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# Schrödinger Invariance and Strongly Anisotropic Critical Systems

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The extension of strongly anisotropic or dynamical scaling to local scale invariance is investigated. For the special case of an anisotropy or dynamical exponent $\theta = z = 2$, the group of local scale transformation considered is the Schrödinger group, which can be obtained as the nonrelativistic limit of the conformal group. The requirement of Schrödinger invariance determines the two-point function in the bulk and reduces the three-point function to a scaling form of a single variable. Scaling forms are also derived for the two-point function close to a free surface which can be either spacelike or timelike. These results are reproduced in several exactly solvable statistical systems, namely the kinetic Ising model with Glauber dynamics, lattice diffusion, Lifshitz points in the spherical model, and critical dynamics of the spherical model with a nonconserved order parameter. For generic values of $\theta$, evidence from higher-order Lifshitz points in the spherical model and from directed percolation suggests a simple scaling form of the two-point function.

**KEY WORDS:** Anisotropic scaling; conformal invariance; Schrödinger invariance; critical dynamics; response function.

## 1. INTRODUCTION

Scale invariance is a central notion in present theories of critical behavior. In the context of two-dimensional, static, and isotropic critical behavior, these ideas have become spectacularly successful in the context of conformal invariance.[1] The main physical idea behind this is the extension of the covariance of correlation functions under length rescaling by a constant factor $\lambda$ to general, space-dependent rescalings $\lambda(\mathbf{r})$. A coordinate trans-

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formation $\lambda(\mathbf{r})$ is conformal if the angles are kept unchanged. In two dimensions, the conformal group is infinite-dimensional. This has led, for example, to the exact calculation of critical exponents and correlation functions and yields a handle for classifying two-dimensional universality classes (for reviews, see, e.g., ref. 2).

Much less is known about nonisotropic scaling. Consider a (connected) correlation (response) function $C(\mathbf{r}; t)$ depending on "space" coordinates $\mathbf{r}$ and a "time" coordinate $t$ which satisfies the scaling relation

$$C(\lambda\mathbf{r}; \lambda^{\theta}t) = \lambda^{-2x}C(\mathbf{r}; t) \tag{1.1}$$

where $x$ is a scaling dimension and $\theta$ is referred to as the *anisotropy exponent*. Systems which satisfy Eq. (1.1) with $\theta \neq 1$ are by definition *strongly anisotropic critical systems*. In fact, dynamical scaling of this kind appears quite commonly in time-delayed averages close to an equilibrium phase transition, where the anisotropy exponent is referred to as the dynamical exponent $z = \theta$,[3,4] or in domain growth problems of systems quenched at or below the equilibrium critical point (see refs. 5–7 and for a recent review ref. 8). Alternatively, strongly anisotropic scaling may arise in statics; examples are provided by directed percolation[9] or by magnetic systems at a Lifshitz point,[10] where the anisotropy exponent $\theta = \nu_{\|}/\nu_{\perp}$ is related to the critical exponents $\nu_{\|,\perp}$ of the correlation lengths $\xi_{\|,\perp}$ parallel and perpendicular to the preferred direction. Equation (1.1) can be recast in the form

$$C(\mathbf{r}; t) = t^{-2x/\theta}\Phi(u) \tag{1.2}$$

which defines the scaling function $\Phi(u)$ and

$$u = \frac{r^{\theta}}{t} \tag{1.3}$$

is the scaling variable and henceforth we shall be always taking the scaling limit $r \to \infty$, $t \to \infty$ where $u$ is kept fixed.

We ask the following question: what can be said about the scaling function $\Phi(u)$? Is it sensible to look beyond global scaling with $\lambda$ constant to a space-time-dependent rescaling factor $\lambda(\mathbf{r}, t)$?

This question had been addressed by Cardy.[11] Assuming dynamical scaling for the dynamical response function, he takes as extended set of local scale transformations $\lambda(\mathbf{r})$ the space-dependent scaling $\mathbf{r} \to \lambda(\mathbf{r})\mathbf{r}$, $t \to \lambda(\mathbf{r})^{\theta}\, t$, where the $\lambda(\mathbf{r})$ are two-dimensional conformal transformations. This means that only systems at a static critical point are considered. The assumed covariance of the response function is used to map the problem from the two-dimensional plane (in the space coordinates) to the strip geometry, with a nonuniform rate. Next, since close to a static critical point the static correlation length of the system in the strip is of the same order of magnitude as its width, it is claimed that "on much larger distances it is permissible to use mean field theory to calculate the dynamic correlation function in the strip."[11] For a system with a nonconserved order parameter the response function then turns out to be[11]

$$G(r, t) \sim t^{-2x/\theta - 1} \exp\left(-\frac{r^\theta}{t}\right) \tag{1.4}$$

where some nonuniversal constants have been suppressed. The case of a conserved order parameter was also treated. While this result is appealingly simple, the assumptions made in deriving it may appear to be quite strong,[2] in particular, the use of mean-field (van Hove) theory. Also, one might wish to reconsider the assumption that $\lambda(\mathbf{r})$ is time-independent. In fact, we shall study the scaling of the two-point function in $(1+1)$-dimensional directed percolation and higher-order Lifshitz points in the spherical model, both with $\theta \neq 2$, and find that the form of the scaling function of these models does not agree with Eq. (1.4) (see Section 5).

Here we propose another group of space-time-dependent local scalings $\lambda(\mathbf{r}, t)$. For definiteness, we shall only consider the case $\theta = 2$, but we do not have to make the restriction to $d = 2$ space dimensions. Although the van Hove theory for a nonconserved order parameter has also $\theta = 2$,[13] we do not make any of the approximations involved in that theory. Rather, it is our aim to find the scaling functions merely from their transformation properties under local scale transformations. By taking $\theta = 2$, we mean to perform the simplest case study of local, albeit not conformal, scaling transformations. The group of the local scaling transformations is the Schrödinger group, which shall be defined in the next section. The approach chosen has the advantage of being close in spirit to the earliest investigations of conformal invariance in critical phenomena.[14] On a more formal level, the comparison of conformal with Schrödinger invariance provides some insight into the characteristic properties of both.

Our results are as follows:

1. If the domain of both time and space coordinates is infinite in extent and the scaling fields transform covariantly under the Schrödinger group, the two-point function is completely determined, while the three-point function is reduced to a scaling form of one variable; see Eqs. (3.12), (3.28). For example, this applies to the calculation of time-delayed correla-

[2] The restriction to two-dimensional space is not really required and could be removed.[12]

tions of systems at equilibrium and at a static critical point, or else to lattice diffusion problems.

2. If the space geometry is semi-infinite, a scaling form for the two-point function will be derived; see Eq. (3.35). This may be relevant to critical dynamics close to a surface; see ref. 15 for an example.

3. For a system in a predefined initial state, it can be shown that critical relaxation *toward* equilibrium displays dynamical scaling already at intermediate times,[6,7] much later than microscopic times, but also well before the late-time regime usually considered. We derive the form of the two-point function; see Eq. (3.48).

4. These results can be reproduced from a variety of exactly solvable models.

5. The Schrödinger group can be extended to an infinite-dimensional group whose Lie algebra contains a Virasoro subalgebra. It can be shown that for systems with local interactions Schrödinger invariance follows from the requirements of translation invariance in both space and time, rotation invariance in space, scale invariance, and Galilei invariance. If no anomalies occur, this even holds for the whole infinite-dimensional group.

The work described in this paper uses background from both conformal field theory and time-dependent statistical mechanics. To make the paper accessible to readers with knowledge in one but not both of these fields, we repeat in Section 2 the definition of the Schrödinger group and recall a few well-known facts about Galilei-invariant theories and dynamical scaling. Section 3 describes the derivation of the two- and three-point functions for either infinite or semi-infinite geometries. The Schrödinger Ward identity is considered as well, and the nonexistence of nonconventional central extensions of the Schrödinger Lie algebra is shown (Appendix B). The discussions of this section follow closely the known derivation of correlation functions from conformal invariance. In Section 4, we test and confirm the predictions from Schrödinger invariance by calculating two- and three-point functions in several exactly solvable and strongly anisotropic critical models. In Section 5, we examine the scaling of the two-point function for some systems with an anisotropy exponent $\theta \neq 2$. Exact and numerical results indicate a disagreement with the result, Eq. (1.4), obtained from two-dimensional conformal invariance and suggest an alternative simple scaling form. Section 6 gives our conclusions.

## 2. BASIC CONCEPTS AND TERMINOLOGY

We begin by recalling some well-known facts about the Schrödinger group, Galilean invariance in field theories, and dynamical scaling.

## 2.1. The Schrödinger Group

The Schrödinger group is defined[16,17] by the following set of transformations:

$$\mathbf{r} \to \mathbf{r}' = \frac{\mathscr{R}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta}, \qquad t \to t' = \frac{\alpha t + \beta}{\gamma t + \delta}; \qquad \alpha\delta - \beta\gamma = 1 \tag{2.1}$$

where $\alpha$, $\beta$, $\gamma$, $\delta$, $\mathbf{v}$ and $\mathbf{a}$ are real parameters and $\mathscr{R}$ is a rotation matrix in $d$ space dimensions. It is apparent that the Schrödinger group can be obtained from the Galilei group by extending the time translations to the full Möbius group $Sl(2, R)$ of fractional real linear transformations in time as given in Eq. (2.1). A faithful matrix representation is given by

$$\mathscr{L}_g = \begin{pmatrix} \mathscr{R} & \mathbf{v} & \mathbf{a} \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix}, \qquad \mathscr{L}_g \mathscr{L}_{g'} = \mathscr{L}_{gg'} \tag{2.2}$$

Niederer[16] showed that this group is the maximal kinematical group which transforms solutions of the free Schrödinger equation

$$\left( i\frac{\partial}{\partial t} + \frac{1}{2m}\frac{\partial^2}{\partial r^2} \right) \psi = 0 \tag{2.3}$$

into other solutions of (2.3), namely $(\mathbf{r}, t) \mapsto g(\mathbf{r}, t)$, $\psi \to T_g \psi$,

$$(T_g \psi)(\mathbf{r}, t) = f_g(g^{-1}(\mathbf{r}, t))\, \psi(g^{-1}(\mathbf{r}, t)) \tag{2.4}$$

where the companion function $f_g$ is[16]

$$f_g(\mathbf{r}, t) = (\gamma t + \delta)^{-d/2} \times \exp\left[ -\frac{im}{2} \frac{\gamma \mathbf{r}^2 + 2\mathscr{R}\mathbf{r} \cdot (\gamma \mathbf{a} - \delta \mathbf{v}) + \gamma \mathbf{a}^2 - \delta t\, \mathbf{v}^2 + 2\gamma \mathbf{a}\mathbf{v}}{\gamma t + \delta} \right] \tag{2.5}$$

Independently, it was shown by Hagen[17] that nonrelativistic free field theory is Schrödinger invariant, treating both scalar and spin-1/2 fields. It was also shown that the operators which appear in the conservations laws associated with the space-time symmetries can be reformulated to allow the statement of Schrödinger group in $d$ space dimensions can be obtained by a group contraction (where the speed of light $c \to \infty$) from the conformal group in $d+1$ dimensions[18] provided the mass is conviently rescaled as well. Its projective representations as required by (2.4) have been studied in detail.[19]

There are many more equations whose kinematical group is isomorphic to the Schrödinger group. It can be shown that the most general potential which can be added in (2.3) such that the kinematical group is still isomorphic to (2.1) is of the form, up to orthogonal transformations, $V(\mathbf{r}) = v^{(2)} \sum_i r_i^2 + \sum_i v_i^{(1)} r_i + v^{(0)}$.[20] Further examples are provided by a nonlinear Schrödinger equation,[21] or, but with a more general transformation law than (2.4), by the Navier–Stokes equation with homogeneous pressure or by Burger's equation.[22] Higher-order symmetry operators of Eq. (2.3) are examined in ref. 23. The Schrödinger group also appears as a dynamical symmetry group for the Dirac monopole or magnetic vortices[24] or in the nonrelativistic $N$-body problem with inverse-square interactions.[25] Similarly, one may treat the diffusion equation by writing $m^{-1} = 2iD$, where $D$ is the diffusion constant. We shall do so for most of this paper. In any case, we shall only consider here the realization of the Schrödinger group provided by the free Schrödinger equation (2.3). Using different realizations will in general lead to different results.

