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FIXING A HOLE

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Movements in space-time: the author composes a hole diffeomorphism for a 3+1-dimensional space-time manifold, which is required in the so-called Hole Argument. The composition applies directly to the universe.

Keys: general relativity, hole argument, diffeomorphism, substantivalism, differential geometry.

1. INTRO (PRESTO)

In foundational analyses of and philosophical debates about the General Theory of Relativity (GTR), Einstein's *Lochbetrachtung* from 1913 has recently become a major issue. Earman and Norton [1] have ingeniously amended this argument, purporting to show that GTR is a *necessarily indeterministic* theory if one assumes the independent existence of *bare* space-time points ('substantivalism'). This 'Hole Argument' has caused and is causing an avalange of responses; see for example Butterfield [2], Stachel [3], Maudlin [4], Rynasiewicz [5] (recommended), and references in Earman [6].

A hole diffeomorphism ϕ_H is a map from one space-time manifold $\mathcal{M}$ to another one $\mathcal{M}'$ (or to itself: $\mathcal{M}' = \mathcal{M}$), meeting a few strong conditions (Section 2), such that ϕ_H acts like the identity outside a hole $H \subset \mathcal{M}$ but acts non-identically inside H. The Hole Argument requires the existence of such a hole diffeomorphism; thus its existence is always asserted. Although not a soul doubts this assertion, the best

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possible reason for the absence of doubt would be the presence of an example; if only for the sake of reference, for one searches in vain for such an example in all writings about the Hole Argument (*supra*). The sole aim of the present Letter is to provide such a reference, which is at least eight years overdue. We shall not be satisfied with some boring trivial example for a 1+0-dimensional manifold, but define our goal as: constructing a non-trivial hole diffeomorphism for a 3+1-dimensional manifold that applies directly to the universe, and that has some vim and spunk.

After a few preliminary mathematical movements (Section 2), we compose a hole diffeomorphism for a 3+1-dimensional space-time manifold, such that all points are shaken in all four dimensions (Section 3). This 3+1-dimensional example applies directly to a Robertson-Walker manifold, which is used by cosmologists to describe the universe; the construction is however not limited to such a manifold.

2. MATHEMATICAL MOVEMENTS (ALLEGRO)

A function $f : \mathbb{R} \to \mathbb{R}$ is *smooth* iff it is continuously differentiable to all orders; $C^\infty[\mathbb{R}]$ is the space of all smooth functions on $\mathbb{R}$, closed under summation, multiplication, and composition whenever defined; see Corollary 2.4 in Boothby [8]. The *mollifier* is the following non-negative, smooth function $\Omega_{(a,b)} \in C^\infty[\mathbb{R}]$ of bounded support (a,b) [7]:

$$\Omega_{(a,b)}(x) \;:=\; 1_{(a,b)}(x) \exp\left[\frac{1}{(x-a)(x-b)}\right] , \qquad (1)$$

where $1_{(a,b)} : \mathbb{R} \to \{0,1\}$ is the characteristic function of the set (a,b); the Greek letter Ω alludes to the graph of the mollifier (see Figure 1). All derivatives of the mollifier $\Omega_{(a,b)}$ vanish at the end-points $x = a$ and $x = b$ whenever $x \downarrow a$ and $x \uparrow b$, respectively. This remarkable property will be exploited vigorously in the present Letter and should therefore stick in the reader's mind onwards. If one were to delete the characteristic function $1_{(a,b)}$ from (1), the remaining function $\exp[1/(x-a)(x-b)]$ would have singularities at $x = a$ and $x = b$ and would have unbounded support $(-\infty, +\infty)$. We leave the appropriate choice of units implicit; whenever x, a and b have the dimension of, say, length, then the 1 in the exponential of (1) has to be replaced by 1 [length]^2; *etc.* Notice that the mollifier is far from bijective, but is symmetric around its maximum. This maximum of $\Omega_{(a,b)}$ lies half-way the interval (a,b) and equals $\exp[-4/(b-a)^2]$; as the interval gets narrower and tends to

