Journal of Statistical Physics, Vol. 87, Nos. 1/2, 1997

Invariant Manifolds Associated to Nonresonant Spectral Subspaces

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Received September 8, 1994; final September 30, 1996

We show that, if the linearization of a map at a fixed point leaves invariant a spectral subspace which satisfies certain nonresonance conditions, the map leaves invariant a smooth manifold tangent to this subspace. This manifold is as smooth as the map—when the smoothness is measured in appropriate scales—but is unique among C^L invariant manifolds, where L depends only on the spectrum of the linearization or on some more general smoothness classes that we detail. We show that if the nonresonance conditions are not satisfied, a smooth invariant manifold need not exist, and we also establish smooth dependence on parameters. We also discuss some applications of these invariant manifolds and briefly survey related work.

KEY WORDS: Invariant manifolds; linearizations; partial linearizations; asymptotic behavior; renormalization group; invariant foliations.

1. INTRODUCTION

Besides their intrinsic appeal, invariant manifold theorems are interesting in dynamics because they provide landmarks which organize the long-time behavior.

From this point of view, having more invariant manifolds is quite desirable, since it means having more tools for the analysis of a dynamical system. In particular, it is often the case that associated to invariant manifolds for operators acting on sections in the tangent bundle one can associate other invariant structures in the manifold itself. For example, one of the standard constructions of stable and unstable foliations[14] includes as one of the steps applying the stable manifold theorem to the operator f_* acting on vector fields by $[f_* v](x) = Df(f^{-1}(x))\, v(f^{-1}(x))$. The invariant manifolds constructed here will also lead to invariant structures on the

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manifold. Nevertheless they are not in general foliations, as shown in ref. 19 (these structures seem to have been considered first in ref. 31). We will discuss some of their uses later. For example, they play a role in rigidity theory. We will also show that they can be used to provide obstructions to regularity of attractive invariant circles with rational rotation number (such circles appear after Hopf bifurcations).

An important motivation for this paper is the study of renormalization group transformations. Even if a precise analytical definition of a renormalization group operator is fraught with technical difficulties (see, e.g., refs. 9 and 29), it is fruitful to study finite-dimensional maps that are caricatures of the real situation (see ref. 4, Chapter 3; ref. 23, Section 5, Appendix E). In that picture, different ways to approach the fixed point correspond to different properties of finite-size scale fluctuations. An example of physical properties that can be described by properties of the approach to the fixed point can be found in ref. 13. In ref. 23 it is argued that the nonresonant invariant manifolds constructed here correspond to the beta functions or renormalization group theory in the case that the latter are smooth (as is assumed in perturbative calculations). The renormalization group in dynamical systems—especially in the period-doubling case—is much better behaved than the renormalization group in statistical mechanics and in that case it is sometimes possible to write well-defined renormalization maps that are analytic in an appropriate space and which have a compact derivative (see, e.g., refs. 5, 6, and 35) (these motivations are why we choose to prove our results in the generality of Banach spaces).

If the system were linear, a very natural invariant set would be an spectral subspace. For a nonlinear system close to a fixed point—and hence approximable by the derivative at the fixed point—one can ask if there are analogues of the spectral subspaces of the derivative which are invariant for the full nonlinear system.

The classical theory of invariant manifolds establishes the existence of invariant manifolds associated with spectral subsets which are disks around the origin or complements of disks around the origin. Usually the manifold associated to $\{z \in \mathbb{C} \mid |z| \leqslant \rho < 1\}$ is called the strong stable manifold, that associated to $\{z \in \mathbb{C} \mid |z| < 1\}$ is called the stable manifold, those associated to sets of the form $\{z \in \mathbb{C} \mid |z| \leqslant 1\}$ are called center stable manifolds, and those associated to sets $\{z \in \mathbb{C} \mid |z| \leqslant \rho > 1\}$ are called pseudostable manifolds. We have used "is" or "are" on purpose to indicate whether uniqueness under local assumptions holds or not. We refer to Section 6 for other results which also include uniqueness under global assumptions.

On the other hand, for some finite-dimensional systems, one can apply the Sternberg linearization theorem and conclude that the system is equivalent to the linearization expressed in another smooth system of coordinates.

Since the spectral subspaces of the derivative are invariant under the linearized dynamics, their image under the change of variables that linearizes the map will be invariant under the full map. Hence, the nonlinear map may leave invariant some smooth manifolds that correspond to any spectral subspaces, in particular to some spectral subspace not considered in the classical invariant manifold theory.

Even if the above argument indicates that the classical theory of invariant manifolds is not as general as possible, the theory based on the Sternberg linearization theorem is not very satisfactory. For example, for infinite-dimensional systems the nonresonance conditions become harder to verify—or even false if the spectrum includes open sets. Even in finite dimensions, the conditions of Sternberg theorem are not C^1 open. Moreover, since the linearizing changes of variables provided by the Sternberg linearization theorem are highly nonunique, it seems that the smooth invariant manifolds produced this way are also not unique.

In this paper we try to obtain a compromise between the Sternberg linearization theorem and the classical invariant manifold theory. The proofs will start as in the Sternberg linearization theorem, using nonresonance assumptions to eliminate undesired terms, but we will switch as soon as possible to the—much easier than linearization—invariant manifold theory.

This will allow us to prove some invariant manifold theorems for spectral subspaces satisfying some mild nonresonance conditions that persist for open sets of problems. Moreover, there will be local conditions that guarantee uniqueness. Once this uniqueness is established, it makes sense to study the dependence on parameters. We show that indeed these manifolds depend smoothly on parameters and compute explicit formulas for the derivatives. These formulas can be used to justify some perturbative calculations of beta functions in renormalization group theory at least in the caricatures where all the operators are well-defined and well-behaved maps.

We will also provide examples that show that if those nonresonance conditions are not met, the conclusions are false.

The ideas presented above are closely related to partial normal forms, that is, showing certain terms are irrelevant for the dynamics since they can be eliminated just by switching to appropriate systems of coordinates. From the renormalization group point of view the discussion of when a term cannot be eliminated is quite interesting. There is a class of theorems that state that if there are maps that agree at the origin to a high enough order, then there is a differentiable mapping that sends one into the other. One can apply these theorems to systems after applying the finite normal form calculations to conclude that there is a true differentiable transformation that maps one into another.

Note that if we express the dynamics in a system of coordinates where some terms in the map are not present, we can sometimes obtain uniform estimates about all the iterates of the map in this system of coordinates. Then these estimates can be read off in the original system. This type of argument is often used in dynamical systems to obtain very uniform control of iterates. It is sometimes also used in renormalization group arguments as a way to control the approach of surfaces defining an interesting phenomenon to critical surfaces.

In Section 5 we just quote some results on these problems of partial linearizations. We point out that, compared to the method of graph transforms developed in Section 3, they have the advantage that they can produce invariant manifolds whose spectral subset straddles the unit circle. Nevertheless, in the case that the spectral subset straddles the unit circle, the invariant manifolds produced by the partial linearization method are much less differentiable than the map. When, as in the case considered in this paper, the spectral subsets are inside the unit circle the partial linearization method produces manifolds that are almost as differentiable as the map (e.g., Theorem 5.1 in ref. 2 produces $C^{r-1+\text{Lipschitz}}$ manifolds for C^r maps, but this can be improved). The partial linearization method seems to require that the maps are invertible. Even if the partial linearization method yields information for a whole neighborhood, it is much more difficult to implement numerically than the graph transform methods described in this paper.

Finally, in Section 6 we review briefly other work on invariant manifolds other than the classical stable, strong stable, etc., and in Section 7 we sketch some applications of the results presented here.

2. NOTATION AND STATEMENT OF THE MAIN RESULTS

In this paper X will be a Banach space (not necessarily separable) over the reals or over the complex. f will be a C^r, $1 \leqslant r \leqslant \infty$, ω, mapping from X to itself such that it has a fixed point, which we will place at the origin [i.e., $f(0)=0$]. We will call $Df(0)=A$ and write $f(x)=Ax+N(x)$.

The question we will address is the existence of invariant manifolds passing though the fixed point, that is, submanifolds W of X such that $f(W)=W$, $0 \in W$.

In the cases that we will consider the problem is equivalent to a local version, so that we just need to assume f is defined in a small neighborhood of the origin.

The guiding idea behind the results presented here is that, on small scales, f is a perturbation of A and the invariant manifolds of f should be very similar to those invariant under A.

The most natural invariant manifolds for a linear operator (they are not the only ones!) are invariant subspaces associated to spectral projections. If X is a complex space, to each closed subset of the spectrum bounded away from the rest of the spectrum we can associate a spectral projection whose range is a closed subspace. (If X is real, the subspaces are associated to subsets of the spectrum as above which are also invariant under complex conjugation).[33,21]

The invariant manifolds we will construct are perturbations of these spaces. As is often the case with perturbation theory, it is very advantageous to try to remove some terms with an appropriate change of variables. Notice that if we could eliminate all of them as in the Sternberg linearization theorem, the result would follow: the invariant manifolds would be the images under the linearizing map of the invariant subspaces. The problem with applying the Sternberg linearization theorem in infinite-dimensional Banach spaces is that the nonresonance conditions fail in open sets of maps. (The spectrum of the linearization may have nonempty interior.)

One can observe that the best known results on the existence of invariant manifolds, the strong stable, stable, center stable, center, center unstable pseudostable manifolds,[15,22,10,16–18,37] amount to considering subsets of the spectrum obtained by intersecting it with disks, complements of disks, or circles.

We will be able to generalize those results, in that we will be able to consider more general sets. (At the end of the paper we will list other results that go in this direction.)

We will not be able to associate invariant manifolds to all the subsets of the spectrum of A, but will have to impose "nonresonance" conditions so that some eliminations can be performed. We will, furthermore, show that if these nonresonance conditions fail, there are counterexamples to the theorems considered here (or slightly stronger versions).

The technique we will use is the "graph transform" coupled with some manipulations standard in "normal form" theory. These manipulations, even if they simplify the proof, are not really necessary and it is possible to construct a proof without using them (we will give a sketch of these alternative methods).

Since ref. 25 (parts of which are reproduced in ref. 28) contains not only an excellent exposition of the graph transform method for invariant manifolds, but also an exposition of normal forms, we refer the reader there for some basic results.

Notation. Given a splitting $X = E^S \oplus E^U$, we will denote by Π_S, Π_U the corresponding projection.

Given a function $N: X \mapsto X$, we will define N_S (resp. N_U) by $N_{S,U}(\Pi_S x, \Pi_U x) = \Pi_{S,U} N(x)$.

If the splitting is invariant under a linear operator A, we will call A_S, A_U the restrictions of A to those subspaces. This is slightly inconsistent with the previous convention, but is also customary and will not lead to confusion.

Finally, we will call $B^S(l)$ the ball of radius l around 0 in the space E^S, and analogously for all spaces. If some of the indexes are clear from the context, we will omit them.

Our main theorem is as follows:

Theorem 2.1. Let X be a real or complex Banach space, f a C^r mapping, $r \in \mathbb{N} \cup \{\infty, \omega\}$, $r \geqslant 1$, $f(0) = 0$, $Df(0) = A$. Assume that there is a decomposition of X into closed subspaces $X = E^S \oplus E^U$ with bounded projections Π^S, Π^U, respectively, such that:

(i) The splitting is invariant under A (use the notation A_S, A_U for the restrictions).