For simplicity, we restrict attention here to fields which are scalar under space rotations (so that it is sufficient to take $\mathcal{R} = 1$) and shall mostly also take just one space dimension $d = 1$. This is not a serious restriction and generalizations are straightforward.

The set $\mathcal{S}_{\text{fin}} = \{X_{-1}, X_0, X_1, Y_{-1/2}, Y_{1/2}, M_0\}$ spans the Lie algebra of the Schrödinger group, Eq. (2.1). The generators read (we take $d = 1$)

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - \frac{n+1}{2} t^n r \partial_r - \frac{n(n+1)}{4} \mathcal{M} t^{n-1} r^2 \\ Y_m &= -t^{m+1/2}\partial_r - \left(m + \frac{1}{2}\right) \mathcal{M} t^{m-1/2} r \\ M_n &= -t^n \mathcal{M} \end{aligned} \tag{2.6}$$

where the terms $\sim \mathcal{M}$ come from the companion function. When $\mathcal{M} = im$ is purely imaginary, this corresponds to the Schrödinger equation where $m$ is the mass, while for $\mathcal{M}$ real, this is the form corresponding to the diffusion equation. The commutation relations are

$$\begin{aligned} [X_n, X_m] &= (n-m) X_{n+m} \\ [X_n, Y_m] &= (n/2 - m) Y_{n+m} \\ [X_n, M_m] &= -m M_{n+m} \\ [Y_n, Y_m] &= (n-m) M_{n+m} \\ [Y_n, M_m] &= [M_n, M_m] = 0 \end{aligned} \tag{2.7}$$

(In more than one space dimension, there are several sets of generators $Y_m^{(i)}$, $i = 1,..., d$, but only one set of $X_n$, where $r\partial_r$ is replaced by $\mathbf{r}\partial_\mathbf{r}$, etc.). The commutation relations (2.7) remain valid when the infinite set of generators $\mathscr{S} = \{X_n, Y_m, M_n\}$, where $n$ is an integer and $m$ is a half-integer, is considered.[26] The Lie algebra can be decomposed, $\mathscr{S} = \mathscr{S}_X \oplus_s \mathscr{S}_Y$, where $\mathscr{S}_X = \{X_n\}$ and $\mathscr{S}_Y = \{Y_m, M_n\}$. As we shall see later, these two subalgebras arise in quite distinct physical situations.

## 2.2. Galilei-Invariant Field Theory

The Galilei Lie algebra (here for the case $d = 1$ only) is generated from the set $\mathscr{G} = \{X_{-1}, Y_{-1/2}, Y_{1/2}\} \subset \mathscr{S}_{\text{fin}}$. Thus any Schrödinger-invariant theory will have to satisfy the constraints following from Galilean invariance as well. These conditions have been well known for a long time[27] and we briefly recall the properties relevant for us. In fact, it is possible to construct a consistent field theory which is Galilei-covariant from the following postulates. The simplest example of this is second-quantized ordinary nonrelativistic quantum mechanics.

States are rays in a Hilbert space and the dynamical variables are operators in the Hilbert space, with the usual rules for the calculation of probabilities. The Galilei group acts by a unitary projective representation $\mathscr{U}(g)$ in the Hilbert space. If $\phi(r, t)$ is a field of the theory, it is required to transform locally

$$\mathscr{U}(g)^{-1}\,\phi(r, t)\,\mathscr{U}(g) = \exp\left[\frac{im}{2}(v^2 t + 2vr)\right]\phi(r + vt + a, t + \beta) \quad (2.8)$$

States are characterized by the Casimir operators, whose eigenvalues are mass and spin (if $d > 1$). Since we deal with projective representations, we obtain a unitary representation of a central extension of $\mathscr{G}$ by a one-dimensional Lie algebra generated by $M_0$; see Eq. (2.7). This implies, since the extension is nontrivial because $\mathscr{G}$ is nonsemisimple, that a physically trivial transformation may result in a modification of the phase of the state vector which depends on the mass of the system. Galilei covariance thus requires the Bargmann superselection rule of the mass,[28] which states that for an interaction of particles of the form

$$A + B + \cdots \to A' + B' + \cdots \quad (2.9)$$

one must have

$$m_A + m_B + \cdots = m_{A'} + m_{B'} + \cdots \quad (2.10)$$

This implies that no Galilean field can be Hermitian unless it is massless. We see that the mass plays quite a distinct role in nonrelativistic theories as compared to relativistic ones. We emphasize that the mass no longer describes the deviation from critical behavior in our context, as it does in relativistic theories. Masses should, in the light of (2.10), rather be regarded as some kind of analog of a charge[27] [this requires the Lagrangian to have a $U(1)$ invariance]. Nonvanishing correlations are of the type

$$\langle \phi_a(\mathbf{r}, t)\, \phi_b^*(\mathbf{r}', t')\rangle \sim \delta_{m_a, m_b} \mathscr{X}_{a,b}(\mathbf{r}-\mathbf{r}', t-t') \tag{2.11}$$

and similarly for higher-order correlations. This also holds by analytic continuation for Euclidean theories. We shall rederive the superselection rule, Eq. (2.10), in several cases below.

A further remark is in order here. In general, in the context of a statistical system, the mass will contain nonuniversal factors which merely serve to define the time scale. Here we are interested in the *ratios* of masses of different scaling fields, which are universal.

While the Bargmann superselection rule provides a restriction not present in relativistic theories, Galilean field theory is considerably less restricted in many other aspects. For example, the consequences of locality in Galilean field theory are no longer sufficient to prove either the *CPT* or the spin–statistics theorem; see ref. 27 for a full discussion.

## 2.3. Dynamical Scaling

Finally, we recall some facts about response functions and dynamical scaling, following refs. 4 and 29. Consider a scaling field $\phi(\mathbf{r}, t)$. In systems described by a Hamiltonian one can define its conjugate field $h(\mathbf{r}, t)$. Then the linear response function $\chi(\mathbf{k}, \omega)$ is defined in momentum–frequency space by

$$\langle \phi(\mathbf{k}, \omega)\rangle = \chi(\mathbf{k}, \omega)\, h(\mathbf{k}, \omega) \tag{2.12}$$

where the field $h$ is taken to be infinitesimal, the average is determined from the time-dependent probability distribution in the presence of $h$, and the system is assumed to start from equilibrium at $t \to -\infty$. The Fourier transforms are

$$h(\mathbf{r}, t) = \int \frac{d\mathbf{k}}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} h(\mathbf{k}, t) \tag{2.13}$$

$$\chi(\mathbf{k}, \omega) = \int_0^{\infty} dt \int d\mathbf{r}\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} G(\mathbf{r}, t) \tag{2.14}$$

We are interested in time-delayed correlation functions

$$C_\phi(\mathbf{r}, t) = \langle \phi(\mathbf{r}, t)\, \phi(\mathbf{0}, 0)\rangle_{h=0} - \langle \phi(\mathbf{r}, t)\rangle_{h=0} \langle \phi(\mathbf{0}, 0)\rangle_{h=0} \tag{2.15}$$

and define its Fourier transform $C_\phi(\mathbf{k}, \omega)$ according to Eq. (2.13) and the equal-time correlation function $C_\phi(\mathbf{k})$ defined as

$$C_\phi(\mathbf{k}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, C_\phi(\mathbf{k}, \omega) \tag{2.16}$$

From causility, it can be shown[29] that the response function $\chi(\mathbf{k}, \omega)$ is an analytic function of the complex frequency $\omega$ in the upper half-plane and its real and imaginary parts satisfy the Kramers–Kronig dispersion relations. For classical systems with a Hamiltonian, the fluctuation-dissipation theorem states[4,29]

$$C_\phi(\mathbf{k}, \omega) = \frac{2k_B T}{\omega}\, \Im\chi(\mathbf{k}, \omega) \tag{2.17}$$

where $\Im$ denotes the imaginary part. The hypothesis of dynamical scaling now asserts that at a static critical point[3,4]

$$\chi(\mathbf{k}, \omega) = \mathscr{A} k^{2x-d} \Phi(\mathscr{B} \omega k^{-\theta}) \tag{2.18}$$

in the scaling limit $\omega \to 0$, $k \to 0$ with $\omega k^{-\theta}$ fixed, where $\theta = z$ is the dynamical (anisotropy) exponent, $x$ is a scaling dimension, $\Phi$ is a universal scaling function, and $\mathscr{A}$, $\mathscr{B}$ are nonuniversal constants.

## 3. MULTIPOINT CORRELATIONS FROM SCHRÖDINGER INVARIANCE

We now derive the consequences of Schrödinger invariance for the correlations. In general, we expect a scaling field $\phi(\mathbf{r}, t)$ to be characterized by its mass $\mathscr{M}$, its scaling dimension $x$, and its spin $s$ (which we take to be zero throughout, but see ref. 17 for the case of spin 1/2). The discussion will be exclusively for $d = 1$, but the extension to arbitrary $d$ is immediate. The transformation of $\phi(r, t)$ will contain terms describing the space-time coordinate change given by $\lambda(r, t)$, the scaling as described by the Jacobian of $\lambda(r, t)$, and the change in the phase which is a peculiar feature of nonrelativistic systems. Under infinitesimal coordinate changes, we have the transformations

$$[X_n, \phi(r, t)] = \left( t^{n+1}\partial_t + \frac{n+1}{2} t^n r \partial_r + \frac{n(n+1)}{4} \mathcal{M} t^{n-1} r^2 + (n+1) \frac{x}{2} t^n \right) \phi(r, t) \tag{3.1}$$

$$[Y_m, \phi(r, t)] = \left( t^{m+1/2}\partial_r + \left( m + \frac{1}{2} \right) \mathcal{M} t^{m-1/2} r \right) \phi(r, t)$$

Taking over the conformal terminology of ref. 1, we call a field *primary* if it satisfies (3.1) for all $n$ integer and all $m$ half-integer. A field is called *quasiprimary* if it satisfies (3.1) for the finite-dimensional subalgebra $\mathcal{S}_{\text{fin}}$ only. We consider multipoint correlators

$$\langle \phi_a(r_a, t_a)\, \phi_b(r_b, t_b) \cdots \phi_y^*(r_y, t_y)\, \phi_z^*(r_z, t_z) \rangle$$

of quasiprimary fields and we derive the restrictions following from the hypothesis of their covariant transformation under $\mathcal{S}_{\text{fin}}$. We shall use the short-hand notation

$$\partial_a = \frac{\partial}{\partial t_a}; \qquad D_a = \frac{\partial}{\partial r_a} \tag{3.2}$$

We shall not consider explicitly the action of the generator $M_0$, because invariance with respect to it follows from the Bargmann superselection rule.

### 3.1. Two-Point Function in the Bulk

We consider the two-point function

$$F = F(r_a, r_b; t_a, t_b) = \langle \phi_a(r_a, t_a)\, \phi_b^*(r_b, t_b) \rangle \tag{3.3}$$

of quasiprimary fields $\phi_{a,b}$ in the infinite geometry in both time and space. Invariance under translations in time and space implies $F = F(r, \tau)$, where $r = r_a - r_b$, $\tau = t_a - t_b$. Invariance under scale transformations generated by $X_0$ requires

$$\left( t_a\partial_a + \frac{1}{2} r_a D_a + \frac{x_a}{2} + t_b \partial_b + \frac{1}{2} r_b D_b + \frac{x_b}{2} \right) F(r, \tau) = 0 \tag{3.4}$$

which is rewritten as, with $x = \frac{1}{2}(x_a + x_b)$,

$$(\tau\partial_\tau + \tfrac{1}{2} r \partial_r + x)\, F(r, \tau) = 0 \tag{3.5}$$

We write the solution in the form

$$F(r,\tau)=\tau^{-x}G\left(\frac{r^2}{\tau}\right) \tag{3.6}$$

which is nothing but the scaling form, Eq. (1.2). New information comes from requiring Galilei invariance ($Y_{1/2}$)

$$\begin{aligned}(t_aD_a+\mathcal{M}_ar_a+t_bD_b-\mathcal{M}_br_b)\,F(r,\tau)\\ =(\tau\partial_r+\mathcal{M}_ar_a-\mathcal{M}_br_b)\,F(r,\tau)=0\end{aligned} \tag{3.7}$$

and we obtain two conditions

$$\begin{aligned}\mathcal{M}_a-\mathcal{M}_b=0\\ (\tau\partial_r+\mathcal{M}_ar)\,F(r,\tau)=0\end{aligned} \tag{3.8}$$

We recognize in the first of these the Bargmann superselection rule, Eq. (2.10). Combining with scale invariance (3.6), we find

$$G(u)=G_0\exp\left(-\frac{\mathcal{M}_a}{2}u\right) \tag{3.9}$$

We remark that the form of $F(r,\tau)$ in (3.6) is not an arbitrary ansatz. This can be seen by first solving the condition of Galilei invariance before using scale invariance.

