^2\rangle}}{cp}l_c=\frac{l_c}{R_1}\ll 1 \tag{2.30}$$

determines the condition of applicability of the considered approximation*. We can see that this condition is reduced to the smallness of the ratio of the random

* The authors are indebted to I. N. Toptygin for valuable remarks which helped in specifying the criterion (2.30).

field correlation radius to the Larmor radius of a particle in the random field $R_1 = (cp/e\sqrt{\langle H_1^2 \rangle})$. The condition means that the particle is scattered on each magnetic field inhomogeneity through a small angle $\theta \sim l_c/R_1$.

The following circumstance will be noted in conclusion of this Section. Until recently, the problems of the cosmic ray propagation theory were not considered from the viewpoint of the dynamical theory and were studied using other methods of which the Fokker–Planck method may be considered traditional (Parker, 1963, 1966; Dorman, 1965, 1967; Jokipii, 1967, 1971; Hall and Sturrok, 1968; Gleeson and Axford, 1967; Kurlsrud and Pearce, 1969; Kurlsrud and Ferrai, 1971). An intrinsic controversy is, however, typical of this method. On the one hand, the particle motion is treated as a certain stochastic process and, on the other hand, the kinetic factors of the equations of this process are calculated according to the motion equations. It is natural that a consistent theory of cosmic ray propagation should be based on dynamical equations supplemented by the corresponding hypotheses about the nature of the random field in which the particles propagate (Klyatskin and Tatarsky, 1973).

In such an approach, the kinetic equation describing the cosmic ray propagation cannot be reduced in the general case to the Fokker–Planck equation (see (2.27)) and turns into it only in the extreme case when the correlation time of the random field is much less than the characteristic time of the change in the particle distribution function. A consistent method for deriving the kinetic equation and the Fokker–Planck equation, which describe the motion of cosmic ray charged particles in random magnetic fields, was first developed by Dolginov and Toptygin (1966). They selectively summarized a definite class of diagrams of the disturbance theory series and obtained the kinetic equation on the assumption of the Gaussian distribution of random magnetic field inhomogeneities.

In this Section we have discussed the functional method of deriving the kinetic equation (Dorman and Katz, 1972) borrowed from the quantum field theory and statistical hydromechanics (Schwinger, 1951; Fradkin, 1965; Hopf, 1952). This method has been intensively developed in recent years in connection with the problems of the theory of wave propagation in media with random inhomogeneities (Tatarsky, 1967; Klyatskin and Tatarsky, 1973). It can be seen that the use of the diagram techniques under the Gaussian distribution, together with the approximation $F_2 = 0$ in the functional method, results in the kinetic equation the collision term of which is determined by the second-rank correlation tensor of the random magnetic field. As noted above, in this case the particles are scattered on each inhomogeneity of the magnetic field through a small angle. To examine the cases when the particles are scattered through large angles, it is necessary to take account of the higher-rank correlators in the kinetic equation. It will be noted that the functional method makes it possible to go beyond the frameworks of the Born approximation and to include the triple correlation (Katz, 1973), i.e. to examine the cases where a particle is scattered through large angles on collision with an individual inhomogeneity of the magnetic field.

3. Diffusion Approximation

Turning to the study of the kinetic equation, consider the problem of derivation of the diffusion approximation equations. These equations were first consistently derived by Dolginov and Toptygin (1966) on the basis of Equation (2.29). If the size of the system is sufficiently large and the particles have time to be strongly scattered, so that their distribution is close to the isotropic one, the distribution function may be series-expanded in spherical harmonics and we may restrict ourselves to the first terms of the expansions

$$\mathcal{F}(\mathbf{r},\mathbf{p},t)=\frac{1}{4\pi}\left[N(\mathbf{r},p,t)+\frac{3\mathbf{p}}{vp}\mathbf{J}(\mathbf{r},p,t)\right], \tag{3.1}$$

where N is the concentration of particles; $\mathbf{J}$ is the density of the particle flux in the momentum space. Substituting (3.1) in (2.29) and integrating the kinetic equation over the angles of vector $\mathbf{p}$, we shall obtain the equation for particle concentration $N(\mathbf{r}, p, t)$. If the kinetic equation is multiplied by vector $\mathbf{p}$ and integrated over angular variables, we shall obtain the equation for the particle flux density $\mathbf{J}(\mathbf{r},p,t)$

$$\frac{\partial N}{\partial t}+\operatorname{div}\mathbf{J}=\frac{u_0^2}{9\varkappa_0}\left[p^2\frac{\partial^2 N}{\partial p^2}+\left(1+\frac{v^2}{c^2}\right)p\frac{\partial N}{\partial p}\right]+\frac{1}{3\varkappa_0}\mathbf{u}_0\left(p\frac{\partial\mathbf{J}}{\partial p}+\frac{v^2}{c^2}\mathbf{J}\right)$$
$$+\frac{1}{vR}[\mathbf{u}_0\mathbf{h}]\left(p\frac{\partial\mathbf{J}}{\partial p}+\left(1+\frac{v^2}{c^2}\right)\mathbf{J}\right), \tag{3.2}$$

$$\frac{\Lambda}{v}\frac{\partial\mathbf{J}}{\partial t}+\mathbf{J}=-\varkappa_0\frac{\partial N}{\partial\mathbf{r}}-\left(\mathbf{u}_0+\frac{\Lambda}{R}[\mathbf{h}\mathbf{u}_0]\right)\frac{p}{3}\frac{\partial N}{\partial p}-\frac{\Lambda}{R}[\mathbf{h}\mathbf{J}], \tag{3.3}$$

where

$$\mathbf{h}=\frac{\mathbf{H}_0}{H_0},\quad R=\frac{cp}{eH_0},\quad \varkappa_0=\frac{v\Lambda}{3}, \tag{3.4}$$

$$\Lambda=\frac{12c^2p^2\nu\Gamma\left(\frac{\nu}{2}-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu/2)l_c\langle H_1^2(\mathbf{r})\rangle}, \tag{3.5}$$

$\Gamma(x)$ is the Euler Γ-function; ν is the exponent of the inhomogeneity spectrum of the interplanetary magnetic field.

It was taken into account when writing (3.2)–(3.5) that $D_{\alpha\lambda}$ is of the form determined by the relation (2.26) and $\psi(r)$ is determined by the expression (Dolginov and Toptygin, 1968)

$$\psi\left(\frac{r}{l_c}\right)=\left(\frac{r}{l_c}\right)^{(\nu-1)/2}\mathcal{K}_{(\nu-1)/2}\left(\frac{r}{l_c}\right), \tag{3.6}$$

where $\mathcal{K}_\mu(x)$ is the McDonald function. According to the observation data, the inhomogeneity spectrum exponent takes on the value $1<\nu\leq 3.8$. It will be noted that the spectral representation (3.6) corresponds to the inhomogeneity power spectrum decreasing in the region of small scales of magnetic field inhomogeneities.

The first term of Equation (3.3) may be neglected for the time intervals $t \gg \Lambda/v$. Then Equation (3.3) can be resolved relative to J and we shall obtain the following expression for the particle flux density

$$J_\alpha = -\varkappa_{\alpha\lambda}\frac{\partial N}{\partial r_\lambda} - u_{0\alpha}\frac{p}{3}\frac{\partial N}{\partial p}, \tag{3.7}$$

where $\varkappa_{\alpha\lambda}$ is the tensor coefficient of particle diffusion in space

$$\varkappa_{\alpha\lambda} = \frac{\varkappa_0 R^2}{R^2+\Lambda^2}\left(\delta_{\alpha\lambda} + \frac{\Lambda^2}{R^2}h_\alpha h_\lambda + \frac{\Lambda}{R}\varepsilon_{\alpha\gamma\lambda}h_\gamma\right). \tag{3.8}$$

The first term in (3.7) is the usual diffusive flux proportional to the concentration gradient, and in the absence of the regular field ($R \to \infty$) $\varkappa_{\alpha\lambda} = \varkappa_0\delta_{\alpha\lambda}$, which corresponds to the isotropic diffusion. Then, (3.7) takes the form

$$\mathbf{J} = -\varkappa_0\frac{\partial N}{\partial \mathbf{r}} - \mathbf{u}_0\frac{\partial N}{\partial p}. \tag{3.9}$$

In this case, Λ has the meaning of the particle transport free path, and $\varkappa_0$ is the scalar diffusion coefficient.

The second term in (3.7) describes the convective flux due to the motion of plasma-frozen magnetic field inhomogeneities at velocity $\mathbf{u}_0$. The expression (3.7) for the particle flux density was first obtained by Dolginov and Toptygin (1966) on the basis of the kinetic theory, and by Parker (1965) on the basis of a phenomenological approach. Substituting (3.7) in (3.2), we obtain the anisotropic diffusion equation for the particle concentration

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial r_\alpha}\varkappa_{\alpha\lambda}(\mathbf{r}, p)\frac{\partial N}{\partial r_\lambda} - u_{0\alpha}\frac{\partial N}{\partial r_\alpha} + \frac{\partial u_\alpha}{\partial r_\alpha}\frac{p}{3}\frac{\partial N}{\partial p}. \tag{3.10}$$

The last term in the right part of (3.10) describes the adiabatic cooling of charged particles (the antifermi mechanism of acceleration), associated with the radial expansion of the solar wind plasma with the frozen-in magnetic field inhomogeneities.

Equation (3.10) was obtained on the assumption that the proper motion of magnetic field inhomogeneities had been neglected, i.e. $\mathbf{u} = 0$. Inclusion of the chaotic component of the magnetic inhomogeneity velocity is equivalent to the appearance of stochastic electric fields resulting in particle acceleration (the Fermi acceleration mechanism). In virtue of the general properties of the Fermi acceleration mechanism (Tverskoy, 1967), the necessary condition of the effectiveness of this mechanism is a high degree of the isotropy of the particle distribution in the momentum space. Therefore, the particle acceleration may be considered in the diffusion approximation. A procedure similar to that used in deriving the Equation (3.10) results in the anisotropic diffusion equation including the particle acceleration effect (Dolginov and Toptygin, 1967):

$$\frac{\partial N}{\partial t}=\frac{\partial}{\partial r_\alpha}\varkappa_{\alpha\lambda}(\mathbf{r},p)\frac{\partial N}{\partial r_\lambda}-u_{0\alpha}\frac{\partial N}{\partial r_\alpha}+\frac{\partial u_{0\alpha}}{\partial r_\alpha}\frac{p}{3}\frac{\partial N}{\partial p}+\frac{1}{p^2}\frac{\partial}{\partial p}p^2D(\mathbf{r},p)\frac{\partial N}{\partial p}, \quad (3.11)$$

where

$$D(\mathbf{r},p)=\frac{\langle u_1^2(\mathbf{r})\rangle p^2}{3v\Lambda(\mathbf{r},p)} \quad (3.12)$$

is the particle diffusion coefficient in the momentum space; $\langle u_1^2\rangle$ is the root-mean-square fluctuation of the velocity.

The question of the relative importance of particle deceleration and acceleration in interplanetary space is one of the most interesting ones in the theory of cosmic ray modulation by solar wind. Some qualitative observations (Dorman, 1971) will be presented prior to consideration of direct solutions to Equations (3.10) and (3.11). It was established from the data on the high-energy solar cosmic ray anisotropy (Dorman, 1963) that the transport path, Λ, of these particles near the Earth's orbit is substantially longer than is expected from diffusion theory. This means that a region with small Λ (essentially smaller than at $r>r_1$) is located at some distance from the Sun ($r_1\approx 2$–3 AU). It is this region that is determinant in the diffusion approach (such a model was proposed by Meyer *et al.* (1956) to explain the effects of the flare of February 23, 1956).

Thus, the values $\Lambda\sim 10^{12}$ cm (for particles with kinetic energy $\varepsilon_k\sim 1$ GeV) obtained in terms of the diffusion theory relate not to the region $r<1$ AU but to the region $r\approx r_1$. This circumstance should also be borne in mind when tackling the problem of deceleration and acceleration of cosmic rays in interplanetary space.

Consider first the question of the relative importance of cosmic ray acceleration and deceleration in the region $r\lesssim r_1$. Because of the radial expansion of the solar wind, the mean change of the particle energy will be

$$\left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right)_{\mathrm{dec}}=-\frac{2}{3}\frac{\varepsilon v^2}{c^2}\frac{u_0}{r} \quad (3.13)$$

i.e., the relative change of the energy decreases when moving away from the Sun and with decreasing u_0. Expression (3.13) is valid in the ranges of both relativistic and non-relativistic energies. It is of importance to note that, according to (3.13), the deceleration effect is independent of Λ. On the other hand, the chaotic motion of magnetic inhomogeneities at velocity u_1 (against the background of their regular radial motion at velocity u_0) results in particle acceleration due to the action of the Fermi acceleration mechanism, the mean change of the particle energy being

$$\left(\frac{d\varepsilon}{dt}\right)_{ac} = \frac{\langle u_1^2\rangle}{c\Lambda}\varepsilon \tag{3.14}$$

for relativistic particles and

$$\left(\frac{d\varepsilon}{dt}\right)_{ac} = \frac{\langle u_1^2\rangle}{\Lambda}(2m\varepsilon)^{\frac{1}{2}} \tag{3.15}$$

for non-relativistic particles.