(ii) $\sigma(A_S)$ bounded away from $\sigma(A_U)$.

(iii) $\sigma(A_U)$ bounded away from zero.

(iv) $\sigma(A_S)$ contained strictly inside the unit disk.

We will call L an integer big enough so that

$$(\sup\{|t| \mid t \in \sigma(A_S)\})^L (\sup\{|t|^{-1} \mid t \in \sigma(A_U)\}) < 1$$

and we will assume that

(v) $L < r$.

(vi) If $1 < i < L$, then $[\sigma(A_S)]^i \cap \sigma(A_U) = \varnothing$. Where $[\sigma(A_S)]^i = \{x_1 \cdot x_2 \cdot \cdots \cdot x_i \mid x_j \in \sigma(A_S),\ 1 \leqslant j \leqslant i\}$.

(vii) The function $N(x) = f(x) - A(x)$ has sufficiently small C^{L+1} norm when restricted to a ball of radius 1 around 0. (The smallness conditions depend only on the spectrum of A.)

Then there is a function $w: B^S(1) \to U$ such that:

(a) The graph of w is invariant under f.

(b) w is $C^{r-1+\text{Lipschitz}}$.

Moreover, w is unique among the C^L functions satisfying (a).

Remark. Given a function $f: X \mapsto X$, $f(0) = 0$, and $Df(0)$ satisfying the assumptions (i)–(vii), $f_\lambda(x) \equiv (1/\lambda) f(\lambda x)$ will satisfy the smallness hypothesis for λ small enough. Indeed, recall that the smallness conditions only depend on $Df_\lambda(0)$, which does not depend on λ. On the other hand,

N_λ gets smaller in the C^r sense as λ gets small. If w_λ is the function whose graph is invariant under f_λ, $w \equiv \lambda w(x/\lambda)$ will have a graph invariant under f. Note that $f_\lambda = A + N_\lambda$ and that N_λ converges to 0 in C^r in the ball as λ tends to zero. Therefore, assuming smallness conditions in N and considering only a small neighborhood are equivalent.

Remark. We call attention to the fact that even though we assumed in (iii) that A_U is invertible, we are not assuming that A_S is invertible. In particular, for compact operators, we can assume that S consists of the most stable eigenvalues and some others (provided, of course, that the nonresonance conditions are met).

Remark. Note that in (iv) we only require that the nonresonance condition holds for $i \geqslant 2$. In particular, S could intersect with U. This arises if we take E^S to be a subset of the full spectral subspace. For example, in finite dimensions we could take S to correspond to an eigenvalue admitting several linearly independent eigenvectors and take E^S the space corresponding to just one of the eigenvalues. We only need to assume the nonresonance conditions for order 2 or higher because we only need to eliminate quadratic and higher order terms. If S and U are disjoint, and hence a finite distance apart, E^S and E^U are the full spectral subspace. We will refer to E^S and E^U as the spectral subspaces associated to S and U even if they may not be the full spectral subspaces. When we discuss smooth dependence on parameters we will require that S, U are disjointed, to be able to establish smooth dependence of the spaces.

Remark. It will be important later that the smallness conditions we impose on N are only in C^L and not in C^r. If we want to consider C^∞ f's, we will have to do a different proof for every finite r. It will be important that we can choose the same λ in all cases so that the function w corresponding to different r's will be defined in the same domain.

Remark. Condition (vi) will be referred to as the "nonresonance" condition. Its interpretation is obvious when the X is finite dimensional; it just means that the products of any set of less than L eigenvalues of A_S are not eigenvalues of A_U.

Remark. We will derive later stronger uniqueness properties than those claimed in the theorem. In particular, Theorem 4.2 implies that the solution is unique among $C^{r_0+\varepsilon}$ with $r_0 = \log \|A_U^{-1}\| / \log \|A_S\|$. Note that $L = [r_0] + 1$.

Remark. Observe that if the spectral subset we consider is given by $\sigma(A) \cap \{z \mid |z| < \alpha\}$, $\alpha < 1$, Theorem 2.1 reduces to the α-stable manifold theorem characterized as the set of points x such that $\alpha^{-n} f^n(x)$ remains bounded. In this case, the nonresonance conditions are obviously satisfied.

Remark. Note that by the definition of L, $[\sigma(A_S)]^i \cap \sigma(A_U) = \varnothing$ when $i \geqslant L$. Hence, with (vi), the intersection is empty for all $i \in \mathbb{N}$. From the presentation in the text, it is clear that the condition holds in a C^1 open set of mappings.

Remark. The conclusion (b) of Theorem 2.1 can be improved to C^r. See the remarks after the proof for details on the finite-dimensional case. C. Pugh pointed out that adapting the considerations in ref. 14, such a result is also true in infinite dimensions. Nevertheless, we will not have space to discuss it in detail in this paper.

There are many equivalent norms we can use in X. Given an operator A, we can choose a norm in such a way that $\|A\| \leqslant \sup\{|t| \mid t \in \sigma(A)\} + \varepsilon$ for any $\varepsilon > 0$. Moreover, we could also choose the norm in such a way that the splittings associated to a finite number of closed subsets of the spectrum have projections of norm 1. We will henceforth assume such a norm, with a sufficiently small ε, has been defined so that the L introduced in assumption (iv) of the theorem also satisfies

$$\|A_S\|^L \, \|A_U^{-1}\| < 1$$

Note that all the smallness assumptions, etc., are to be understood in this norm. Since it is equivalent to the original one, all "sufficiently small" requirements in this norm are implied by "sufficiently small" in the original one.

Theorem 2.1 will be derived from the following theorem, which we will prove first:

Theorem 2.2. In the same setup as Theorem 2.1, do not assume (iv), but assume instead

(iv′) $N_U(s, u) = 0\,(\|u\|^{L+1}, \|s\|^2)$ near 0.

Then, the same conclusions as in Theorem 2.1 hold, but we moreover have

(c) $\|w(s)\| = \mathcal{O}(\|s\|^{L+1})$.

Note that, under the conditions of Theorem 2.2, denoting by (s, u) the projections on the stable and unstable component, we have

$$f(s, u) = (A_S s + N_S(s, u), A_U u) + \mathcal{O}(\|u\|^{L+1}, \|s\|^2) \tag{2.1}$$

Observe that if we ignore the high-order terms, the set corresponding to points with $u = 0$ is invariant under the dynamics. In a neighborhood of this set the terms ignored are a very small perturbation and we will be able to construct the invariant manifold as a perturbation of the set $\{(s, 0)\}$.

To simplify the arguments, we have stated the results only for integer regularities. The same argument applies to fractional regularities.

Although we stated Theorem 2.1 and Theorem 2.2 for maps, there are corresponding statements for flows whose proof follows from the statement for maps.

If V is a C^r vector field defined in a neighborhood of 0 and such that $V(0)=0$, it is possible to define a local flow F_t for $|t| \leqslant \alpha$ defined in a neighborhood of 0. Since $DV(0)$ is a bounded operator and $DF_t(0) = \exp(tDV(0))$, we have

$$\operatorname{Spec}(DF_t(0)) = \exp(t \operatorname{Spec} DV(0)) \tag{2.2}$$

and the spectral subspaces for $DV(0)$ are spectral subspaces for $DF_t(0)$. Hence, if $S \subset \operatorname{Spec}(DV(0))$ is contained in $\{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}$, it admits the spectral subspace E^S and it satisfies, for $1 < j \leqslant L$,

$$s_1,..., s_j \in S \Rightarrow s_1 + \cdots + s_j \notin \operatorname{Spec}(DV(0))$$

[where L is such that $L \max_{z \in S} \operatorname{Re}(z) < \min_{z \in \operatorname{Spec}(DV(0))} \operatorname{Re}(z)$]. Then $\exp(tS)$ satisfies the hypothesis of Theorem 2.1 for F_t. Since the spectral subspaces are the same, observing that a manifold invariant for $F_{t/n}$ is also invariant for F_t, $|t|$, $|t/n| \leqslant t_0$, $n \in \mathbb{Z}$, using the uniqueness statements of Theorem 2.1, we can conclude that the invariant manifolds produced by Theorem 2.1 for F_t and $F_{(t/n)}$ are the same whenever $|t|$, $|t/n| \leqslant t_0$, $n \in \mathbb{N}$. Hence, those produced for F_t and $F_{t(m/n)}$, $m, n \in \mathbb{Z}$, are the same. By continuity of F_t, we conclude that the manifolds are the same for all $|t| \leqslant t_0$. Hence the manifold is invariant under the whole flow.

The above argument generalizes without change to the situation where F_t is a smooth semigroup generated by an unbounded vector field satisfying (2.2). This situation arises very frequently with regard to partial differential equations. For unbounded operators, (2.2) may fail, but it holds for several classes of unbounded operators of interest in connection with PDEs (e.g., normal).

3. PROOF OF THEOREM 3.1 AND THEOREM 3.2

As in ref. 25, the proof will consist in writing the invariant manifold as the graph of a function $w: B^S(1) \to E^U$. The fact that the manifold is invariant will be equivalent to the fact that the function w is a fixed point of an operator $\mathcal{T}$ acting on an appropriate space of smooth functions. The existence of such a fixed point will be established using a variant of the contraction mapping principle.

First, we will prove the theorem when $r < \infty$. We will take $\mathscr{T}$ to be

$$\mathscr{T}[w](x) = A_U^{-1}(w(A_S x + N_S(x, w(x))) - N_U(x, w(x))) \tag{3.1}$$

This operator was used in ref. 26, not in ref. 25. It is not exactly the graph transform, but is somewhat more manageable. The reason we define the operator is that if we compute f on a point of the graph of the w, we have

$$f(x, w(x)) = (A_S x + N_S(x, w(x)), A_U w(x) + N_U(x, w(x))) \tag{3.2}$$

This transformed point belongs to the graph of w if and only if the second coordinate is the result of applying w to the first. That is,

$$w(A_S x + N_S(x, w(x))) = A_U w(x) + N_U(x, w(x))$$

which, provided that all the compositions can be defined, is equivalent to w being a fixed point of $\mathscr{T}$.

The operator $\mathscr{T}$ is well defined if $\|A_S\| + \|N_S\|_{L^\infty} < 1$. We will assume that N is small enough that indeed $\mathscr{T}$ is defined on functions on the unit ball of S.

We will consider $\mathscr{T}$ as acting on the following spaces:

$$\begin{aligned} \chi^r_{\varepsilon_1 \cdots \varepsilon_{r-L}} = \{w\colon B^S(1) \mapsto E^U \text{ such that} \\ &\text{(a)} \quad w \in C^r \\ &\text{(b)} \quad D^k w(0) = 0,\ 0 \leqslant k \leqslant L \\ &\text{(c)} \quad \sup_{x \in B^S(1)} \|D^k w(x)\| \leqslant 1,\ 0 \leqslant k \leqslant L \\ &\text{(d)} \quad \sup_{x \in B^S(1)} \|D^{L+i} w(x)\| \leqslant \varepsilon_i,\ 1 \leqslant i \leqslant r - L\} \end{aligned} \tag{3.3}$$

We call attention to the fact that these spaces are convex subsets of C^r. Note that, because of condition (b), the $w \in \chi$ is determined uniquely by $D^L w$. We will therefore consider χ endowed with the topology given by the norm

$$\|w\| = \|D^L w\|_{L^\infty(B^S(1))}$$

We also point out that by condition (b) we also have

$$\|D^i w\|_{L^\infty(B^S(1))} \leqslant (1/(L-i)!)\ \|D^L w\|_{L^\infty(B^S(1))}$$

for $0 \leqslant i \leqslant L$.

