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The Simplified Fermi Accelerator in Classical and Quantum Mechanics

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We review the simplified classical Fermi acceleration mechanism and construct a quantum counterpart by imposing time-dependent boundary conditions on solutions of the "free" Schrödinger equation at the unit interval. We find similiar dynamical features in the sense that limiting KAM curves, respectively purely singular quasienergy spectrum, exist(s) for sufficiently smooth "wall oscillations" (typically of $\mathscr{C}^2$ type). In addition, we investigate quantum analogs to local approximations of the Fermi map both in its quasiperiodic and irregular phase space regions. In particular, we find pure point q.e. spectrum in the former case and conjecture that "random boundary conditions" are necessary to model a quantum analog to the chaotic regime of the classical accelerator.

KEY WORDS: Schrödinger equation with time-periodic boundary conditions; absence of absolutely continuous quasienergy spectrum; correspondence to analogous (chaotic) classical model.

1. THE CLASSICAL FERMI MAP

The *Fermi acceleration model* is an example of a Hamiltonian system that can be represented by an area-preserving mapping and which illustrates the nature of stochastic trajectories in systems with two degrees of freedom. The mapping results from an idealization of the one-dimensional motion of a ball bouncing between a fixed and a periodically oscillating wall. (The original model goes back to Fermi,[1] who sought an explanation for the acceleration of cosmic rays in the combined earth—moon gravitational field.)

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The mathematical description of the accelerator best starts in the pointwise rest frame of the oscillating plate. There we observe

$$\mathfrak{H} = p^2 \qquad \text{with momentum} \quad p = \dot{q}/2, \quad 0 \leqslant q \leqslant a(t) \tag{1}$$

together with the reflection condition for the nth impact: $\dot{q}_n^{(i)} = -\dot{q}_n^{(f)}$. Under the canonical transformation[2]

$$(q, p) \mapsto (\bar{q} = q/a(t), \bar{p} = p \cdot a(t)) \tag{2}$$

the Hamilton function (1) becomes

$$\bar{\mathfrak{H}}(t) = \bar{p}^2/a^2(t) - (\dot{a}/a)(t)\, \bar{p} \cdot \bar{q}$$

or, after introducing the auxiliary variable $\mathscr{A}(t) = \hat{t} := \int_0^t d\tau\, a(\tau)^{-2}$, we obtain from (1)

$$\hat{\mathfrak{H}}(\hat{t}) = \hat{p}^2 - (\dot{a}a)[\mathscr{A}^{-1}(\hat{t})]\, \hat{p} \cdot \hat{q} \tag{3}$$

Accordingly, the elastic impact condition to (1) is recovered as

$$\hat{q}^{(i)} - \dot{a}a = -\hat{q}^{(f)} + \dot{a}a \tag{4}$$

(here $\circ$ indicates the derivative with respect to $\hat{t}$).

The main technical difficulty in solving problems (1), respectively (3)–(4), is the dependence of the flight times between the nth and $(n+1)$th collisions of the ball and the oscillating plate from the value of the phase of the moving wall at the nth impact. Therefore Ulam[3] and later Lichtenberg and Lieberman[4] proposed a simplified version of the Fermi accelerator in which the walls are kept at constant distance from the very beginning; however, one of the plates imparts energy to the ball via some nontrivial relation similiar to (4). Since then, it has been confirmed in several numerical studies[4–6] that the simplified version approximates the full model sufficiently well in a certain range of physical parameters. (For instance, the combination "fast" particle versus "slow" wall works.)

In terms of the system (3)–(4), the simplification consists in disregarding the drift term $(\dot{a}a)(\mathscr{A}^{-1}(\hat{t}))\, \hat{p} \cdot \hat{q}$ in (3) while keeping the inelastic transfer (4). Expressed as a dynamical system, the *simplified Fermi accelerator* is therefore described by the following difference equations[4, 5]:

$$\begin{aligned} u_{n+1} &= |u_n - (d\mathfrak{F}/d\phi)(\phi_n)| \\ \phi_{n+1} &= \phi_n + 2\, d\omega(u_{n+1})^{-1} \end{aligned} \tag{5}$$

Here $(u_n, \phi_n) \in \mathbb{R}^+ \times S^1$ are conjugate action–angle variables right before the nth impact. In particular, $\phi = \omega t$ is the phase of the wall oscillation,

$u = \dot{q}/2\omega\gamma$ corresponds to the normalized particle velocity, with $\dot{q} := dq/dt$ and γ the amplitude of the wall oscillation. The latter is described by $a(t) = 1 + \gamma\mathfrak{F}(\omega t)$, $\mathfrak{F}$ being 2π-periodic, with $\mathfrak{F}_{\max} = -\mathfrak{F}_{\min} = 1$. Thus, $2d(u_{n+1})^{-1}$ is the time between the nth and $(n+1)$th collisions between bouncing ball and "oscillating" table with $d = 1$ denoting the distance between the plates. Given (5) and a sufficiently powerful computer, millions of iterations ($\triangleq$ collisions) can be performed for various initial conditions.