Then from (3.13) and (3.14) we get for galactic cosmic rays of moderate energies:

$$\left|\left(\frac{d\varepsilon}{dt}\right)_{dec}\right| : \left|\left(\frac{d\varepsilon}{dt}\right)_{ac}\right| = \frac{2}{3}\frac{u_0}{c}\frac{v^2}{\langle u_1^2\rangle}\frac{\Lambda}{r} \equiv \alpha. \tag{3.16}$$

If $\Lambda = \Lambda_{\oplus}(r/r_{\oplus})$, then

$$\alpha = \frac{2}{3}\frac{u_0}{c}\frac{v^2}{\langle u_1^2\rangle}\frac{\Lambda_{\oplus}}{r_{\oplus}}.$$

Thus, if Λ increases when moving away from the Sun in proportion to r, the value α should be independent of r (at constant $\langle u_1^2\rangle$) in the region of supersonic expansion of the solar wind where $u_0 = \text{const}$. At a moderate solar activity, $\Lambda_{\oplus} \approx 10^{12}$ cm, which follows from the analysis of the observation data on solar cosmic rays. Assuming $u_0 \approx 4\times 10^7$ cm s^{-1} in (3.16), we get $\alpha \approx 2\times 10^4$. Thus, under the said assumptions, the particle acceleration by the Fermi statistical mechanism may be neglected in the entire region of interplanetary space. In this case, if $\Lambda = \text{const}$, α decreases when moving away from the Sun. If Λ and $\langle u_1^2\rangle$ are considered distance-independent, we get $\alpha \approx 2\times 10^4$ $(r_{\oplus}/r)$ for the above values of u_0 and $\langle u_1^2\rangle$, i.e., $\alpha \lesssim 1$ only at distances $r \gtrsim 2\times 10^4$ AU. Since, on the other hand, the effective boundary of the modulation region is limited by a distance $r_0 \lesssim 10^2$ AU, this means that particle deceleration by radially expanding streams of solar wind plasma is the predominant factor throughout the modulation region. The same conclusion follows from Equation (3.11) (Toptygin, 1973). By additionally multiplying (3.11) by εp^2 and integrating (3.11) over p, this equation will be reduced (at $\varkappa = \text{const}$) to the form

$$\frac{\partial W}{\partial t} + \text{div}\,\mathbf{J}_w = \frac{n}{\tau_a}\left[4\overline{pv} + \overline{pv\left(\frac{mv^2}{\varepsilon}\right)^2} - \overline{pv}\tau_a\,\text{div}\,\mathbf{u}_0\right], \tag{3.17}$$

where the horizontal line denotes the means over the particle ensemble;

$$W = \int_0^{\infty} dp p^2 \varepsilon N(\mathbf{r}, p, t) = \bar{\varepsilon} n(\mathbf{r}, t) \tag{3.18}$$

is the density of particle energy;

$$\mathbf{J}_W = -\int \mathrm{d}pp^2 \varkappa \varepsilon \nabla N + W\mathbf{u}_0 \tag{3.19}$$

is the flux density; and

$$\tau_a = 9\varkappa/\langle u_1^2 \rangle. \tag{3.20}$$

The value in the right part of Equation (3.17) is the energy q per unit volume acquired by particles in unit time. The particle acceleration corresponds to the positive sign of the right part. In the non-relativistic case

$$q = \frac{n\overline{pv}}{\tau_a}\left(5 - \frac{2u_0\tau_a}{r}\right) \tag{3.21}$$

and in the ultrarelativistic case

$$q = \frac{n\overline{pv}}{\tau_a}\left(4 - \frac{2u_0\tau_a}{r}\right). \tag{3.22}$$

The acceleration effect is predominant at

$$\tau_a < 5r/2u_0 \quad \text{and} \quad \tau_a < 2r/u_0 \tag{3.23}$$

respectively, i.e. at

$$\frac{\langle u_1^2 \rangle}{u_0^2} > \frac{2}{3}\frac{v\Lambda}{u_0 r}. \tag{3.24}$$

Then, at $V = 3 \times 10^9$ cm s^{-1}, $u_0 = 4 \times 10^7$ cm s^{-1}, $\Lambda = 10^{12}$ cm, and $\langle u_1^2 \rangle = u_0^2$ we find that the acceleration of particles of such low energies may prevail over their deceleration at $r \gtrsim 10$ AU.

It can be seen from the above estimates that the favourable conditions for particle acceleration can be realized only at great distances from the Sun if it is taken into account that u_0 decreases when moving away from the Sun (whereas u_1 is either constant or even increases slightly owing to the conversion of the energy of the directed plasma motion into the turbulent motion energy). If at such distances $u_0 \sim u_1 \approx 3 \times 10^6$ cm s^{-1}, then $\alpha \approx 1.5 \times 10^3 (r_\odot/r)$ according to (3.16), i.e. $\alpha > 1$ practically throughout the modulation region. Thus, the acceleration will be appreciable only in regions where Λ is essentially smaller than the adopted values. One of such regions is the so-called 'buffer layer', the transient zone between the solar wind and the interstellar medium (Dorman and Dorman, 1968). In this region of space the solar wind becomes subsonic (i.e. u_0 abruptly decreases as r^{-2} and the effectiveness of the adiabatic cooling of particles becomes considerably smaller), a developed small-scale turbulence appears (the random component of velocity u_1 increases), and favourable conditions for particle acceleration are created (Dorman and Dorman, 1968). Another region of interplanetary space is that behind the shock wave front where a pronounced hydromagnetic turbulence is generated (Tverskoy, 1967a).

We shall now discuss in brief the applicability conditions of the diffusion approximation equations. As usual, the diffusion approximation applies better to the description of particle propagation, the slower the particle flux concentration and density vary at distances of the order of the free transport path. In particular, the expression for the particle flux density were derived using the condition

$$\frac{\Lambda}{v}\left|\frac{1}{J}\frac{\partial J}{\partial t}\right| \ll 1 \tag{3.25}$$

which means that the diffusion approximation describes the particle propagation in time intervals $t \gg \Lambda/v$.

Besides, in deriving the diffusion approximation equations it was assumed that $u_0 \ll v$, and the terms of the form $u_0 J/v^2$ and $u_0^2 N/v^2$ were included when series-expanding the collision integral of the kinetic equation in powers of u_0/v. The series-expansion of the distribution function in spherical harmonics and the inclusion of only two expansion terms means that the condition $J/Nv \ll 1$ should be satisfied. It will be noted that, according to the experimental data in the energy range $\gtrsim 10^3$ MeV, this criterion is satisfied with a large margin. For example, the data on the diurnal cosmic ray variations give $J/Nv \lesssim 10^{-2}$ and only at some periods of large Forbush-effects $J/Nv \sim (3\text{–}5) \times 10^{-2}$.

If the above-said conditions are not satisfied (for example, for solar cosmic ray propagation, cases are sometimes observed when $J/Nv \sim 1$), a kinetic examination of the evolution of the cosmic ray distribution function is necessary.

Turning to the specific application of Equation (3.10), consider the question of stationary propagation of solar cosmic rays. Solar cosmic ray propagation is known to be an essentially transient process. However, if the frequency of occurrence of solar flares is sufficiently high, solar cosmic rays accumulate in interplanetary space and a quasistationary background of charged particles is formed.

The stationary Equation (3.10) may be used to study such a quasistationary background of solar cosmic rays at a high disturbance of the interplanetary magnetic field ($\Lambda \ll 1$ AU). In accordance with (Toptygin, 1973), we shall consider a spherically symmetrical model of charged particle propagation which takes into account diffusion, convective transfer, and adiabatic cooling. The particle diffusion coefficient $\varkappa(r, p)$ will be assumed to be a power function of distance and an arbitrary function of particle momenta

$$\varkappa(r, p) = \left(\frac{r}{r_0}\right)^{\beta} \varphi(p), \tag{3.26}$$

where r_0 and β are constants. This dependence of the diffusion coefficient on coordinates and momenta is in good agreement with the experimental data on the magnetic inhomogeneity spectrum and is consistent with the general expression for the particle diffusion coefficient (3.4). Under the above assumptions, Equation

(3.10) takes the form

$$\frac{\partial \mathcal{Y}}{\partial p}-\frac{3r}{2p}\frac{\partial \mathcal{Y}}{\partial r}=-\frac{3\bar{\varphi}(p)}{2u_0 p r_0^\beta}\frac{1}{r}\frac{\partial}{\partial r}r^{2+\beta}\frac{\partial \mathcal{Y}}{\partial r}-\frac{Q}{r_0 p_0^3}\delta(r-r_0)\,\delta(p-p_0), \tag{3.27}$$

where

$$Q=\tfrac{3}{8}(\tau l_c^2 c(mc)^3 u_0)^{-1};\quad \bar{\varphi}(p)=\frac{\varphi(p)}{c l_c}. \tag{3.28}$$

Equation (3.27) is written in dimensionless variables (momentum is measured in mc, velocity in c, and spatial scales in l_c) for the Green's function of the equation for stationary isotropic propagation of cosmic rays.

Replacement of the independent variables

$$p=\zeta,\quad r=\left(\frac{\xi}{\zeta}\right)^{\frac{3}{2}} \tag{3.29}$$

reduces (3.27) to the form

$$\frac{\partial \mathcal{Y}}{\partial \zeta}=-\gamma(\zeta)\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\xi^{\frac{1}{2}(5+3\beta)}\frac{\partial \mathcal{Y}}{\partial \xi}-\tfrac{2}{3}Q(r_0^{\frac{2}{3}}p_0)^{-2}\delta(\xi-p_0 r_0^{\frac{2}{3}})\delta(\zeta-p_0). \tag{3.30}$$

where

$$\gamma(\zeta)=\frac{2}{3}\frac{\bar{\varphi}(\zeta)}{u r_0^\beta}\zeta^{\frac{1}{2}(1-3\beta)}. \tag{3.31}$$

Integrating (3.30) over ζ from $p_0-\varepsilon$ to $p_0+\varepsilon(\varepsilon>0)$, we obtain the additional condition

$$\mathcal{Y}|_{\zeta=p_0}=\tfrac{2}{3}Q(r_0^{\frac{2}{3}}p_0)^{-2}\delta(\xi-p_0 r_0^{\frac{2}{3}}) \tag{3.32}$$

which makes it possible to formulate Cauchy's problem for Equation (3.30):

$$\frac{\partial \mathcal{Y}}{\partial \zeta}=-\gamma(\zeta)\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\xi^{\frac{1}{2}(5+3\beta)}\frac{\partial \mathcal{Y}}{\partial \xi} \tag{3.33}$$

$$\mathcal{Y}|_{\zeta=p_0}=\tfrac{2}{3}Q(r_0^{\frac{2}{3}}p_0)^{-2}\delta(\xi-p_0 r_0^{\frac{2}{3}}) \tag{3.34}$$

$$\mathcal{Y}|_{\zeta>p_0}=0 \tag{3.35}$$

Equation (3.33) has partial solutions of the form

$$\mathcal{Y}_\lambda=[\tfrac{3}{4}(1-\beta)x]^{-\nu}\exp\{-\lambda^2 q^2(\zeta)\}J_\nu(\lambda x) \tag{3.36}$$

where

$$q^2(\zeta)=\int_\zeta^{p_0} d\zeta\gamma(\zeta);\quad x=[\tfrac{3}{4}(1-\beta)]^{-1}\xi^{(\frac{3}{4}(1-\beta))} \tag{3.37}$$

$$\nu=\frac{1+\beta}{1-\beta} \tag{3.37a}$$

and $J_\nu(z)$ is a Bessel function.

The general solution of Equation (3.33) may be written in the form

$$\mathcal{Y} = \int_0^\infty \mathrm{d}\lambda \lambda^{\frac{1}{2}} \psi(\lambda) \mathcal{Y}_\lambda, \tag{3.38}$$

where $\mathcal{Y}_\lambda$ is determined by relation (3.36).

Using the additional condition (3.34) and applying the Fourier–Bessel theorem, from (3.38) we obtain the expression for the function

$$\psi(\lambda) = \tfrac{2}{3} Q[\tfrac{3}{4}(1-\beta)]^{-2(1-\beta)} \lambda^{\frac{1}{2}} x_0^{-\nu} J_\nu(\lambda x_0), \tag{3.39}$$

where

$$x_0 = [\tfrac{3}{4}(1-\beta)]^{-1} (r_0 p_0^{\frac{3}{2}})^{(1-\beta)/2}. \tag{3.40}$$

Substituting the expression for $\psi(\lambda)$ in (3.38), we obtain Green's function of the stationary equation for isotropic diffusion with regard to the adiabatic particle deceleration at $\beta \neq 1$:

$$\mathcal{Y} = C \frac{(x_0 x)^{-\nu}}{2q^2(\zeta)} \exp\left\{-\frac{x_0^2 + x^2}{4q^2(\zeta)}\right\} J_\nu\left(\frac{x_0 x}{2q^2(\zeta)}\right); \quad C = \tfrac{2}{3} Q[\tfrac{3}{4}(1-\beta)]^{-(3+\beta)/(1-\beta)}, \tag{3.41}$$

where $J_\nu(z)$ is a modified Bessel function, $q^2(\zeta)$ is determined by expression (3.37); and ν is determined by (3.37a). At $\beta = 1$, Equation (3.33) may be written in the form

$$\frac{\partial \mathcal{Y}}{\partial \tau} = \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \xi^4 \frac{\partial \mathcal{Y}}{\partial \xi}, \tag{3.42}$$

where

$$\tau = q^2(\zeta)|_{\beta=1}. \tag{3.43}$$

Replacement of the independent variable $\xi = e^z$ and of the sought function $\mathcal{Y} = \xi^{-\frac{3}{2}} e^{-\frac{9}{4}\tau} g(\xi, \tau)$ permits Equation (3.42) to be reduced to the heat conductivity equation

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial z^2} \tag{3.44}$$

with Green's function

$$g = \frac{1}{2\sqrt{\pi\tau}} \exp\left\{-\frac{(z-z_0)^2}{4\tau}\right\}, \tag{3.45}$$

where $z_0 = p_0 r_0^{\frac{2}{3}} = \xi_0$.

Hence

$$\mathcal{Y} = \frac{1}{2\sqrt{\pi\tau}} \xi^{-\frac{3}{2}} \exp\left\{-\tfrac{9}{4}\tau + \frac{\ln^2 \frac{\xi}{\xi_0}}{4\tau}\right\}. \tag{3.46}$$

The expressions obtained for Green's function by Toptygin (1973) were used to examine the effect of adiabatic cooling on solar cosmic ray propagation. Numerical calculations have shown that low-energy particles lose a considerable portion of their energy and that their spectrum becomes very deformed during the propagation.

4. Multiple Scattering of Particles by Magnetic Inhomogeneities in a Strong Regular Magnetic Field

Experimental studies of low-energy cosmic rays (1–10 MeV) (Vernov *et al.*, 1968) have shown that the transport path of such particles exceeds 1 AU. Besides, a pronounced anisotropy was found in the particle angular distribution. Direct use of the kinetic equation is necessary to study the propagation of such particles. The processes of the low-energy cosmic ray propagation were first theoretically studied by Tverskoy (1967a; 1969) who paid particular attention to the analysis of the particle acceleration effects in interplanetary space. However, the formulation of the problem proposed by Tverskoy, with some modifications, formed the basis of most of the subsequent studies dealing with the charge particle propagation in space. The multiple scattering of low-energy charged particles on random inhomogeneities of the magnetic field was studied in detail by Galperin *et al.* (1971) and Toptygin (1973a). The kinetic equation is used in the present Section to examine the low-energy charged particle motion through the weakly turbulized and magnetized solar wind plasma in which Alfvén waves are excited. It will be noted that the presence of Alfvén waves in the solar wind plasma is confirmed by direct measurements (Belcher and Davis, 1969, 1971). When analyzing the charged particle motion, the energy exchange between magnetic field turbulent pulsations and charged particles due to particle interactions with stochastic electric fields of Alfvén waves will be taken into account simultaneousely with particle scattering on magnetic field turbulent pulsations.