For ease of notation we will suppress the $B^S(1)$ from the spaces, but L^∞ is to be understood always to be on unit ball of E^S.

To apply a fixed-point theorem, our first task is to find a space that gets mapped into itself by $\mathscr{T}$.

Proposition 3.1. Given some smallness assumptions on $\|D^k N_S\|_{L^\infty}$, $0 \leqslant k \leqslant L$, it is possible to find $\varepsilon_1, \ldots, \varepsilon_{r-L} > 0$ in such a way that

$$\mathscr{T}(\chi^r_{\varepsilon_1, \ldots, \varepsilon_{r-L}}) \subset \chi^r_{\varepsilon_1, \ldots, \varepsilon_{r-L}}$$

These ε's can be found in a recursive consistent way: that is, if $\bar{r} > r$ and we have found $\varepsilon_1 \cdots \varepsilon_{r-L}$, we can find $\varepsilon_{r-L+1} \cdots \varepsilon_{\bar{r}-L+1}$ so that

$$\mathscr{T}(\chi^{\bar{r}}_{\varepsilon_1, \ldots, \varepsilon_{r-L}, \varepsilon_{r-L+1}, \ldots, \varepsilon_{\bar{r}-L}}) \subset \chi^{\bar{r}}_{\varepsilon_1, \ldots, \varepsilon_{r-L}, \varepsilon_{r-L+1}, \ldots, \varepsilon_{\bar{r}-L}}$$

Proof. The fact that $\mathscr{T}[w]$ satisfies (a), (b) if w does is quite easy. Since for functions satisfying (a), (b) we have for $k < L$

$$\|D^k w(x)\| \leqslant \sup_{y \in B^S(\|x\|)} \|D^L w(y)\| \frac{\|x\|^{L-k}}{(L-k)!}$$

to check (c), it suffices to check it for $k = L$.

Taking L (recall $L \geqslant 2$) derivatives of (4.1), we obtain

$$\begin{aligned} D^L \mathscr{T}[w](x) = {} & A_U^{-1} D^L w(A_S x + N_S(x, w(x)))[A_S + D_1 N_S(x, w(x)) \\ & + D_2 N_S(x, w(x))\, Dw(x)]^{\otimes L} \\ & + A_U^{-1} Dw(A_S x + N_S(x, w(x)))\, D_2 N_S(x, w(x))\, D^L w(x) \\ & - D_2 N_U(x, w(x))\, D^L w(x) \\ & + R \end{aligned} \tag{3.4}$$

where the remainder R is a polynomial all of whose terms contain a factor which is a derivative of N and derivatives of w up to order $L-1$.

Formula (3.4) can be established by observing that

$$\begin{aligned} D\mathscr{T}[w](x) = {} & A_U^{-1} Dw(A_S x + N_S(x, w(x)))[A_S + D_1 N_S(x, w(x)) \\ & + D_2 N_S(x, w(x))\, Dw(x)] \\ & + A_U^{-1} Dw(A_S x + N_S(x, w(x)))\, D_2 N_S(x, w(x))\, Dw(x) \\ & - D_2 N_U(x, w(x))\, Dw(x) \end{aligned} \tag{3.5}$$

If we compute higher derivatives using the chain rule and the product rule, we see that the only way to produce terms involving L derivatives in $D^L\mathscr{T}[w](x)$ is to take derivatives in the factors Dw in (3.5).

Since all the derivatives of w of order up to L are bounded by 1, we can make this remainder as small as we please by making smallness assumptions on $\|N\|_{C^L}$.

The first term of (3.4) can be bounded by $\|A_U^{-1}\| \, (\|A_S\|+2\,\|N\|_{C^1})^L$ and, by the assumption (iv), this can be made strictly smaller than 1 by assuming smallness conditions in $\|N\|_{C^1}$. The other terms can be bounded by $2\,\|A_U^{-1}\|\,\|N\|_{C^1}$, which clearly can be made as small as desired by assuming that $\|N\|_{C^1}$ is sufficiently small.

In order to adjust conditions (d), we will not make further smallness assumptions for N, but rather adjust the ε's. This is the recursion alluded to above that allows us to determine the ε's.

Taking $L+i$ derivatives, we have, in a way similar to (3.4),

$$\begin{aligned} D^{L+i}\mathscr{T}[w](x) = A_U^{-1} & Dw(A_S x + N_S(x, w(x)))[A_S + D_1 N_S(x, w(x)) \\ & + D_2 N_S(x, w(x))\, Dw(x)]^{\otimes(L+i)} \\ & + A_U^{-1} Dw(A_S x + N_S(x, w(x)))\, D_2 N_S(x, w(x))\, D^{L+1} w(x) \\ & - D_2 N_U(x, w(x))\, D^{L+i} w(x) \\ & + R \end{aligned} \tag{3.6}$$

where R, again, is a polynomial expression that involves only derivatives of w up to order $L+i-1$ and derivatives of N. Hence, if N is fixed and w satisfies the inequalities in (d) up to $i-1$, we can bound

$$\begin{aligned} \|D^{L+i}\mathscr{T}[w](x)\| \leqslant [\, & \|A_U^{-1}\|(\|A_S\|+2\,\|N\|_{C^1})^{L+i} \\ & + (\|A_U^{-1}\|+1)\,\|N\|_{C^1}]\,\varepsilon_i + R(\varepsilon_{i-1}\cdots\varepsilon_1) \end{aligned} \tag{3.7}$$

Under the assumption that $\|N\|_{C^1}$ is small enough, we can ensure that for all $i \in \mathbb{N}$,

$$[\,\|A_U^{-1}\|(\|A_S\|+2\,\|N\|_{C^1})^{L+i} + (\|A_U^{-1}\|+1)\,\|N\|_{C^1}] \leqslant \delta < 1$$

In such a case, the factor multiplying ε_i on the right-hand side of (3.7) is smaller than 1; assuming we have already found $\varepsilon_{i-1}\cdots\varepsilon_1$, we can choose ε_i big enough so that the right-hand side of the previous inequality is smaller than ε_i. ∎

Remark. We emphasize that the choices of ε_i are always possible provided that $\|N\|_{C^1}$ is sufficiently small. The smallness assumptions on

$\|N\|_{C^L}$ are just to adjust that the derivatives of w to order L are less than 1. We also call attention to the fact that ε_i only depends on $\|D^jN\|_{L^\infty}$, $1 \leqslant j \leqslant L+i$, and that it can be made arbitrarily small by assuming that $\|D^jN\|_{L^\infty}$, $L+1 \leqslant j \leqslant L+i$, are sufficiently small.

The following propositions, consequences of more general results whose proofs can be found in the references indicated, will allow us to study fixed points in χ by a small modification of the contraction mapping theorem.

We start by recalling Lemma 2.5 in ref. 25.

Proposition 3.2. If we give the χ's defined before the topology of the distance

$$d(w, w') = \sup_{x \in B^S(1)} \|D^L w(x) - D^L w'(x)\|$$

then the sequence closure of $\chi^r_{\varepsilon_1, \ldots, \varepsilon_{r-L}}$ is contained in

$\{w\colon B^S(1) \mapsto U$, such that

(a) $w \in C^{r-1+\text{Lipschitz}}$

(b) $D^k w(0) = 0,\ 0 \leqslant k \leqslant L$

(c) $\sup_{x \in B^S(1)} \|D^k w(x)\| \leqslant 1$

(d) $\sup_{x \in B^S(1)} \|D^{L+i} w(x)\| \leqslant \varepsilon_i,\ 1 \leqslant i \leqslant r-L-1$

(e) $\mathrm{Lip}(D^r w) \leqslant \varepsilon_{r-L}\}$

Proof. This is a particular case of Lemma 2.5 of ref. 25. We refer to that paper for the astute proof. We call attention to two subtle points of the proof:

1. We are not considering the closure of the set χ, only the closure under sequences. In a nonseparable Banach space they could be different. Looking ahead to the argument, we note that for the contraction mapping principle what matters is that the sets that get mapped into themselves are sequentially closed.

2. Even if this looks very similar to the Ascoli–Arzelá theorem, it is not necessary to assume that the space has a countable base.

(Of course, the proof can be greatly simplified if the space X is assumed to be separable and, if the space is finite dimensional, the proof reduces to the Ascoli–Arzelá theorem.) ∎

Our next task is to prove that the map $\mathscr{T}$ is a contraction in an appropriate norm. We will do this by showing it is differentiable and then estimate the norm of the derivative.

The fact that $\mathscr{T}$ is differentiable can be proved using the same methods that ref. 16 used to show that the composition operator is differentiable in the appropriate spaces. We note that, as is well known, the composition is only differentiable in C^L on functions which are more differentiable. Since $\mathscr{T}$ is obtained by applying the composition operator twice, it is clear that, invoking ref. 16, we can obtain that $\mathscr{T}$ is differentiable in spaces of more differentiable functions. On the other hand, if, rather than quoting ref. 16, we just apply the same method there to our particular case, we can get by with using a derivative less. This is the content of the following proposition.

Proposition 3.3. The mapping $\mathscr{T}$ considered as a mapping from $C^L(E^S, E^U)$ to itself is differentiable in the Frechet sense at all the points of χ^r (recall that we assumed $r > L$ and both integers). Its differential is

$$\begin{aligned}[D\mathscr{T}(w)]\,\eta(x) = A_U^{-1}[&\eta(A_S x + N_S(x, w(x))) \\ &+ Dw(A_S x + N_S(x, w(x)))\, D_2 N_S(x, w(x))\, \eta(x) \\ &- D_2 N_U(x, w(x))\, \eta(x)]\end{aligned} \tag{3.8}$$

Proof. For every x we have the formula

$$\begin{aligned}\mathscr{T}[w+\eta](x) = \mathscr{T}[w](x) &+ \int_0^1 A_U^{-1}\eta(A_S x + N_S(x, (w+\lambda\eta)(x))) \\ &+ \int_0^1 A_U^{-1} Dw(A_S x + N_S(x, (w+\lambda\eta)(x))) \\ &\times \eta(x)\, D_2 N_S(x, (w+\lambda\eta)(x))\, \eta(x)\, d\lambda \\ &- \int_0^1 A_U^{-1} D_2 N_U(x, (w+\lambda\eta)(x))\, \eta(x)\, d\lambda\end{aligned} \tag{3.9}$$

The desired result follows from interpreting the above formula as a formula in Banach space of functions and estimating the continuity of the remainders. This is the same method used in ref. 25 and we refer to that paper for details. ▮

Remark. We note that the composition operator, considered as an operator in C^L, is differentiable in the Gateaux sense only at functions which are at least C^{L+1} and with a uniform continuity of the derivatives of high order. Other than that, it is only Frechet.