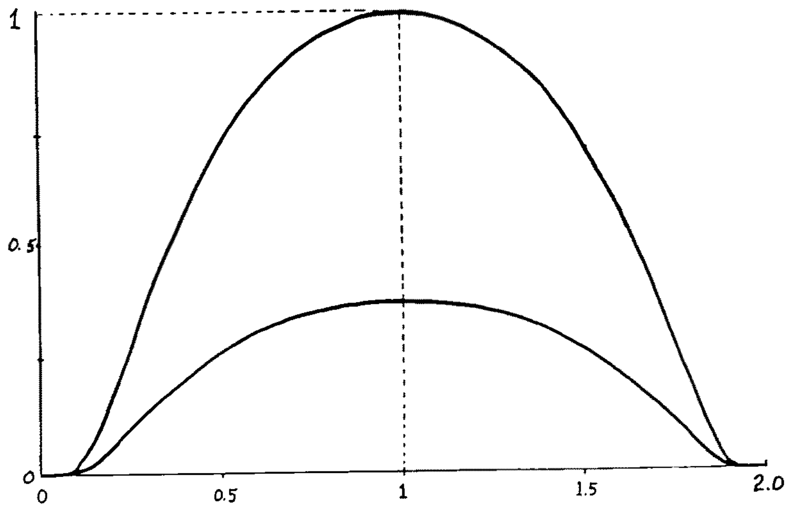

Figure 1. Graph of the mollifier Ω_2 and the normed mollifier $\overline{\Omega}_2$.

$(a, a) = \emptyset$, the maximum drops to 0; as the interval grows and tends to either $(-\infty, b)$ or $(a, +\infty)$ or $(-\infty, +\infty)$, the maximum does not grow indefinitely but approaches 1. A normalizing factor $\exp[4/(b-a)^2]$ in front of the exponential in def. (1) makes the maximum equal to 1 regardless of the support (a, b) (Figure 1):

$$\overline{\Omega}_{(a,b)}(x) \; := \; \exp\left[\frac{4}{(b-a)^2}\right] \Omega_{(a,b)}(x) \; . \tag{2}$$

Denote the Area under the graph of the mollifier $\Omega_{(a,b)}$ by $A(a, b)$:

$$A(a,b) \; := \; \int_a^b \Omega_{(a,b)}(y)\, dy \; . \tag{3}$$

Consider the indefinite integral of the mollifier $\Omega_b := \Omega_{(0,b)}$, sometimes referred to as a Legendre Transformation, where $A_b := A(0, b)$:

$$J_b(x) \; := \; 1_{[0,\infty)}(x)\, A_b^{-1} \int_0^x \Omega_b(y) dy \; , \tag{4}$$

which we take to be function $\mathbb{R} \to [0, 1]$ for each fixed $b > 0$. The graph of J_b connects the X-axis from the left smoothly to the horizontal line of ordinate 1 during the interval $(0, b)$. The partial derivatives of J_b is:

$$\partial_x J_b(x) \; = \; A_b^{-1}\, \Omega_b(x) \; , \tag{5}$$

so that J_b is indeed smooth on $\mathbb{R}$. On (a subset of) the interval $[0, b]$ the function J_b increases monotonically and therefore is bijective.

We shall work with an n-dimensional differentiable, topological, metrical C^∞-manifold $\mathcal{M}$ [8]; we call its finite, connected, open subsets *regions* and its finite, simply-connected, open subsets *holes*. A hole in a n-dimensional manifold is therefore topologically equivalent to an open ball B^n in n-dimensional Euclidean space (say $\mathbb{R}^n$). The word 'hole' echoos Einstein's original *Lochbetrachtung*, where it referred to a tiny matter-free spatial region, where by implication the energy-momentum tensor vanished. Earman and Norton's hole has little to do with matter distributions, but arguably is just what we defined above [1]. The possible tiny size of the hole contributes to the sensational character of their Hole Argument. Further, it serves to mention that the hole has nothing to do with *deleting* some subset from the manifold, as the topologically inclinded mind might have suspected.