We shall proceed from Equation (2.28) with the correlation tensor $B_{\alpha\lambda}$ determined by relation (2.12a). The effect of induction electric fields associated with the motion of the plasma as a whole was considered in Section 3. To identify the effects due to stochastic fields, we shall set $\mathbf{u}_0 = 0$. Whereas the regular magnetic field may be considered homogeneous at distances of the order of the correlation radius of the random field, the changes in the radius-vector $\Delta\mathbf{r}(\tau)$ and particle momentum $\Delta\mathbf{p}(\tau)$ in the regular field are determined by the relations ($\Omega = ecH_0/\varepsilon$ is the Larmor frequency):

$$\Delta\mathbf{r}(\tau) \equiv \mathbf{R}(0, \tau, \mathbf{r}, \mathbf{p}) = \mathbf{r} + (\mathbf{vh})\mathbf{h}\tau + [[\mathbf{vh}]\mathbf{h}]\frac{\sin\Omega\tau}{\Omega} + [\mathbf{vh}]\frac{1-\cos\Omega\tau}{\Omega}$$

$$\Delta\mathbf{p}(\tau) \equiv \mathbf{P}(0, \tau, \mathbf{p}) = (\mathbf{ph})\mathbf{h} + [\mathbf{ph}]\mathbf{h}\sin\Omega\tau - [[\mathbf{ph}]\mathbf{h}]\cos\Omega\tau. \quad (4.1)$$

Using (4.1) and taking into account the action of operator $\exp(-L_0\tau)$ Equation (2.28) can be written in the form

$$\frac{\partial\mathcal{F}}{\partial t}+\mathbf{v}\frac{\partial\mathcal{F}}{\partial\mathbf{r}}+\frac{e}{c}[\mathbf{v}\mathbf{H}_0]\frac{\partial\mathcal{F}}{\partial\mathbf{p}}=\mathcal{L}_\alpha\int_0^\infty \mathrm{d}\tau D_{\alpha\lambda}(\mathbf{R}(0,\tau,0,\mathbf{p}),\mathbf{P}(0,\tau,\mathbf{p});\mathbf{p},\mathbf{r})$$
$$\times(\mathcal{L}_\lambda\mathcal{F}(\mathbf{r},\mathbf{p},t))^{\mathbf{r}\to\mathbf{R}}_{\mathbf{p}\to\mathbf{P}}. \quad (4.2)$$

Following Galperin *et al.* (1971), we shall assume the regular magnetic field to be sufficiently intense. This means that the disturbance of the particle motion by stochastic electromagnetic fields of pulsations is weak at time intervals of the order of $1/\Omega$. In this case the distribution function may be averaged over the angle of the particle rotation around a force line of the regular field, i.e. the drift approximation may be used. Here, in the zero approximation over $1/\Omega$, the distribution function depends on the longitudinal and transverse (relative to the magnetic field) component of the particle momentum $\mathcal{F}(\mathbf{p})=\mathcal{F}(p_\perp,\ p_\parallel)$. The expression for the left part of the kinetic equation in this approximation is known from the drift theory (Chandrasekhar *et al.*, 1958; Volkov, 1964), and the kinetic equation takes the form

$$\frac{\partial\mathcal{F}}{\partial t}+v_\parallel\frac{\partial\mathcal{F}}{\partial z}-\tfrac{1}{2}(\operatorname{div}\mathbf{h})v_\perp\left(p_\parallel\frac{\partial}{\partial p_\perp}-p_\perp\frac{\partial}{\partial p_\parallel}\right)\mathcal{F}=\langle St\mathcal{F}\rangle_\varphi, \quad (4.3)$$

where z is the coordinate along the force line of the regular magnetic field and

$$\langle\cdots\cdots\rangle_\varphi\equiv\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi(\cdots\cdot)$$

denotes the averaging over the angle of particle rotation.

The collision integral $St\mathcal{F}$ of the kinetic Equation (4.3) may be written as

$$St\mathcal{F}=\frac{\partial}{\partial p_\alpha}\int_0^\infty \mathrm{d}\tau D_{\alpha\lambda}(\mathbf{R}(0,\tau,0,\mathbf{p}),\mathbf{P}(0,\tau,\mathbf{p});\mathbf{p},\mathbf{r})\frac{\mathrm{d}p_\nu}{\mathrm{d}P_\lambda}\frac{\partial\mathcal{F}}{\partial p_\nu}. \quad (4.4)$$

If Alfvén pulsations with a frequency

$$\omega(\mathbf{k})=u_a k_\parallel-i\gamma, \quad (4.5)$$

(where $u_a=H_0/\sqrt{4\pi\rho}$ is the Alfvén velocity; ρ is the plasma density; $k_\parallel$ is the component of the wave vector of pulsation in the regular field direction; and γ is the decrement of Alfvén wave extinction) are excited in the plasma, the electric and magnetic fields of the pulsations are related by

$$\mathbf{E}_1(\mathbf{k},\omega)=\frac{u_a^2}{c\omega}[\mathbf{h}[\mathbf{h}[\mathbf{k}\mathbf{H}_1(\mathbf{k},\omega)]]]. \quad (4.6)$$

If these waves are isotropically distributed in space, stationary in time, and statistically independent, then the Fourier-image of the second-rank correlation tensor of the random magnetic field is of the form:

$$\mathcal{B}_{\gamma\nu}(\mathbf{k}, \omega; \mathbf{k}', \omega') = \langle H_{1\gamma}(\mathbf{k}, \omega) H_{1\nu}(\mathbf{k}', \omega')\rangle$$
$$= \delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\delta(\omega-\omega(\mathbf{k}))\left(\delta_{\gamma\nu} - \frac{k_\gamma k_\nu}{k^2}\right)\mathcal{B}(k), \quad (4.7)$$

where $\mathcal{B}(k)$ is the spectral function of the random magnetic field. Passing in (2.12a) from the spatial correlators to their spectral representations and using (4.5)–(4.7), we write the kinetic equation in cylindrical coordinates in the momentum space with the Z-axis in the regular magnetic field direction:

$$\frac{\partial \mathcal{F}}{\partial t} + v_\| \frac{\partial \mathcal{F}}{\partial z} - \frac{1}{2}(\operatorname{div}\mathbf{h}) v_\perp \left(p_\| \frac{\partial}{\partial p_\perp} - p_\perp \frac{\partial}{\partial p_\|}\right)\mathcal{F} = \langle St\mathcal{F}\rangle_\varphi$$
$$\equiv \left(\frac{p_\|}{p_\perp}\frac{\partial}{\partial p_\perp} p_\perp - p_\perp \frac{\partial}{\partial p_\|}\right)$$
$$\times D_1 \left(p_\| \frac{\partial}{\partial p_\perp} - p_\perp \frac{\partial}{\partial p_\|}\right)\mathcal{F} + \frac{1}{p_\perp}\frac{\partial}{\partial p_\perp} p_\perp D_2 \left(p_\perp \frac{\partial}{\partial p_\|} - p_\| \frac{\partial}{\partial p_\perp}\right)\mathcal{F}$$
$$+ \left(p_\perp \frac{\partial}{\partial p_\|} - \frac{p_\|}{p_\perp}\frac{\partial}{\partial p_\perp} p_\perp\right) D_2 \frac{\partial \mathcal{F}}{\partial p_\perp} + \frac{1}{p_\perp}\frac{\partial}{\partial p_\perp} p_\perp D_3 \frac{\partial \mathcal{F}}{\partial p_\perp}$$

$$D_1 = \frac{e^2 c^2}{\varepsilon} \int_0^\infty \mathrm{d}\tau \int \mathrm{d}\mathbf{k}\,\xi(\mathbf{k}, \tau)\left[\cos \Omega\tau + \left(\frac{k_\perp}{k}\right)^2 \sin(\varphi - \varphi_1)\right.$$
$$\left.\times \sin(\varphi - \varphi_1 + \Omega\tau)\right]\mathcal{B}(k)$$

$$D_2 = \frac{e^2 u_a^2}{\varepsilon} \int_0^\infty \mathrm{d}\tau \int \mathrm{d}\mathbf{k}\,\xi(\mathbf{k}, \tau)\frac{k_\|}{\omega}\cos \Omega\tau \mathcal{B}(k)$$

$$D_3 = \left(\frac{e u_a^2}{c}\right)^2 \int_0^\infty \mathrm{d}\tau \int \mathrm{d}\mathbf{k}\,\xi(\mathbf{k}, \tau)[k_\perp^2 \sin(\varphi - \varphi_1)\sin(\varphi - \varphi_1 + \Omega\tau)$$
$$+ k_\|^2 \cos \Omega\tau]\mathcal{B}(k) \quad (4.8)$$

$$\xi(\mathbf{k}, \tau) = \exp\{i(k_\| v_\| - \omega)\tau + i\lambda[\sin(\varphi - \varphi_1 + \Omega\tau) - \sin(\varphi - \varphi_1)]\}, \quad (4.9)$$

where φ and φ_1 are the azimuth angles of vectors $\mathbf{p}$ and $\mathbf{k}$, respectively; $\lambda = (k_1 v_1/\Omega)$.

For the purpose of further examination, it is necessary to set the form of the spectral function of the random magnetic field $\mathcal{B}(k)$. If the spatial correlation function of the random magnetic field is determined by relation (3.21), the corresponding Fourier-image is of the form

$$\mathcal{B}(k)=\frac{\mathcal{A}_\nu}{(k_0^2+k^2)^{\nu/2+1}};\quad \mathcal{A}_\nu=\frac{\nu\Gamma\left(\frac{\nu}{2}\right)\langle H_1^2\rangle}{4\pi^{\frac{3}{2}}\Gamma\left(\frac{\nu-1}{2}\right)l_c^{\nu-1}};\quad k_0=l_c^{-1}. \tag{4.10}$$

Substituting (4.10) in (4.9) and using the known expansion

$$\exp i(\lambda\sin\varphi)=\sum_{n=-\infty}^{\infty}\exp i(n\varphi)J_n(\lambda)$$

($J_n\,(x)$ is a modified Bessel function), we obtain the following expressions for the factors D_1, D_2, D_3 (Galperin *et al.*, 1971; Dorman and Katz, 1972a):

$$D_1=\left(\frac{ec}{\varepsilon}\right)^2\left\{\left(\frac{v_\perp}{\Omega}\right)^\nu\left[\frac{2\mathcal{A}_\nu}{v_\|-u_a}\sum_{n=0}^{\infty}n^2\int_0^\infty \mathrm{d}\lambda\frac{\lambda J_n^2(\lambda)}{(\mu_n^2+\lambda^2)^{\nu/2+2}}+\frac{\mathcal{A}_\nu v_\perp^2}{(v_\|-u_a)^3}\right.\right.$$
$$\left.\left.\times\sum_{n=0}^{\infty}(n+1)^2\int_0^\infty \mathrm{d}\lambda\frac{\lambda(J_n^2(\lambda)+J_{n+2}^2(\lambda))}{(\mu_{n+1}^2+\lambda^2)^{\nu/2+2}}\right]+\beta\right\}$$

$$D_2=\frac{e^2u_a}{\varepsilon}\left\{\frac{\mathcal{A}_\nu}{v_\|-u_a}\left(\frac{v_\perp}{\Omega}\right)^\nu\sum_{n=0}^{\infty}\int_0^\infty \mathrm{d}\lambda\frac{\lambda(J_n^2(\lambda)+J_{n+2}^2(\lambda))}{(\mu_{n+1}^2+\lambda^2)^{\nu/2+1}}+\beta\right\}$$

$$D_3=\left(\frac{eu_a}{c}\right)^2\left\{\frac{\mathcal{A}_\nu(v_\|-u_a)}{v_1^2}\left(\frac{v_\perp}{\Omega}\right)^2\left[\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\int_0^\infty \mathrm{d}\lambda\frac{\lambda(J_n^2(\lambda)+J_{n+2}^2(\lambda))}{(\mu_{n+1}^2+\lambda^2)^{\nu/2}}\right.\right.$$
$$\left.\left.-2\sum_{n=1}^{\infty}\int_0^\infty \mathrm{d}\lambda\frac{\lambda J_n^2(\lambda)}{(\mu_n^2+\lambda^2)^{\nu/2+1}}\right]+\beta\right\}, \tag{4.11}$$

where

$$\beta=\int \mathrm{d}\mathbf{k}\frac{\gamma(k)\mathcal{B}(k)J_n^2(\lambda)}{k_\|^2(v_\|-u_a)^2+\gamma^2(k)};\quad \mu_m=\frac{mv_\perp}{v_\|-u_a},\quad m=0,1,2,\ldots. \tag{4.12}$$

The argument of the Bessel functions $J_n(\lambda)$ entering into the collision integral $\lambda=(k_\perp v_\perp/\Omega)$ is the ratio of the Larmor circle length $2\pi v_1/\Omega$ to the transverse (relative to magnetic field) length of pulsation wave $2\pi/k_\perp$. For great wavelengths, $\lambda\ll 1$ and it may be set that $J_n(\lambda)=\lambda^n/2^n n!$. In this case, the expressions for the factors D_1, D_2, D_3 take the form

$$D_1=\left(\frac{ec}{\varepsilon}\right)^2\left[\frac{2\mathcal{A}_\nu}{(\nu+2)\Omega^\nu}(v_\|-u_a)^{\nu-1}+\beta\right],\quad D_2=\frac{e^2u_a}{\varepsilon}\left[\frac{\mathcal{A}_\nu}{\nu\Omega^\nu}(v_\|-u_a)^{\nu-1}+\beta\right],$$

$$D_3=\left(\frac{eu_a}{c}\right)^2\left[\frac{\mathcal{A}_\nu\Gamma\left(\frac{\nu}{2}-\frac{1}{2}\right)}{2\Gamma(\nu/2)}(v_\|-u_a)^{\nu-1}+\beta\right]. \tag{4.13}$$

It can be seen from the expression for D_3 that D_3 diverges at the value of the exponent of the turbulence spectral function $\nu = 2$. This is associated with the fact that at $\nu = 2$ a particle interacts with waves of arbitrarily small amplitude and, at a sufficiently weak dependence of the wave amplitude on the pulsation scale, the effective time of particle interaction with wave vanishes. In reality, when integrating over k, the integral should be cut off at the wavelength corresponding to the Larmor radius of the particle. At $\nu = 2$ this results in a logarithmic dependence of the particle diffusion coefficient in the momentum space on the momentum (Tverskoy, 1967). In the opposite extreme case, $\lambda \gg 1$, we obtain the following expression for D_1, D_2, and D_3:

$$D_1 = \left(\frac{ec}{\varepsilon}\right)^2 \left[\frac{\mathcal{A}_\nu(\nu+1)}{2\sqrt{\pi}}\zeta(\nu+1)\frac{(v_\parallel - u_a)^\nu}{\Omega^\nu v_\perp} + \beta\right]$$

$$D_2 = \frac{e^2 u_a}{\varepsilon}\left[\frac{\mathcal{A}_\nu \Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)}{\sqrt{\pi}\nu\Gamma(\nu/2)}\zeta(\nu+1)\frac{(v_\parallel - u_a)^\nu}{\Omega^\nu v_\perp} + \beta\right]$$

$$D_3 = \left(\frac{eu_a}{c}\right)^2\left[\frac{\mathcal{A}_\nu \Gamma\left(\frac{\nu}{2}-\frac{1}{2}\right)}{\sqrt{\pi}\nu\Gamma(\nu/2)}\zeta(\nu+1)\frac{(v_\parallel - u_a)^\nu}{\Omega^\nu v_\perp} + \beta\right], \tag{4.14}$$

where $\zeta(x)$ is the Riemann ζ-function.

The asymptotic of the Bessel functions with $\cos^2 x$ replaced by its mean value $\overline{\cos^2 x} = \frac{1}{2}$ (Galperin *et al.*, 1971) was used to derive (4.14).

The addend β describes particles in the Čerenkov resonance with waves. Generally speaking, since we deal with particles whose velocity is much in excess of the Alfvén velocity ($u_a \approx 60\ \text{km s}^{-1}$ in the solar wind plasma), the Čerenkov resonance would seem of small significance, and the particle–wave interaction is due to cyclotron resonances of all orders. The Čerenkov resonance is, however, necessary for estimating the time of particle isotropization and acceleration. The cyclotron addends were calculated using the limiting transition $\gamma \to 0$ in the factors D_1, D_2, D_3.

When calculating β, however, this approximation cannot be applied, since the limiting transition $\gamma \to 0$ means that the effective time of particle–wave interaction turns out to be infinitely great. In fact, the presence of the imaginary part of the frequency gives a finite width of the region of interaction between the individual Fourier-harmonic of the wave and the moving particles. The importance of inclusion of the Čerenkov resonance was first indicated by Galperin *et al.* (1971) (see also Vedenov *et al.*, 1962).