Remark. We also note that a similar but simpler argument can be used to prove that $\mathscr{T}$ is differentiable with respect to N if N is given the C^r norm and $\mathscr{T}(w, N)$ the C^L norm. Similarly, we note that assuming that w, N are given the C^{L+k+2} norm, it is possible to show that $\mathscr{T}$ is C^k with respect to parameters. The proof is again obtained by writing a formula for the derivatives and checking that by integration it gives the right answer. We will perform these arguments in detail in the section on dependence on parameters.

Proposition 3.4. Assume that N is C^r and that A satisfies the assumptions of Theorem 2.2. Denote by $\chi^r_{\varepsilon_1,\dots,\varepsilon_{r-L}}$ a set of the form (3.3) such that $\mathscr{T}(\chi^r_{\varepsilon_1,\dots,\varepsilon_{r-L}}) \subset \chi^r_{\varepsilon_1,\dots,\varepsilon_{r-L}}$. Assume that $\|N\|_{C^L}$ and that ε_1 are sufficiently small, where the smallness conditions depend only on the spectrum of the linearization, and in particular are independent of r. Then the derivative in (3.8) is a contraction in $\chi^r_{\varepsilon_1,\dots,\varepsilon_{r-L}}$ in the norm of the supremum of $D^L w$.

We call attention to the fact that if the sets χ^r are produced using Proposition 3.1, the hypotheses of Proposition 3.4 are implied by $\|N\|_{C^{L+1}}$ being sufficiently small (since, as argued in the proof of Proposition 3.1, when $\|N\|_{C^{L+1}}$ is small, we can take ε_1 sufficiently small). Again, we recall that the assumption that $\|N\|_{C^{L+1}}$ is sufficiently small amounts to considering a sufficiently small neighborhood.

In summary, if $\|N\|_{C^{L+1}}$ is sufficiently small, we can use Proposition 3.1 to obtain a set that $\mathscr{T}$ maps into itself. We can use Proposition 3.4 to conclude that $\mathscr{T}$ is a contraction restricted to it.

Remark. The fact that the smallness assumptions are independent of r will be crucial in the proof of the result for C^∞ manifolds. This independence is to be expected since we are considering a norm that only involves L derivatives.

Proof. Since the spaces χ^r are convex, we can use the formula $\mathscr{T}(u) - \mathscr{T}(v) = \int_0^1 D\mathscr{T}(v + s(u-v))(u-v)\,dt$, and hence, to prove that $\mathscr{T}$ is a contraction, it suffices to prove that the norm of the derivative is less than 1 [this formula is, of course, just (3.9)].

In the formula (3.8) for the differential of $\mathscr{T}$ take L derivatives with respect to x, and expand using the chain rule and the rules for sums and products (tensor products) of derivatives as often as possible. We get $D^L_x[(D_w\mathscr{T})[w]\,\eta](x)$ as a sum of terms. The only term in this sum that does not contain a derivative of N is

$$A_U^{-1} D^L\eta(A_S x + N_S(x + w(x)))\, A_S^{\otimes L} \tag{3.10}$$

All the other terms contain a factor which is a derivative of N of order not bigger than L. The factors other than derivatives of N of order up to L that appear in the other terms are either derivatives of η of order not bigger than L or derivatives of w of order not bigger than $L+1$.

The only terms that include a derivative of w of order $L+1$ are those resulting from expanding

$$\begin{aligned}&A_U^{-1}(D^{L+1}w)(A_S x+N_S(x, w(x)))\\&\quad\times[A_S+(D_1N_S(x, w(x))+D_2N_S(x, w(x))\,Dw(x))]^{\otimes L}\\&\quad\times D_2N_S(x, w(x))\,\eta(x))\end{aligned}\tag{3.11}$$

Therefore, except for (3.10) and (3.11), all the other terms involve derivatives of η of order not more than L, derivatives of w of order not more than L, and at least one factor which is a derivative of N of order not more than L.

The term (3.10) can be bounded by $\lambda\,\|D^L\eta\|_{L^\infty}$, where $0<\lambda\equiv\|A_S\|^L\,\|A_U^{-1}\|<1$ by assumption (iv). Hence, the linear operator defined in (3.10) has a norm which is strictly smaller than 1.

To bound (3.11), we note that the $L+1$ derivative of w is bounded by ε_1 and all the other derivatives of w are bounded by 1. Recalling that, since $D^i\eta(0)=0$ for $0\leqslant i\leqslant L-1$, we can bound the supremum of η by $1/L!\,\|D^L\eta\|_{L^\infty}$. Therefore, we can bound the norm of the linear operator in (3.11) by

$$(1/L!)\,\varepsilon_1\,\|A_U^{-1}\|(\|A_S\|+2\,\|DN\|_{L^\infty})\,\|DN\|_{L^\infty}$$

This, clearly, can be made arbitrarily small by assuming that $\|DN\|_{L^\infty}$ is sufficiently small.

Since all the other terms include at least a factor which is a derivative of N of order not bigger than L and derivatives of w of order not larger than L and derivatives of η up to order L, the norm of all the other terms can be bounded by $\rho\,\|\eta\|_{C^L}$, where ρ can be made as close to zero as desired by making $\|N\|_{C^L}$ sufficiently small. That is, we can estimate the norm of the derivative by a number which is strictly smaller than 1 and a finite number of other terms that can be made arbitrarily small by assuming that $\|N\|_{C^L}$ is sufficiently small. ∎

Since, using Proposition 3.1 and Proposition 3.4 under the conditions of Theorem 2.2 we have produced a set that gets mapped into itself by $\mathscr{T}$ and shown that $\mathscr{T}$ is a contraction on it, applying the contraction mapping theorem, we conclude that there is a unique fixed point in the sequence closure of χ^r. This finishes the proof of Theorem 2.2 except for the C^∞, C^ω

cases. (Notice that this method automatically produces the uniqueness statement claimed in the theorem.)

To prove the C^∞ result, the key observation (standard in the theory; see refs. 25, 37) is that all the fixed points in different χ^r have to agree.

We note that given a C^∞ N, Proposition 3.1 produces a sequence ε_i in such a way that all the sets $\chi^r \equiv \chi^r_{\varepsilon_1,\dots,\varepsilon_{r-L}}$ are mapped into themselves by $\mathscr{T}$. Proposition 3.4 shows that $\mathscr{T}$ is contractive on these sets. Hence, there is a fixed point in the sequence closure characterized by $w_r = \lim \mathscr{T}^n w$ for any $w \in \chi^r$. Since for $r' > r$, $\chi^{r'} \subset \chi^r$, we conclude that $w_r = w_{r'}$. Hence, the unique fixed point has to be in all the sequence closures of χ^r and, by Proposition 3.2, it has to be C^∞.

In Section 4 we develop some other uniqueness statements that will also show that the different fixed points in all the regularities have to agree. They provide an alternative route for this part of the proof of Theorem 2.2 for infinite regularity.

We emphasize that here we use essentially that S is contained in the unit circle. Indeed, there are examples in which the spectrum of A_S contains the unit circle and in which there is no C^∞ invariant manifold.

The proof of C^ω regularity is simpler. We have to consider $\mathscr{T}$ acting on a space of analytic functions vanishing up to order L at the origin with the C^L norm in a complex neighborhood of the space. The same argument used here shows it is a contraction (the properties of the absolute value are the same, be it real or complex) and the uniform limit of analytic functions is analytic. (We refer to ref. 21, Chapter 7, for the proof of these results in infinite dimensions. The finite-dimensional cases are quite well known.) ▮

Proposition 3.5. If S, S' are spectral subsets both of which satisfy the assumption of the theorem and $S \subset S'$, then $W^S \subset W^{S'}$.

Proof. If we consider the derivative at zero of $f|_{W^{S'}}$ we see that its spectrum is precisely S' and clearly S satisfies the nonresonance assumptions. Hence, we can find a C^r manifold associated to it in the restriction to $W^{S'}$.

Now, this manifold can be considered as a submanifold of X, and it fulfills the assumptions of the uniqueness theorem. So it is W^S, hence W^S is contained in $W^{S'}$.

Remark. When X is finite dimensional the conclusion (b) of Theorem 2.1 can be improved to C^r using the same proof. The idea for this improvement is to show that the sequence of functions $\{D^r\mathscr{T}^n[w](x)\}_{n=0}^{\infty}$ is equicontinuous and equibounded in n. This can be shown by introducing a modulus of continuity for $D^r w$ and showing that this modulus of continuity is preserved under $\mathscr{T}$. This is not very difficult, given the bounds that we have proved already, and a similar calculation is in ref. 25. Unfortunately, this argument does not work in infinite-dimensional spaces because uniform continuity on the unit ball does not follow from continuity. So another argument is needed. For the stable manifold case, C^r regularity even in infinite-dimensional Banach spaces is proved by a different argument in ref. 14. It seems that this argument can be adapted to our case, but since this borderline regularity in infinite-dimensional spaces seems specialized, we postpone for future work the discussion of this point. (We thank C. Pugh for bringing this point to our attention and providing a sketch of the proof.) We also note that the same circle of ideas can be used to prove the result without loss of differentiability when r is not an integer (C^r in that case means $[r]$ derivatives satisfying an $r-[r]$ Hölder condition). Since we have not included these in the statement of our theorems, we will not try to give a proof, but we remark that it requires only minor modifications from the proofs presented here.

The following proposition, whose proof is well known, will show that Theorem 2.1 follows from Theorem 2.2.

Proposition 3.6. Given a C^r function f satisfying assumptions (ii), (iii), (v), and (vi) of Theorem 2.1, there is a C^ω map ϕ with a C^ω local inverse such that

(i) $\phi(0)=0$.

(ii) $D\phi(0)=Id$.

(iii) $\phi^{-1}\circ f\circ\phi$ satisfies the assumptions of Theorem 2.2.

Remark. It is possible to take ϕ to be a polynomial.

Remark. We emphasize that Proposition 3.6 does not require assumption (iv) of Theorem 2.1. That is, we do not require that $\sigma(A_S)$ is contained inside the unit disk. This will become crucial when we discuss pseudostable nonresonant sets.

Proof. (See refs. 25, 34, 30, or 3, among many others, for very similar computations.)

We try to write $\phi=\phi^2\circ\cdots\circ\phi^L$, where each of the ϕ^i can be written as $\phi^i=Id+\phi^i_U$ and $\phi^i_U(x,y)$ only depends on the first argument and is multilinear of order i.

The implicit function theorem shows that ϕ^i has a local inverse $(\phi^i)^{-1}(x,y)=(x,y)-\phi^i_U(x)+\mathcal{O}(\|x\|^{i+1})$.

Our goal is to determine ϕ^i_i so that

$$\Pi_U(\phi^i)^{-1}\circ\cdots\circ(\phi^i)^{-1}\circ f\circ\phi^i\cdots\phi^1(x,y)=\Pi_U f(y)+\mathcal{O}(\|x\|^{i+1})$$

This can be achieved by finding ϕ^i_U recursively, under the inductive hypothesis that ϕ^l_U, $l<i$, are already known. Substituting in the equation in Proposition 3.6, we see that ϕ^i_U should satisfy an equation of the form

$$\phi^i_U(A_S(x)) - A_U\phi^i(x) = h^i(x)$$

where h^i is a multilinear function of degree i that can be computed out of the previously known ones.