Suppose we have two local coordinate frames, F and F' respectively, of a region $G \subset \mathcal{M}$. Then the local coordinate transformation $T : F[G] \to F'[G]$ is a *diffeomorphism* iff T is a bijection and both T and T^{inv} are smooth; see Boothby [8]. Whenever T acts like the identity outside the image $F[H] \subset \mathbb{R}^n$ of a *hole* $H \subseteq G$ and acts non-identically inside $F[H]$, and maps $F[H]$ onto $F'[H]$, we call T a *hole diffeomorphism*. The local coordinate transformation $T : F[G] \to F'[G]$, which results in a mere relabelling of the same points in G, can be taken to induce a point transformation $\phi : G \to G''$, where $G \subset \mathcal{M}$ and $G'' \subset \mathcal{M}'$; and *vice versa*. We shall define a *global* point transformation $\phi_H : \mathcal{M} \to \mathcal{M}$, which maps not only the manifold onto itself but also region G as well as the hole H, such that the coordinates of $p' := \phi_H(p) \in G$ in frame F' are identical to the coordinates of $p \in G$ in frame F:

$$(\, r(p),\, \theta(p)\,) \;=\; (\, r'(p'),\, \theta'(p')\,) \;=\; (\, r'(\phi_H(p)),\, \theta'(\phi_H(p))\,)\,. \qquad (6)$$

This leads to the following definition:

$$\phi_H(p) \;:=\; 1_H(p)[F^{\mathrm{inv}} \circ T^{\mathrm{inv}} \circ F](p) + [1 - 1_H(p)]I(p)\,, \qquad (7)$$

where $I : \mathcal{M} \to \mathcal{M}$ is the identity and $1_H : \mathcal{M} \to \{0, 1\}$ is the characteristic function of the hole. With a slight abuse of language we call a global point transformation ϕ_H (7) induced by the local hole diffeomorphism T a *global hole diffeomorphism*, for it actively shifts points smoothly around in the hole H and leaves points outside the hole untouched. Formula (7) is, we claim, the cruxial formula for defining any hole diffeomorphism.

Finally, we mention the Inverse Mapping Theorem, one of the pillars of Analysis on manifolds. The Inverse Mapping Theorem reads, in a suitable formulation for the present purpose, as follows [8]. Suppose $\mathcal{M}$ is an n-dimensional C^∞-manifold; $F : G \to F[G] \subset \mathbb{R}^n$ and $F' : G \to F'[G] \subset \mathbb{R}^n$ are coordinate frames and $G \subset \mathcal{M}$ is an open set; suppose further that smooth coordinate transformation $T : F[G] \to F'[G]$ has a non-singular Jacobian matrix $\mathsf{J}(T; x)$ at point $x \in F[G]$. Then according to the Inverse Mapping Theorem there exists a neighbourhood $U(x) \subset F[G]$ such that T maps $U(x)$ *diffeomorphically* onto a neighbourhood of $T(x)$: $T[U(x)] \subset F'[G]$; and the Jacobian matrix of the inverse T^{inv} at $T(x)$ equals the inverted Jacobian matrix of T at x:

$$\mathsf{J}(T^{\text{inv}}; T(x)) = \mathsf{J}^{\text{invt}}(T; x) \,. \tag{8}$$

Thus for every smooth coordinate transformation having a non-singular Jacobian matrix at a given point $x \in \mathbb{R}^n$, its inverse exists on some neighbourhood of x and that inverse is, remarkably, smooth too.

That a vanishing derivative of a bijection destroys the smoothness of the inverse, is easily understood if we consider some bijective function $f : (a, b) \to \mathbb{R}$ such that $f'(x_\circ) = 0$ in $x_\circ \in (a, b)$, and recall that the derivative is the slope of the tangent. The graph of f^{inv} is the mirror-image of the graph of f in the line $y = x$. Then the horizontal tangent to f in $x = x_\circ$ becomes a vertical tangent to f^{inv} in $y_\circ = f(x_\circ)$. But a vertical tangent means a diverging derivative of f^{inv} in $(y_\circ, x_\circ)$. Hence for a differentiable inverse f^{inv}, the derivative of bijection f is not allowed to vanish anywhere, that is, f must either increase or decrease monotonically everywhere in (a, b). For mappings T from some region $U \subset \mathbb{R}^n$ to $T[U] \subset \mathbb{R}^n$ the previous understanding generalizes to the requirement that some but not all partial derivatives of each component are allowed to vanish. That is, suppose $T(x_1, x_2, ..., x_n) = (y_1, y_2, ..., y_n)$; then for each $j \in \{1, 2, ..., n\}$, at least for one $k \in \{1, 2, ..., n\}$, one must have that $\partial_k y_j(x) \neq 0$, for all $x \in U$. Indeed, if not, then a row of zero's would appear in the Jacobian matrix, making its determinant vanish.