It is known (Braginsky, 1963) that the decrement of the Alfvén wave extinction is

$$\gamma(\mathbf{k}) = \frac{c^2}{4\pi\sigma_\perp}k_\parallel^2, \tag{4.15}$$

where $\sigma_\perp$ is the coefficient of transverse conductivity of the plasma. Using (4.15) and (4.12), we obtain the following expression for β:

$$\beta = \frac{c_\nu}{v_\| - u_a}\left(\frac{v_\perp}{\Omega}\right)^{\nu-1}; \quad c_\nu = \frac{c^2 \mathcal{A}_\nu}{2\pi\sigma_\perp}\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \frac{xJ_1^2(x)}{(x^2+y^2)^{\nu/2+1}}. \tag{4.16}$$

Another possible cause of the broadening of the Čerenkov resonance is the particle scattering considered in detail by Galperin *et al.* (1971).

Equation (4.8) together with (4.9) and (4.11)–(4.16) which determine the factors D_1, D_2, and D_3, completely describe the charged particle propagation in a weakly turbulent magnetized plasma in which Alfvén waves are excited. Let us consider some specific problems of the theory of cosmic ray propagation which are solved using Equation (4.8).

5. Particle Scattering in the Magnetic Field. The Green's Function of the Kinetic Equation

Let the Equation (4.8) be written in spherical coordinates in the momentum space ($p_\| = p\cos\theta$, $p_\perp = p\sin\theta$)

$$\frac{\partial \mathcal{F}}{\partial t} + v\cos\theta\frac{\partial \mathcal{F}}{\partial z} - \frac{1}{2}(\mathrm{div}\,\mathbf{h})v\sin\theta\frac{\partial \mathcal{F}}{\partial \theta} = \frac{1}{p^2\sin\theta}\frac{\partial}{\partial\theta}$$
$$\times \sin\theta\cdot D_{\theta\theta}\frac{\partial \mathcal{F}}{\partial\theta} + \frac{1}{p\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\cdot D_{\theta p}\frac{\partial \mathcal{F}}{\partial p}$$
$$+\frac{1}{p^2}\frac{\partial}{\partial p}pD_{p\theta}\frac{\partial \mathcal{F}}{\partial\theta} + \frac{1}{p^2}\frac{\partial}{\partial p}p^2 D_{pp}\frac{\partial \mathcal{F}}{\partial p} \tag{5.1}$$

$$\begin{aligned} D_{\theta\theta} &= p^2 D_1 + \cos\theta(\cos\theta\cdot D_3 - 2pD_2) \\ D_{\theta p} &= D_{p\theta} = \sin\theta(\cos\theta\cdot D_3 - pD_2) \\ D_{pp} &= \sin\theta\cdot D_3. \end{aligned} \tag{5.2}$$

The factors D_1, D_2, and D_3 are determined by relations (4.11)–(4.16) with the corresponding replacement of variables.

At $D_{\theta p} = D_{p\theta} = D_{pp} = 0$ Equation (5.1) describes the particle diffusion in angular space for the case of energy conservation. Consider the solution of (5.1) for this case. In the general case, at arbitrary values of particle pitch-angle θ no solution to Equation (5.1) can be obtained. For the angular range $\theta \ll 1$, however, an analytic solution to (5.1) exists. At $\theta \ll 1$, the coefficient of diffusion in the angular space is determined by relations (4.14), where $v_\| = v\cos\theta$. Using (4.14) and (5.2), we can write Equation (5.1) for the stationary case including a point source with coordinates z_0 and θ_0 in the right hand part of this equation, i.e. we consider the equation for the Green's function of the kinetic equation:

$$\frac{\partial \mathcal{G}_\|}{\partial z} - \frac{1}{2}(\mathrm{div}\,\mathbf{h})\theta\frac{\partial \mathcal{G}_\|}{\partial\theta} = \frac{1}{\Lambda_\|(z)}\frac{1}{\theta}\frac{\partial}{\partial\theta}\cdot\theta\cdot\frac{\partial \mathcal{G}_\|}{\partial\theta} + \frac{1}{\theta}\delta(\theta-\theta_0)\delta(z-z_0) \tag{5.3}$$

where $\mathcal{G}_{\|}=\mathcal{G}_{\|}(z, \theta; z_0, \theta_0)$ is the Green's function of the kinetic equation and the value (Galperin *et al.*, 1971)

$$\Lambda_{\|}(z)=\frac{4(\nu+2)\Gamma\left(\frac{\nu}{2}-\frac{1}{2}\right)}{\sqrt{\pi}\nu\Gamma(\nu/2)}\frac{H_0^2}{\langle H_0^2\rangle}\left(\frac{l_c}{R}\right)^{\nu-2} l_c \tag{5.4}$$

is the transport free path of particles along the force lines of the regular magnetic field. To solve (5.3) it is convenient to replace the variables $\rho'\to\frac{1}{2}\ln H_0(z)$, $\theta\to e^{\rho-\xi'}$. The resultant equation is

$$\frac{\partial\mathcal{G}_{\|}}{\partial\rho'}=\exp\left[2(\xi'-\rho')\right]\varphi(\rho')\frac{\partial^2\mathcal{G}_{\|}}{\partial\xi'^2}+\delta(\xi'-\ln\theta_0)\delta(\rho')$$

$$\times\varphi(\rho')=\left[\Lambda_{\|}(z)\frac{\partial\rho'}{\partial z}\right]^{-1}\Bigg|_{z=z(\rho')} \tag{5.5}$$

The solution to (5.5) is of the following form (for details see Dorman and Katz, 1974):

$$\mathcal{G}_{\|}(z, \theta; z_0, \theta_0)=\frac{H_0(z_0)}{\pi H_0(z)\overline{\theta^2(z)}}\exp\left\{-\frac{\theta_0^2+\theta^2}{\overline{\theta^2(z)}}\right\}J_0\left(\frac{2\theta_0\theta}{\overline{\theta^2(z)}}\right) \tag{5.6}$$

$$\overline{\theta^2(z)}=4\int_{z_0}^{z}\mathrm{d}z'\frac{H_0(z)}{H_0(z')\Lambda_{\|}(z')}. \tag{5.7}$$

In (5.6) and (5.7) we returned to the variables $\{z, \theta\}$; $J_0(x)$ is a modified Bessel function.

At $\theta_0=0$, the expression for the Green's function (5.6) transforms into the expression first found by Galperin *et al.* (1971) who used it to explain the cases of anisotropic propagation of particles with energies of 1–5 MeV, observed in direct measurements in interplanetary space. Thus, if the random and regular magnetic fields vary in space and time following the same power law $H_0\sim H_1\sim(z_0/z)^{\alpha}$, the value $\overline{\theta^2(z)}$, which represents the mean squared particle scattering angle at $\alpha=2$ (which corresponds to the interplanetary magnetic field model developed by Parker (1963)) and at the exponent of magnetic pulsation spectrum $\nu=1.5$, is independent of z. Then, if $\overline{\theta^2(z)}<1$, the spatial-angle distribution of particles becomes markedly anisotropic. At $\nu=2$ the mean square of the particle scattering angle increases in proportion to the distance from the cource, and the anisotropy degree decreases.

The Green's function (5.6) describes the distribution of particles emitted by a point source. The real function of the source is so far unknown. It is possible that the conditions imposed on the particle propagation by the conservation of adiabatic invariant $\sin^2\theta/H$ make the character of particle angular distribution in the source insignificant. However, the knowledge of Green's function (5.6) becomes absolutely necessary when analyzing more subtle problems in cosmic ray kinetics pertaining to fluctuation effects arising during the cosmic ray motion in interplanetary magnetic fields (see Section 7).

Green's function of the kinetic equation may also be found in the transient case and in the case where particles propagate diffusively across the direction of the regular magnetic field. Write down the final expression

$$\mathcal{Y} = \mathcal{Y}_{\parallel}(z, \theta; z_0, \theta_0)\mathcal{Y}_{\perp}(x, y, z; x_0, y_0, z_0), \tag{5.8}$$

where x_0, y_0, z_0 are the coordinates of the source; $\mathcal{Y}_{\parallel}$ is Green's function of the longitudinal particle motion determined by relation (5.6);

$$\mathcal{Y}_{\perp} = \frac{3}{4\pi \int_{z_0}^{z} dz'\Lambda_{\perp}(z')} \exp\left\{-\frac{3[(x-x_0)^2+(y-y_0)^2]}{4\int_{z_0}^{z} dz'\Lambda_{\perp}(z')}\right\} \tag{5.9}$$

is Green's function of the transverse particle motion; $\Lambda_{\perp}$ is the particle transport path across the force lines of the regular field (Toptygin, 1973a).

Consider the particle scattering in the angular range where the inequality $|\cos\theta| = x \ll 1$ (Galperin *et al.*, 1971) is satisfied. The diffusion coefficient in the angular space $D_{\theta\theta}$ as a function of θ decreases slightly with increasing θ, which is due to the decreasing contribution from cyclotron resonances of higher orders. At the same time, the presence in $D_{\theta\theta}$ (4.14) of the second addend β, associated with the Čerenkov resonance, leads, starting from some value of θ, to an increase in $D_{\theta\theta}$. Such behaviour of the diffusion coefficient is also repeated at $\theta > \pi/2$, so that $D_{\theta\theta}(\theta) = D_{\theta\theta}(\pi-\theta)$. Thus the $D_{\theta\theta}(\theta)$ curve exhibits two peaks at $x = x_0 = |\cos\theta_0|$, whose position and depth depend on the relation between the Čerenkov and cyclotron addends. It can be shown (Toptygin, 1973) that x_0 is determined by the relation

$$x_0 = (\gamma_0/\Omega)^{1/(\nu+2)}, \quad \gamma_0 = \gamma(k_{\parallel} = R^{-1}). \tag{5.10}$$

The values of the particle transport path and isotropization time turn out to be substantially different for the cases $x \approx 1$ and $x_0 \ll 1$. In the first case, the minimum is not deep or is quite absent and hardly affects the particle scattering. Since in the Čerenkov resonance range the scattering is rapid, the isotropization time is mainly determined by the angular range $0 \leq \theta \ll 1$, i.e. by the region where the expression for D_1 determined by Formula (4.14) is applicable (or, which is the same, the transport path is determined by Formula (5.4)). In this case, the isotropization time is determined by the relation $\tau_3 = \Lambda_{\parallel}/v$. Note the characteristic dependence of the transport path on the particle momentum (Galperin *et al.*, 1971). At $\nu > 2$, the path decreases with increasing particle momentum. This is due to the fact that, as the Larmor radius increases, the particles are scattered by inhomogeneities of ever increasing scale, whose number grows. At $\nu = 2$, the transport path ceases to be dependent on momentum. This circumstance was first noted by Dorman and

Miroshnichenko (1965) who used data on cosmic ray propagation from solar flares.

In the case of a narrow Čerenkov resonance, $x_0 \ll 1$, the particle pitch-angle scattering in the region $\cos\theta \approx x_0$ weakens drastically, which leads to a considerable increase in the isotropization time and transport path of particles. In this case, an analytical solution to (5.1) can be obtained at $x_0 \leqslant x \leqslant 1$, $H_0 = \text{const}$:

$$\frac{1}{v}\frac{\partial \mathcal{F}}{\partial t} \pm x\frac{\partial \mathcal{F}}{\partial z} = \frac{1}{l}\frac{\partial}{\partial x}x^{\nu}\frac{\partial \mathcal{F}}{\partial x}. \tag{5.11}$$

The sign $\pm$ corresponds to $\cos\theta > 0$ or < 0, respectively, and the value

$$l = \frac{4(\nu+2)}{(\nu^2-1)\zeta(\nu+1)}\frac{H_0^2}{\langle H_1^2\rangle}\left(\frac{l_c}{R}\right)^{\nu-2} l_c \tag{5.12}$$

differs by a factor of the order of unity from $\Lambda_{\parallel}$ in (5.4).

Consider the case of $\nu = 2$. Neglect the spatial inhomogeneity of the system ($\partial\mathcal{F}/\partial z = 0$, $l = \text{const}$) and trace the filling of the angular range between $x = x_0$ and $x = x_1$ ($x_0 \ll x_1 \ll 1$). Equation (5.11) takes the form

$$\frac{\partial \mathcal{F}}{\partial \tau'} = x^2\frac{\partial^2 \mathcal{F}}{\partial x^2} + 2x\frac{\partial \mathcal{F}}{\partial x}, \quad (\tau' = vt/l). \tag{5.13}$$

Impose the boundary conditions $\mathcal{F}(x_1) = \mathcal{F}_1$ and $\mathcal{F}(x_0) = 0$ on the distribution function. We have $\mathcal{F}_1 = (2\pi)^{-1}$, if the region $x > x_1$ is filled by particles and the distribution function is normalized to unity.

The second condition corresponds to the assumption that the particles arriving at the boundary $x = x_0$ are instantaneously carried away to the rear hemisphere of the angular space. Such an approximation is sufficient for estimating the order of the isotropization time.

The solution to Equation (5.13) with the aforesaid boundary conditions is of the form

$$\mathcal{F}(x, \tau') = \mathcal{F}_1\left(1 - \frac{x_0}{x}\right) + x^{-\frac{1}{2}}\sum_{n=1}^{\infty}\mathcal{A}_n \exp\left\{-\left(\tfrac{1}{4} + \lambda_n^2\right)\tau'\right\} \times \sin\left(\lambda_n \ln\frac{x}{x_0}\right) \tag{5.14}$$

where $\lambda_n = \pi n \ln(x_1/x_0)$; the factors λ_n are determined by the initial condition. It follows from (5.14) that the time of filling of the region $x_0 < x < x_1$ is of the order of

$$\tau_1 \approx \left\{\frac{1}{4} + \left[\frac{\pi}{\ln(x_1/x_0)}\right]^2\right\}^{-1}. \tag{5.15}$$

This time varies from zero at a broad resonance $x_0 \approx x_1 \approx 1$ to $\tau_1 = 4$ at $x_0 \to 0$. The time τ_0 of particle scattering through angle $\theta \sim 1$ is of the order of unity, according to the previous results. If the initial function of particle distribution is such that the particle number at $\theta<\pi/2$ and $\theta>\pi/2$ is approximately the same, the isotropization time $\tau_s \approx \tau_0+\tau_1$. If, however, the distribution function is pronouncedly anisotropic at the initial moment, the isotropization time is much greater. This is due to the fact that the particle penetration from the forward to the rear hemisphere is slower at small x_0. The rate of particle transit to the rear hemisphere can be obtained by integrating (5.13) over x

$$\frac{\mathrm{d}N}{\mathrm{d}\tau}=-x_0^2\left(\frac{\partial\mathscr{F}}{\partial x}\right)\Big|_{x=x_0}. \tag{5.16}$$

From this, in the order of magnitude we have

$$\tau_s\approx-\left(2\pi\frac{\mathrm{d}N}{\mathrm{d}\tau}\right)^{-1}. \tag{5.17}$$

If $\tau_s \gg \tau_0+1$, then at $\tau_s\gg\tau\gg\tau_1$ a single term remains in the right part of (5.14) to give the quasistationary particle distribution in the forward hemisphere. Using this value of $\mathscr{F}$ from (5.17), we obtain

$$\tau_s=x_0^{-1}\gg 1 \tag{5.18}$$

It follows from the above-presented estimates that during the time interval $t_s=l/vx_0$ the corpuscular stream is of peculiar structure: the forward hemisphere is completely filled by particles, whereas the rear hemisphere contains few particles, and a pronounced gradient of angular distribution exists near $x=x_0$.