These equations can be solved because the operator induced by A_S on the multilinear functions has spectrum contained in the ith set product of the spectrum of A_S.

In the case that X is finite dimensional and A diagonalizable one can choose a basis for multilinear functions as the monomials of degree i in the coordinates in a basis of eigenvectors. The result, however, can be proved easily even if A is not diagonalizable and X is infinite dimensional. See, e.g., ref. 30. ∎

Remark. We notice for future reference that the ϕ^i and the ϕ can be chosen in such a way that they depend in a C^m fashion on N, A_S, A_U.

Remark. It is not necessary to prove Theorem 2.1 using Theorem 2.2 and performing the normal form calculation that reduces one to the other. Alternatively, one could write

$$w(x) = \sum_{i=2}^{L} \frac{1}{i!} D^i w(0)\, x^{\otimes i} + w^{[>L]}(x)$$

and, taking derivatives in $\mathscr{T}[w] = w$ at zero, derive equations for the $D^i w(0)$. In effect, taking i derivatives, $i \geqslant 2$, as in (3.4) and evaluating at 0, we obtain

$$D^i w(0) = A_U^{-1} D^i w(0)\, A_s^{\otimes i} + R \tag{3.12}$$

where R is an expression that involves derivatives of w evaluated at 0 of order not larger than $i-1$ and of N of order not larger than i. Since the spectrum of $\gamma \to A_U^{-1}\gamma A^i_S$ is $S^i U^{-1}$ (see ref. 30), it does not contain 1 by our nonresonance assumptions. Hence (3.12) can be solved.

It is easy to see that if the $D^i w(0)$ solve these equations, then

$$\mathscr{T}\left[\sum^{L} \frac{1}{i!} D^i w(0)\, x^{\otimes i} + w^{[>L]}\right](x) = \sum^{L} \frac{1}{i!} D^i w(0)\, x^{\otimes i} + \tilde{\mathscr{T}}[w^{[>L]}](x)$$

The operator $\tilde{\mathscr{T}}$ can be studied by the same methods used to study $\mathscr{T}$ here, but the computations are much more cumbersome. Nevertheless, it seems that the dependence on parameter results could be improved.

Remark. We also point out that the method of calculation outlined in (3.12) is quite practical for numerical calculations. Also, it is related, as we will see later, so the perturbative calculations of β functions in renormalization group theory.

4. DEPENDENCE OF THE MANIFOLDS ON THE MAP AND UNIQUENESS RESULTS

We can think of Theorem 2.2 as defining a mapping that, given any N, produces w. Since our point of view was to think of these results as perturbations of $N \equiv 0$, it is quite natural to investigate the dependence of w on N. Our first result on dependence on parameters is as follows.

Theorem 4.1. Assume the conditions of Theorem 2.2 as well as $S \cap U = \Phi$ and smallness assumptions in $\|N\|_{C^r}$. If we give the w's the C^{L+k} norm and the N's the C^r norm, the mapping $N \to w(N)$ is $C^{r-(L+k+3)}$ provided $r > L+k+3$. An analogous result holds for the invariant manifolds constructed in Theorem 2.1.

Remark. It seems that the loss of differentiability we incur in this theorem is not optimal.

Proof. If we were going to prove that the mapping was differentiable, the most natural thing to do would be to write down explicitly the N dependence in $\mathscr{T}$ and apply the implicit function theorem to the functional equation $\mathscr{T}(w, N) = w$.

Unfortunately, this fails because in order to compute $D_w \mathscr{T}$, the first derivative of w enters [see (3.4)]. However, to prove differentiability, it is possible to follow the strategy of the proof of the implicit function theorem, observing that at the solutions, the mapping $\mathscr{T}$ is differentiable with respect to w and, moreover,

$$D\mathscr{T}\colon C^{L+k} \to C^{L+k}$$

is a contraction. (Recall that this smallness follows from C^{L+k+1} smallness assumptions in N and in w; the latter are implied by C^{L+k+2} smallness assumptions in N.) A very similar argument occurs in ref. 24.

For example, to establish that $w \mapsto w(N)$ is differentiable, we will use the differentiability of $\mathscr{T}$ in these spaces to construct a guess for $w'(N)$ in such a way that

$$\|\mathscr{T}(w(N) + w'(N)\,\Delta, N + \Delta) - w(N) - w'(N)\,\Delta\|_{C^r} \leqslant K\,\|\Delta\|^2_{C^{L+4}}$$

Then, by the fact that $\mathscr{T}$ is a contraction with a uniform constant, we conclude that there is a fixed point for $\mathscr{T}(\cdot, N)$ which differs from $w(N) + w'(N)\,\Delta$ by an amount not bigger than $K\,\|\Delta\|^2_{C^{L+4}}$. That is,

$$\|w(N+\Delta) - w(N) - w'(N)\,\Delta\|_{C^r}\, K\,\|\Delta\|^2_{C^{L+4}}$$

Proceeding formally, we conclude that the candidate for the derivative has to satisfy $D_w\mathscr{T}(w, N)\,w'(N) + D_N\mathscr{T} = w'(N)$. Hence,

$$w'(N) = (1 - D_w\mathscr{T})^{-1}\,D_N\mathscr{T} = \sum_k^{\infty}\,[D_w\mathscr{T}]^k\,D_N\mathscr{T}$$

when w is a solution. The convergence of the sum follows by repeating the argument that lead us to conclude that $D_w\mathscr{T}$ is a contraction in $C^L \to C^L$. Since the sum $\sum_k [D_w\mathscr{T}]^k$ converges, the map w' is well defined and is a continuous function.

Once the formal solution can be found we need to argue that it is a true derivative. The fact that $w'(N)$ is the true derivative comes from the fact that it is continuous and, if we integrate back,

$$\tilde{w}(N_\lambda) = w\left(N_0 + \int_0^\lambda w'(N_0 + t(N - N_0))\right) dt$$

satisfies

$$\mathscr{T}(\tilde{w}(N_0), N_0) - \tilde{w}(N_0) = 0$$

$$\frac{d}{d\lambda}\,[\mathscr{T}(\tilde{w}(N_\lambda), N_\lambda) - \tilde{w}(N_\lambda)] = 0$$

so that, by the uniqueness properties established in Theorem 2.2, we have that $\tilde{w} \equiv w$ or that w' is a bona fide derivative of w considered as a function of N.

The existence of higher derivatives can be obtained in a similar manner. We proceed by induction, assuming that we have already established that the formal expressions for the first j derivatives are indeed the true derivatives, and we will show that the formal expression for the $j+1$ derivative is a true derivative.

Again, we start by observing that the $j+1$ derivative should satisfy

$$D_w\mathscr{T}(w, N)\,D^{j+1}w + D_N^{j+1}\mathscr{T}(w, N) + R = D^{j+1}w \tag{4.1}$$

where R is an expression involving derivatives of $\mathscr{T}$ of order not bigger than j and derivatives of w of order not bigger than j.

Since $D_w \mathscr{T}$ is a contraction in C^L, it is possible to find an expression for $D^{j+1}w$ involving all the previously computed ones. Since this D^{j+1} satisfies (4.1), it is easy to show that the term of $\Delta^{\otimes(j+1)}$ in the expansion of

$$\begin{aligned}&\mathscr{T}(w(N) + w'(N) + \cdots + [1/(j+1)!]\, w^{(j+1)}(N)\, \Delta^{\otimes(j+1)}, N + \Delta)\\ &\quad - w(N) + w'(N)\, \Delta^{\otimes(j+1)} + \cdots + [1/(j+1)!]\, w^{(j+1)} \qquad (4.2)\end{aligned}$$

vanishes.

The validity of such an expansion can again be justified by taking derivatives of the expressions with respect to a parameter and noting that the derivatives involved are only derivatives of w up to order $L + j + 2$. The errors can be estimated uniformly by the C^{L+j+3} norm of w. But, as we argued before, this can be estimated by the C^{L+j+4} norm of N.

Therefore, the C^L norm of (4.2) can be estimated by $\|\Delta\|^{j+2}_{C^{L+j+4}}$.

Hence, the fixed point of $\mathscr{T}$ satisfies similar estimates and therefore our formal candidate is a true derivative.

This establishes the smooth dependence on parameters for Theorem 2.2. To prove smooth dependence on parameters in Theorem 2.1, we need only to prove that the calculations done to reduce Theorem 2.1 to Theorem 2.2—choosing coordinates along the spectral subspaces—depend smoothly on the map. But the smooth dependence of the spectral spaces on the linear map is a standard result in functional analysis (see, e.g., ref. 21). Similarly, the eliminations depend smoothly on the nonlinearity, since they just involve solving linear equations on the jets. ▌

Note that $S \cap U = \phi$ is used only to ensure smooth dependence of the subspaces. The smooth dependence of the manifold with respect to the nonlinear part does not need this hypothesis.

The uniqueness part of Theorem 2.2 can be considerably strengthened. Since this may be useful for other developments—in particular, it gives a second proof of the fact that fixed points in different χ^r produced in Proposition 3.1 agree, and therefore proves the C^∞ conclusion of Theorem 2.2, we will formulate it precisely and present a proof.

Denote by r_0 a number, not necessarily an integer, such that

$$\|A_U^{-1}\| \cdot \|A_S\|^{r_0} < 1$$

(that is, $r_0 < \log \|A_U^{-1}\| / \log \|A_S\|$). Then set

$$\begin{aligned}\chi^\delta = \{ w\colon B^S(1) \to E^U \mid\, & w(0) = 0;\ w \text{ Lipschitz};\\ & \operatorname{Lip}(w\,|_{B^S(r)})/r^{(r_0 - 1)} \leqslant \delta,\ 0 < r \leqslant 1\} \qquad (4.3)\end{aligned}$$

where Lip denotes the Lipschitz constant and for ease of notation we supress the dependence of χ^δ on r_0.

We observe that then, for all w in this space,

$$|||w||| = \sup_{x \neq 0} \frac{\|w(x)\|}{\|x\|^{r_0}}$$

is finite and, if we topologize χ^δ with this norm, it is complete. (Note that convergence in $|||\cdot|||$ implies pointwise convergence and that the uniform Lipschitz bounds are preserved under pointwise limits.) Notice also that all spaces of the form (3.3) are contained in some χ^δ. Nevertheless, all the χ^δ contain functions which are not even differentiable and are therefore larger sets than the $\chi^r_{\varepsilon_1,\ldots,\varepsilon_r - L}$ that we introduced before. Proving uniqueness in (4.3) is stronger than proving uniqueness in any of the spaces in (3.3).

Theorem 4.2. In the assumptions of Theorem 2.2 there exists a δ such that $\mathscr{T}(\chi^\delta) \subset \chi^\delta$ and $\mathscr{T}$ is a contraction there. Hence, the w of Theorem 2.2, which is actually smooth, is the only function in χ^δ satisfying $\mathscr{T} w = w$.