3. MOVEMENT IN 3+1-DIMENSIONAL SPACE-TIME (ANDANTE)

3.1 Looking like a glass onion

The space-time manifold of the universe is supposed to be a 3+1-dimensional Robertson-Walker manifold $\mathcal{M}_{\text{RW}}$; see Weinberg [9]. The construction below is however not limited to these particular manifolds. Consider global coordinate frame

$$\begin{aligned} F: \quad \mathcal{M}_{\rm RW} &\to [0,\infty) \times [0,\infty) \times [0,1) \times [0,1/2]\,, \\ p \quad &\mapsto F(p) = (ct, r, \varphi, \theta)\,, \end{aligned} \tag{9}$$

where c is the speed of light, t is the time coordinate, and r, φ, θ are the spatial coordinates, expressed in spherical coordinates; and where we have left the argument $p \in \mathcal{M}_{\rm RW}$ of the coordinate functions implicit. We measure angles in windings: 1 winding $\equiv$ 2π radians $\equiv$ $360°$. To prevent the spatial spherical coordinate frame from breaking down at the poles and the origin, and to deal with the periodic character of the angular coordinates in one single sweep, we should, from a rigorous point of view, work with equivalence classes that partition the Cartesian product set $[0,\infty) \times \mathbb{R}^2$. All triples of the type

$$(r,\ \varphi + n,\ \theta + m/2)\ \in\ [0,\infty) \times \mathbb{R}^2\,,$$

with n, m arbitrary integers, form an equivalence class and refer to one point in a flat 3-dimensional manifold. Obviously the collection of all these equivalence classes are in one-to-one correspondence to the elements of $[0,\infty) \times [0,1) \times [0,1/2)$; thus the elements of the latter can be used to label the equivalence classes in an obvious way. The poles and the origin correspond to the following classes: all coordinates $(r, \varphi, 0)$, with φ arbitrary, refer to the North pole of a sphere $S^2(r)$; similarly all coordinates $(r, \varphi, 1/2)$ refer to the South pole of $S^2(r)$; and all coordinates $(0, \varphi, \theta)$ refer to the origin. For the sake of expediency, we shall from now on equivocate between (r, φ, θ) as to denote a label of, and as to denote an element of, an equivalence class.

The components of the Robertson-Walker metric, which fixes the geometry of $\mathcal{M}_{\rm RW}$, expressed in the coordinates (9), are the coefficients of the following squared differential line-element [9]:

$$ds^2 \;=\; c^2 dt^2 - R^2_{\rm univ}(t) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \right]\,, \tag{10}$$

where $k \in \{-1, 0, +1\}$ and where $R_{\rm univ}(t)$ is a time-dependent parameter related to the Gauss curvature scalar. In the case $k = 1$, $R_{\rm univ}(t)$ is the radius of the 'closed universe' at time t. We whish to emphasize that the 3-dimensional spatial universe at time t is *not* like a gigantic ball $B^3(R_{\rm univ}(t))$, but is like a gigantic 3-dimensional sphere $S^3(R_{\rm univ}(t))$, which is the surface of a gigantic spatial 4-dimensional ball $B^4(R_{\rm univ}(t))$.

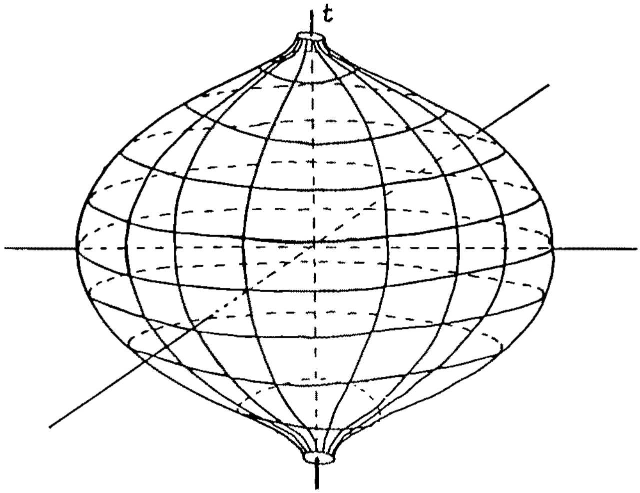

Figure 2. Space-time hole H (12); one spatial dimension suppressed.