At $\nu\neq 2$, the qualitative features of the isotropization process are the same as in the case of $\nu=2$. The estimate $\tau_s\approx x_0^{1-\nu}$ valid at $x_0^{1-\nu}\gg 1$ is obtained for the isotropization time. According to this estimate and the Formulas (5.10) and (5.12), the path relative to the scattering through angle π is of the order of

$$\Lambda=l\tau_s\approx l(\Omega/\gamma_0)^{(\nu-1)/(\nu+2)} \tag{5.19}$$

The path Λ is additionally increased if the regular field H_0 is inhomogeneous and the particles move in the direction of its decrease. The particle focusing due to the conservation of the adiabatic invariant $\sin^2\theta/H_0$ hampers the particle penetration to the rear hemisphere of the angular space. The value of τ_s may be estimated for this case in the following way. In a weakly inhomogeneous field, Equation (5.1) for the stationary case takes the form:

$$x^{\nu}\frac{\mathrm{d}^2\mathscr{F}}{\mathrm{d}x^2}+(\nu x^{\nu-1}-\theta_1)\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}x}=0;\quad x_0<x\ll 1, \tag{5.20}$$

where $\theta_1=(l/2)\operatorname{div}\mathbf{h}=\text{const}$; $\theta_1>0$ if the particles move towards the decrease in H_0. Then, the solution of (5.20) with the same boundary conditions as for (5.11) at $\theta\ll\nu x_0^{\nu-1}$ gives the same result as in the case of $H_0=\text{const}$, and at $\theta_1\gg\nu x_0^{\nu-1}$ we get

$$\mathscr{F}(x)=\mathscr{F}_1\left\{1-\exp\left[\frac{\theta_1(x^{\nu-1}-x_0^{\nu-1})}{(\nu-1)x_0^{1-\nu}x^{\nu-1}}\right]\right\}. \tag{5.21}$$

The estimate of isotropization time gives

$$\tau_s = \frac{x_0^{2-\nu}}{\theta_1} \exp\left\{\frac{\theta_1}{(\nu-1)x_0^{\nu-1}}\right\}. \tag{5.22}$$

Hence, an additional factor $(x_0/\theta_1)\exp\{\theta_1/(\nu-1)x_0^{\nu-1}\}$ will appear in the expression for the transport path Λ in (5.19). The low-energy particle transport path in interplanetary space can be estimated on the basis of the obtained relations using the observation data on the magnetic field inhomogeneity spectrum. However, the experimental data obtained by various authors at various times are markedly different. Using the data of Jokipii and Coleman (1968) and estimating the collision width of the Čerenkov resonance, we find that $x_0 = 0.9$ for 1 MeV protons. This means that at $x = x_0$ the scattering is but slightly weakened. The estimate of the transport path according to (5.4) gives a value of the order of 0.1 AU. Such a value of $\Lambda_\parallel$ agrees with the observation data presented by Vernov *et al.* (1968) for the low-energy particle diffusive propagation in interplanetary space. On the other hand, the transport path calculated according to the data of Sari and Ness (1969) exceeds 1 AU. It should be noted that, if the magnetohydrodynamic discontinuities make the major contribution to the observed spectrum of magnetic field inhomogeneities (as is assumed in Sari and Ness, 1969), the theory developed may prove inapplicable, because a particle may be immediately scattered through a large angle when passing through a discontinuity.

6. Charged Particle Acceleration by Small-Amplitude Magnetohydrodynamic Waves

The theory of interaction between the cosmic ray charged particles and the solar wind plasma, developed in the previous Sections, covers the drift, scattering, diffusion, and adiabatic cooling of particles due to cosmic plasma expansion. Charged particle acceleration was considered in brief only in Section 3 where particle acceleration by 'magnetic clouds' (the Fermi acceleration mechanism) was discussed. As was pointed out in previous Sections, however, excitation of the magnetized solar wind plasma and other plasma formations in space cannot be always realized in the form of chaotic motion of individual clusters of matter with a magnetic field frozen into it. Development of various kinds of instabilities in cosmic plasma results in formation of a broad spectrum of turbulent pulsations whose stochastic electromagnetic fields give rise to particle acceleration. The magnetohydrodynamic turbulence, i.e. excitations in the form of Alfvén and magnetosound waves, are probably most characteristic of the cosmic plasma. As noted in Section 4, Alfvén waves were first found in the solar wind plasma as a result of direct measurements; they constitute up to 50% of the total disturbance of the interplanetary magnetic field. That is why studies of charged particles acceleration by magnetohydrodynamic turbulence is of considerable interest to cosmic ray physics. The various problems related to this range of phenomena were examined by many authors (Kurlsrud and Pearce, 1969; Kurlsrud and Ferrai, 1971; Tverskoy, 1967, 1967a, 1969; Toptygin, 1973a; Tsytovich, 1966, 1971; Kaplan and Tsytovich, 1972; Ginzburg *et al.*, 1972; Dorman, 1972).

The kinetic equation is used in the present Section to examine the acceleration of cosmic ray charged particles during their interactions with the present spectrum of the small-amplitude magnetohydrodynamic waves with the wave vectors distributed isotropically in space.

We shall proceed from Equation (5.1). Assume the regular magnetic field to be homogeneous in space. Then, Equation (5.1) takes the form

$$\frac{\partial \mathcal{F}}{\partial t}+v\cos\theta\frac{\partial \mathcal{F}}{\partial z}=\frac{1}{p^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta D_{\theta\theta}\frac{\partial\mathcal{F}}{\partial\theta}+\frac{1}{p\sin\theta}\frac{\partial}{\partial\theta}$$
$$\times\sin\theta D_{\theta p}\frac{\partial\mathcal{F}}{\partial p}+\frac{1}{p^2}\frac{\partial}{\partial p}pD_{p\theta}\frac{\partial\mathcal{F}}{\partial\theta}+\frac{1}{p^2}\frac{\partial}{\partial p}p^2D_{pp}\frac{\partial\mathcal{F}}{\partial p}\equiv St\mathcal{F}, \tag{6.1}$$

and the factors $D_{\theta\theta}$, $D_{p\theta}$, $D_{\theta p}$ and D_{pp} are determined by relations (5.2). Consider the collision integral of kinetic Equation (6.1). If the various terms in the collision integral are intercompared, the determinant contribution to the collision integral will be from the term containing the components $D_{\theta\theta}$ of tensor $D_{\alpha\lambda}$, which describe the elastic particle scattering on turbulent pulsations of magnetic field. It follows then that the effective frequency of charged particle scattering on the turbulent pulsations of the magnetic field $\nu_0=p^{-2}D_{\theta\theta}$ is considerably in excess of the particle collision frequency in the inelastic processes described by the factors $D_{\theta p}$, $D_{p\theta}$ and D_{pp} which include the energy exchange between turbulent pulsations and charged particles. This means that the particles are highly scattered at time intervals of the order of ν_0^{-1} and the subsequent particle diffusion in the momentum space is described by the isotropic distribution function.

In accordance with this, a solution to (6.1) will be sought in the form (Rutov, 1969; Alfvén and Feldhammer, 1967)

$$\mathcal{F}=\bar{f}+\delta f;\quad \bar{f}=\frac{1}{2\pi}\int_0^{\pi}\mathrm{d}\theta\sin\theta\mathcal{F}, \tag{6.2}$$

where the anisotropic part $\delta f(\theta)\ll\bar{f}$. Using (6.2) and employing the averaging over the angular variables, Equation (6.1) will be broken into two equations for $\bar{f}$ and δf. Combination of the two equations gives a single equation for the isotropic part of the distribution function $\bar{f}$:

$$\frac{\partial\bar{f}}{\partial t}=\frac{\partial}{\partial z}\varkappa(z)\frac{\partial\bar{f}}{\partial z}+\frac{1}{p^2}\frac{\partial}{\partial p}p^2D(p)\frac{\partial\bar{f}}{\partial p}, \tag{6.3}$$

where $\varkappa(z)=(\Lambda_{\|}v/3)$ is the coefficient of particle diffusion along the force line of the regular magnetic field;

$$\Lambda_{\|}(z)=\frac{3}{4\pi}vp^2\int_0^{\pi}\mathrm{d}\theta\sin\theta\cos\theta\int_0^{\theta}\mathrm{d}\theta\frac{\sin\theta}{D_{\theta\theta}(\theta)} \tag{6.4}$$

is the transport free path of particles along the force lines of the regular magnetic field. Calculation of $\Lambda_{\|}$ on the basis of Formula (6.4) gives the value of $\Lambda_{\|}$ coinciding with Expression (5.4). The value $D(p)$ in (6.3) is the particle diffusion coefficient in the momentum space:

$$D(p)=\frac{1}{2\pi}\int_0^{\pi}\mathrm{d}\theta\sin\theta\left(D_{pp}-\frac{D_{p\theta}^2}{D_{\theta\theta}}\right). \tag{6.5}$$

At $D_{p\theta}=0$, the diffusion coefficient $D(p)$ (together with Equation (6.3)) transforms into the expression for $D(p)$ obtained by Tverskoy (1967, 1967a). The term in (6.5) associated with $D_{p\theta}$ is due to the presence of the terms $\sim\langle E_{1\alpha}H_{1\lambda}\rangle$ in the correlation tensor $D_{\alpha\lambda}$.

Inclusioh of only the first term in the right hand part of (6.5) corresponds to the case where the distribution function of accelerated particles is completely isotropic. The effect of the crossed terms of the correlation tensor $D_{\alpha\lambda}$ results, however, in the fact that the particles are always distributed against a certain small anisotropic background. In this case, in accordance with (6.5), the diffusion coefficient in the momentum space decreases but remains always positive. In order to show this (Toptygin, 1973), we shall note that the diffusion in both momentum and conventional spaces is an irreversible process which leads to an increase of the entropy S of the particle system:

$$\frac{\mathrm{d}S}{\mathrm{d}t}>0,\quad S(t)=-\int\mathrm{d}pp^2\bar{f}\ln\bar{f}. \tag{6.6}$$

Differentiating (6.6) in time and using expression (6.3), we get

$$\frac{\mathrm{d}S}{\mathrm{d}t}=\int\mathrm{d}pp^2D(p)\frac{1}{\bar{f}}\left(\frac{\partial\bar{f}}{\partial p}\right)^2. \tag{6.7}$$

For the monotonic momentum dependence of $D(p)$ taking place in this case, the entropy increase is possible only if $D(p)>0$.

Specific calculations of acceleration processes in interplanetary space for the case where the Alfvén turbulence is excited in the solar wind plasma were first carried out by Tverskoy (1967a, 1969) (see also Tverskoy, 1967; Toptygin, 1973a). Direct calculations of the spectrum of the accelerated particle intensity on the basis of (6.3) at the exponent of the magnetic inhomogeneity spectrum $\nu=2$ show that in this case an exponential spectrum is asymptotically formed which satisfactorily describes the distribution of low-energy particles (1–10 MeV) accelerated in interplanetary space. Interest has recently been shown again in the particle acceleration mechanism for an alternating magnetic field (the Alfvén magnetic pumping) (Alfvén and Feldhammer, 1967). A consistent theory of the magnetic pumping mechanism was developed by Bakhareva *et al.* (1970; 1973). The quasilinear kinetics equations were used to solve the self-consistent problem and to find the turbulence spectrum and the distribution function of accelerated particles. The results obtained were used to propose a mechanism permitting the features of the magnetobremsstrahlung of the Crab Nebula to be explained. Toptygin (1973a) examined the general case when the solar wind plasma contains, apart from the (constant) regular magnetic field, the electric and magnetic fields varying slowly in time. The low-frequency part of the Alfvén wave spectrum may represent such fields in the solar wind. In (Toptygin, 1973a) the kinetic equation was averaged over the largescale harmonics of the random field, and the particle propagation was analyzed in detail.

Another region where the particle acceleration is possible is the so-called 'buffer layer', considered in (Dorman and Dorman, 1968) (see Section 3), which lies at the boundary between the solar wind and the interstellar medium. In this region of space where the solar wind becomes subsonic (the effectiveness of adiabatic deceleration decreases abruptly), a pronounced small-scale turbulence arises and favourable conditions for particle acceleration are created. An interesting study has recently appeared (Bakhareva, 1975) devoted to the theory of particle acceleration in the buffer layer.

7. Fluctuation Effects in Cosmic Rays

A number of fluctuation events associated with multiple scattering of charged particles on random magnetic inhomogeneities occur in interplanetary space during the propagation of cosmic ray charged particles in turbulent magnetic fields. It should be noted that these fluctuations in the particle angular-space distribution are of stochastic nature and differ markedly from quasiperiodic cosmic ray variations which are due to the entire complex of electromagnetic conditions in interplanetrary space correlating with the known recurrence of solar activity. Even the first experimental studies of the cosmic ray intensity fluctuations (Danju and Sarabhai, 1967; McCoy and Anderson, 1968) showed that they were probably caused by stochastic pulsations of the interplanetary magnetic field. It is not surprising, therefore, that the theory of cosmic ray fluctuation effects was developed on the assumption that these effects are related to interplanetary magnetic field pulsations. The first theoretical studies of the cosmic ray fluctuations were carried out by Shishov (1968). It turned out that the knowledge of the mean function of charged particle distribution was insufficient to analyze the fluctuation events and that the characteristics of the fluctuations of the exact distribution function, which are due to the fluctuations of interplanetary magnetic field, had to be found. The Dolginov-Toptygin diagram technique (Dolginov and Toptygin, 1966) was used by Shishov (1968) to derive the equation for the correlation function of the fluctuations of the cosmic ray distribution function (the second moment of the exact distribution function) from the collisionless kinetic equation. The equation obtained was then used to analyze the fluctuations in the high-energy range of charged particles ($R = (cp/eH) \gg l_c$). In this case, the effect of the regular magnetic field on the charged particle motion may be neglected, and the particle scattering on random inhomogeneities of the magnetic field is determined by the 'rough' characteristics of the random magnetic field, such as the mean square of the random field $\langle H_1^2 \rangle$ and its correlation radius l_c. The equation for the correlation function

$$g_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \langle f_{\mathbf{p}_1}(\mathbf{r}_1, t_1) f_{\mathbf{p}_2}(\mathbf{r}_2, t_2) \rangle$$

of fluctuations of the cosmic ray correlation function $f_{\mathbf{p}}(\mathbf{r}, t)$ is of the following form (Shishov, 1968; Dorman and Katz, 1974) (see Appendix I):