Typical surface-of-section plots for various choices of $\mathfrak{F}$ can be found in refs. 4–7. Generically, the (u, ϕ) space is divided into three regions: (i) A region of predominantly irregular motion for small values of u, in which all primary periodic solutions to (5) appear to be unstable. See below for more details. (ii) An interval of intermediate u values, where islands of stability are embedded in the chaotic sea. (iii) High u values, for which most of the trajectories are regular.

As pointed out in refs. 5 and 7, there is the following important consequence of the KAM theorem about the existence of invariant limiting curves: *Based on numerical studies it has been conjectured in ref. 5 that for all* $a \in \mathscr{C}^2(S^1)$ *there exists a limiting KAM curve in the* (u, ϕ) *surface-of-section which restricts the energy gain of the bouncing particle.* (This feature will find its quantum analog in Section 2.)

As the dynamics in the vicinity of period-one points of (5) has significance for the quantum model discussed in Section 2, we briefly describe the linear stability of the simplified accelerator following ref. 7, Section 4.1b. For definiteness we choose $\mathfrak{F} = \sin\phi$ from now on. Then the mapping (5) has period-one fixed points situated at $\phi = \pi/2$, respectively $\phi = 3\pi/2$, and for the former

$$2\, d\omega/u_0 = (2m + 1/2)\pi, \qquad m \in \mathbb{Z} \tag{6}$$

Putting $u_n = u_0 + \Delta u_n$ and introducing the angle $\theta_n = \phi_n - \pi$, $-\pi < \theta_n < \pi$, we obtain from (5) and (6) the celebrated *standard map* (see refs. 5, 7, and 8 for instance):

$$\begin{aligned} \mathfrak{Q}_{n+1} &= \mathfrak{Q}_n + K\cos\theta_n \\ \theta_{n+1} &= \theta_n + 2\mathfrak{Q}_{n+1} \end{aligned} \tag{7}$$

with coupling parameter $K = \omega/u_0^2$, $(\mathfrak{Q}, \theta) \in \mathbb{R} \times S^1$, and $\mathfrak{Q}_n := \mathfrak{Q}(n-)$. Henceforth, the kick period is normalized to unity and it replaces the time intervals between the nth and $(n+1)$th impacts of the ball at the (moving) plate.

The standard map has been extensively studied (see refs. 7 and 8 and references therein). One of its peculiarities is the 2π-periodicity of both

angle and action variables. For K values small enough the (Ω, θ) surface-of-section is reminiscent of a gravitational pendulum in discrete time. Increasing K leads to an overlap of the separatrices around adjacent primary resonance islands. Once this happens, certain orbits are no longer trapped inside the islands, a feature commonly called transition to chaos.

Remark that the particular choice of the period-one point $(u_0, \pi/2)$ influences the actual value of K in (7). Thus, decreasing particle velocity increases the coupling in the standard map. As described in refs. 5 and 7, there is a critical value u_s such that all primary fixed points of (5) with $u_0 < u_s$ are unstable. Hence, the approximation of the simplified Fermi map (5) in form of the standard map (7) loses its significance for $u_0 < u_s$.

An appropriate tool for the description of the $u_0 < u_s$ portion of phase space is a random phase approximation for the phase coordinate ϕ: Rather than discussing the surface-of-section map (5), Lichtenberg and Lieberman investigated the "time" evolution of the distribution function $\mathscr{P}(u, n)$ in terms of a Markov process in u:

$$\mathscr{P}(u, n + \Delta n) = \int d(\Delta u)\, \mathscr{P}(u - \Delta u, n)\, \mathscr{W}_t(u - \Delta u, n, \Delta u, \Delta n) \tag{8}$$

where $\mathscr{W}_t(u, n, \Delta u, \Delta n)$ is the transition probability for a change in action after the time Δn for an ensemble of phase points having action u at time n. (Technical) details such as the employment of a Fokker–Planck-type equation and the validity of the random phase approximation can be found in refs. 4, 5, and 7. We only remark that the ergodic hypothesis appears to be satisfied, i.e., when computed over sufficiently long times, (8) leads to a uniform density distribution in the corresponding region of phase space.