We limit our attention to a coordinatized region $G \subset \mathcal{M}_{\text{RW}}$, say:

$$G := F^{\text{inv}}\Big[(ct, r, \varphi, \theta) \in \mathbb{R}^4 \mid t \in [0, 1000),\ r \in [0, 1000)\Big] , \quad (11)$$

where the number 1000 is arbitrarily chosen; by choosing appropriate units of time and length, region G can be as small or as large, physically speaking, as one whishes. We define the *space-time hole* $H \subset G$ as follows:

$$H := F^{\text{inv}}\Big[\{ B^3(\varepsilon(t)) \mid t \in (0, \tau) \}\Big] . \quad (12)$$

$F[H]$ is a non-denumerable collection of spatial balls of continuously varying radii $\varepsilon(t)$ from $\varepsilon_{\text{min}}(t) = \delta$ to $\varepsilon_{\text{max}}(t) = \rho + \delta$, where ρ is a small positive number and δ is some ridiculously small positive number. (For the reason why $\delta > 0$, *vide infra* §3.2.) We choose $\delta :=$ googolplex^{-1} mm, where [10]:

$$\text{googolplex} := 10^{\text{googol}} \quad \text{and} \quad \text{googol} := 10^{100} . \quad (13)$$

The idea is now to perform a spatial coordinate transformation inside ball $B^3(\varepsilon(t))$ of radius $\varepsilon(t)$, at each time $t \in (0, \tau)$. The function $\varepsilon(t)$ determines the *shape* of the space-time hole. We choose the following normed mollifier: $\varepsilon : [0, \infty) \to [\delta, \rho + \delta]$ (ρ is fixed: $\rho \gg \delta > 0$):

$$\varepsilon(t) := \rho\,\overline{\Omega}_\tau(t) + \delta . \quad (14)$$

With one spatial coordinate suppressed, the hole is then like the solid of revolution of the normed mollifier of Figure 1 around the X-axis (time axis), with tiny parts sliced off near the end-points along a parallel of latitude of radius δ; the hole is looking like a glass onion (Figure 2). The shape of the 3+1-dimensional hole (12) is then the 4-dimensional analogue of Figure 2.

Concerning the *size* of the hole, we must of course always choose τ and ρ such that:

$$R_{\text{univ}}(t) \approx R_{\text{univ}}(t+\tau) \quad \text{and} \quad \rho \ll R_{\text{univ}}(t) . \tag{15}$$

Thus the shaking of points can happen all during a split second (one second splitted into, say, one thousand parts; $\tau := 1$ ms), inside a spatial ball of the maximum size of a pin-head (say, $\rho := 1$ mm). Relative to the length and time scales typical for GTR, and in particular relative to the estimated age and radius of the Robertson-Walker universe we seem to inhabit (about 10^{23} ms and 10^{28} mm, respectively), this is an extremely tiny hole indeed.

3.2 Explication of the hole diffeomorphism

The following local coordinate transformation shakes points inside the space-time hole H (12) all together now. We define:

$$T_4 : (\,ct,\, r,\, \varphi,\, \theta) \;\mapsto\; (\,ct',\, r',\, \varphi',\, \theta') \,, \tag{16}$$

where the domains are as in (9), as follows:

$$\begin{aligned} t'(t) &:= t\,, & (17)\\ r'(r) &:= r\,, & (18)\\ \varphi'(t,r,\varphi) &:= \varphi + N\,\Omega_\tau(t)\,\Omega_{\varepsilon(t)}(r)\,, & (19)\\ \theta'(t,\theta) &:= \theta - \overline{\Omega}_\tau(t)\left[1 - J_{1/2}(\theta)\right]\theta\,. & (20) \end{aligned}$$

The global hole diffeomorphism $\psi_H : \mathcal{M}_{\text{RW}} \to \mathcal{M}_{\text{RW}}$ obtains by taking hole (12) and coordinate transformation T_4 (16) in def. (7). We emphasize that a non-trivial hole diffeomorphism already obtains whenever we take for the latitude transformation (20) the identity too; and that more intricate hole diffeomorphisms obtain whenever we take for the time and radial transformations not the identity. We explicate and illustrate the coordinate transformation T_4 (16) and establish that T_4 (16) is hole diffeomorphism in a separate Section (§3.3).