Finally, invariance under the special Schrödinger transformation $X_1$ gives

$$(t_a^2\partial_a+t_ar_aD_a+\mathcal{M}_ar_a^2+x_at_a+t_b^2\partial_b+t_br_bD_b-\mathcal{M}_br_b^2+x_bt_b)\,F(r,\tau)=0 \tag{3.10}$$

which is seen as before to lead to the conditions

$$\begin{aligned}x=x_a=x_b\\ \mathcal{M}_a-\mathcal{M}_b=0\\ G'+\tfrac{1}{2}\mathcal{M}_aG=0\end{aligned} \tag{3.11}$$

where the prime denotes the derivative. The last two of these had already been obtained before. The final result is, where $\Phi_0$ is a normalization constant,

$$F=\delta_{x_a,x_b}\delta_{\mathcal{M}_a,\mathcal{M}_b}\Phi_0\,(t_a-t_b)^{-x_a}\exp\left[-\frac{\mathcal{M}_a}{2}\frac{(r_a-r_b)^2}{t_a-t_b}\right] \tag{3.12}$$

which should be understood in the scaling limit. This had been announced before[26] for the special case of equal masses. Since for a nonconserved order parameter van Hove theory leads to a dynamic exponent $z=2$, it is not surprising that we recover in this case the form Eq. (1.4) as found for $d=2$ by conformal invariance.[11]

### 3.2. Three-Point Function in the Bulk

Consider the three-point function

$$F=F(r_a, r_b, r_c; t_a, t_b, t_c)=\langle \phi_a(r_a, t_a)\, \phi_b(r_b, t_b)\, \phi_c^*(r_c, t_c)\rangle \quad (3.13)$$

Translation invariance in both space and time let $F$ only depend on distances, $F=F(r, s; \tau, \sigma)$, where

$$r=r_a-r_c, \qquad s=r_b-r_c, \qquad \tau=t_a-t_c, \qquad \sigma=t_b-t_c \quad (3.14)$$

Scale invariance requires that

$$\sum_{i=a}^{c} (t_i\partial_i+\tfrac{1}{2}r_iD_i+\tfrac{1}{2}x_i)\, F(r, s; \tau, \sigma)$$
$$=[\tau\partial_\tau+\sigma\partial_\sigma+\tfrac{1}{2}r\partial_r+\tfrac{1}{2}s\partial_s+\tfrac{1}{2}(x_a+x_b+x_c)]\,F=0 \quad (3.15)$$

Making the ansatz

$$F(r, s; \tau, \sigma)=\tau^{-\rho_1}\sigma^{-\rho_2}(\tau-\sigma)^{-\rho_3}\, G(r, s; \tau, \sigma) \quad (3.16)$$

as motivated by the corresponding result for the three-point function as obtained from conformal invariance,[14] we find

$$\begin{gathered}\rho_1+\rho_2+\rho_3=\tfrac{1}{2}(x_a+x_b+x_c)\\ (\tau\partial_\tau+\sigma\partial_\sigma+\tfrac{1}{2}r\partial_r+\tfrac{1}{2}s\partial_s)\, G(r, s; \tau, \sigma)=0\end{gathered} \quad (3.17)$$

Note that the relation found between the exponents $\rho_i$ and the scaling dimensions $x_i$ must be satisfied for any scale-invariant system. From Galilei invariance we get, with $\varepsilon_a=\varepsilon_b=-\varepsilon_c=1$ because of Eq. (2.8)

$$\sum_{i=a}^{c} (t_iD_i+\varepsilon_i\mathcal{M}_ir_i)\, F(r, s; \tau, \sigma)$$
$$=[\tau\partial_r+\sigma\partial_s+\mathcal{M}_a r+\mathcal{M}_b s+(\mathcal{M}_a+\mathcal{M}_b-\mathcal{M}_c)\, r_c]\, F(r, s; \tau, \sigma)=0 \quad (3.18)$$

which leads to the conditions

$$\begin{gathered}\mathcal{M}_a + \mathcal{M}_b - \mathcal{M}_c = 0 \\ (\tau\partial_r + \sigma\partial_s + \mathcal{M}_a r + \mathcal{M}_b s)\, G(r, s; \tau, \sigma) = 0\end{gathered} \tag{3.19}$$

and we recognize again the Bargmann superselection rule for the masses. The equation for $G$ can be further simplified by setting

$$G(r, s; \tau, \sigma) = \exp\left(-\frac{\mathcal{M}_a}{2}\frac{r^2}{\tau} - \frac{\mathcal{M}_b}{2}\frac{s^2}{\sigma}\right) H(r, s; \tau, \sigma) \tag{3.20}$$

Since the first factor is scale invariant, $H$ satisfies the same equation (3.17) as does $G$, but the second equation (3.19) becomes

$$(\tau\partial_r + \sigma\partial_s)\, H(r, s; \tau, \sigma) = 0 \tag{3.21}$$

Finally, invariance under the special Schrödinger transformation requires

$$\sum_{i=a}^{c} (t_i^2\partial_i + t_i r_i D_i + \tfrac{1}{2}\varepsilon_i \mathcal{M}_i r_i^2 + t_i x_i)\, F(r, s; \tau, \sigma) = 0 \tag{3.22}$$

First, we use the form (3.16). Then we get $(\mathcal{T} + \mathcal{D})G = 0$, where $\mathcal{D}$ is a differential operator and

$$\mathcal{T} = t_a(-\rho_1 - \rho_3 + x_a) + t_b(-\rho_2 - \rho_3 + x_b) + t_c(-\rho_1 - \rho_2 + x_c) \tag{3.23}$$

The requirement that $\mathcal{T}$ vanishes leads together with scale invariance (3.17) to

$$\rho_2 = \tfrac{1}{2}(x_b + x_c - x_a), \quad \rho_1 = \tfrac{1}{2}(x_a + x_c - x_b), \quad \rho_3 = \tfrac{1}{2}(x_a + x_b - x_c) \tag{3.24}$$

Second, we use Eq. (3.20) and get $(\mathcal{T}' + \mathcal{D}')H = 0$, where

$$\mathcal{T}' = \tfrac{1}{2} r_c^2 (\mathcal{M}_a + \mathcal{M}_b - \mathcal{M}_c) \tag{3.25}$$

This vanishes due to the Bargmann superselection rule. The differential operator $\mathcal{D}'$ finally takes the form, where we use that $H$ is scale as well as Galilei-invariant,

$$(\tau^2\partial_\tau + \sigma^2\partial_\sigma + \tau r\partial_r + \sigma s\partial_s)\, H(r, s; \tau, \sigma) = 0 \tag{3.26}$$

It remains to solve the resulting system of first-order linear partial differential equations. This is done in Appendix A, with the result

$$H(r, s; \tau, \sigma) = \Psi\left(\frac{(r\sigma - s\tau)^2}{(\sigma - \tau)\,\sigma\tau}\right) \tag{3.27}$$

where $\Psi = \Psi_{ab,c}$ is an arbitrary function. The final result is, with the appropriate scaling limits understood,

$$\begin{aligned} F = & \delta_{\mathcal{M}_a + \mathcal{M}_b, \mathcal{M}_c} (t_a - t_c)^{-(x_a + x_c - x_b)/2} \\ & \times (t_b - t_c)^{-(x_b + x_c - x_a)/2} (t_a - t_b)^{-(x_a + x_b - x_c)/2} \\ & \times \exp\left[ -\frac{\mathcal{M}_a}{2} \frac{(r_a - r_c)^2}{t_a - t_c} - \frac{\mathcal{M}_b}{2} \frac{(r_b - r_c)^2}{t_b - t_c} \right] \\ & \times \Psi\left( \frac{[(r_a - r_c)(t_b - t_c) - (r_b - r_c)(t_a - t_c)]^2}{(t_a - t_b)(t_a - t_c)(t_b - t_c)} \right) \end{aligned} \tag{3.28}$$

In particular, it follows that any three-point function $\langle \phi\phi\phi^* \rangle$ of a massive field with itself vanishes.

The results, Eqs. (3.12) and (3.28), deserve some comments. It is instructive to compare them with those for the two-point and the three-point functions obtained from the requirement of conformal invariance by Polyakov[14] (see also refs. 2 and 30). Conformal invariance completely specifies the form of the two- and three-point functions in any number of space dimensions. Also, two-point correlations of quasiprimary fields must vanish if the scaling dimensions are different. We reproduce this result in Eq. (3.12), but add the stronger requirement of the Bargmann mass selection rule. The exponential behavior of the two-point scaling function is a consequence of Galilei invariance, while $\Phi_0$ is merely a normalization. Our results do depend on the explicit realization of the Galilei transformation as given by the generator $Y_{1/2}$. If, for example, we had considered the Schrödinger group with a potential present, the realization of the generators is different[20] and we would have found a different form of the correlations. Turning to the three-point function, Eq. (3.28), we observe that the purely time-dependent factors reproduce the familar form of the conformal three-point function which is completely symmetric in the times $t_a$, $t_b$, $t_c$. We then note that this symmetry is not met by the exponential factors, determined from Galilei invariance, and we also note the appearance of the Bargmann mass selection rule, not present in conformal invariance. Generalization of our results to higher space dimension $d$ merely requires a check for rotation invariance, which is obviously satisfied.

### 3.3. Two-Point Function in Semi-infinite Space

Having studied correlations of quasiprimary fields in the infinite geometry, we now consider the effect of surfaces. Consider a free surface at $r = 0$. It is kept invariant under the transformations generated by the subalgebra $\mathscr{S}_X$, but space translations and Galilei transformations will no longer leave the system invariant. Nevertheless, it is known that conformal invariance can be used in analogous situations to constrain the two-point correlation function.[31] For quasiprimary fields, we require covariance only under the subalgebra $\mathscr{S}_X \cap \mathscr{S}_{\text{fin}} = \{X_{-1}, X_0, X_1\}$.

Consider the two-point function of quasiprimary fields

$$F = F(r_a, r_b; t_a, t_b) = \langle \phi_a(r_a, t_a)\, \phi_b^*(r_b, t_b) \rangle \tag{3.29}$$

and we require space points to be in the right half-plane, i.e., $r_a, r_b \geqslant 0$. Time translation invariance gives $F = F(r_a, r_b; \tau)$, with $\tau = t_a - t_b$. From scale invariance we obtain

$$\sum_{i=a}^{b} (t_i \partial_i + \tfrac{1}{2} r_i D_i + \tfrac{1}{2} x_i) F = (\tau \partial_\tau + \tfrac{1}{2} r_a D_a + \tfrac{1}{2} r_b D_b + x) F = 0 \tag{3.30}$$

where $x = \frac{1}{2}(x_a + x_b)$. On the other hand, from the invariance under the special Schrödinger transformation we have, with $\varepsilon_a = -\varepsilon_b = 1$,

$$\begin{aligned} &\sum_{i=a}^{b} (t_i^2 \partial_i + t_i r_i D_i + \tfrac{1}{2} \varepsilon_i \mathscr{M}_i r_i^2 + t_i x_i) F \\ &\quad = [\tau^2 \partial_\tau + \tau r_a D_a + t_b(2\tau \partial_\tau + r_a D_a + r_b D_b) \\ &\qquad + \tfrac{1}{2} \mathscr{M}_a r_a^2 - \tfrac{1}{2} \mathscr{M}_b r_b^2 + t_a x_a + t_b x_b] F \\ &\quad = (\tau^2 \partial_\tau + \tau r_a D_a + \tfrac{1}{2} \mathscr{M}_a r_a^2 - \tfrac{1}{2} \mathscr{M}_b r_b^2 + \tau x_a) F = 0 \end{aligned} \tag{3.31}$$

where in the last equation the scale invariance of $F$ was used. Now, we make the ansatz

$$F(r_a, r_b; \tau) = \tau^{-x} G(u, v), \qquad u = \frac{r_a^2}{\tau}, \qquad v = \frac{r_b^2}{\tau} \tag{3.32}$$

which solves for scale invariance, while Eq. (3.31) gives

$$\begin{aligned} x &= x_a = x_b \\ (u\partial_u - v\partial_v + \tfrac{1}{2}\mathscr{M}_a u - \tfrac{1}{2}\mathscr{M}_b v)\, G(u, v) &= 0 \end{aligned} \tag{3.33}$$

The general solution of this is found using the method of characteristics,[32]

$$G(u, v) = \chi(uv) \exp(-\tfrac{1}{2}\mathscr{M}_a u - \tfrac{1}{2}\mathscr{M}_b v) \tag{3.34}$$

where $\chi = \chi_{a,b}$ is an arbitrary function. The final result is

$$F = \delta_{x_a, x_b} (t_a - t_b)^{-x_a} \chi\left(\frac{r_a r_b}{t_a - t_b}\right) \exp\left(-\frac{\mathscr{M}_a}{2} \frac{r_a^2}{t_a - t_b} - \frac{\mathscr{M}_b}{2} \frac{r_b^2}{t_a - t_b}\right) \tag{3.35}$$

We note that analogously to the conformal result,[31] the scaling dimensions have to agree, while in this case we do not have a constraint on the masses $\mathscr{M}_{a,b}$, since the system is *not* Galilei-invariant.