2. QUANTIZATION

The aim of the first part of this section is the construction of a suitable quantum analog to the simplified classical accelerator. In the second part we discuss dynamical features like quasienergy spectrum, etc., of the obtained quantum model and compare the results to the classical counterpart. In order to introduce a *simplified quantum Fermi accelerator*, we return to the very beginning and define a formal Schrödinger equation $i\partial_t \phi = -\partial_x^2 \phi$ on the family of intervals $\mathscr{I}(a) = \{(0, a(t)\}_{t \in [0, \mathscr{T}]}$. {"Formal," since the LHS involves $\phi(t_1) - \phi(t_2)$ with $\phi(t_i) \in L^2([0, a(t_i)], dx)$, $i = 1, 2$, and in general $a(t_1) \neq a(t_2)$.} The scaling $(x, t) \to (y := x/a(t), t' = t)$ yields the transformed equation on the unit interval as

$$i\partial_t \psi = \{ -[a(t)]^{-2} \partial_y^2 + i(\dot{a}/a)(t)[y\partial_y + 1/2] \} \psi \tag{9}$$

After using again $\mathscr{A}(t)=\hat{t}:=\int_0^t d\tau\, a(\tau)^{-2}$ as an auxiliary time, we see that the periodic family

$$\hat{H}(\hat{t})=-\partial_x^2+i(\dot{a}a)[\mathscr{A}^{-1}(\hat{t})][x\partial_x+1/2] \tag{10}$$

of Hamiltonians on $h:=L^2([0,1],dx)$ emerges. Hence, by $(-i\partial_x, x)\mapsto(\hat{p},\hat{q})$, this formula yields the exact analog to (3). Integration by parts shows that a symmetric family $\{\hat{H}(\hat{t})\}$ is obtained with

$$\begin{aligned}\psi_x(\hat{t}, x=1)&=\{\alpha(\hat{t})+i(\dot{a}a)[\mathscr{A}^{-1}(\hat{t})]/2\}\,\psi(\hat{t}, x=1)\\ \psi_x(\hat{t}, x=0)&=\beta(\hat{t})\,\psi(\hat{t}, x=0)\end{aligned} \tag{11}$$

with numbers $-\infty<\alpha(\hat{t}), \beta(\hat{t})\leqslant\infty$. Note that merely for $\alpha(\hat{t})=\beta(\hat{t})=\infty$ for all $\hat{t}$, i.e., *Dirichlet conditions* at both $x=0$ and $x=1$, (10) is a self-adjoint Laplacian plus on its domain symmetric perturbation. In addition, only the *Dirichlet condition* is dilation-invariant. Thus, choosing the *Dirichlet Laplacian* in (10) implies *Dirichlet conditions* on $\mathscr{I}(a)$ as well.

However, as our aim is the construction of a quantum counterpart to the simplified Fermi map (5), we now disregard the drift term in (10), which in turn requires the neglect of the factor $i(\dot{a}a)[\mathscr{A}^{-1}(\hat{t})]/2$ in (11). Henceforth, the quantum analog to the energy transfer between the classical ball and the (hypothetically) oscillating plate has to be formulated in terms of the logarithmic derivatives resulting from (11).

Remark that this procedure disqualifies the *Dirichlet Laplacian*, since in that case it leads to an "oversimplified model" at the unit interval. Nevertheless, there are several studies of (9) under Dirichlet conditions at both $x=0$ and $x=1$.[9-11]

In view of these features, we therefore introduce the *simplified quantum Fermi accelerator* as the following arrangement:

Definition. A formal Schrödinger equation $-i\partial_t\psi=\partial_x^2(\alpha(t),\beta(t))\psi$ is defined on (0, 1) and the set of solutions obeys the boundary conditions $\psi_x(t, x=0)=\beta(t)\cdot\psi_x(t, x=0)$, $\psi_x(t, x=1)=\alpha(t)\cdot\psi(t, x=1)$ with α, β sufficiently smooth; see below. [Here $-\partial_x^2(\alpha(t),\beta(t))$ is the Laplacian defined on functions obeying the stated boundary conditions.]

What are physically relevant choices of α and β? Intuitively, both functions should be related to the (hypothetical) wall motions. As the "wall" at $x=0$ is at rest, we therefore adopt $\beta(t)=0$ for all $t\in[0,\mathscr{T}]$, whereas the detailed structure of $\alpha(t)$ is left open for the time being; see Conjecture 4. (The usefulness of the Neumann condition in relation to a classical limit of certain time-dependent Schrödinger operators at the unit interval shows up in ref. 12 as well.)

We defer the proof of existence and uniqueness of the propagator $\mathscr{U}_\alpha(t, s)$ solving $-i\partial_t\psi = \partial_x^2(\alpha(t), \beta(t))\psi$ to Theorem 3 and turn directly to the spectral properties of its one-period evolution $\mathscr{U}_\alpha(\mathscr{T}, 0)$. Based on recent work by Howland[13] and Nenciu,[14] it is a straightforward task to determine the absence of $\sigma_{ac}(\mathscr{U}_\alpha(\mathscr{T}, 0))$.

Theorem 1. Let $\alpha \in \mathscr{C}^2((0, \mathscr{T}))$ with $\alpha(t=0) = \alpha(t=\mathscr{T}) = 0$. Then $\mathscr{U}_\alpha(\mathscr{T}, 0)$ is purely singular.