Longitudinal transformation. The normalizing factor $N > 0$ is fixed by choosing some $\varphi_\circ \in (0,1)$ such that:

$$N := \varphi_\circ \exp[4/c^2\tau^2 + 4/(\rho+\delta)^2]\,; \tag{21}$$

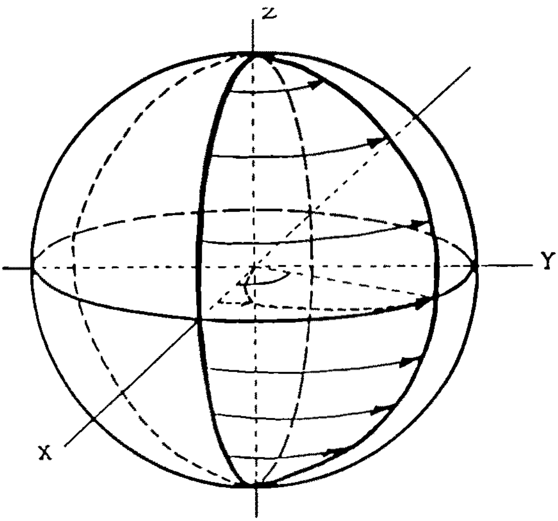

Figure 3. Longitudinal transformation (19) on a sphere $S^2(r)$.

then φ_o is the maximum angle of longitudinal rotation. For the result of $\varphi' = \varphi'(t, r, \varphi)$ (19) is that parallels of longitude are mapped onto parallells of longitude (see Figure 3). In a given ball $B^3(r)$ of radius $r \in (0, \varepsilon(t))$ at a given time $t \in (0, \tau)$, the longitudinal rotation is maximal at $r = \varepsilon(t)/2$ and vanishes smoothly as $r \downarrow 0$ and as $r \uparrow \varepsilon(t)$ (see Figure 4). The factor $\Omega_\tau(t)$ guarantees that the entire second term in (19) vanishes smoothly outside the hole ($t \geq \tau$); without this factor, T_4 (16) would keep on stirring all points inside a hypercylinder around the time-axis of radius $\delta > 0$ across the universe. The smoothness of $\varphi(t, r, \varphi)$ (19) is manifest; some care must however be exercised in addressing the smoothness in t of $\varphi(t, r, \varphi)$. For in addressing that case, r and $\varepsilon(t)$ exchange their role as variable and parameter in the mollifier $\Omega_{\varepsilon(t)}(r)$, which results in a smooth function of unbounded support. Specifically, the resulting function is an instance of $f_a : \mathbb{R} \to [0, 1)$, defined as:

$$f_a(x) := 1_{(a,\infty)}(x) \exp\left[\frac{1}{a(a-x)}\right] , \quad \text{for } a > 0 , \tag{22}$$

and $f_a(x) := 0$ for $a \leq 0$. The graph of f_1 crawls towards the horizontal asymptote of ordinate 1 as $x \to \infty$. The domain of f_a is in our case more restricted: $(\delta, \delta + \rho]$ instead of $\mathbb{R}$, due to the identifications $a = r$ and $x = \varepsilon(t)$. The smoothness of $\varepsilon(t)$ is a necessity for the smoothness of f_a and consequently for the smoothness of T_4.

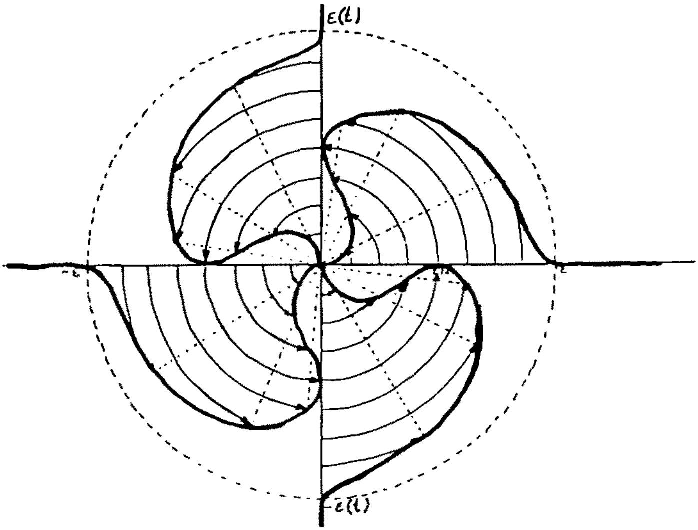

Figure 4. Horizontal cross-section of the ball $B^3(\varepsilon(t))$ of Figure 3

Now we answer a question we posed above: for what reason have we not chosen $\delta = 0$ in def. (14), so that the glass onion hole starts to grow smoothly from a point and shrinks back to a point again? The reason is that in such a scenario the support $(0, \varepsilon(t))$ of the mollifier in (16) vanishes for $t \downarrow 0$ and for $t \uparrow \tau$, which necessitates the existence and vanishing of the limit