The function $\chi$ is partially determined from a consistency conditions. We should expect to recover the bulk behavior for large distances to the surfaces, that is, for $r_a r_b/\tau \to \infty$. Therefore, up to normalization,

$$\chi(u) \simeq \delta_{\mathscr{M}_a, \mathscr{M}_b} e^{\mathscr{M}_a u}; \qquad u \to \infty \tag{3.36}$$

On the other hand, for a free surface where the field vanishes, we expect the absence of any correlations and thus

$$\chi(0) = 0 \tag{3.37}$$

Indeed, this is exactly the behavior obtained from the method of images. We have for the surface correlation $G_s$ in terms of bulk correlations $G_b$

$$\begin{aligned} G_s(r, r'; \tau) &= G_b(r - r', \tau) - G_b(r + r', \tau) \\ &= G_0 \tau^{-x} \left\{ \exp\left[ -\frac{\mathscr{M}}{2} \frac{(r - r')^2}{\tau} \right] - \exp\left[ -\frac{\mathscr{M}}{2} \frac{(r + r')^2}{\tau} \right] \right\} \\ &= 2G_0 \tau^{-x} \sinh\left( \mathscr{M} \frac{rr'}{\tau} \right) \exp\left( -\frac{\mathscr{M}}{2} \frac{r^2 + r'^2}{\tau} \right) \end{aligned} \tag{3.38}$$

and we identify $\chi(u) = 2G_0 \sinh(\mathscr{M} u)$ in agreement with the consistency conditions (3.36), (3.37).

Finally, if we were to impose in addition translation invariance in space, invariance under Galilei transformations is also implied and we do recover the bulk result (3.12) for the two-point function.

### 3.4. Two-Point Function for a Nonstationary State

We now consider a situation with a boundary condition at a fixed time. Boundary conditions of this type will be kept invariant by the subalgebra $\mathscr{S}_Y$ together with both the scale transformation $X_0$ as well as the special Schrödinger transformation $X_1$ (provided the initial state is massless), or rather its finite-dimensional subalgebra $\{X_0, X_1, Y_{-1/2}, Y_{1/2}, M_0\}$ for quasiprimary fields. For example, this may correspond to the situation when a system is in a predefined initial state and relaxes toward its equilibrium state.[6,7] Consider the two-point function of quasiprimary fields

$$F = F(r_a, r_b; t_a, t_b) = \langle \phi_a(r_a, t_a)\, \phi_b^*(r_b, t_b) \rangle \tag{3.39}$$

Invariance under space translations implies $F = F(r; t_a, t_b)$ with $r = r_a - r_b$. We next demand invariance under Galilei transformations, with $\varepsilon_a = -\varepsilon_b = 1$:

$$\sum_{i=a}^{b} (t_i D_i + \varepsilon_i \mathcal{M}_i r_i) F = [(t_a - t_b)\,\partial_r + \mathcal{M}_a r_a - \mathcal{M}_b r_b] F = 0 \tag{3.40}$$

In analogy to what was done before, this implies the Bargmann superselection rule $\mathcal{M}_a = \mathcal{M}_b$ and

$$F(r; t_a, t_b) = G(t_a, t_b) \exp\left( -\frac{\mathcal{M}_a}{2} \frac{r^2}{t_a - t_b} \right) \tag{3.41}$$

Scale invariance demands that

$$\sum_{i=a}^{b} (t_i \partial_i + \tfrac{1}{2} r_i D_i + \tfrac{1}{2} x_i) F = 0 \tag{3.42}$$

Inserting (3.41), we find

$$G(t_a, t_b) = t_a^{-(x_a + x_b)/2} \Phi(t_a / t_b) \tag{3.43}$$

where $\Phi(v)$ is a yet undetermined function. So far we have

$$F = \delta_{\mathcal{M}_a, \mathcal{M}_b} t_a^{-(x_a + x_b)/2} \Phi\left(\frac{t_a}{t_b}\right) \exp\left[ -\frac{\mathcal{M}_a}{2} \frac{(r_a - r_b)^2}{t_a - t_b} \right] \tag{3.44}$$

Note that we have no condition on the exponents $x_{a,b}$ here. This is the form of the two-point function which follows from just Galilei and scale invariance.

We now add the requirement of covariance under the special Schrödinger transformation generated by $X_1$, provided the initial state has vanishing mass. If that is the case, we have the additional condition

$$\sum_{i=a}^{b} (t_i^2 \partial_i + t_i r_i D_i + \tfrac{1}{2} \varepsilon_i \mathcal{M}_i r_i^2 + t_i x_i) F = 0 \tag{3.45}$$

We use the result (3.44) obtained so far and find an equation for $\Phi(v)$,

$$v(v-1)\,\Phi'(v) + [\tfrac{1}{2}(x_a - x_b) v + x_b]\,\Phi(v) = 0 \tag{3.46}$$

which has the solution

$$\Phi(v) = \Phi_0 v^{(x_b - x_a)/2} (1 - v^{-1})^{-(x_a + x_b)/2} \tag{3.47}$$

where $\Phi_0$ is a normalization constant. So the two-point function covariant under the algebraically closed set $\{X_0, X_1, Y_{\pm 1/2}, M_0\}$, but not under $X_{-1}$, finally is

$$F = \delta_{\mathcal{M}_a, \mathcal{M}_b} \Phi_0 \left(\frac{t_a}{t_b}\right)^{-(x_a - x_b)/2} (t_a - t_b)^{-(x_a + x_b)/2} \times \exp\left[-\frac{\mathcal{M}_a}{2} \frac{(r_a - r_b)^2}{t_a - t_b}\right] \tag{3.48}$$

The exponents $x_a$, $x_b$ are not constrained, since time translation invariance was not assumed. In Section 4, we shall consider the relationship of this two-point function with response functions in the context of the relaxation kinetics of the spherical model.

### 3.5. Ward Identity

We now consider the effect of arbitrary coordinate transformations on correlations. We suppose that the system under consideration is described by a local action in $d+1$ dimensions. This is obviously satisfied for static, but strongly anisotropic systems with local interactions. For many dynamical problems in $d$ space dimensions which are at first defined via their equation of motion, there exists an equivalent equilibrium problem in $d+1$ dimensions, usually supplemented with "disorder conditions" to maintain the strong anisotropy.[33,34] Then, considering the change of the action induced by an arbitrary coordinate transformation, the following identity holds (see, e.g., ref. 2):

$$\sum_{a=1}^{n} \langle \phi_1(\mathbf{r}_1, t_1) \cdots \delta\phi_a(\mathbf{r}_a, t_a) \cdots \phi_n(\mathbf{r}_n, t_n) \rangle + \int d\mathbf{R}\, dT \, \langle \phi_1(\mathbf{r}_1, t_1) \cdots \phi_n(\mathbf{r}_n, t_n) \, T_{ij}(\mathbf{R}, T) \rangle \, \partial_i(\delta \mathbf{r}_j)(\mathbf{R}, T) = 0 \tag{3.49}$$

where we implicitly assume that in the correlators the Bargmann superselection rule is satisfied, the time $T$ is denoted as the zeroth component of the coordinate $\mathbf{R}$, and $T_{ij}$ is the stress-energy tensor. As Eq. (3.49) is written, we assume $\delta\phi$ to contain all variations of the field $\phi$ (including changes of its phase) and $T_{ij}$ to describe all changes of the action used to calculate the averages $\langle \cdots \rangle$. The discussion presented here remains at the formal level of the equations of motion. We discard the possibility of anomalies which may arise from renormalization effects. Detailed discussions of these are available for conformal invariant theories,[1,2] but for Schrödinger invariance, the analogous developments have not yet been done.

Since correlations are supposed to be invariant under infinitesimal Schrödinger transformations, we obtain a few constraints on the stress-energy tensor in complete analogy with the corresponding results for conformal invariance.[2] The form of the $\delta \mathbf{r}_j$ is taken from the generators, Eq. (2.6). Rotation invariance implies that $T_{ij}$ is symmetric in space,

$$T_{ij} = T_{ji}, \qquad i, j = 1,..., d \tag{3.50}$$

Scale invariance gives the "trace condition" (here written in Euclidean form)

$$2T_{00} + \sum_{i=1}^{d} T_{ii} = 0 \tag{3.51}$$

which is the analog of the vanishing trace condition in conformal invariance[2] and the factor 2 comes from $\theta = 2$. Equation (3.51) is satisfied in free nonrelativistic field theory[17] (where it is written in Minkowskian form). Interacting nonrelativistic field theories may give rise to anomalies.[35] From Galilei invariance we find

$$T_{0i} = 0, \qquad i = 1,..., d \tag{3.52}$$

The requirement of special Schrödinger invariance does not add any further condition on $T_{ij}$. We have thus seen that

$$\left.\begin{array}{l}\text{translation invariance in space and time}\\ \text{rotation invariance in space}\\ \text{anisotropic scale invariance with } \theta = 2\\ \text{Galilei invariance}\\ \text{local interactions}\end{array}\right\} \Rightarrow \text{Schrödinger invariance} \tag{3.53}$$

in analogy to the conformal result.[2] In fact, using formally the equations of motion, we may even verify invariance under the entire infinite algebra $\mathscr{S}$. This is for the time being the only indication that the generalization beyond $\mathscr{S}_{\text{fin}}$ might be sensible. Again, the same type of result also holds for conformal invariance (when the central charge vanishes).

We do not go into a discussion of the possible anomaly structure here. As a preliminary exercise to that, we show in Appendix B that the Schrödinger algebra (2.7) does not admit any nonconventional central extension besides the familiar Virasoro form for the generators $X_n$.

## 3.6. SUMMARY

The main results of this section are the explicit expressions for the two- or three-point Schrödinger-covariant correlations in either an infinite or a semi-infinite geometry as given in Eqs. (3.12), (3.28), (3.35), (3.48). Although the general form is quite similar to the corresponding results found from conformal invariance,[14,30,31] there are some properties which come from the nonrelativistic nature of the symmetry. The first one is the Bargmann superselection rule[28] for the masses. Second, the space dimension $d$ has mainly the role of a parameter, at least for the quasiprimary fields only considered here, whereas the nontrivial group structure capable of extension to an infinite-dimensional algebra only occurs in the "time" coordinate.

We remark that there is a certain analogy between Schrödinger invariance and conformal invariance close to a free surface.[31] In both cases, the pair of *complex* linear projective transformations characteristic for full conformal invariance gets replaced by the subgroup of a single *real* linear projective transformation.

The results obtained only use (a peculiar realization of) the finite-dimensional algebra. It remains an open problem how to extend Schrödinger invariance to the full infinite-dimensional algebra and find the scaling functions.

## 4. TESTS OF SCHRÖDINGER INVARIANCE IN EXACTLY SOLVABLE MODELS

We now test the predictions obtained in the last section for some correlations in the context of some exactly solvable strongly anisotropic critical systems. We do not include here among the tests the well-known fact that the Green's functions of the free Schrödinger equation and the diffusion equation reproduce Eq. (3.12) in any space dimension with $x = d/2$. We also refrain from discussing the range of possible applications of the models considered.