$$\begin{aligned} g_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = {} & \mathcal{F}_{\mathbf{p}_1}(\mathbf{r}_1, t_1)\mathcal{F}_{\mathbf{p}_2}(\mathbf{r}_2, t_2) \\ & + \int \mathrm{d}\boldsymbol{\rho}_1\, \mathrm{d}\mathbf{p}_1'\, \mathrm{d}\tau_1\, \mathrm{d}\boldsymbol{\rho}_2\, \mathrm{d}\mathbf{p}_2'\, \mathrm{d}\tau_2\, \mathcal{Y}_{\mathbf{p}_1\mathbf{p}_1}(\mathbf{r}_1, t_1; \boldsymbol{\rho}_1, \tau_1) \\ & \times \mathcal{Y}_{\mathbf{p}_2\mathbf{p}_2'}(\mathbf{r}_2, t_2; \boldsymbol{\rho}_2, t_2)\hat{\mathcal{L}}_{\alpha\mathbf{p}_1'}\mathcal{B}_{\alpha\lambda}(\boldsymbol{\rho}_1, \tau_1; \boldsymbol{\rho}_2, \tau_2) \\ & \times \mathcal{L}_{\lambda\mathbf{p}_2'} g_{\mathbf{p}_1'\mathbf{p}_2'}(\boldsymbol{\rho}_1, \tau_1; \boldsymbol{\rho}_2, \tau_2), \end{aligned} \tag{7.1}$$

where

$$\mathcal{F}_{\mathbf{p}}(\mathbf{r}, t) = \langle f_{\mathbf{p}}(\mathbf{r}, t) \rangle; \quad \mathcal{Y}_{\mathbf{p}\mathbf{p}'}(\mathbf{r}, t; z', t')$$

is Green's function of the kinetic equation for the distribution function $\mathcal{F}_{\mathbf{p}}(\mathbf{r}, t)$ averaged over the random field; $\hat{\mathcal{L}}_{\alpha\mathbf{p}} = (ec/\varepsilon)[\mathbf{p}(\partial/\partial\mathbf{p})]_\alpha$ is the operator of the angular moment in the momentum space; $\mathcal{B}_{\alpha\lambda}(\mathbf{r}, t; \mathbf{r}', t') = \langle H_{1\alpha}(\mathbf{r}, t) H_{1\lambda}(\mathbf{r}', t') \rangle$ is the second-rank correlation tensor of the random magnetic field. The mean Green's function $\mathcal{Y}_{\mathbf{p}\mathbf{p}'}$ of the kinetic equation has to be known to solve Equation (7.1). The stationary Green's function in the small-angle approximation is of the following form (Dolginov and Toptygin, 1966):

$$\mathcal{Y}_{\mathbf{p}\mathbf{p}'}(r - r') = \frac{3}{4\pi q|\mathbf{r} - \mathbf{r}'|^3} \exp\left(-\frac{3\theta^2 v}{4\pi q|\mathbf{r} - \mathbf{r}'|}\right) \tag{7.2}$$

$$q = \frac{\sqrt{\pi} e^2 l_c \langle H_1^2 \rangle}{12 m^2 c^2 v}, \tag{7.3}$$

where θ is the polar angle characterizing the momentum direction relative to the vector $\mathbf{r} - \mathbf{r}'$. Assume that the dependence of the correlation function $g_{\mathbf{p}\mathbf{p}'}(\mathbf{r}, t; \mathbf{r}', t')$ on coordinates and momenta is so weak (the first of these limitations can be avoided, see Appendix II) that, when solving Equation (7.1) by the iteration method, one may limit oneself to the first-order iteration and take $\mathcal{F}_{\mathbf{p}}(\mathbf{r}, t)$ outside the integral sign. Then, using (7.2) and expressions (2.26) and (3.6) for $\mathcal{B}_{\alpha\lambda}$, we shall get, after integrating over x and y, the following expression for the correlation function g:

$$g = \mathcal{F}^2 + \frac{\pi^{\frac{1}{2}}}{6} l_c^3 \langle H_1^2 \rangle \int_0^\infty \frac{dz}{l_c^2 + \xi_0} \left(1 - \frac{\xi_0}{l_c^2 + \xi_0}\right), \tag{7.4}$$

where $\xi_0 = 8qz^3/3v$; z is the coordinate in the direction of cosmic ray reception.

Since the integrand in (7.4) decreases appreciably with distance, the fluctuations are formed only in a small region of space adjoining the observation point and limited to the dimensions

$$x \leqslant (\tfrac{4}{3}q\zeta^3/v)^{\frac{1}{2}} = \frac{l_c}{\sqrt{2}}; \quad y \leqslant (\tfrac{4}{3}q\zeta^3/v)^{\frac{1}{2}} = \frac{l_c}{\sqrt{2}}; \quad z \leqslant \zeta = (3l_c^2 v/8q)^{\frac{1}{3}}. \tag{7.5}$$

In this case the fluctuation amplitude is

$$A \approx (l_c q/v)^{\frac{2}{3}} = \left(\frac{\pi^{\frac{1}{2}} l_c^2 e^2 \langle H_1^2 \rangle}{12m^2c^2v^2}\right)^{\frac{2}{3}}. \tag{7.6}$$

Time fluctuations with characteristic periods $T_\parallel \sim \zeta/u_0$ and $T_\perp \sim (l_c/\sqrt{2}\, u_0)$ will be observed owing to the solar wind motion at velocity u_0 for observations in different directions. According to the observations of Danju and Sarabhai (1967), $A \sim 4 \times 10^{-4}$ for galactic cosmic rays, and it follows from (7.5) and (7.6) that

$$l_c q = A^{\frac{3}{2}} v; \quad \frac{l_c^2}{q} = \frac{8\zeta^3}{3v} = \frac{8u_0^3 T_\parallel^3}{3v} \tag{7.7}$$

whence

$$l_c = \left(\frac{8}{3}\right)^{\frac{1}{3}} u_0 T_\parallel; \quad q = \frac{A^{\frac{3}{2}} v}{l_c} \tag{7.8}$$

and, for A and $T_\parallel$, according to the data of Danju and Sarabhai (1967), we get $l_c = 1.5 \times 10^{10}$ cm and $q = 2 \times 10^{-5}\ \mathrm{s}^{-1}$, which is in a satisfactory agreement with the results of direct measurements in interplanetary space. If the mean distribution function is axially symmetrical, one can obtain the expression for the correlation function

$$g = \mathcal{F}^2 + 3^{\frac{1}{3}} \left(\frac{l_c q}{v}\right)^{\frac{2}{3}} \left(\frac{\partial \mathcal{F}}{\partial \theta}\right)^2 \tag{7.9}$$

where θ is the polar angle measured from the symmetry axis. It can be seen from (7.9) that the charged-particle scattering on magnetic field random inhomogeneities results in fluctuations of the particle angular distribution, these fluctuations being due to the random variations $\delta\theta \approx (q\zeta/v)$ of the particle scattering angle.

Expressions (7.7)–(7.9) show that knowledge of the fluctuation correlation function gives new information (as compared with knowledge of the mean distribution function $\mathcal{F}$) about the parameters of the medium where the particles propagate. As can be seen from (7.2), $\mathcal{F}$ is a function of $q \propto \langle H_1^2 \rangle$ only, and observations of the fluctuation scale immediately give the inhomogeneity size l_c. Comparison of the mean square of fluctuations with the mean distribution function gives the parameter $ql_c \propto \langle H_1^2 \rangle l_c$. If no fluctuations are observed, limitations may be imposed on the maximum size of inhomogeneities and the minimum field intensity in them.

We have examined the case $R \gg l_c$ when the effect of the regular magnetic field on particle propagation may be neglected. In the opposite extreme case $R \ll l_c$ the effect of the regular magnetic field on the charged particle motion must be included. At $R \ll l_c$, in the small-angle approximation, Green's function of the kinetic equation is determined by relations (5.8) and (5.9). If the correlation function $g_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)$ depends little on the momentum variables, it may be taken outside the integral sign when integrating over $\mathbf{p}_1'$ and $\mathbf{p}_2'$. Substituting (5.8) and (5.9) in (7.1) and taking account of the fact that, owing to the rapid convergence of the integral, the upper integration limit may be assumed to be infinite when integrating over angular variables,

$$\int_0^\infty d\theta_0 \theta_0 \mathcal{G}_\parallel(z, \theta; z_0, \theta_0) = \frac{H_0(z)}{2\pi H_0(z_0)}, \tag{7.10}$$

so that from Equation (7.1) written for the stationary case we obtain

$$g_{\mathbf{p}}(\mathbf{r}_1, \mathbf{r}_2) = \mathcal{F}_{\mathbf{p}}(\mathbf{r}_1)\mathcal{F}_{\mathbf{p}}(\mathbf{r}_2) + \int d\boldsymbol{\rho}_1\, d\boldsymbol{\rho}_2 \mathcal{G}_\perp(\mathbf{r}_1, \boldsymbol{\rho}_1) \times \mathcal{G}_\perp(\mathbf{r}_2, \boldsymbol{\rho}_2)\hat{\mathcal{L}}_{\alpha\mathbf{p}}\mathcal{B}_{\alpha\lambda}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)\hat{\mathcal{L}}_{\lambda\mathbf{p}} g_{\mathbf{p}}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2). \tag{7.11}$$

In (7.11) the sought function has been replaced according to the rule

$$g_{\mathbf{p}}(\mathbf{r}_1, \mathbf{r}_2) \to \frac{1}{H_0(z_{\mathbf{r}_1})H_0(z_{\mathbf{r}_2})} g_{\mathbf{p}}(\mathbf{r}_1, \mathbf{r}_2) \tag{7.12}$$

and the case $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$ is considered for the sake of simplicity.

To go further in solving Equation (7.11), the specific form of the particle transport path across the regular magnetic field $\Lambda_\perp$ must be determined which, in turn, determines the form of Green's function of the particle transverse motion (see (5.9)). It is shown by Toptygin (1972, 1973) that the transport path $\Lambda_\perp$ across the force line of the regular magnetic field is determined by the relation

$$\Lambda_\perp(z) = C \frac{\langle H_1^2 \rangle}{H_0^2} l_c, \tag{7.13}$$

where C is the numerical constant of the order of unity.

If the mean square of random magnetic field and the square of regular magnetic field vary in space according to the same power law (see Section 5), the transport path $\Lambda_\perp$ is independent of coordinates and Green's function $\mathcal{G}_\perp$ is a function of difference in the coordinates. Then, if the correlation function $g_\mathbf{p}(\mathbf{r}_1, \mathbf{r}_2)$ varies gradually from one part of the turbulent region to another, i.e. the function $g_\mathbf{p}(\mathbf{r}_1, \mathbf{r}_2)$ varies markedly with its arguments at distances much in excess of the correlation radius of the random magnetic field, we obtain the equation of spectral densities

$$g_\mathbf{p}(\mathbf{k}_1, \mathbf{k}_2) = \mathcal{F}_\mathbf{p}(\mathbf{k}_1)\mathcal{F}_\mathbf{p}(\mathbf{k}_2) + (2\pi)^6 \mathcal{Y}_\perp(\mathbf{k}_1)\mathcal{Y}_\perp(\mathbf{k}_2) \times \hat{\mathcal{L}}_{\alpha\mathbf{p}} \mathcal{B}_{\alpha\lambda}\left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2}\right) \hat{\mathcal{L}}_{\lambda\mathbf{p}} \int d\mathbf{k}\, g_\mathbf{p}(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}, \mathbf{k}) \quad (7.14)$$

from Equation (7.11) using the Fourier transformation. Since we examine the cases where the regular magnetic field is sufficiently intense, so that the particle motion disturbance by random magnetic fields within a time of the order of the cyclotronic rotation period is small, Equation (7.14) should be averaged over the angle of particle rotation around a force line of the regular field (see Sections 4 and 5; it will be remembered that the mean Green's function $\mathcal{Y}$ was also obtained as a solution to the kinetic equation in the drift approximation). Assuming that the random magnetic field is statistically isotropic and averaging (7.14) over the rotation angle, we get

$$g_\mathbf{p}(\mathbf{k}_1, \mathbf{k}_2) = \mathcal{F}_\mathbf{p}(\mathbf{k}_1)\mathcal{F}_\mathbf{p}(\mathbf{k}_2) + (2\pi)^6 \mathcal{Y}_\perp(\mathbf{k}_1)\mathcal{Y}_\perp(\mathbf{k}_2)\mathcal{B}\left(\frac{|\mathbf{k}_1 - \mathbf{k}_2|}{2}\right)\hat{\mathcal{L}}_\alpha^2 \times \int d\mathbf{k}\, g_\mathbf{p}(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}, \mathbf{k}) \quad (7.15)$$

where $\hat{\mathcal{L}}_\alpha^2$ is the squared operator of angular momentum in the momentum space.

Similarly to the case $R \gg l_c$ considered above, Equation (7.15) can be solved by the iteration method. We shall refrain from writing down the corresponding relations, which are similar to the previous ones, and note that (7.15) gives an important relation

$$\mathcal{B}\left(\frac{|\mathbf{k}_1 - \mathbf{k}_2|}{2}\right) = \frac{\mathcal{F}_\mathbf{p}(\mathbf{k}_1)\mathcal{F}_\mathbf{p}(\mathbf{k}_2) - g_\mathbf{p}(\mathbf{k}_1, \mathbf{k}_2)}{(2\pi)^6 \mathcal{Y}_\perp(\mathbf{k}_1)\mathcal{Y}_\perp(\mathbf{k}_2)\hat{\mathcal{L}}_\alpha^2 \int d\mathbf{k}\, g_\mathbf{p}(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}, \mathbf{k})} \quad (7.16)$$

permitting the spectral function of the random magnetic field $\mathcal{B}(\mathbf{k})$ to be obtained from the given spectral densities of the mean distribution function $\mathcal{F}_\mathbf{p}(\mathbf{k})$ and of the correlation function $g_\mathbf{p}(\mathbf{k}, \mathbf{k}')$ (Dorman and Katz, 1974).

It can be seen that studies of the fluctuation effects in cosmic rays make it possible to find the detailed characteristics of interplanetary magnetic fields. This is of special importance to studies of individual regions of the solar system which cannot at present be investigated by more direct methods.

Conclusions

The theory of cosmic ray propagation is being intensively developed at present. We did not consider a wide range of problems in the cosmic ray kinetics associated with the presence of strong shock waves in interplanetary space. The available results cannot be reviewed within the framework of the present paper and we limit ourselves to pointing out the appropriate literature (Dorman and Shogenov, 1974).

To compare the theory developed and the observation data, the above-presented results should be specified. First of all, it is necessary to go beyond the small-angle approximation when solving the kinetic equation and to find the function of particle distribution in the presence of the regular magnetic field (at $R \gg l_c$). Calculations of the correlation function in the low-energy range, where the most appreciable fluctuations of cosmic ray intensity are expected, are of special interest for the study of fluctuation effects.

It will be noted in conclusion that, although the kinetic theory of cosmic ray propagation was set forth specifically for the solar wind plasma, the theory developed makes it possible to describe the processes of particle propagation in the magnetized plasma of the Galaxy and other cosmic objects.

Appendix I

Consider the derivation of Equation (7.1) for the correlation function of distribution function fluctuations (Dorman and Katz, 1974; Shishov, 1968). It is convenient to start with the equation for Green's function of the kinetic equation. In this case, Equation (2.15) will be written in the form

$$\{\hat{L}_0 - i\mathscr{L}_\alpha V_\alpha[\boldsymbol{\eta}; x_1]\}\mathscr{Y}[\boldsymbol{\eta}; x_1, x_1'] - i\mathscr{L}_\alpha \frac{\delta \mathscr{Y}[\boldsymbol{\eta}; x_1, x_1']}{\delta \eta_\alpha(x_1)} = -\delta(x_1 - x_1')$$

$$\hat{L}_0 = \frac{\partial}{\partial t} + L_0; \quad V_\alpha[\boldsymbol{\eta}; x_1] = \frac{\delta \ln \phi[\boldsymbol{\eta}]}{\delta \eta_\alpha(x_1)}; \quad \{x\} \to \{\mathbf{r}, \mathbf{p}, t\} \to \{\mathbf{x}, t\}$$

$$\mathscr{Y}[\boldsymbol{\eta}; x_1, x_1'] = \langle G(x_1, x_1') \exp i(\boldsymbol{\eta}\mathbf{H}_1)\rangle, \tag{A.I.1}$$

where $G(x_1, x_1')$ is Green's function of the collisionless kinetic equation (2.10).