Proof. The proof is a quite straightforward calculation. We first show $\mathscr{T}$ is a contraction,

$$\begin{aligned}
&[\mathscr{T} u](x) - [\mathscr{T} w](x) \\
&\quad = A_U^{-1}[(u(A_S x + N_S(x, u(x))) - w(A_S x + N_S(x, u(x)))) \\
&\qquad + (w(A_S x + N_S(x, u(x))) - w(A_S x + N_S(x, w(x))) \\
&\qquad + (N_u(x, w(x))) - N_u(x, u(x)))]
\end{aligned}$$

Taking norms and dividing by $\|x\|^{r_0}$, we can estimate the first term by inserting in the numerator and denominator $\|A_S x + N_S(x, u(x))\|^{r_0}$; the second one uses that

$$\|N_S(x, u(x)) - N_S(x, w(x))\| \leqslant (\text{Lip}\, N_S(x, u))\ \|u(x) - w(x)\|$$

and that this factor can be made as small as we wish by assumption, and a similar argument works for the last.

The proof that $\mathscr{T}(\chi^\delta) \subset \chi^\delta$ follows easily from the chain rule for Lipschitz constants; we have

$$\begin{aligned}
\text{Lip}\, \mathscr{T}[w]\,|_{B^S(r)} \leqslant \|A_U^{-1}\|(\|A_S\| + \text{Lip}\, N_S(1+\delta))\ \text{Lip}\, w\,|_{B^S(r^*)} \\
+ \text{Lip}\, N_U(r_0 + \delta)
\end{aligned}$$

where

$$r^* \geqslant \sup_{|x|>r} \|A_S x + N_S(x, w(x))\|$$

which up to errors arbitrarily small by suitable smallness assumptions on $\|N\|_{\mathcal{L}^\infty}$ is just $\|A_S\|\, r$.

If we now divide both sides by r^{r_0-1}, we get

$$\frac{\operatorname{Lip}\mathscr{T}[w]\,|_{B^S(r)}}{r^{r_0-1}} \leqslant \|A_U^{-1}\|(\|A_S\|+\gamma)\,\frac{\operatorname{Lip} w\,|_{B^S(r^*)}}{r^{*\,r_0-1}}\,\frac{r^{*\,r_0-1}}{r^{r_0-1}}$$

$$\leqslant \|A_U^{-1}\|(\|A_S\|+\gamma)^{r_0}\,\frac{\operatorname{Lip} w\,|_{B^S(r^*)}}{r^{*\,r_0-1}}$$

where γ denotes a number which can be made arbitrarily small by assuming smallness conditions on Lip N. ▌

The following characterization of invariant manifolds is more restrictive in the conditions we impose on the spectrum, but, on the other hand, does not make any regularity assumptions.

This characterization roughly says that if we only consider orbits for which the components are bounded by a power of the S component, the S component determines all of them. In other words, restricted to this "parabolic region," the set of points that converge is a graph. Again, we remark that this uniqueness result gives another proof of the agreement of the fixed points in different χ^r and hence establishes the C^∞ part of Theorem 2.2.

This is quite analogous to the usual proof of the fact that the stable manifold (characterized only by topological properties) is indeed the graph of a function.

Theorem 4.3. Let f, A, S, U be as in Theorem 2.1. Write

$$\Gamma_{\rho,C,l} = \{x \in B^S(l) \mid \|\Pi_U x\| \leqslant C\,\|\Pi_S x\|^\rho\}$$

If ρ satisfies $\rho < \ln \|A_U^{-1}\|^{-1}/\ln \|A_S\|$, we can find $l^*(\rho)$ in such a way that if two points x_1, x_2 satisfy $\Pi_S x_1 = \Pi_S x_2$ and $\{f^n(x_i)\}_{n=0}^\infty \subset \Gamma_{\rho,C,l}$ [any $C \geqslant l < l^*(\rho)$], then $x_1 = x_2$.

Proof. We will denote $\varepsilon = \sup_{x\in B^S(l)} \|DN(x)\|$. We will show that if ε is small enough, which amounts to l small enough, we get the conclusions of the theorem. We have, if $x, y \in B^S(l)$,

$$\|\Pi_U f(x) - \Pi_U f(y)\| \geqslant \|A_U^{-1}\|^{-1}\,\|\Pi_U(x-y)\| - \varepsilon\,\|\Pi_S(x-y)\|$$

$$\|\Pi_S f(x) - \Pi_S(f_y)\| \leqslant \|A_S\| \cdot \|\Pi_S(x-y)\| + \varepsilon\,\|\Pi_U(x-y)\|$$

If we consider

$$\begin{pmatrix} \|A_S\| & \varepsilon \\ -\varepsilon & \|A_U^{-1}\| \end{pmatrix}^n \begin{pmatrix} \|\Pi_S(x-y)\| \\ \|\Pi_U(x-y)\| \end{pmatrix}$$

the first component gives an upper bound for $\|\Pi_S f^n x - f^n(y)\|$ and the second a lower bound for $\|\Pi_U i(f^n(x) - f^n(y))\|$.

If we diagonalize the matrix, we can see that

$$\|\Pi_S f^n(x) - f^n(y)\| \leqslant (\|A_S\| + \varepsilon')^n (\|\Pi_S(x-y)\| + \varepsilon' \|\Pi_U(x-y)\|)$$

$$\|\Pi_U f^n(x) - f^n(y)\| \geqslant (\|A_U^{-1}\| - \varepsilon')^n (\|\Pi_U(x-y)\| - \varepsilon' \|\Pi_S(x-y)\|)$$

(ε' depends only on ε and is as small as we wish with ε).

If we now apply this result taking $x = x_i$, $y = 0$, we get

$$\|\Pi_S f^n(x_i)\| \leqslant (\|A_S\| + \varepsilon')^n \|\Pi_S x_i\|$$

and if we apply it with $x = x_1$, $y = x_2$, we obtain

$$\|\Pi_U(f^n(x_1) - f^n(x_2))\| \geqslant (\|A_U^{-1}\|^{-1} - \varepsilon)^n \|\Pi_U(x-y)\|$$

Unless $\|\Pi_U(x-y)\| = 0$, this is a contradiction with the assumption about the orbits of x_1, x_2 satisfying $\|\Pi_U f(x_i)\| \leqslant C \|\Pi_S f(x_i)\|^\rho$. ∎

Remark. We note that if we have a map satisfying the conditions of Theorem 2.2 and it is $C^{r_0+\varepsilon}$, in a sufficiently small ball it has to be in χ^δ. The reason is because, by the nonresonance argument that we had before, all the derivatives up to order $[r_0]$ have to vanish. Then the fact that the map is in $C^{r_0+\varepsilon}$ implies that the remainder of the Taylor expansion of the derivative has to make it be in χ^δ.

The existence and uniqueness results developed so far can be counterpointed with the following examples:

Example 4.4. The mapping $(x, y) \mapsto (1/2x, 1/4y)$, besides the spectral subspaces, has (x, x^2) as an invariant manifold. This manifold is clearly analytic.

Therefore, in general there could be other invariant manifolds besides the ones we consider here; notice how this example violates the assumptions of both of our uniqueness theorems. This example shows that the parameters appearing in our uniqueness theorems cannot be lowered.

Example 4.5. Let A be diagonalizable. If the relation between eigenvalues $\lambda_{i_1} \cdots \lambda_{i_r} = \lambda_1$ holds, the map

$$x \to Ax + \underline{l}_1 x_{i_1} \cdots x_{i_r}$$

where $\underline{l}_1$ denotes the eigenvector corresponding to λ_1 and x_{i_j} denotes the coordinates along the directions of the eigenvectors corresponding to λ_{i_j}, does not have a C^r invariant manifold tangent to the invariant subspace spanned by the eigenvectors of $\lambda_{i_1} \cdots \lambda_{i_r}$.

Remark. We are not assuming $\lambda_{i_1} \cdots \lambda_{i_r}$ different.

Proof. Since in a sufficiently small neighborhood we would have that the manifold would have to be a graph, it suffices to show there is no C^r solution of $\mathscr{T} w = w$ (as in Theorem 2.1).

If this w had a Taylor expansion of order r, we could match powers. Substituting the definitions, we see this is impossible.

The construction can be easily modified to produce a similar counterexample when the matrix is not diagonalizable. So the nonresonance assumptions of our theorem are sharp. ∎

5. PARTIAL LINEARIZATIONS AND PSEUDO-STABLE MANIFOLDS

The following result is proved in ref. 2. (Even if the statement of Theorem 1.1 of ref. 2 is only for $\mathbb{R}^n$, the remarks along the proof make it clear that the result is true also for a general Banach space which admits smooth cutoff functions.)

We recall that a cutoff function is a function that takes the value one on a ball and the value zero outside a bigger ball. For finite dimensional spaces, the existence of smooth cut-off functions can be proved very easily. On the other hand, for infinite dimensional Banach spaces, it is a nontrivial assumption on the space. For example, the space of continuous functions on the interval does not admit a C^1 cutoff function.[20] More generally, a separable Banach space admits a Frechet differentiable cutoff function if and only if its dual is separable.[27] Any Hilbert space admits smooth cutoff functions.

Theorem 5.1. Let f, g be C^r, $r \in \mathbb{N}$, diffeomorphisms of a Banach space, admitting smooth cutoff functions. $f(0) = g(0) = 0$, and let A and B be numbers computed explicitly in the proof which depend only on the spectrum of $Df(0)$.

Assume

(i) $D^i f(0) = D^i g(0)$, $i = 0, \ldots, k < r - 1$.

(ii) Spec $Df(0) \subset \{z \in \mathbb{C} \mid \lambda_-^{-1} \leqslant |z| \leqslant \lambda_+\} \cup \{z \in \mathbb{C} \mid \mu_-^{-1} \leqslant |z| \leqslant \mu_+\}$ for some $0 < \lambda_-^{-1} < \lambda_+ < 1 < \mu_-^{-1} < \mu_+$.

Then, provided that $1 \leqslant l < kA - B$, for some integer l, we can find a C^l diffeomorphism h such that

$$h^{-1} \circ f \circ h = g$$

on a neighborhood of the origin, $h(0) = 0$, $Dh(0) = Id$.

This paper also contains explicit expressions for the numbers A and B in terms of $\lambda_\pm, \mu_\pm$, which undoubtedly are not optimal (indeed, ref. 2 sketches the proof of some better number for finite-dimensional spaces), but there are examples that show that one cannot get the conjugating map to be as smooth as the order of tangency.

The way that these cutoff functions enter in the proof is in the observation that, if ϕ is a cutoff function, setting

$$\tilde{f}(x) = \phi(x) f(x) + (1 - \phi(x))(N(x) - f(x))$$

we find that the function $\tilde{f}$ is identical with f in a neighborhood of the origin and is globally close to N. In particular, when f is tangent to N to a high order in the origin, $\tilde{f}$ is tangent to a high order to N in the origin and globally close to N.

In the case that we have considered in this paper the graph transform operator mapped functions defined in a ball into functions defined in a bigger set. If A_S was not strictly contractive, this would not be the case and then we would have to deal with functions defined everywhere. But then it is necessary to have global proximity assumptions.