### 4.1. Kinetic Ising Model with Glauber Dynamics

Consider the time-dependent one-dimensional model with the classical spin Hamiltonian $\mathscr{H} = -J\sum_{i=1}^{L} s_i s_{i+1}$ and $s_i = \pm 1$. To describe the time dependence, following Glauber,[36] consider the probability distribution function $P(s_1,..., s_L; t)$. It is often convenient to describe the evolution of $P$ in terms of a master equation (see refs. 29 and 37 for a detailed discussion)

$$\frac{\partial}{\partial t} P(s_1,..., s_L; t) = -\left(\sum_i w_i(s_i)\right) P(s_1,..., s_L; t) + \sum_i w_i(-s_i)\, P(s_1,..., -s_i,..., s_l; t) \qquad (4.1)$$

where the $w_i$ are the rates describing the transitions between spin configurations. The following consistency conditions have to be kept. The first is probability conservation when summing over all configurations $\{s\} = (s_1, ..., s_L)$:

$$\sum_{\{s\}} P(\{s\}; t) = 1 \tag{4.2}$$

to be kept at all times. Second, the equilibrium distribution $P_{eq} \sim e^{-\beta \mathscr{H}}$ has to be a stationary solution of the master equation, where $\beta$ is the inverse temperature. This is usually implemented via detailed balance

$$\frac{w_i(-s_i)}{w_i(s_i)} = \frac{\exp[-\beta J s_i(s_{i-1} + s_{i+1})]}{\exp[\beta J s_i(s_{i-1} + s_{i+1})]} \tag{4.3}$$

Finally, averages are obtained from

$$\langle X \rangle(t) = \sum_{\{s\}} X(\{s\}) \, P(\{s\}; t) \tag{4.4}$$

Glauber[36] showed that the particular choice

$$w_i(s_i) = \frac{\alpha}{2} \left[ 1 - \frac{\gamma}{2} s_i(s_{i-1} + s_{i+1}) \right] \tag{4.5}$$

where $\alpha$ is the constant transition rate and $\gamma = \tanh(2\beta J)$, renders the model completely integrable. We are interested here in his result for the time-delayed (connected) two-point function when the system is in thermal equilibrium at temperature $T$,[36]

$$G(r_1 - r_2, t_1 - t_2) = \langle s_{r_1}(t_1) \, s_{r_2}(t_2) \rangle_c = e^{-\alpha t} \sum_l \eta^{|r-l|} I_l(\alpha \gamma t) \tag{4.6}$$

where $r = r_1 - r_2$, $t = t_1 - t_2$, $\eta = \tanh(\beta J)$, $I_l$ is a modified Bessel function, and the sum extends over the whole lattice. If there are no correlations between spins in the initial state, only the term with $l = r$ would be present.

To analyze this, recall the asymptotic form, as $x \to \infty$,[38]

$$I_l(x) \simeq (2\pi x)^{-1/2} \exp\left(x - \frac{l^2}{2x}\right) [1 + \mathcal{O}(x^{-1})] \tag{4.7}$$

and obtain in the scaling limit $r \to \infty$, $t \to \infty$ with $u = r^2/t$ fixed

$$\begin{aligned} G(r, t) &\simeq \{\exp[-(1-\gamma)\alpha t]\}(2\pi\gamma\alpha t)^{-1/2} \\ &\quad \times \left\{ \exp\left(-\frac{r^2}{2\gamma\alpha t}\right) + \sum_{l \neq 0} \eta^{|l|} \exp\left(-\frac{(r+l)^2}{2\gamma\alpha t}\right) \right\} \\ &\sim (2\pi\alpha t)^{-1/2} \exp\left(-\frac{r^2}{2\alpha t}\right) \end{aligned} \tag{4.8}$$

where in the last equation we performed the zero-temperature limit (since the one-dimensional Ising model has its critical point at $T=0$). This holds exactly for vanishing correlations in the initial state and up to scaling corrections otherwise. This is indeed in agreement with the predicted two-point function (3.12) and we identify $x=1/2$ and $\mathscr{M}=1/(2\alpha)$.

## 4.2. Lattice Diffusion with Exclusion

Consider a system of many particles performing random walks on a chain of $L$ sites. Each site can be either empty or occupied. The dynamics is defined as follows.[34] First, pair all neighboring sites, which can be done in two ways, to be labeled $\mathscr{A}$, chosen at odd times, and $\mathscr{B}$, which is chosen at even times. At every time step, the dynamics of each pair is as follows. If both sites of a pair are either occupied or empty, the state of the pair is unchanged. If one site is occupied and one empty, the particle moves to the empty site with probability $p$ or stays where it is with probability $1-p$. This stochastic rule for updating is applied in parallel to all pairs.

The time-delayed particle–particle correlation is

$$G(r,t)=\langle n(r,t)\,n(0,0)\rangle_c \tag{4.9}$$

where $n(r,t)=1$ if the site $r$ is occupied at time $t$ and $n(r,t)=0$ otherwise. The average is first over realizations of all possible time developments and second over an ensemble of initial states that is stationary with respect to the stochastic process.[34] $G$ was calculated exactly first for $p=1/2$,[34] and later for $p$ arbitrary,[39] with the result, for even times $t$ and even sites $r$ (similar results are known for the other cases, see ref. 39)

$$\begin{aligned}G(r,t)=\rho(1-\rho)\Bigg[p^t\delta_{-r,t}+\sum_{k=1}^{(t-|r|)/2}\binom{(t-r)/2}{k}\\ \times\binom{(t-2+r)/2}{k-1}p^{t-2k}(1-p)^{2k}\Bigg]\end{aligned} \tag{4.10}$$

where $\rho=N/L$ is the particle density and $N$ is the number of particles on the lattice. The limit $L\to\infty$ such that $\rho$ is kept fixed is understood. In the scaling limit $r, t\to\infty$ such that $u=r^2/t$ is kept fixed, this simplifies to[34,39]

$$G(r,t)=\rho(1-\rho)(2\pi\,Dt)^{-1/2}\exp\left(-\frac{r^2}{2Dt}\right) \tag{4.11}$$

and the diffusion constant $D=p/(1-p)$. This agrees with the Schrödinger invariance expectation (3.12), and we have $x=1/2$ and $\mathscr{M}=1/D$. Similar results were obtained when quenched weak disorder is added, and it was shown that if the spatial disorder correlations decay rapidly enough, the same scaling form (4.11) results with a modified value of the diffusion constant $D$; see ref. 39 for details.

These results were derived[34-39] by mapping the system onto a six-vertex model satisfying a disorder condition. From the transfer matrix $\mathscr{T} = e^{-\tau H}$ the quantum Hamiltonian can be obtained. In the time continuum limit $\tau \to \infty$ such that $p\tau$ remains constant, the quantum Hamiltonian is found[39] to agree with the quantum Hamiltonian obtained directly from the master equation written in the form $\partial_t P = -HP$, and reads[40]

$$H = -\frac{1}{2} \sum_{i=1}^{L} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + (1-\Delta)(\sigma_i^z + \sigma_{i+1}^z) + \Delta - 2\right] \tag{4.12}$$

where the $\sigma^{x,y,z}$ are Pauli matrices and with $\Delta = 1$ for lattice diffusion. We note that this Hamiltonian, but now with $\Delta = 0$, is also obtained[40,41] from the master equation of the kinetic Ising model at $T = 0$ considered above. The disorder condition for the vertex model formulation of lattice diffusion above makes the model undergo a transition of Pokrovsky–Talapov type.[42] In view of the common scaling form for both $\Delta = 0$ or 1, and because the spectrum of $H$ is known to be $\Delta$-independent,[40] we should expect this scaling form to hold independently of the value of $\Delta$.

### 4.3. Lifshitz Point in the Spherical Model

We now consider a static, but strongly anisotropic system. The model is the anisotropic next-nearest-neighbor spherical (ANNNS) model (see ref. 43 and references therein). Conventionally, it is defined by the Hamiltonian, on a hypercubic lattice,

$$\mathscr{H}_{\mathrm{SM}} = -\sum_{i,j} J_{\mathbf{ij}} \sigma_{\mathbf{i}} \sigma_{\mathbf{j}} + \beta^{-1} \zeta \sum_{i} \sigma_{\mathbf{i}}^2 \tag{4.13}$$

where $\sigma_i$ are real numbers, and the spherical parameter $\zeta$ is determined from the constraint $\langle \sum_i s_i^2 \rangle = \mathcal{N}$, where $\mathcal{N}$ is the number of sites. Consider the model in $D = d' + d$ dimensions. The couplings $J_{\mathbf{ij}}$ are defined as follows. First, in all $D$ dimensions, there is a ferromagnetic nearest-neighbor interaction of energy $J > 0$. Second, in the $d$ "parallel" directions, there is along the axes an interaction of energy $\kappa J$ between next nearest neighbors. The Fourier transform of $J_{\mathbf{ij}}$ is

$$J(\mathbf{k}) = 2J \left( \sum_{i=1}^{D} \cos k_i + \kappa \sum_{i=1}^{d} \cos 2k_i \right) \tag{4.14}$$

If a phase transition occurs, there is at $\kappa=\kappa_c=-1/4$ a meeting point between a paramagnetic, a ferromagnetic, and an ordered incommensurate phase, which is referred to as a Lifshitz point,[10] which is a strongly anisotropic critical point. When $d'=1$, one has $\theta=2$ and those coordinates in the $d$ "parallel" directions will be referred to as "space," while the one remaining direction will be referred to as "time."

In order to make the Galilei invariance explicit, we consider here a variant of this model, which gives the same thermodynamics. The Hamiltonian is

$$\mathscr{H} = -\sum_{i,j} \tfrac{1}{2} J_{ij}(s_i^* s_j + s_i s_j^*) + \beta^{-1}\zeta \sum_i |s_i|^2 \tag{4.15}$$

where now $s_i$ is complex and the spherical constraint is

$$\left\langle \sum_i |s_i|^2 \right\rangle = 2\mathscr{N} \tag{4.16}$$

We introduce the Fourier transform of the spin variable

$$s_{\mathbf{a}} = (2\pi)^{-D/2} \int d\mathbf{k}\, \mu_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{a}} \tag{4.17}$$

and get

$$\mathscr{H} = -\int d\mathbf{k}\, [J(\mathbf{k}) - \beta^{-1}\zeta]\, |\mu_{\mathbf{k}}|^2 = -\int d\mathbf{k}\, \Lambda(\mathbf{k})\, |\mu_{\mathbf{k}}|^2 \tag{4.18}$$

The partition function is $\mathscr{Z} = \int \mathscr{D}s\, e^{-\beta\mathscr{H}}$. Since

$$\langle \mu_{\mathbf{k}} \rangle = \langle \mu_{\mathbf{k}}^2 \rangle = 0, \qquad \langle |\mu_{\mathbf{k}}|^2 \rangle = \Lambda^{-1}(\mathbf{k}) \tag{4.19}$$

the spherical constraint (4.16) becomes $\int d\mathbf{k}\, \Lambda^{-1}(\mathbf{k}) = 2\mathscr{N}$, which is exactly the same as obtained from $\mathscr{H}_{\text{SM}}$ (see, e.g., ref. 44) and the free energy is $F = -\beta^{-1} \ln \mathscr{Z} = 2F_{\text{SM}}$, where $F_{\text{SM}}$ is the free energy obtained from the Hamiltonian $\mathscr{H}_{\text{SM}}$. Consequently, the model defined by $\mathscr{H}$ is in the same universality class as the one defined by $\mathscr{H}_{\text{SM}}$. In particular, the critical point is characterized by the condition $\zeta = \beta J D$. The lower critical dimension is at $D=3$, the upper critical dimension at $D=7$, provided $d'=1$, which we assume from now on.