The procedure borrowed from (Tatarsky, 1967; Bonch-Bruevich and Tyablikov, 1961) can be conveniently used for further consideration. Let the functional argument in (A.I.1) be replaced according to the rule

$$\frac{\delta}{\delta \eta_\alpha(x_1)} \to \int dz \frac{\delta V_\beta[\boldsymbol{\eta}; z]}{\delta \eta_\alpha(x_1)} \frac{\delta}{\delta V_\beta[\boldsymbol{\eta}; z]} \tag{A.I.2}$$

and, using the orthogonality property of the functional $\mathscr{Y}[\mathbf{V}; x_1, x_1']$

$$\int dz \mathscr{Y}[\mathbf{V}; z, x_1]\mathscr{Y}^{-1}[\mathbf{V}; z, x_1'] = \delta(x_1 - x_1'), \tag{A.I.3}$$

Equation (A.I.1) will be written in the form

$$\{\hat{L}_0 - i\mathscr{L}_\alpha V_\alpha[\boldsymbol{\eta}; x_1]\}\mathscr{Y}[\mathbf{V}; x_1, x_1'] - \int dz M(x_1, z)\mathscr{Y}(\mathbf{V}; z, x_1'] = -\delta(x_1 - x_1') \tag{A.I.4}$$

where

$$M(x_1, z) = -i\mathcal{L}_\alpha \int dy\, dy_1 \mathcal{Y}[\mathbf{V}; x_1, y]\Gamma_\lambda[\mathbf{V}; y, z, y_1]D_{\alpha\lambda}[\boldsymbol{\eta}; y_1, x_1];$$

$$\Gamma_\lambda[\mathbf{V}; y, z, y_1] = \frac{\delta\mathcal{Y}^{-1}[\mathbf{V}; y, z]}{\delta\eta_\lambda(x_1)}; \quad D_{\alpha\lambda}[\boldsymbol{\eta}; y_1, x_1] = \frac{\delta V_\lambda[\boldsymbol{\eta}; y_1]}{\delta\eta_\alpha(x_1)} \tag{A.I.5}$$

$$D_{\alpha\lambda}[0; y_1, x_1] = D_{\alpha\lambda}(y_1, x_1). \tag{A.I.6}$$

For further consideration, it is convenient to use the disturbance theory and regard a random field as a disturbance. To the first non-vanishing approximation of the disturbance theory

$$\mathcal{Y}[\mathbf{V}; x_1, y_1]_{\mathbf{V}\to 0} \to \exp\{-L_0(t_{\mathbf{x}_1} - t_{\mathbf{y}_1})\}\delta(\mathbf{x}_1 - \mathbf{y}_1) \tag{A.I.7}$$

(where $t_\mathbf{x}$ and $\mathbf{x}$ are the time and spatial coordinates of the point $x \to \{\mathbf{x}, t_\mathbf{x}\}$), and the functional Γ_β may be determined from (A.I.4). Multiplying (A.I.4) on the right-hand side by $\mathcal{Y}^{-1}[\mathbf{V}; x_1, y]$, we integrate the obtained relation over x_1', taking account of (A.I.3). Neglecting the second-order value for the random field, we get

$$\{\hat{L}_0 - iV_\alpha[\boldsymbol{\eta}; x_1]\mathcal{L}_\alpha\}\delta(x_1 - y_1) = \mathcal{Y}^{-1}[\mathbf{V}; x_1, y]. \tag{A.I.8}$$

From this, using definition (A.I.6), we have

$$\Gamma_\lambda^0[\mathbf{V}; x_1, y, z] = i\delta(x_1 - z)\mathcal{L}_\lambda\delta(x_1 - y). \tag{A.I.9}$$

To this approximation, the functional M is

$$M(x_1, z) = \mathcal{L}_\lambda\{\exp[-L_0(t_\mathbf{x} - t_\mathbf{z})]\delta(\mathbf{x} - \mathbf{z})\mathcal{L}_\lambda D_{\alpha\lambda}[\boldsymbol{\eta}; z, x]\}^{x = x_1} \tag{A.I.10}$$

where we must set $x = x_1$ after the action of the operator $\exp[-L_0(t_\mathbf{x} - t_\mathbf{z})]$. Substituting (A.I.10) in (A.I.4) and setting $\boldsymbol{\eta} \to 0$, we obtain Equation (2.24), which was derived in Section 2 using another method. Now we turn to the derivation of the equation for the correlation function. Similarly to the derivation of the equation for the mean distribution function, we shall proceed from the collisionless kinetic equation. Then the Green's function $G(x_1, x_1')$ obeys the equation

$$\hat{L}_0 G(x_1, x_1') + \mathcal{L}_\alpha H_{1\alpha}(x_1) G(x_1, x_1') = -\delta(x_1 - x_1'). \tag{A.I.11}$$

Multiplying (A.I.11) by

$$G(x_2, x_2') \exp i(\boldsymbol{\eta}\mathbf{H}_1)$$

and averaging the obtained equation over a statistical ensemble corresponding to the random magnetic field, we shall obtain the functional derivative equation

$$\left\{\hat{L}_1 - i\mathcal{L}_\alpha \frac{\delta}{\delta\eta(x_1)}\right\} g[\boldsymbol{\eta}; x_1, x_1'; x_2, x_2'] = -\delta(x_1 - x_1')\mathcal{Y}[\boldsymbol{\eta}; x_2, x_2'] \tag{A.I.12}$$

$$\hat{L}_1 = \hat{L}_0 - i\mathcal{L}_\alpha V_\alpha[\boldsymbol{\eta}; x_1] \tag{A.I.13}$$

relative to the functional

$$g[\boldsymbol{\eta}; x_1, x_1'; x_2, x_2'] = \frac{\langle G(x_1, x_1')G(x_2, x_2') \exp i(\boldsymbol{\eta}\mathbf{H}_1)\rangle}{\phi[\boldsymbol{\eta}]} \tag{A.I.14}$$

whose value coincides at $\boldsymbol{\eta}=0$ with the correlation function of distribution function fluctuations

$$g[0; x_1, x_1'; x_2, x_2'] \equiv g(x_1, x_1'; x_2, x_2') = \langle G(x_1, x_1')G(x_2, x_2')\rangle. \tag{A.I.15}$$

For further consideration, it is convenient to write equation (A.I.12) in the matrix form

$$\mathcal{Y}_2^{-1}\left\{\hat{L}_1 - i\mathcal{L}_{1\alpha}\frac{\delta}{\delta\eta_{1\alpha}}\right\}g_{1\alpha} = -\mathcal{T}_{12} \tag{A.I.16}$$

where $\mathcal{T}_{12}$ is the single matrix and the subscripts indicate the variables affected by the corresponding operators and the arguments on which the corresponding functionals depend. Equation (A.I.4) written in the matrix form

$$(\hat{L}_1 - M_1)\mathcal{Y}_1 = -1 \tag{A.I.17}$$

gives the relation

$$\hat{L}_1 = M_1 - \mathcal{Y}_1^{-1}. \tag{A.I.18}$$

Substituting (A.I.18) in (A.I.16), we obtain the equation

$$\mathcal{Y}_2^{-1}\left(\mathcal{Y}_1^{-1} - M_1 + i\mathcal{L}_{1\alpha}\frac{\delta}{\delta\eta_{1\alpha}}\right)g_{12} = \mathcal{T}_{12}. \tag{A.I.19}$$

Let the operator $\hat{Q}_{12}$ be determined by the following equality

$$(\mathcal{Y}_1^{-1}\mathcal{Y}_2^{-1} - \hat{Q}_{12})g_{12} = \mathcal{T}_{12}. \tag{A.I.20}$$

Comparing (A.I.19) with (A.I.20), we get

$$\hat{Q}_{12}g_{12} = \mathcal{Y}_2^{-1}M_1g_{12} - i\mathcal{L}_{1\alpha}\frac{\delta}{\delta\eta_{1\alpha}}(\mathcal{Y}_2^{-1}g_{12}) + i\mathcal{L}_{1\alpha}\frac{\delta\mathcal{Y}_2^{-1}}{\delta\eta_{1\alpha}}g_{12} \tag{A.I.21}$$

On the basis of (A.I.20) we have

$$\frac{\delta}{\delta\eta_{1\alpha}}(\mathcal{Y}_2^{-1}g_{12}) = \mathcal{Y}_1\frac{\delta}{\delta\eta_{1\alpha}}(\hat{Q}_{12}g_{12}) - \mathcal{Y}_1\frac{\delta\mathcal{Y}_2^{-1}}{\delta\eta_{1\alpha}}\mathcal{Y}_2^{-1}g_{12} \tag{A.I.22}$$

Substituting (A.I.22) in (A.I.21), we get

$$\hat{Q}_{12}g_{12} = \mathcal{Y}_2^{-1}M_1g_{12} - i\mathcal{L}_{1\alpha}\mathcal{Y}_1\frac{\delta}{\delta\eta_{1\alpha}}(\hat{Q}_{12}g_{12}) + i\mathcal{L}_{1\alpha}\left(\mathcal{Y}_1\frac{\delta\mathcal{Y}_1^{-1}}{\delta\eta_{1\alpha}}\mathcal{Y}_2^{-1} + \frac{\delta\mathcal{Y}_2^{-1}}{\delta\eta_{1\alpha}}\right)g_{12}. \tag{A.I.23}$$

On the other hand, it follows from (A.I.6) that

$$i\mathscr{L}_{1\alpha}\mathscr{Y}_1\frac{\delta\mathscr{Y}_1^{-1}}{\delta\eta_{1\alpha}}=-M_1, \tag{A.I.24}$$

$$i\mathscr{L}_{1\alpha}\frac{\delta\mathscr{Y}_2^{-1}}{\delta\eta_{1\alpha}}g_{12}=i\mathscr{L}_{1\alpha}\Gamma_{2\lambda}D_{1\alpha\lambda}g_{12} \tag{A.I.25}$$

and Equation (A.I.23) can be written in the form

$$\hat{Q}_{12}g_{12}=i\mathscr{L}_{1\alpha}\Gamma_{2\lambda}D_{1\alpha\lambda}g_{12}-i\mathscr{L}_{1\alpha}\frac{\delta}{\delta\eta_{1\alpha}}(\hat{Q}_{12}g_{12}). \tag{A.I.26}$$

Using the expression (A.I.9) for $\Gamma_{2\lambda}$ to the first approximation for the random field we obtain from (A.I.26):

$$\hat{Q}_{12}g_{12}=i\mathscr{L}_{1\alpha}\Gamma^0_{2\lambda}D_{1\alpha\lambda}g_{12}. \tag{A.I.27}$$

Substitution of (A.I.27) in (A.I.20) immediately gives the closed equation for the correlation function of fluctuations of the cosmic ray distribution function:

$$(\mathscr{Y}_1^{-1}\mathscr{Y}_2^{-1}-i\mathscr{L}_{1\alpha}\Gamma^0_{2\alpha}D_{1\alpha\lambda})g_{12}=\mathscr{T}_{12} \tag{A.I.28}$$

or, explicitly writing out all the arguments and specifying the matrix denominations, we obtain, after setting $\boldsymbol{\eta}=0$, the equation

$$g(x_1,x_1';x_2,x_2')=\mathscr{Y}(x_1,x_1')\mathscr{Y}(x_2,x_2')+\int \mathrm{d}y_1\,\mathrm{d}y_2\,\mathscr{Y}(x_1,y_1)\times\mathscr{Y}(x_2,y_2)\mathscr{L}_{1\alpha}\mathscr{B}_{\alpha\lambda}(y_1,y_2)\mathscr{L}_{2\lambda}g(x_1',y_1;x_2',y_2) \tag{A.I.29}$$

which was used in Section 7 for examining the cosmic ray fluctuations.

Appendix II

Considered in this Appendix is the solution of the equation for the cosmic ray correlation function $g_{\mathbf{pp}'}(\mathbf{r},t;\mathbf{r}',t')$ in the high-energy range of particles ($R\gg l_c$). In this case, we obtain a solution free of some of the limitations imposed in Section 7 when solving Equation (7.1) by the iteration method.

We shall use the method developed by Tatarsky (1967). To solve Equation (7.1), it is convenient to replace the variables of integration in the integral term of this equation according to the relations

$$\boldsymbol{\rho}=\tfrac{1}{2}(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2);\quad \mathbf{r}=\boldsymbol{\rho}_1-\boldsymbol{\rho}_2;\quad t=\tfrac{1}{2}(\tau_1+\tau_2);\quad \tau=\tau_1-\tau_2. \tag{A.II.1}$$

Then Equation (7.1) takes the form

$$g_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\mathscr{F}_{\mathbf{p}_1}(\mathbf{r}_1,t_1)\mathscr{F}_{\mathbf{p}_2}(\mathbf{r}_2,t_2)+\int \mathrm{d}\boldsymbol{\rho}\,\mathrm{d}\mathbf{p}_1'\,\mathrm{d}t\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p}_2'\,\mathrm{d}\tau \times\mathscr{Y}_{\mathbf{p}_1\mathbf{p}_2'}\left(\mathbf{r}_1-\boldsymbol{\rho}-\frac{\mathbf{r}}{2},t_1-t-\frac{\tau}{2}\right)\mathscr{Y}_{\mathbf{p}_2\mathbf{p}_2'}\left(\mathbf{r}_2-\boldsymbol{\rho}+\frac{\mathbf{r}}{2},t_2-t+\frac{\tau}{2}\right)\hat{\mathscr{L}}_{\alpha\mathbf{p}_1'}\mathscr{B}_{\alpha\lambda}(\mathbf{r},\tau)\times\mathscr{L}_{\lambda\mathbf{p}_\lambda'}g_{\mathbf{p}_1'\mathbf{p}_2'}\left(\boldsymbol{\rho}+\frac{\mathbf{r}}{2},t+\frac{\tau}{2};\boldsymbol{\rho}-\frac{\mathbf{r}}{2},t-\frac{\tau}{2}\right). \tag{A.II.2}$$

The Fourier transformation of this equation gives

$$g_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{k}_1, \omega_1; \mathbf{k}_2, \omega_2) = \mathcal{F}_{\mathbf{p}_1}(\mathbf{k}_1, \omega_1)\mathcal{F}_{\mathbf{p}_2}(\mathbf{k}_2, \omega_2)$$
$$+ (2\pi)^6 \int d\mathbf{p}_1' \, d\mathbf{p}_2' \mathcal{Y}_{\mathbf{p}_1\mathbf{p}_1'}(\mathbf{k}_1, \omega_1)\mathcal{Y}_{\mathbf{p}_2\mathbf{p}_2'}(\mathbf{k}_2, \omega_2)$$
$$\times \int d\mathbf{k} \, d\omega \hat{\mathcal{L}}_{\alpha\mathbf{p}_1}\mathcal{B}_{\alpha\lambda}(\mathbf{k}, \omega)\mathcal{L}_{\lambda\mathbf{p}_2'}$$
$$\times g_{\mathbf{p}_1'\mathbf{p}_2'}(\mathbf{k}_1 - \mathbf{k}, \omega_1 - \omega; \mathbf{k}_2 + \mathbf{k}, \omega_2 + \omega). \quad \text{(A.II.3)}$$