$$\lim_{b \downarrow 0} \Omega_b(r) , \tag{23}$$

which equals

$$1_{(0,0)}(r) \exp[1/r^2] , \tag{24}$$

which does vanish for all $r > 0$ by virtue of $(0,0) = \emptyset$, but blows up in $r = 0$; the limit $r \downarrow 0$ for (24) leads to a mathematical infidelity: $0 \cdot \infty$. Hence a vanishing support ($\delta = 0$) for this mollifier is detrimental to the smoothness of T_4 and needs to be prevented at all costs.

Latitudinal transformation. We should choose a similar transformation for the parallels of latitude (for θ, with fixed t, r and φ), as (19) for the parallels of longitude (for φ), not a second time, for it invites the danger of some parallel of latitude (a non-denumerable set) being mapped on the South pole (a singleton set), which would destroy the bijectivity of T_4. Transformation $\theta' = \theta'(t, \theta)$ (20), with domain $[0, \infty) \times [0, 1/2]$, averts this danger. We start by taking the difference

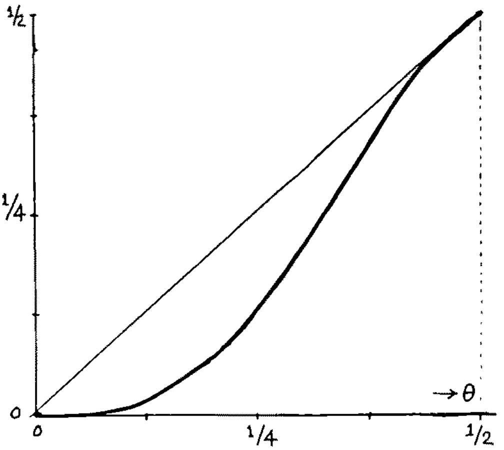

Figure 5. Graph of the function $J_{1/2}(\theta)\,\theta$

between the identity $I(\theta) = \theta$ and the function $J_{1/2}(\theta)\theta$, which is positive in the interval $(0, 1/2)$. See Figure 5. Whenever we substract this positive difference from the identity again, we evidently retrieve the function we started with:

$$J_{1/2}(\theta)\theta \;=\; I(\theta) - \left[I(\theta) - J_{1/2}(\theta)\theta\right] , \tag{25}$$

which is the latitudinal transformation (20) for $t = \tau/2$. For all other values of $t \in (0, \tau)$, the term substracted from the identity in (20) obtains a positive factor smaller than 1 (by virtue of the normed mollifier $\overline{\Omega}_\tau(t)$), which vanishes smoothly as $t \uparrow \tau$ and as $t \downarrow 0$.[1] Notice that (20) maps the North and South pole on themselves at any time: $\theta'(t, 0) = 0$ and $\theta'(t, 1/2) = 1/2$, respectively, for all $t \in [0, \infty)$.

3.3 Proof of the hole diffeormorphism

To prove that the coordinate transformation T_4 (16) is a diffeomorphism, we have to prove that it is smooth, invertible and that its inverse is smooth too. The smoothness of T_4 is manifest, since it consists of sums, products and compositions of manifestly smooth functions; and the space of smooth functions is closed under these operations (see Section 2). We address (a) the bijectivity of T_4 and then (b) the smoothness of its inverse.

(a) *Bijectivity.* We focus on the coordinate transformation formulae one by one.