Consider the two-point function $C(\mathbf{a}-\mathbf{b}) = \Re\langle s_{\mathbf{a}} s_{\mathbf{b}}^* \rangle$, where $\Re$ denotes the real part and

$$\langle s_{\mathbf{a}} s_{\mathbf{b}}^* \rangle = \frac{k_B T}{\mathscr{Z}(2\pi)^D} \int \mathscr{D}\mu \int d\mathbf{k}\, d\boldsymbol{l}\, \mu_{\mathbf{k}}^* \mu_{\boldsymbol{l}} \times \exp[i(\boldsymbol{l} \cdot \mathbf{a} - \mathbf{k} \cdot \mathbf{b})] \exp\left[ - \int d\mathbf{m}\, \Lambda(\mathbf{m})\, |\mu_{\mathbf{m}}|^2 \right]$$

$$= (2\pi)^{-D} k_B T \int d\mathbf{k}\, \Lambda^{-1}(\mathbf{k}) \exp[i\mathbf{k} \cdot (\mathbf{a} - \mathbf{b})] \tag{4.20}$$

This has been calculated for integer dimensions $D$.[45] In the scaling limit $r \to \infty$, $t \to \infty$ with $r^2/t$ fixed ($\mathbf{r}$ and $t$ are the distances in space and time between the points $a$ and $b$) the result is

$$C(\mathbf{a} - \mathbf{b}) = C(\mathbf{r}, t) = \mathscr{A}^2 t^{-(D-3)/2} \Psi\left(\frac{D-3}{4}, \frac{r^2}{2t}\right)$$
$$\mathscr{A}^2 = \frac{k_B T_c}{J} \left(\frac{2^{1-D}}{\pi^{2D-3}}\right)^{1/2} \left[\Gamma\left(\frac{1}{4}\right)\right]^{D-2} \tag{4.21}$$

The function $\Psi(a, x)$ is given in Table I.[45]

Comparison with Schrödinger invariance, Eq. (3.12), shows agreement for the case $D = 6$, while the scaling function has a different form in the other cases. Recall that the system given by the Hamiltonian $\mathscr{H}$ has a dispersion relation of the form

$$E^2 - \frac{1}{4m^2} k^4 = \left(E + \frac{1}{2m} k^2\right)\left(E - \frac{1}{2m} k^2\right) = 0 \tag{4.22}$$

rather than $E = k^2/(2m)$, which was used for making the Schrödinger-invariance predictions. We can only expect to recover the results of Schrödinger invariance if the propagator actually solves the free Schrödinger

**Table I. Scaling Function $\Psi(a, x)$ at the Lifshitz Point of the Spherical Model and Leading Asymptotic Behavior for $x \to \infty$[a]**

| $a$ | $\Psi(a, x)$ | Asymptotics |
|---|---|---|
| $\frac{1}{4}$ | $x^{1/2}[I_{-1/4}(x/2) + I_{1/4}(x/2)]\, K_{1/4}(x/2)$ | $2x^{-1/2}$ |
| $\frac{1}{2}$ | $\cdot(\pi^2 x/2)^{1/4}\, [I_{-1/4}(x) - \mathbf{L}_{-1/4}(x)]$ | $[2/\Gamma(1/4)]\, x^{-1}$ |
| $\frac{3}{4}$ | $e^{-x}$ | |
| 1 | $1/\Gamma(3/4) + \sqrt{\pi}\, (x/2)^{3/4}\, [\mathbf{L}_{1/4}(x) - I_{1/4}(x)]$ | $-1/[2\Gamma(3/4)]\, x^{-2}$ |

[a] $I_n$ and $K_n$ are modified Bessel functions and $\mathbf{L}_n$ is as modified Struve function.

equation, and not just a more general fourth-order differential equation. We see this to be the case for $D=6$ from Table I and concentrate on this from now on.

In order to test the prediction for two- and three-point correlations, we consider several scaling fields as defined in Table II. We also give the scaling dimensions and the masses (in units of the mass of the field $\sigma$) of these fields. Concerning the field $\sigma$, we can confirm its values for $x_\sigma$ and $v_\sigma$ from Eq. (4.21).

The calculation of the other correlations is simple because, since $\Lambda(-\mathbf{k})=\Lambda(\mathbf{k})$, the imaginary part of $\langle s_a s_b^* \rangle$ vanishes and we have $C(\mathbf{a}-\mathbf{b})=\langle s_\mathbf{a} s_\mathbf{b}^* \rangle$.

Consider the two-point functions first. [From now on, $a$, $b$, $c$ denote space-time vectors and we also write $a=(\mathbf{r}_a, t_a)$, etc.] Obviously, $\langle \varepsilon_a \rangle = 0$. Then

$$\begin{aligned}\langle \varepsilon_a \varepsilon_b^* \rangle &= \langle s_a s_{a'} s_b^* s_{b'}^* \rangle = C(a-b)\, C(a'-b') + C(a-b')\, C(a'-b) \\ &\simeq 2[C(a-b)]^2 = 2\mathscr{A}^4 t^{-2x_\sigma} e^{-r^2/t} \end{aligned} \tag{4.23}$$

where in the second line the scaling limit was taken and we confirm the result given in Table II. Next, we have $\langle \eta_a \rangle = C(a-a') = \text{const}$. Then the connected two-point function is

$$\langle \eta_a \eta_b \rangle_c = \langle s_a^* s_{a'} s_b^* s_{b'} \rangle_c = C(a-b')\, C(b-a') \simeq t^{-2x_\sigma} \mathscr{A}^4 \tag{4.24}$$

whereas the exponential terms cancel. Finally, we have $\langle \Sigma_a \rangle = 0$ and

$$\langle \Sigma_a \Sigma_b^* \rangle = \langle s_a s_{a'} s_{a''} s_b^* s_{b'}^* s_{b''}^* \rangle \simeq 6[C(a-b)]^3 = 6\mathscr{A}^6 t^{-3x_\sigma} e^{-(3/2)\, r^2/t} \tag{4.25}$$

**Table II. Scalar Scaling Fields Arising at the Lifshitz Point of the Spherical Model at $D=6$[a]**

| $\phi$ | | $x_\phi$ | $\mathscr{M}_\phi/\mathscr{M}_\sigma$ | $v_\phi$ |
|---|---|---|---|---|
| $\sigma_\mathbf{a}$ | $s_\mathbf{a}$ | 3/2 | 1 | $\mathscr{A}$ |
| $\varepsilon_\mathbf{a}$ | $s_\mathbf{a} s_{\mathbf{a}'}$ | 3 | 2 | $\sqrt{2}\,\mathscr{A}^2$ |
| $\eta_\mathbf{a}$ | $s_\mathbf{a}^* s_{\mathbf{a}'}$ | 3 | 0 | $\mathscr{A}^2$ |
| $\Sigma_\mathbf{a}$ | $s_\mathbf{a} s_{\mathbf{a}'} s_{\mathbf{a}''}$ | 9/2 | 3 | $\sqrt{6}\,\mathscr{A}^3$ |

[a] Also given are scaling dimensions $x_\phi$, masses $\mathscr{M}_\phi$ in units of the mass $\mathscr{M}_\sigma$, and the normalization such that $v_\phi^{-1}\phi$ has $\Phi_0=1$ in Eq. (3.12). $\mathbf{a}'$ denotes a space-time point on a neighboring site of the point given by $\mathbf{a}$.

and we have verified all entries in Table II. It is straightforward to verify that

$$\langle \sigma_a \varepsilon_b^* \rangle = \langle \sigma_a \eta_b \rangle = \langle \sigma_a \Sigma_b^* \rangle = \langle \varepsilon_a \eta_b \rangle = \langle \varepsilon_a \Sigma_b^* \rangle = \langle \eta_a \Sigma_b^* \rangle = 0 \quad (4.26)$$

This fully confirms the Schrödinger two-point function, Eq. (3.12).

At this stage, we clearly recognize the difference between the models described by $\mathscr{H}_{\mathrm{SM}}$ and $\mathscr{H}$, respectively, from the point of view of Schrödinger (in fact already Galilean) invariance. The two fields $\varepsilon$ and $\eta$ have the same scaling dimension, but different masses. In the context of the model described by $\mathscr{H}_{\mathrm{SM}}$, however, these two distinct fields get lumped together into just one field, $\tilde{\varepsilon}$, say. Correlations of $\tilde{\varepsilon}$ will in general *not* satisfy Galilean invariance. On the other hand, there is just a single order-parameter field $\sigma$. Since in many applications one is only interested in the order-parameter correlations with itself, it is for that restricted purpose enough to stay with the conventional form of $\mathscr{H}_{\mathrm{SM}}$, rather than go to $\mathscr{H}$ with the correct Galilean transformation properties.

We now turn to the three-point functions. To check the Bargmann superselection rule, one may verify that, for example

$$\langle s_a s_b s_c^* \rangle = \langle \varepsilon_a \varepsilon_b \varepsilon_c^* \rangle = \langle \eta_a \sigma_b \varepsilon_c^* \rangle_c = \langle \sigma_a \sigma_b \Sigma_c^* \rangle = \langle \varepsilon_a \varepsilon_b \Sigma_c^* \rangle = 0 \quad (4.27)$$

A nonvanishing correlation is

$$\begin{aligned} \langle \sigma_a \sigma_b \varepsilon_c^* \rangle &= \langle s_a s_b s_c^* s_{c'}^* \rangle \\ &= C(a-c)\, C(b-c') + C(a-c')\, C(b-c) \\ &\simeq 2C(a-c)\, C(b-c) \end{aligned} \quad (4.28)$$

Since $x_\varepsilon = 2x_\sigma$, this agrees indeed with the prediction (3.28) from Schrödinger invariance and we identify the scaling function $\Psi_{\sigma\sigma,\varepsilon} = \sqrt{2}$, which is a constant, and we have used the normalizaion given in Table II. Another check of the Bargmann superselection rule is provided by showing that $\langle \varepsilon_a \sigma_b \sigma_c^* \rangle = 0$. Furthermore,

$$\begin{aligned} \langle \sigma_a \varepsilon_b \Sigma_c^* \rangle &= \langle s_a s_b s_{b'} s_c^* s_{c'}^* s_{c''}^* \rangle \\ &\simeq 6C(a-c)[C(b-c)]^2 \\ &= 6\mathscr{A}^6 (t_a - t_c)^{-x_\sigma} (t_b - t_c)^{-2x_\sigma} e^{-(r_a - r_c)^2/2(t_a - t_c)} e^{-(r_b - r_c)^2/(t_b - t_c)} \end{aligned} \quad (4.29)$$

which from Table II is seen to agree with (3.28). For normalized fields we identify $\Psi_{\sigma\varepsilon,\Sigma} = \sqrt{3}$. Apparently, the massive scaling fields do reproduce the predictions of Schrödinger invariance.

Finally, we look at some examples with the massless field $\eta$ whose correlations are not immediately zero. For example,

$$\langle \eta_a \sigma_b \sigma_c^* \rangle_c = \langle s_a^* s_{a'} s_b s_c^* \rangle_c \simeq C(a-b)\, C(c-a) \tag{4.30}$$

We denote $\mathbf{a} = (\mathbf{r}_a, t_a)$ and get

$$\langle \eta_a \sigma_b \sigma_c^* \rangle_c = \mathscr{A}^4 (t_a - t_b)^{-x_\sigma} (t_a - t_c)^{-x_\sigma} e^{-(r_a - r_b)^2/2(t_a - t_b)} e^{+(r_a - r_c)^2/2(t_a - t_c)} \tag{4.31}$$

which indeed agrees with (3.28) [recall from Eq. (2.8) that $\sigma^*$ picks up a phase opposite to $\sigma$], since $x_\eta = 2x_\sigma$. Using normalized fields, we identify $\Psi_{\eta\sigma,\sigma} = 1$. Next, consider

$$\begin{aligned} \langle \eta_a \varepsilon_b \varepsilon_c^* \rangle_c &= \langle s_a^* s_{a'} s_b s_{b'} s_c^* s_{c'}^* \rangle_c \\ &\simeq 4C(b-a)\, C(a-c)\, C(b-c) \\ &= 4\mathscr{A}^6 [(t_b - t_a)(t_a - t_c)(t_b - t_c)]^{-x_\sigma} \\ &\quad \times \exp(-\tfrac{1}{2}[\Delta_{b,a} + \Delta_{a,c} + \Delta_{b,c}]) \end{aligned} \tag{4.32}$$

where $\Delta_{a,b} = (r_a - r_b)^2/(t_a - t_b) = -\Delta_{b,a}$. We verify that

$$\alpha := \Delta_{a,b} + \Delta_{c,a} + \Delta_{b,c} = \frac{[(r_a - r_c)(t_b - t_c) - (r_b - r_c)(t_a - t_c)]^2}{(t_a - t_b)(t_a - t_c)(t_b - t_c)} \tag{4.33}$$

has exactly the form of the argument of the scaling function occurring in (3.28). Then the argument of the exponential in the last line in (4.32) becomes $\alpha/2 - \Delta_{b,c}$. Comparing now with (3.28), we see that the powers of the $t_i$ agree with $x_\varepsilon = x_\eta = 2x_\sigma$ and the masses agree with $\mathcal{M}_\varepsilon = 2$, $\mathcal{M}_\eta = 0$ as expected form Table II. We finally identify the scaling function $\Psi_{\eta\varepsilon,\varepsilon}(u) = 2\exp(u/2)$ for normalized fields. Finally, we consider

$$\begin{aligned} \langle \eta_a \eta_b \eta_c \rangle_c &= \langle s_a^* s_{a'} s_b^* s_{b'} s_c^* s_{c'} \rangle_c \\ &\simeq C(a-b)\, C(c-a)\, C(b-c) + C(a-c)\, C(b-a)\, C(c-b) \\ &= \mathscr{A}^6 [(t_a - t_b)(t_c - t_a)(t_b - t_c)]^{-x_\sigma} \\ &\quad \times \{\exp(-\tfrac{1}{2}[\Delta_{a,b} + \Delta_{c,a} + \Delta_{b,c}]) \\ &\quad + \exp(-\tfrac{1}{2}[\Delta_{b,a} + \Delta_{a,c} + \Delta_{c,b}])\} \end{aligned} \tag{4.34}$$

The very fact that this correlation does not vanish again confirms that the field $\eta$ is massless. We use again (4.33) and find, identifying $\Psi_{\eta\eta,\eta}(u) = 2\cosh(u/2)$, complete agreement with the prediction, Eq. (3.28).