The meaning of Theorem 5.1 is that, if we get maps which are hyperbolic and tangent to one another to a high enough order, we can make a change of variables that is moderately smooth in such a way that they become exactly the same.

Recall that given the nonresonance conditions, Proposition 3.6 allowed us to make a change of variables in our original map in such a way that it had the form (2.1)

$$f(s, u) = (A_S s + N_S(s, u), A_S u + N_U(s, u)) + O(|u|^{L+1}, |s|^2)$$

We can apply Theorem 5.1 to f and to g defined by

$$g(s, u) = (A_S s + N_S(s, u), A_U u + N_U(s, u))$$

Notice that, as already noted in the remark after Theorem 2.2, the map g leaves invariant the manifold $u=0$. Hence the manifold $W=h(\{(s,0)\})$ is invariant under f.

Note that the method of partial linearization has the advantages that it gives more detailed dynamical information and that it can deal with non-resonant subsets that have components on both sides of the unit circle.

On the other hand, note that the partial linearization method requires that the map we consider is hyperbolic and invertible, the manifolds thus considered are not unique under natural hypotheses, and the regularity of the invariant manifolds is only a fraction of the regularity of the map.

As we have mentioned, the regularity conclusions of the partial linearization method can be considerably improved in the case that the non-resonant spectral subsets are inside the unit circle. In ref. 2, Theorem 5.1 shows that the regularity of the conjugacy—and *a fortiori* that of the invariant manifolds—is C^{r-} for C^r mappings. Presumably this is not the limit of the method, since ref. 2 was more concerned with the preservation of geometric structures than with regularity issues. In the case that the spectral subset straddles the unit circle, presumably the regularity of the invariant manifolds is not any better than a fraction of that of the original map, even if the exact value of the fraction is better than that of the partial linearization—and undoubtedly better than that in Theorem 5.1.

From the numerical or perturbative point of view the graph transform method only requires that we deal with functions in E^S, whereas the partial linearization method requires we deal with functions defined on X.

6. OTHER RESULTS

In this section we discuss other results in the literature that are also concerned with obtaining invariant manifolds on spectral subspaces that are not disks or complements of disks.

In ref. 7 there is a proof of Theorem 2.1 in the particular case that the invariant subspace is one-dimensional. The method used there, very different from ours, gives not only the invariant manifold, but also a parametrization of it in which the motion is linear. The method presented in this paper can be readily implemented on a computer for analytic mappings.

Pöschel[32] considers finite-dimensional analytic maps and sets S of eigenvalues that satisfy a Diophantine condition,

$$|\lambda_{i_1}\cdots\lambda_{i_n}-\lambda_j|\geqslant\Omega(n),\qquad i_1,\dots,i_n\in S,\quad j\in[0,d]\tag{6.1}$$

where Ω is a decreasing function satisfying the so-called Brjuno condition,

$$\sum_{n=1}^{\infty} 2^{-n} \ln \Omega(2^n) < \infty$$

[For example, $\Omega(n) = Kn^{-r}$, the standard condition of the Diophantine approximation, satisfies the Brjuno condition.] The conclusions are not only that there exists an invariant manifold, but also that the motion on it is conjugate to its linear part. Note that the above result applies even when the eigenvalues in S lie on the unit circle. In that case, condition (6.1) amounts to a Diophantine condition among the angles of the rotation. It also applies to situations when some eigenvalues λ are inside the unit circle and others outside. In the case that all the eigenvalues are inside the unit circle, the conclusions are stronger than those of Theorem 2.1—they also include equivalence to the linear part—but so are the hypotheses.

Sometimes one can even get invariant manifolds corresponding to eigenspaces of eigenvalues 1. Since 1 leads to resonances, Example 2 shows that one cannot have a general theorem concluding the existence of an invariant manifold for all such maps. On the other hand, if the system contains several parameters or if there is an internal symmetry, one can sometimes have results for some values of the parameter. (They are harder to prove, since one does not have that the linear part is contractive.) Such extra parameter problems or symmetries appear naturally in celestial mechanics, for example, in the study of "invariant manifolds at infinity." With these motivations, the case of a nilpotent block corresponding to an eigenvalue one is considered in ref. 7. Fontich[12] finds necessary and sufficient conditions for the existence of analytic invariant manifolds tangent to eigenvalues equal to 1 in Hénon-like mappings. Again they are of finite codimension. It seems that similar phenomena appear in the applications of renormalization group in field theory. In the language of field theory, the eigenvalues with modulus 1 are called "marginal." Very often they are precisely 1. (See, e.g., ref. 13 for more details about this problem in field theory.)

7. SOME APPLICATIONS

In this section we describe three applications of these nonresonant manifolds.

Two of them (beta functions of renormalization group and intermediate foliations) are explained in the literature. The third one—lack of smoothness of invariant circles in Hopf bifurcation—is presumably known to experts. Hence, we will be somewhat sketchy.

7.1. Renormalization Group in Field Theory

As a first application we call attention to ref. 23, where some applications to field theory/statistical mechanics are described (as a matter of fact, the present paper was partly motivated by this application). To a large extent, in this section we will just summarize some of the points made in ref. 23. We apologize in advance for any excessive oversimplification or inaccuracy.

Since the precise definition of renormalization operators as differentiable maps in appropriate Banach spaces has not been achieved except in some restricted class of models, it is worthwhile to consider finite-dimensional situations that have features similar to the full renormalization group. This can be justified by the belief that the number of relevant parameters of a model is finite and, indeed, small. This belief is one of the main conclusions of renormalization group theory and, indeed, has been verified in many cases either empirically or by studying specific models rigorously.

Hence, following ref. 23 and many other papers in renormalization theory, we will present some rigorous results for these finite-dimensional applications and will not discuss how they can be derived from more fundamental models. Since the gist of the results is that things are more pathological than sometimes expected, it seems that there is hope that similar results will hold in more realistic models.

Hence, we consider a renormalization transformation defined on a finite-dimensional space. This transformation is supposed to describe how the relevant parameters of the model change when we change the scale of the description. This transformation will be assumed to have one trivial fixed point that describes the high-temperature phase and a nontrivial fixed point that describes a phase transition.

When a model converges to a nontrivial fixed point it is natural to consider relevant parameters which lie along the eigenspaces of the linearization. As we converge to the fixed point all the relevant parameters can be expressed as a function of the one that is decreasing more slowly. This function is usually called the β function. We note that these β functions give the behavior of the model close to the renormalization. That is, they give the leading order in the fluctuations when the renormalization has been applied many times. Given their importance, these β functions have been computed in many models. The favorite method of calculation assumes a power series expansion and matching coefficients.

In a more dynamic language, the β functions are such that their graphs contain trajectories approaching the fixed point. That is, they are invariant manifolds. Since we are using the parameter that is converging the slowest to the fixed point as an independent variable, these manifolds will be tangent to the eigenspace corresponding to this parameter.

The results of this paper show that, assuming that the nonresonance conditions of Theorem 2.1 are met by the renormalization group transformation at the fixed point, these smooth β functions are uniquely determined and are the smooth invariant manifolds that we have described. Indeed, the calculation of the coefficients of the β functions is done in the way that we have described in (3.12).

The unfortunate part is that, if we take the calculation of coefficients of the β function too far—which amounts to considering very smooth invariant manifolds—our uniqueness result apply. Hence, these smooth invariant manifolds are the smooth invariant manifolds we have considered. In particular, if we pick a model at random, it is very unlikely that it will be on this manifold. That is, its behavior under renormalization will not correspond to all the coefficients of the β function. In other words, for most of the models, the β function will not be very differentiable at the origin.

This argument does not exclude that the smooth β function can be used to describe the leading behavior of models, but, of course, the error terms would have to be big enough not to conflict with the previous argument. (We indeed think that some results along these lines could be proved, but this will take us far from the present goal.)

We refer to ref. 23 for further details and clarifications on these finite-dimensional models and their relevance.

7.2. Lack of Smoothness of Invariant Circles with Periodic Orbits

As a second application, we consider invariant circles in the plane consisting of heteroclinic orbits.

These circles appear naturally after a Hopf bifurcation. When the rotation number of the invariant circle is rational, there is at least one periodic orbit and, indeed, one expects to have two periodic orbits, one stable and one unstable. This will be the situation that we consider. We point out that our considerations apply without any change when there are more than two hyperbolic periodic orbits, but the case of just two is the one that appears most often and the notation is somewhat simpler.

We will show that, using the theory developed in this paper, we should expect these circles to be only finitely differentiable and that indeed there are finite calculations—which can be carried out either perturbatively or with a finite-precision computer—that exclude certain regularities.

More precisely, we will consider a circle which consists of two periodic orbits $P=\{p_1,..., p_n\}$ and $Q=\{q_1,..., q_n\}$ and manifolds connecting them as described below.

We will have $f(p_i)=p_{i+1}, f(q_i)=q_{i+1}$ with the sums in the subindices understood mod n. Hence $f^n(p_i)=p_i$, $f^n(q_i)=q_i$. We will assume that the orbit P has one stable direction and an unstable one, while the orbit Q is completely stable:

$$\mathrm{Spec}(Df^n(p_i))=\{\lambda^p, \mu^p\}$$
$$\mathrm{Spec}(Df^n(q_i))=\{\lambda^q, \mu^q\}$$

We will assume

$$|\lambda^p|, |\lambda^q|<1, \qquad |\mu^q|<1, \qquad |\mu^p|>1, \qquad |\lambda^p|\cdot|\mu^q|<1$$

To avoid unnecessary complications, we will aslso assume that the map f is C^∞.

We will furthermore assume that the unstable manifold of P is contained in the stable manifold of Q. Hence, the unstable manifolds of P together with Q form an invariant circle.

Such situations arise after a Hopf bifurcation. If the rotation number on the invariant circle is a rational number, we have a periodic orbit, and if it is hyperbolic, we should have another one. In a generic situation we will just have two. (There are constructive calculations that in explicitly known cases can lead to the conclusion that there are just two periodic orbits.)

We recall that the standard proof of the Hopf bifurcation for maps consists in making transformations to a normal form that can be expressed in polar coordinates as $(r'=f(r, \varepsilon), \theta'=\theta+\omega(r, \varepsilon))$ and the error terms are of the order r^N, where N is the order of the smallest resonance. In this case we have

$$\lambda^p=1-c_1\sqrt{\varepsilon}+O(\varepsilon); \qquad \lambda^q=1-c_1\sqrt{\varepsilon}+O(\varepsilon)$$
$$|\mu^{p,q}-1|=O(\varepsilon^{N/2})$$

where ε is the bifurcation parameter, N is the order of the smallest resonance in the bifurcation (it is at least 4), and c_1 is a positive constant. Hence, the assumption $|\lambda^p|\cdot|\mu^q|<1$ is certainly true for small values of the perturbation parameter.