To directly solve equation (A.II.3), it will be taken into account that at $R \gg l_c$ the spectral tensor of the random magnetic field varies weakly at spatial distances corresponding to the characteristic scale of the change of the correlation function $g_{\mathbf{p}\mathbf{p}'}(\mathbf{r}, t; \mathbf{r}', t')$ and Green's function $\mathcal{Y}_{\mathbf{p}\mathbf{p}'}(\mathbf{r}, t; \mathbf{r}', t')$. Then, replacing the variables $\mathbf{k}_1 \to \mathbf{k}_1 - \mathbf{k}'$, $\omega_1 \to \omega_1 - \omega'$ and $\mathbf{k}_2 \to \mathbf{k}_2 + \mathbf{k}'$, $\omega_2 \to \omega_2 + \omega'$ in (A.II.3), we act on the obtained equation by the operator $\hat{\mathcal{L}}_{\alpha\mathbf{p}_1}\mathcal{B}_{\alpha\lambda}(\mathbf{k}', \omega')\hat{\mathcal{L}}_{\lambda\mathbf{p}_2}$ and integrate it over $\mathbf{k}'$, ω'

$$\int d\mathbf{k}' \, d\omega' \hat{\mathcal{L}}_{\alpha\mathbf{p}_1}\mathcal{B}_{\alpha\lambda}(\mathbf{k}', \omega')\mathcal{L}_{\lambda\mathbf{p}_2} g(\mathbf{k}_1 - \mathbf{k}', \omega_1 - \omega'; \mathbf{k}_2 + \mathbf{k}', \omega_2 + \omega')$$
$$= \int d\mathbf{k}' \, d\omega' \hat{\mathcal{L}}_{\alpha\mathbf{p}_1}\mathcal{B}_{\alpha\lambda}(\mathbf{k}', \omega')\hat{\mathcal{L}}_{\lambda\mathbf{p}_2}\mathcal{F}_{\mathbf{p}_1}(\mathbf{k}_1 - \mathbf{k}', \omega_1 - \omega')\mathcal{F}_{\mathbf{p}_2}(\mathbf{k}_2 + \mathbf{k}', \omega_2 + \omega')$$
$$+ (2\pi)^6 \int d\mathbf{p}_1' \, d\mathbf{p}_2' \int d\mathbf{k}' \, d\omega' \hat{\mathcal{L}}_{\mu\mathbf{p}_1}\mathcal{B}_{\mu\nu}(\mathbf{k}', \omega')\hat{\mathcal{L}}_{\nu\mathbf{p}_2}\mathcal{Y}_{\mathbf{p}_1\mathbf{p}_1'}(\mathbf{k}_1 - \mathbf{k}', \omega_1 - \omega')$$
$$\times \mathcal{Y}_{\mathbf{p}_2\mathbf{p}_2'}(\mathbf{k}_2 + \mathbf{k}', \omega_2 + \omega') \int d\mathbf{k} \, d\omega \hat{\mathcal{L}}_{\alpha\mathbf{p}_1'}\mathcal{B}_{\alpha\lambda}(\mathbf{k}, \omega)\hat{\mathcal{L}}_{\lambda\mathbf{p}_2'}$$
$$\times g(\mathbf{k}_1 - \mathbf{k}' - \mathbf{k}, \omega_1 - \omega' - \omega; \mathbf{k}_2 + \mathbf{k}' + \mathbf{k}, \omega_2 + \omega' + \omega). \quad \text{(A.II.4)}$$

Replace the variables $\mathbf{k}' + \mathbf{k} = \mathbf{k}''$ and $\omega' + \omega = \omega''$ in the integral over $d\mathbf{k} \, d\omega$ in the second addend in the right hand part of (A.II.4). Then, the difference between $\mathcal{B}_{\alpha\lambda}(\mathbf{k}'', \omega'')$ and $\mathcal{B}_{\alpha\lambda}(\mathbf{k}'' - \mathbf{k}', \omega'' - \omega')$ in the obtained integral may be neglected in virtue of the slow change of the variation tensor.

The resultant equation is

$$T_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{k}_1, \omega_1; \mathbf{k}_2, \omega_2) = \int d\mathbf{k} \, d\omega \hat{\mathcal{L}}_{\alpha\mathbf{p}_1}\mathcal{B}_{\alpha\lambda}(\mathbf{k}, \omega)\hat{\mathcal{L}}_{\lambda\mathbf{p}_2}\mathcal{F}_{\mathbf{p}_1}(\mathbf{k}_1 - \mathbf{k}, \omega_1 - \omega)$$
$$\times \mathcal{F}_{\mathbf{p}_2}(\mathbf{k}_2 + \mathbf{k}, \omega_2 + \omega) + (2\pi)^6 \int d\mathbf{p}_1' \, d\mathbf{p}_2' \int d\mathbf{k} \, d\omega \hat{\mathcal{L}}_{\alpha\mathbf{p}_1}\mathcal{B}_{\alpha\lambda}(\mathbf{k}, \omega)$$
$$\times \hat{\mathcal{L}}_{\lambda\mathbf{p}_2}\mathcal{Y}_{\mathbf{p}_1\mathbf{p}_1'}(\mathbf{k}_1 - \mathbf{k}, \omega_1 - \omega)\mathcal{Y}_{\mathbf{p}_2\mathbf{p}_2'}(\mathbf{k} + \mathbf{k}, \omega_2 + \omega)$$
$$\times T_{\mathbf{p}_1'\mathbf{p}_2'}(\mathbf{k}_1, \omega_1; \mathbf{k}_2, \omega_2).$$
$$T_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{k}_1, \omega_1; \mathbf{k}_2, \omega_2) = \int d\mathbf{k} \, d\omega \hat{\mathcal{L}}_{\alpha\mathbf{p}_1}\mathcal{B}_{\alpha 1}(\mathbf{k}, \omega)$$
$$\times \mathcal{L}_{\lambda\mathbf{p}_2} g_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{k}_1 - \mathbf{k}, \omega_1 - \omega; \mathbf{k}_2 + \mathbf{k}, \omega_2 + \omega). \quad \text{(A.II.5)}$$

If the function $T_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{k}_1, \omega_1; \mathbf{k}_2, \omega_2)$ is assumed to vary slowly in the momentum space, Equation (A.II.5) can be solved with respect to $T_{\mathbf{p}_1\mathbf{p}_2}$. Substituting the obtained equation in the right part of (A.II.2), we obtain the expression for the correlation function of cosmic ray fluctuations

$$g_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{k}_1, \omega_1; \mathbf{k}_2, \omega_2) = \mathcal{F}_{\mathbf{p}_1}(\mathbf{k}_1, \omega_1)\mathcal{F}_{\mathbf{p}_2}(\mathbf{k}_2, \omega_2) + (2\pi)^6 \int \mathrm{d}\mathbf{p}_1' \, \mathrm{d}\mathbf{p}_2' \mathcal{Y}_{\mathbf{p}_1\mathbf{p}_1}(\mathbf{k}_1, \omega_1)\mathcal{Y}_{\mathbf{p}_2\mathbf{p}_2'}(\mathbf{k}_2, \omega_2)$$

$$\times \left(\frac{\int \mathrm{d}\mathbf{k} \, \mathrm{d}\omega \hat{\mathcal{L}}_{\alpha\mathbf{p}_1} \mathcal{B}_{\alpha\lambda}(\mathbf{k}, \omega) \hat{\mathcal{L}}_{\lambda\mathbf{p}_2} \mathcal{F}_{\mathbf{p}_1}(\mathbf{k}_1 - \mathbf{k}, \omega_1 - \omega) \mathcal{F}_{\mathbf{p}_2}(\mathbf{k}_2 + \mathbf{k}, \omega_2 + \omega)}{1 - \int \mathrm{d}\mathbf{k} \, \mathrm{d}\omega \mathcal{L}_{\alpha\mathbf{p}_1} \mathcal{B}_{\alpha\lambda}(\mathbf{k}, \omega) \mathcal{L}_{\lambda\mathbf{p}_2} \int \mathrm{d}\mathbf{p}_1' \, \mathrm{d}\mathbf{p}_2' \mathcal{Y}_{\mathbf{p}_1\mathbf{p}_1'}(\mathbf{k}_1 - \mathbf{k}, \omega_1 - \omega) \mathcal{Y}_{\mathbf{p}_2\mathbf{p}_2'}(\mathbf{k}_2 \times \mathbf{k}, \omega_2 + \omega)} \right). \tag{A.II.6}$$

If the integral term in the denominator is $\ll 1$, this expression turns into the solution obtained by Shishov (1968) (see Section 7) using the iteration method.

Appendix III

The theory of cosmic ray propagation makes use of the various forms of the anisotropic diffusion equation, depending on the choice of the variables (momentum p, rigidity $\rho = (cp/eZ)$, energy ε) describing the charged particle propagation. When using these equations specifically, the necessity arises to interrelate the values contained in these equations. The present Appendix gives the various forms of the anisotropic diffusion equation and establishes the relationships between the phase densities of particles expressed in different variables.

We shall start with Equation (3.10), which, after the evident transformation, can be written in the following form

$$\frac{\partial N(\mathbf{r}, p, t)}{\partial t} = \frac{\partial}{\partial r_\alpha}\left(\varkappa_{\alpha\lambda}(\mathbf{r}, p)\frac{\partial}{\partial r_\lambda} - u_{0\alpha}(\mathbf{r})\right) N(\mathbf{r}, p, t) + \tfrac{1}{3} \operatorname{div} \mathbf{u}_0(\mathbf{r}) \frac{1}{p^2}\frac{\partial}{\partial p} p^3 N(\mathbf{r}, p, t). \tag{A.III.1}$$

If the rigidity $\rho = (cp/eZ)$ (Z is the charge number) is chosen as the energy variable, Equation (A.III.1) takes the form

$$\frac{\partial N(\mathbf{r}, p, t)}{\partial t} = \frac{\partial}{\partial r_\alpha}\left(\varkappa_{\alpha\lambda}(\mathbf{r}, \rho)\frac{\partial}{\partial r_\lambda} - u_{0\alpha}(\mathbf{r})\right) N(\mathbf{r}, \rho, t) + \tfrac{1}{3} \operatorname{div} \mathbf{u}_0(\mathbf{r}) \frac{1}{\rho^2}\frac{\partial}{\partial \rho} \rho^3 N(\mathbf{r}, \rho, t). \tag{A.III.2}$$

Using the relation between the particle density $n(\mathbf{r}, t)$ and the particle density in the momentum space $N(\mathbf{r}, p, t)$

$$n(\mathbf{r}, t) = \int_{p_0}^{\infty} \mathrm{d}p p^2 N(\mathbf{r}, p, t) \tag{A.III.3}$$

it can be shown that

$$N(\mathbf{r}, p, t) = \left(\frac{c}{eZ}\right)^3 N(\mathbf{r}, \rho, t) \tag{A.III.4}$$

and

$$n(\mathbf{r}, t) = \int_{\rho_0}^{\infty} d\rho \rho^2 N(\mathbf{r}, \rho, t). \tag{A.III.5}$$

If the energy $\varepsilon = c(m_0^2c^2 + p^2)^{1/2}$ is chosen as the energy variable, Equation (A.III.1) can be written as

$$\frac{\partial N(\mathbf{r}, \varepsilon, t)}{\partial t} = \frac{\partial}{\partial r_\alpha}\left(\varkappa_{\alpha\lambda}(\mathbf{r}, \varepsilon)\frac{\partial}{\partial r_\lambda} - u_{0\alpha}(\mathbf{r})\right)N(\mathbf{r}, \varepsilon, t) + \tfrac{1}{3}\operatorname{div}\mathbf{u}_0(\mathbf{r})\frac{\partial}{\partial \varepsilon}\frac{(\varepsilon^2 - \varepsilon_0^2)}{\varepsilon}N(\mathbf{r}, \varepsilon, t) \tag{A.III.6}$$

where $\varepsilon_0 = m_0c^2$, and the phase densities $N(\mathbf{r}, p, t)$ and $N(\mathbf{r}, \varepsilon, t)$ are related by

$$N(\mathbf{r}, p, t) = \frac{c^2}{\varepsilon p} N(\mathbf{r}, \varepsilon, t) \tag{A.III.7}$$

and

$$n(\mathbf{r}, t) = \int_{\varepsilon_0}^{\infty} d\varepsilon N(\mathbf{r}, \varepsilon, t). \tag{A.III.8}$$

Passing over to the variable

$$T = \varepsilon - \varepsilon_0, \tag{A.III.9}$$

(where T is the kinetic energy of a particle), we can write Equation (A.III.6) in the form often used in the literature (Jokipii, 1971):

$$\frac{\partial N(\mathbf{r}, T, t)}{\partial t} = \frac{\partial}{\partial r_\alpha}\left(\varkappa_{\alpha\lambda}(\mathbf{r}, T)\frac{\partial}{\partial r_\lambda} - u_{0\alpha}(\mathbf{r})\right)N(\mathbf{r}, T, t) + \tfrac{1}{3}\operatorname{div}\mathbf{u}_0(\mathbf{r})\frac{\partial}{\partial T}\alpha(T)TN(\mathbf{r}, T, t), \tag{A.III.10}$$

where

$$\alpha = \frac{T + 2\varepsilon_0}{T + \varepsilon_0}. \tag{A.III.11}$$

Consider the ultrarelativistic case $\varepsilon \gg \varepsilon_0$, when it may be set that $\alpha = 1$. Then

Equation (A.III.6) takes the form

$$\frac{\partial N(\mathbf{r}, \varepsilon, t)}{\partial t} = \frac{\partial}{\partial r_\alpha}\left(\varkappa_{\alpha\lambda}(\mathbf{r}, \varepsilon)\frac{\partial}{\partial r_\lambda} - u_{0\alpha}(\mathbf{r})\right)N(\mathbf{r}, \varepsilon, t) + \tfrac{1}{3}\operatorname{div}\mathbf{u}_0(\mathbf{r})\frac{\partial}{\partial \varepsilon}\varepsilon N(\mathbf{r}, \varepsilon, t). \quad \text{(A.III.12)}$$

Since at $\varepsilon \gg \varepsilon_0$ the energy is related to the momentum p by $\varepsilon \approx cp = eZ\rho$, it follows from (A.III.4) and (A.III.7) that

$$N(\mathbf{r}, \varepsilon, t) = \frac{\rho^2}{eZ} N(\mathbf{r}, \rho, t) \quad \text{(A.III.13)}$$

and Equation (A.III.12) takes the form (A.III.2).

It follows from (A.III.13) that, if the cosmic ray spectrum is of power form $N(\varepsilon) \sim \varepsilon^{-\gamma}$ and $N(\rho) \sim \rho^{-\beta}$, the spectrum exponents are interrelated in the ultrarelativistic case as

$$\beta = \gamma + 2. \quad \text{(A.III.14)}$$

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