### 4.4. Relaxation Kinetics of the Spherical Model

Having studied in some detail the case of a setting of infinite extent in both time and space directions, we now look into one example with a macroscopically prepared initial state which is not the equilibrium state and the system is then allowed to relax toward equilibrium. While for very long times we are back to the dynamical scaling considered so far, it was realized[6,7] that already the intermediate stages of the relaxation process display universal behavior. Since we are merely interested in the special case of a dynamical exponent $z = 2$, we concentrate here on the example of the $n \to \infty$ limit of the $O(n)$ vector model, with a nonconserved order parameter. Since for nonequilibrium systems it is the response function which satisfies the analyticity in frequency space (see Section 2), rather than the correlation function, we shall look for a correspondence of the two-point function obtained in Eq. (3.48) with response functions. We are interested in the two-time response function $G_{\mathbf{k}}(t, t')$ which measures the response of the field $\phi$ at time $t$ to a thermal noise at time $t'$.[7] Starting from an initial state without correlations[3] and quenching the system to a temperature $T \leqslant T_c$, we find the response function (with time ordering $t > t'$ understood)[6,7]

$$G_{\mathbf{k}}(t, t') = (t/t')^{\sigma} \exp[-\lambda k^2 (t - t')] \tag{4.35}$$

where $\lambda$ is a constant and, depending on the final temperature,

$$\sigma = \begin{cases} 1 - d/4 & \text{if} \quad T = T_c{}^{(6)} \\ d/4 & \text{if} \quad T = 0^{(7)} \end{cases} \tag{4.36}$$

In the spherical model limit, thermal noise can be shown to be irrelevant below the critical point[7] and this is thought to hold in general. Fourier transformation in space gives

$$G(\mathbf{r}; t, t') = \left(\frac{\pi}{\lambda}\right)^{d/2} \left(\frac{t}{t'}\right)^{\sigma} (t - t')^{-d/2} \exp\left(-\frac{1}{4\lambda^2} \frac{r^2}{t - t'}\right) \tag{4.37}$$

This is indeed fully consistent with the result (3.48) for the two-point function. We can identify $\mathcal{M} = 1/(2\lambda^2)$ and the exponents

$$\begin{aligned} x_a &= \frac{d}{4}, & x_b &= \frac{3d}{4} & \text{if} \quad T &= 0 \\ x_a &= \frac{3d}{4} - 1, & x_b &= 1 + \frac{d}{4} & \text{if} \quad T &= T_c \end{aligned} \tag{4.38}$$

and we explicitly see that $x_a$ and $x_b$ are in general distinct.

[3] The case of a correlated initial state before the quench to $T = 0$ leads for the $n \to \infty$ limit to the same type of result and only affects the value of $\sigma$.[7]

An interesting case occurs when $x_a = x_b$. We can then take the limit $t_b \to 0$ and find from Eq. (3.48) for this particular two-point function

$$F(r_a, r_b; t, 0)\,|_{x_a = x_b} = \delta_{\mathcal{M}_a, \mathcal{M}_b} \Phi_0 t^{-x_a} \exp\left[-\frac{\mathcal{M}_a}{2}\frac{(r_a - r_b)^2}{t}\right] \quad (4.39)$$

This should correspond to the single-time response function $G_{\mathbf{k}}(t)$, which measures the response of the field $\phi$ at time $t$ to a fluctuation in $\phi$ at time 0.[7] In the spherical model, this takes the form, for a quench to $T = 0$,[7]

$$G(\mathbf{r}; t, 0) = \left(\frac{t}{t_0}\right)^{-d/4} \exp\left(-\frac{1}{4\lambda^2}\frac{r^2}{t}\right) \quad (4.40)$$

where $t_0$ is some constant (which serves to define the scaling regime $t \gg t_0$). From Eq. (4.39) we read off $x_a = d/4$, in agreement with the first of Eqs. (4.38), as it should be.

## 5. SOME REMARKS BEYOND $\theta = 2$

Since for generic values of the anisotropy exponent $\theta$ there is at present no general approach available, we content ourselves with a few results from selected models. We do not investigate here whether it might be possible to achieve Galilean invariance [which at least in the form used here requires a $U(1)$ symmetry in the Lagrangians of the respective theories], but merely ask for the phenomenological behavior of the two-point function. The only aim of this section is to submit the conformal invariance result (1.4)[11] to a test. The fact that there are systems with $\theta \neq 2$ which appear not to satisfy that prediction had been the origin of this whole investigation.

### 5.1. Lifshitz Points of Higher Order in the Spherical Model

We come back to the ANNNS model introduced earlier. Now, we add further interaction terms along the axes denoted as "space" dimensions. Since we are only interested here in the spin–spin correlation, it is sufficient for us to consider the real Hamiltonian $\mathscr{H}_{SM}$, Eq. (4.13). The Fourier transform of the couplings now is

$$J(\mathbf{k}) = 2J\left(\sum_{i=1}^{D} \cos k_i + \sum_{j=1}^{d} \sum_{i=1}^{n} \kappa_i \cos[(i+1)\,k_j]\right) \quad (5.1)$$

Previously, we had taken $\kappa=\kappa_1$ to be the only nonvanishing coupling. With several of the $\kappa_i$ nonzero, the phase diagram will contain lines of Lifshitz points (also called Lifshitz points of first order, where $L=2$) which end in a Lifshitz point of second order (with $L=3$), in analogy to the definition of multicritical points; see ref. 43 for a review. Lifshitz points of higher order are defined analogously. At a Lifshitz point of order $L-1$, we have

$$J(\mathbf{k}) \simeq 2JD + d\sum_{i=1}^{n} \kappa_i - \tfrac{1}{2}\sum_{j=d+1}^{D} k_j^2 - c_L \sum_{j=1}^{d} k_j^{2L} + \cdots \tag{5.2}$$

which defines the readily calculable constant $c_L$. When $d'=1$, the anisotropy exponent is $\theta=L$. We are interested in the critical correlation function $C(\mathbf{a}-\mathbf{b})=\langle\sigma_a\sigma_b\rangle$. This can be calculated exactly.[45] As we had already seen for Lifshitz points of first order above, the correlations of the model considered here will only in special cases solve the dispersion relation $E\sim k^\theta$, rather than $E^2\sim k^{2\theta}$. This should be the case if the scaling function does not just show a power-law behavior for large values of $u$. This is the case if[4]

$$D=L+2+2m, \qquad m=1,\ldots, L-1 \tag{5.3}$$

Then the critical two-point function is, in the scaling limit $r\to\infty$, $t\to\infty$ with $u=r^\theta/t$ fixed,[45]

$$C(\mathbf{r}, t) = \mathscr{A}_{L,D}\, t^{-(2m+1)/L}\, \Xi\left(L, \frac{2m+1}{2L}; \frac{2^{1/L}}{4Lc_L^{1/L}} \frac{r^2}{t^{2/L}}\right) \tag{5.4}$$

where $\Xi$ can be expressed as a finite sum of generalized hypergeometric functions and $\mathscr{A}_{L,D}$ is a known nonuniversal constant. Here we merely consider the behavior for large values of the scaling variable $u$, to leading order,[45]

$$\begin{aligned} C(\mathbf{r}, t) \simeq{}& \mathscr{B}_{L,D}\, t^{-(2m+1)/L} u^{(D-3\theta)/[2\theta(\theta-1)]} \\ &\times \exp\left[\mathscr{C}_L \cos\left(\frac{\pi}{2}\frac{L}{L-1}\right) u^{1/(\theta-1)}\right] \\ &\times \cos\left[\mathscr{C}_L \sin\left(\frac{\pi}{2}\frac{L}{L-1}\right) u^{1/(\theta-1)} + \frac{\pi}{2}\frac{L}{L-1}\frac{D-3L}{2L}\right] \end{aligned} \tag{5.5}$$

[4] Equation (2.22) in ref. 45, contains a typographic error and correctly reads $a=\frac{1}{2}d-[(L-1)/2L]m-1=\frac{1}{2}(d-d_-)$.

where $\mathscr{B}_{L,D}$ and $\mathscr{C}_L > 0$ are known nonuniversal constants. We remark that only the power prefactor of the scaling function depends on $D$, the rest of the scaling function depends in the large-$u$ limit only on $L$.

Let us compare this result with the prediction (1.4) following from conformal invariance.[11] While conformal invariance gives for a nonconserved order parameter a simple exponential behavior for the scaling function, we rather find for the higher-order Lifshitz points ($L \geqslant 3$) a stretched-exponential behavior. The two forms only agree for $\theta = 2$. On the other hand, for a conserved order parameter the conformal invariance scaling function for $u$ large is of the form[11]

$$\psi(u) \sim u^{-2x/3z} e^{-u^{1/3}} \cos(\sqrt{3}\, u^{1/3})$$

which for $z = 4$ is the van Hove theory. It is interesting to note that for $L = 4$, the ANNNS model reproduces the same behavior.

Finally, we look into the case where $\theta = 1/2$. This is realized if $d = 1$ and $L = 2$. Now the direction parallel to the next-nearest-neighbor interaction will be interpreted as "time" and the other directions are referred to as "space." For $D = 4$, we find again an exponential-like behavior for the spin–spin two-point function[45]

$$C(\mathbf{r}, t) \sim t^{-3/2} u^{-3} \exp(-\tfrac{1}{2} u^{1/(\theta - 1)}) \tag{5.6}$$

in the scaling limit with $u = r^{1/2}/t$ fixed and $\theta = 1/2$. Again, this is different from the conformal invariance prediction (1.4).

## 5.2. Directed Percolation

As a further example for a strongly anisotropic critical system we consider directed percolation in $1 + 1$ dimensions (for a review, see ref. 9). In percolation, sites or bonds are filled at random with probability $p$ and percolation proceeds along paths between occupied nearest-neighbor links. In directed percolation, there is in addition a preferred direction and percolation is allowed to proceed only in one sense along this direction. This preferred direction is called "time" and the orthogonal ones "space." Consider the pair connectedness $G(\mathbf{r}, \mathbf{r}')$, which is a measure of the probability that sites at $\mathbf{r}$ and $\mathbf{r}'$ are connected by a percolating path. It is well known that there is a critical value $p_c$ such that one has the scaling form

$$G(r, t) = \mathscr{A} t^{-2\beta/\nu_{\|}} \Phi(\mathscr{B} v), \qquad v = r/t^{\nu_\perp/\nu_{\|}} \tag{5.7}$$

where $\beta$, $\nu_\perp$, and $\nu_{\|}$ are critical exponents, the anisotropy exponent is $\theta = \nu_{\|}/\nu_\perp$, and $r$ and $t$ measure the "space" and "time" distances, respectively.