The standard theory of normally hyperbolic manifolds[10,11,36,37] shows that this situation is structurally stable. In particular, the invariant circles continue to exist. Moreover, we claim that the invariant circle is $C^{r_0-\delta}$,

$\forall\delta>0$, $r_0=\log|\lambda^q|/\log|\mu^q|$. This follows directly from the method of refs. 10, 11. The method of ref. 37 seems to produce slightly smaller results if applied naively. The numerator will just be the logarithm of the smallest normal contraction, which could involve the periodic orbit P. Nevertheless, we can argue as follows. Consider the invariant set that is outside a small neighborhood of P. This set is invariant. By replacing the map by a sufficiently high iterate, we can obtain that the normal contraction and the tangential contraction are given by numbers as close to the stable and unstable eigenvalues of Q as desired. This establishes the claim of regularity outside of a neighborhood of P, but on a neighborhood of P the circle has to agree with the unstable manifold of P, hence it is C^∞.

We want to show how the theory of nonresonant manifolds developed in this paper allows us to obtain computable conditions that show that this manifold is not $C^{r_0+\delta}$.

Since our goal is to exclude that the circle produced by Hopf bifurcation is $C^{r_0+\varepsilon}$ in concrete cases, we proceed by contradiction.

We assume that it is C^{r_0} and then derive numerical facts that should hold. Given a concrete map, these numerical facts can be refuted by a finite-precision calculation. We can also show that they hold only in sets of infinite codimension.

We will distinguish two cases, $\ln|\lambda^p|/\ln|\mu^p|\notin\mathbb{N}$ and $\ln|\lambda^p|/\ln|\mu^p|\in\mathbb{N}$. The first case is the generic one. We will refer to it as the nonresonant case.

The linear map $Df^n(p_1)$ has exactly two invariant subspaces, one of them associated to λ^p and another one to μ^p. In the nonresonant case there are two one-dimensional invariant subspace for Df^n. Each of them has a one-dimensional nonresonant invariant manifold.

Of course the manifold associated to λ^p is the well-known strong stable manifold. We also note that in the circles that appear after the Hopf bifurcation the unstable manifold of Q does not agree with the strong stable manifold of P.

As we argued in Theorem 2.1, these are the only C^L invariant manifolds in a neighborhood of P. To show that the circle is not C^L, we just need to show that these two nonresonant manifolds of P do not agree with the unstable manifold for Q.

For the point of view of numerical applications we point out that, given an explicit map, the nonresonant manifolds and the unstable ones are numerically computable with high accuracy by a finite calculation and it is possible to show that they do not agree using also a finite calculation. In practice, this is not even a difficult calculation and using the algorithms in ref. 8, one can get very high precision using only moderate effort. Similarly, a perturbative calculation carried out with error estimates can be used to establish that they do not agree.

From the theoretical point of view, since we have formulas for the derivatives with respect to the map, it is easy to show that even if they agree for a particular map, this agreement will be destroyed for generic perturbations. (Note that the derivatives with respect to N at one point are expressed as sums of functions related to the perturbation evaluated at iterates of the map. For the stable manifold we need forward iterates and for the unstable ones we need backward iterates. It is clear that the sums of perturbations in forward iterates do not agree with the sums over backward iterates.)

The resonant case can be handled similarly, but one needs to distinguish different possibilities. The strong stable manifold of P still exists and is smooth, but it is possible to check that it does not agree with the unstable manifold of Q. Unless there is certain combination of derivatives that vanish, there is no C^{r_0} intermediate invariant manifold and, of course, we are done showing that the circle is not $C^{r_0+\delta}$. If this combination of derivatives vanishes, then, as we showed, there is an intermediate nonresonant manifold, but the same arguments as in the nonresonant case may be used to exclude that it agrees with the unstable manifold of Q and, as before, this leads to the circle not being $C^{r_0+\delta}$.

7.3. Nonresonant Invariant Foliations

As a third application we discuss the possibility of extending these results to invariant foliations. Given a diffeomorphism f on a compact manifold, it is a standard construction[14,37,38] to consider the operator $\tilde{f}$ acting on C^0 vector fields by

$$(\tilde{f}v)(x) = \exp_x^{-1} f(\exp_{f^{-1}(x)}(v(f^{-1}(x)))$$

where $\exp_x v(x)$ denotes the differential geometry exponential map obtained by flowing for a unit of time the geodesic with initial conditions x, $v(x)$. [It is useful to think of $\exp_x v(x)$ as $x + v(x)$. Indeed this is what it amounts to in Euclidean space.]

It is not difficult to check that if $\|v\|_{C^0}$ is sufficiently small, $\tilde{f}$ is well defined.

Moreover,

$$[D\tilde{f}(0)] = f^*$$

where f^* is the pushforward

$$[f^*v](x) = Df(f^{-1}x)\, v(f^{-1}(x))$$

The spectrum of f^* in the complexification of C^0 vector fields (to study spectral properties, it is much better to have a complex space) has been intensively studied since ref. 39. In that paper it is shown that if f is an Anosov system, the spectrum consists of annuli. Moreover, quite remarkably, the spectral projections associated to each of the annuli correspond to projections over a subbundle. As a corollary of this last property, we obtain that the number of annuli is at most the dimension of the space.

Hence, if a subset of these annuli satisfies the nonresonance conditions of Theorem 2.1 we can obtain nonresonant invariant manifolds for $\tilde{f}$. For the case of an annulus this nonresonant invariant manifold is constructed directly in ref. 31. In the case where the nonresonant set is the whole stable component (resp. the annuli within a ball of radius $\rho < 1$) this is the way that stable (resp. ρ stable) foliations are constructed in refs. 14, 37. The theorems in this paper allow us to carry out these constructions for non-resonant sets of the Mather spectrum that consist of several annuli.

Unfortunately, the geometric interpretation of the nonresonant invariant manifolds for $\tilde{f}$ is more complicated than that of the stable (or ρ stable ones).

The nonresonant invariant manifolds for $\tilde{f}$ correspond to "invariant leaf fields," that is, maps that to each point x associate a leaf L_x—a diffeomorphic image of a disk—in such a way that

$$\begin{aligned} f(L_x) &\subset L_f(x) \\ T_x L_x &= E^s_x \end{aligned} \tag{7.1}$$

(The proof consists in walking through the proof of the nonresonant manifold checking that all the steps are bundle maps. Fuller details can be found in ref. 31 for the one-annulus case or in ref. 19 or in lecture notes by the author. In any case, these details can be more or less found in ref. 38.)

The regularity and uniqueness statements in Theorem 2.1 carry through to show that the leaf fields are characterized uniquely by (7.1) and by having C^L leaves. The result in Theorem 2.1 implies that the leaves L_x are $C^{r-1+\mathrm{Lip}}$ if the map is C^r. (It can be improved to C^r.)

In the case of the stable manifold (ρ-stable manifold) it is possible to show that these leafs are a foliation because

$$y \in \bigcup_{n \geqslant 0} f^{-n}(L_x) \Leftrightarrow d(f^n(x), f^n(y)) \leqslant K_{x,y}\lambda^n$$

(resp. $\leqslant K_{x,y}\rho^n$) and this is clearly an equivalence relation.

Unfortunately, this argument does not carry through for general nonresonant manifolds and indeed the conclusions are false. In ref. 19 there are examples where these leaf fields fail to be a foliation in a very strong sense. For a generic map f any neighborhood contains intersections of leaves.

There is another twist to the discussion of invariant foliations corresponding to these invariant sets. There is another construction of invariant manifolds that correspond to spectral subsets. For example, Irwin[16,17] (a more modern version with several extensions is given in ref. 24) constructs invariant manifolds (usually called pseudo-stable) associated to spectral sets of the form $\{z \in \mathbb{C} \mid |z| \leqslant \rho\}$ with $\rho > 1$ for maps that are globally close to linear. Moreover, it is also shown that these manifolds admit characterizations by the speed at which points escape to infinity. Indeed, we have $x \in W_\rho \Leftrightarrow \|f^n(x)\| \leqslant K_x \rho^n$.

By taking intersections of these manifolds with strong stable manifolds of the inverse it is possible to obtain invariant manifolds associated to spectral subsets of the form $\{z \in \mathbb{C} \mid \rho_- \leqslant |z| \leqslant \rho_+\}$. We emphasize that the construction of Irwin manifolds does not involve nonresonance conditions.

The somewat surprising fact is that these Irwin manifolds are not the same as those constructed in this paper even in the case where the manifolds in this paper can be defined. In ref. 24 one can find examples where these Irwin manifolds are not smooth and therefore do not coincide with the smooth nonresonant manifolds constructed in this paper.

The Irwin construction can be lifted to maps on manifolds to produce invariant foliations (we will anon justify why this is the case). This seems to require extra properties of the manifold. Sufficient conditions are that the manifold has $\mathbb{R}^n$ as universal cover and that the map is globally close to linear. This is in contrast with the nonresonant leaf fields constructed using the results in this paper. These nonresonant leaf fields can be constructed in any manifold and such that they have for any map that whose Mather spectrum satisfies the nonresonance conditions. An example when both constructions can be carried out is perturbations of linear automorphisms of the torus with a nonresonant spectrum.

When the Irwin construction can be carried out for maps on a manifold, it leads to foliations (it turns out that $y \in W_x^{s,\mathrm{Irwin}} \Leftrightarrow d(f^n(x) - f^n(y)) \leqslant C_{x,y} \rho_+^n$, $n \geqslant 0$, with f in the universal cover, which is an equivalence relation, so that it indeed is a foliation), but the leaves may be significantly less smooth than the map—the degree of differentiability is related to the gaps of the Mather spectrum. Note also that the characterization above shows that any homeomorphism of the manifold conjugating the dynamics of two diffeomorphisms sends the Irwin manifolds of one into those of the other.

In summary, the construction of nonresonant invariant manifolds in this paper can be lifted to all manifolds; it produces leaf fields of smooth leaves that, in a generic case, fail to be foliations. Moreover, there is another natural construction (Irwin's) that only works on certain manifolds and for certain maps, but which produces foliations with leaves that are not smooth. We refer to ref. 24 for the theory of Irwin manifolds and to refs. 24 and 14 for examples of nonsmooth Irwin manifolds of non-resonant leaf fields that are not foliations. These examples occur even in situations where both constructions can be carried out. Some of these examples are rather explicit and were used in ref. 25 as counterexamples to other questions.

ACKNOWLEDGMENTS

This paper would not have been written without the friendly prodding of A. Sokal, who raised the problem in 1987, explained the statistical mechanics motivation, and has kept encouraging me to write the answer. Independently of that, I was driven to the same problems by an effort to construct more invariant foliations to be used in rigidity questions. Y. Yomdin pointed out ref. 31, which uses similar ideas (see also ref. 19). I am also grateful to H. Rosenthal for bringing to my attention refs. 20 and 27, and also to C. Pugh for encouragement and pointing out that C^r regularity was also possible. I also thank an anonymous referee and M. El Bialy for comments that greatly improved the exposition. This research has been supported by NSF grants and a Centennial Fellowship from AMS and a URI from the University of Texas.

NOTE ADDED IN PROOF

I would like to call to the attention of the reader that M. El Bialy has written a proof of C^r regularity for the manifolds rather than the $C^{r-1+\mathrm{Lip}}$ in this paper. I also became aware of the paper: T. R. Young, C^k smoothness of invariant arcs in a global saddle-node bifurcation, *Jour. Diff. Eq.* **126**:62–86 (1996), which studies the application in Section 7.2 by other methods but reaches similar conclusions.

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