Latitudinal transformation. The latitudinal transformation $\theta'(t,\theta)$ (20) only depends on t and θ. Given t' and θ', is then θ in (ct, r, φ, θ) fixed? Since the time transformation $t'(t)$ is the identity, t' alone fixes t trivially. Thus do t and θ' fix θ? They do, because the graph of $\theta'(t,\theta)$ as a function of θ, for any given t, is like the graph of $J_{1/2}(\theta)\theta$ (Figure 5), but lies closer to the identity, and coincides with the identity at $t = 0$ and $t \geq \tau$. We can work it out rigorously that $\theta'(t,\theta)$ is monotonically increasing on $[0, 1/2]$, by showing the positivity of its relevant partial derivative:

$$\begin{aligned}\partial_\theta\, \theta'(t,\theta) &= \left[1 - \overline{\Omega}_\tau(t)\right] + \\ &\quad \overline{\Omega}_\tau(t)\left[A_{1/2}^{-1}\,\Omega_{1/2}(\theta)\,\theta + J_{1/2}(\theta)\right] \;>\; 0\,. \qquad (26)\end{aligned}$$

Ineq. (26) holds manifestly for all $\theta \in [0, 1/2]$, for each fixed $t \in [0, \infty)$.

Longitudinal transformation. Since the time, the radial and the latitude transformations are all bijective functions, any given quadruple of coordinates $(ct', r', \varphi', \theta')$ immediately yields unambiguously ct, r and θ. Whenever the longitudinal coordinate φ' is given, the transformation formula $\varphi'(t, r, \varphi)$ (19) constitutes a linear equation in φ. Linear equations have a unique solution:

$$\varphi = \varphi' - N\,\Omega_\tau(t')\,\Omega_{\epsilon(t')}(r')\,. \qquad (27)$$

So there is a bijective correspondence between the quadruples (ct, r, φ, θ) and $(ct', r', \varphi', \theta')$ by means of T_4 (16); hence T_4^{inv} exists.

(b) *Smoothness of the inverse.* Pencil and paper will convince the reader that no closed expressions for the inverse transformation can be squeezed out of the latitudinal coordinate transformation formula (20); whence it follows that the smoothness of T_4^{inv} cannot be verified by inspection. The boat of Elementary Real Analysis is going down with the dinghies. Help!

The yellow submarine of Real Analysis on Manifolds comes to the rescue, having the Inverse Mapping Theorem as one of its engines. (Section 2). An immediate Corollary of interest of the Inverse Mapping Theorem (6.7 in Boothby [8]) reads for our case: the smooth coordinate transformation T_4 (16) is a diffeomorphism iff T_4 is bijective and has a non-singular Jacobian everywhere in $F[G] \subset \mathbb{R}^4$. The smoothness and bijectivity of T_4 we have just verified; hence all I've got to do is to verify that the determinant of the Jacobian matrix never vanishes. It won't be long.

For points outside the hole, T_4 is the identity and therefore trivially diffeomorphic; observe that for all these points $x \in F[G] \setminus F[H]$ the

Jacobian is the unit matrix, whose determinant equals 1. We next consider the points $x \in F[H]$, inside the hole; they have coordinates (ct, r, φ, θ) where r is restricted to $[0, \varepsilon(t))$ for each fixed $t \in [0, \tau)$. The Jacobian of T_4 (16) at (ct, r, φ, θ) is the following 4×4 matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \partial_t\varphi'(t,r,\varphi) & \partial_r\varphi'(t,r,\varphi) & 1 & 0 \\ \partial_t\theta'(t,\theta) & 0 & 0 & \partial_\theta\theta'(t,\theta) \end{pmatrix} . \tag{28}$$

By making two successive Laplacian expansions in minors and cofactors, we readily obtain:

$$\det \mathrm{J}\,(T_4;\ (t, r, \varphi, \theta)) \;=\; \partial_\theta\theta'(t,\theta)\ . \tag{29}$$

The positivity of (29) we have already verified in (a) addressing the bijectivity of T_4 — *vide* ineq. (26). Having thus established the nonsingularity of the Jacobian matrix (28) everywhere, the mentioned Corollary of the Inverse Mapping Theorem allows us to infer from the smoothness and bijectivity of T_4 (16) that T_4 is a diffeomorphism.

We have reached the goal set in the Intro. Hello goodbye.

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NOTE

1. P.E. Vermaas (private communication; Utrecht TS, June 1995) suggested to look upon transformation (20) as the convex sum of the identity $I(\theta)$ and $J_{1/2}(\theta)I(\theta)$, such that one convex coefficient smoothly grows and vanishes in time at the expense of the other.

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