Precise numerical values for $p_c$ on various lattices and for the exponents have been obtained.[46,47] For our purposes it is enough to notice that in $(1+1)$ dimensions, to which we restrict ourselves here, we have $\theta \simeq 1.5807$.[47]

Benzoni[46] studied the pair connectedness by calculating numerically the moments

$$\chi^{(n)} = \mathscr{A} \int_{-\infty}^{\infty} dv \, |v|^n \, \Phi(\mathscr{B} v) \tag{5.8}$$

and verified that the ratios

$$C = \frac{[\chi^{(1)}]^2}{\chi^{(0)}\chi^{(2)}}, \qquad F = \frac{[\chi^{(2)}]^2}{\chi^{(0)}\chi^{(4)}}, \qquad G = \frac{[\chi^{(2)}]^2}{\chi^{(1)}\chi^{(3)}} \tag{5.9}$$

are independent of the nonuniversal scale factors $\mathscr{A}$ and $\mathscr{B}$ and should therefore be universal. A careful numerical computation[46] then yields numerical values for $C$, $F$, and $G$ for various lattices of both directed site and directed bond percolation. The results are in full agreement with universality.[46]

We proceed to analyze these results in the following way. We try the ansatz for the scaling function $\Phi(v)$

$$\Phi(v) = |v|^b \exp(-v^a) \tag{5.10}$$

where $a$, $b$ are constants to be determined. Then $\chi^{(n)} = 2a^{-1}\Gamma((n+b+1)/a)$. We now fit this form to Benzoni's[46] numerical results for $C$, $F$, and $G$ and find

$$\begin{aligned} a &= 2.49 \pm 0.16, \qquad b = -0.016 \pm 0.03 \qquad \text{from } C \\ a &= 2.58 \pm 0.12, \qquad b = -0.023 \pm 0.03 \qquad \text{from } F \\ a &= 2.62 \pm 0.17, \qquad b = -0.023 \pm 0.07 \qquad \text{from } G \end{aligned} \tag{5.11}$$

with the mean values $a = 2.56(7)$ and $b = -0.02(3)$. Since the scaling functions $\Phi(v)$ is finite for $v = 0$,[46] we interpret this result as implying that $b = 0$. In Table III we give the results for $a$ as found using the ansatz (5.10) with $b = 0$ from the ratios $C$, $F$, and $G$ and various realizations of directed percolation. Note that the estimates for $a$ obtained from different moments and different lattice realizations of directed percolation are the same, which means that the chosen ansatz does indeed describe the available data. From all this we conclude

$$a = 2.6 \pm 0.2 \tag{5.12}$$

**Table III. Exponent $a$ Determined from Moment Ratios $C$, $F$, $G$ for Various Realizations of Directed Percolation[a]**

| Ratio | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $C$ | 2.41 | 2.56 | 2.49 | 2.24 | 2.39 |
| $F$ | 2.50 | 2.69 | 2.69 | 2.32 | 2.49 |
| $G$ | 2.54 | 2.74 | 2.65 | 2.37 | 2.54 |

[a] 1, square bond; 2, square site; 3, square site-band; 4, triangular bond; 5, triangular site.

Making contact with our previous results, formulated in terms of the scaling variable $u = v^{\theta} = r^{\theta}/t$, we obtain from (5.10)

$$a = \frac{\theta}{\theta - 1} \simeq 2.72... \tag{5.13}$$

using the known value of $\theta$. Comparison of (5.12) and (5.13) implies that also in this class of models the two-point function scaling function appears to be consistent with the *same* stretched-exponential form as already observed for the Lifshitz points of the spherical model and in disagreement with conformal invariance, Eq. (1.4). It is remarkable that, although the Lagrangians of the ANNNS model and directed percolation are quite distinct [in particular, the Lagrangian of Reggeon field theory, which is in the same universality class as directed percolation, certainly has no apparent $U(1)$ symmetry], these distinctions are not reflected in the large-$u$ behavior of the two-point functions.

## 6. CONCLUSIONS

In this paper we have examined the simplest consequences of the hypothesis of local dynamical scaling with space-time-dependent local rescaling factors $\lambda(\mathbf{r}, t)$. We have seen that for the special case of an anisotropy (or dynamical) exponent $\theta = 2$, the Schrödinger group, which is the nonrelativistic limit of the conformal group, is a sensible candidate for a group of local scale transformations. The treatment of Schrödinger invariance (of quasiprimary fields) is in many respects quite analogous to conformal invariance. However, there are a few distinctions, the main one being the role of the phase transformation, which is not present in the conformal group. We hope that the experience obtained in this simplest nonconformal case may become useful for the extension of the method to generic anisotropy exponents $\theta$. We have derived the form of the two-point and three-point functions for both infinite space and time [Eqs. (3.12) and (3.28)] and for the two-point function also if either space or time is restricted to the half-infinite space [Eqs. (3.35) and (3.48)]. The results obtained are in agreement with and extend those following from the weaker restrictions of Galilean invariance. We observe the relationship with *correlation functions* for static but strongly anisotropic systems, but with the *response functions* for systems out of equilibrium.

The Lie algebra of the Schrödinger group can be naturally extended to an infinite-dimensional one. We have not solved the problem of how to use this infinite algebra to calculate the critical exponents and the scaling functions in the correlations which are left undermined in this work. We hope to come back to this in the future.

Several exactly solvable statistical models with anisotropy exponent $\theta = 2$ were seen to reproduce the results of Schrödinger invariance for the two- and three-point functions. In particular, we have seen that due attention must be paid for correctly implementing the changes of the phases of scaling fields as demanded by Galilean invariance. This requires the Lagrangians of the models to be considered to have a global $U(1)$ symmetry. Since many known systems, although having $\theta = 2$, do not have this property, it remains to be seen whether the concept of Galilean invariance can be conveniently extended to deal with these more general situations.

Evidence from some models with anisotropy exponent $\theta \neq 2$ suggests that, at least for large values of the scaling variable $u = r^{\theta}/t$, the two-point scaling correlation function might behave as

$$\Phi(u) \sim \exp(-u^{1/(\theta-1)}) \tag{6.1}$$

(where oscillating and power-law prefactors as well as nonuniversal scale factors were suppressed). We have found examples for $\theta = n$, with any integer $n \geqslant 3$, for $\theta \simeq 1.58...$ and for $\theta = 1/2$ [for which the exact scaling function (5.6) is known]. This finding is in disagreement with the form (1.4) suggested by using conformal invariance in space.

## APPENDIX A. SOLUTION OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

We derive the general solution $H = H(r, s; \tau, \sigma)$ of the following system of differential equations:

$$\begin{aligned} (\tau\partial_\tau + \sigma\partial_\sigma + \tfrac{1}{2}r\partial_r + \tfrac{1}{2}s\partial_s)H &= 0 \\ (\tau\partial_r + \sigma\partial_s)H &= 0 \\ (\tau^2\partial_\tau + \sigma^2\partial_\sigma + \tau r\partial_r + \sigma s\partial_s)H &= 0 \end{aligned} \tag{A.1}$$

The technique consists of subsequent solution and resubstitution.[32] The second of Eqs. (A.1) is solved by

$$H = \tilde{H}(t; \tau, \sigma), \qquad t = \frac{r}{\tau} - \frac{s}{\sigma} \tag{A.2}$$

while the other two equations become

$$\begin{aligned} (\tau\partial_\tau + \sigma\partial_\sigma - \tfrac{1}{2}t\partial_t)\tilde{H} &= 0 \\ (\tau^2\partial_\tau + \sigma^2\partial_\sigma)\tilde{H} &= 0 \end{aligned} \tag{A.3}$$

The first of those is solved by

$$\tilde{H} = \tilde{\tilde{H}}(u, v), \qquad u = \tau t^2, \qquad v = \sigma t^2 \tag{A.4}$$

and the second one becomes

$$(u^2\partial_u + v^2\partial_v)\tilde{\tilde{H}} = 0 \tag{A.5}$$

with the solution $\tilde{\tilde{H}} = \Psi(u^{-1} - v^{-1})$, where $\Psi$ is an arbitrary function. Backsubstitution then yields the result (3.27) in the text.

## APPENDIX B. IMPOSSIBILITY OF NONCONVENTIONAL CENTRAL EXTENSIONS

Consider the centrally extended (infinite) Schrödinger algebra

$$\begin{aligned} [X_n, X_m] &= (n-m)\, X_{n+m} + \frac{c}{12}(n^3 - n)\, \delta_{n+m,0} \\ [X_n, Y_m] &= \left(\frac{n}{2} - m\right) Y_{n+m} + D(n, m) \\ [X_n, M_m] &= -mM_{n+m} + E(n, m) \\ [Y_n, Y_m] &= (n-m)\, M_{n+m} + A(n, m) \\ [Y_n, M_m] &= F(n, m) \\ [M_n, M_m] &= Kn\delta_{n+m,0} \end{aligned} \tag{B.1}$$

where $A$, $D$, $E$, and $F$ are numbers and $c$ and $K$ are constants. The special form of the central extensions for $X_n$ and $M_n$ is well known. We show that, with the only exception of $c$, these central extensions either have to vanish or can be reabsorbed into the generators.

This follows from the Jacobi identities. Begin with $D(n, k)$. Consider $[X_n, [X_m, Y_k]]$ and their cyclic permutations. This implies

$$\left(\frac{m}{2}-k\right) D(n, m+k)-\left(\frac{n}{2}-k\right) D(m, n+k)-(n-m)\, D(n+m, k)=0 \tag{B.2}$$

Let $n=0$. It follows that [besides $D(0, 0)=0$]

$$D(m, k)= -\frac{1}{2}\,\frac{m-2k}{m+k}\, D(0, m+k)=: -\left(\frac{m}{2}-k\right) d(m+k) \tag{B.3}$$

which defines $d(k)$. Turning to $E(n, m)$, consider $[X_n, [X_m, M_l]]$ and their cyclic permutations. This implies

$$-lE(n, m+l)+lE(m, n+l)-(n-m)\, E(n-m, l)=0 \tag{B.4}$$

Now take $m=0$ in (B.4) and then either $l=0$ or $n+l=0$, implying $E(n, 0)=0$ for all $n$. We therefore write $E(n, m)=m\mathscr{E}(n, m)$. Inserting in (B.4) and taking $m=0$, we find that $\mathscr{E}(n, l)=\mathscr{E}(0, n+l)+\delta_{n+l,0}\,\varepsilon(n)+\delta_{l,0}\,\tilde{\varepsilon}(n)$. With the definitions $\mathscr{E}(0, n)=e(n)$ and $\eta(n)= -n\varepsilon(n)$, we have $E(n, m)=me(n+m)+\eta(n)\,\delta_{n+m,0}$. Backsubstitution into (B.4) implies

$$(n+m)[\eta(n)-\eta(m)]=(n-m)\,\eta(n-m) \tag{B.5}$$

Let $n=0$ and get $\eta(n)=\eta(-n)$. Let $n+m=0$ to see that $\eta(2n)=0$ for all $n$. Taking $m=2n$, we get $\eta(n)= -\frac{1}{3}\eta(-n)$. Consequently, $\eta(n)=0$ for all $n$. Turning to $A(m, k)= -A(k, m)$, consider $[X_n, [Y_m, Y_k]]$ and permutations. We get

$$(m-k)\, E(n, m+k)+\left(\frac{n}{2}-k\right) A(n+k, m)-\left(\frac{n}{2}-m\right) A(n+m, k)=0 \tag{B.6}$$

We use the result for $E(n, m)$, the antisymmetry of $A(m, k)$, insert in (B.6), put $n=0$, and divide by $m+k$ to get $A(k, m)=(m-k)\, e(m+k)+a(k)\,\delta_{k+m,0}$, with $a(k)= -a(-k)$. Backsubstitution into (B.6) then implies $(3n/2+m)\, a(-m)=(n/2-m)\, a(n+m)$. We now choose $n=2m$ and get $a(m)=0$. To see that $K=0$, consider $[Y_n, [Y_m, M_k]]$ and its permutations to get $(n-m)\, Kn\delta_{n+m+k,0}=0$. Finally, we turn to $F(m, k)$ and consider $[X_n, [Y_m, M_k]]$ and permutations to obtain

$$kF(m, n+k)-\left(\frac{n}{2}-m\right) F(n+m, k)=0 \tag{B.7}$$

Put $n=0$ to find $F(m,k)=f(m)\,\delta_{m+k,0}$. We now have to distinguish two cases. (i) The index $m$ of $Y_m$ is half-integer. Since the index $k$ of $M_k$ is always integer, we directly have $F(m,k)=0$. (ii) The index $m$ of $Y_m$ is integer. Backsubstitution into (B.7) then gives

$$(n+m)\,f(m)+\left(\frac{n}{2}-m\right)f(n+m)=0 \tag{B.8}$$

Let $n+m=0$ and find $f(0)=0$. Then, let $m=0$ and get $f(n)=-2f(0)=0$.

Consequently, the only surviving terms are given by $d(m)$ and $e(n)$. These can be absorbed into the generators by defining $\tilde{M}_n=M_n-e(n)$ and $\tilde{Y}_m=Y_m-d(m)$. The only central term remaining is the one parametrized by $c$. This proves the assertion.

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