Proof. The proof of the claim follows entirely from Nenciu's article[14] once the analog to his estimate (2) is established. In order to do this, remark that the general solution to

$$-\partial_x^2(\alpha(t), \beta = 0)\,\psi_{n,\alpha}(t) = E_{n,\alpha}(t)\,\psi_{n,\alpha}(t), \qquad t \in [0, \mathscr{T}]$$

reads

$$\psi_{n,\alpha}(t, x) = c_{n,\alpha}(t)\cos[E_{n,\alpha}(t)^{1/2}\,x]$$

with

$$c_{n,\alpha}(t) = \|\cos[E_{n,\alpha}(t)^{1/2}\,x]\|_h^{-1}, \qquad E_{n,\alpha}(t) = (n\pi)^2 + 2\alpha(t) + \mathscr{O}(n^{-1})$$

$$\text{for} \quad n \in \mathbb{N} \text{ sufficiently large}$$

(see ref. 15, p. 2314). Hence, the spectral projection $P_{n,\alpha}(t)$ of $-\partial_x^2(\alpha(t), \beta = 0)$ corresponding to $E_{n,\alpha}(t)$ is given by

$$P_{n,\alpha}(t) = c_{n,\alpha}^2(t)\langle\cos[E_{n,\alpha}(t)^{1/2}\cdot], \cdot\rangle_h \cos[E_{n,\alpha}(t)^{1/2}\cdot]$$

and a short calculation reveals that for $n \in \mathbb{N}$ sufficiently large there is an n-uniform constant $\mathscr{C}$ such that

$$\|(dP_{n,\alpha}/dt)(t)\|_{\mathscr{B}(h)} \leqslant \mathscr{C} n^{-1}$$

Here $\mathscr{B}(h)$ denotes the space of bounded, everywhere defined operators on h.

The remainder of the proof is identical to the corresponding part of ref. 14. For convenience, we recall the main idea, which consists of a two-step iterative adiabatic approximation to the Floquet operator, which has a pure point spectrum. As the difference is shown to be trace-class, the Birman–Krein theorem[16] implies the absence of $\sigma_{ac}(\mathscr{U}_\alpha(\mathscr{T}, 0))$.

To be more detailed, denote by $B(t) := i\sum_{n\in\mathbb{N}} P_n(t)(dP_n/dt)(t)$ the self-adjoint operator on h obeying

$$i(dP_n/dt)(t) = [P_n(t), B(t)]$$

with $[A, B] := AB - BA$. All claims about the operators involved are proven in ref. 14; we also skip the α (sub)index.

In addition, define $H_1(t) := -\partial_x^2 + B(t)$ and corresponding spectral projection $P_{1,n}(t)$ as well as the self-adjoint trace class operator $B_1(t)$:

$$B_1(t) = i \sum_{n \in \mathbb{N}} P_{1,n}(t)\{i(dP_{1,n}/dt)(t) + [\partial_x^2, P_{1,n}(t)]\}$$

Finally, let $\mathscr{V}(t, s)$ be determined by

$$-i[d\mathscr{V}(\cdot, s)/dt](t) = [\partial_x^2 + B_1(t)]\,\mathscr{V}(t, s), \qquad \mathscr{V}(s, s) = \mathbb{1}$$

and consider the *Moeller operator* corresponding to the pair $-\partial_x^2$ and $H_1(t)$, i.e.,

$$\Omega(t, s) := \mathscr{V}(t, s)^* \mathscr{U}_\alpha(t, s)$$

It turns out that $\Omega(t, s) - \mathbb{1}$ is trace-class and the Floquet operator $\mathscr{U}_\alpha(\mathscr{T}, 0)$ can be written as

$$\mathscr{U}_\alpha(\mathscr{T}, 0) = \mathscr{V}(\mathscr{T}, 0)[\mathbb{1} + \Omega(\mathscr{T}, 0) - \mathbb{1}]$$

As $\mathscr{V}(\mathscr{T}, 0)$ is pure point due to $P_{1,n}(\mathscr{T}) = P_{1,n}(0)$ being rank one and $[\mathscr{V}(\mathscr{T}, 0), P_{1,n}(\mathscr{T})] = 0$, the standard argument on trace-class differences of unitary operators[16] finishes the proof. ∎

Remark. The content of Theorem 1 is very encouraging, as it is the general consensus that $\sigma_{ac}(\mathscr{U}_\alpha(\mathscr{T}, 0)) = \varnothing$ implies a limited energy gain of the quantum system in question[17] [i.e., a best some diffusive increase in energy due to $\sigma_{sc}(\mathscr{U}_\alpha(\mathscr{T}, 0)) \neq \varnothing$ might occur].

Henceforth, quantum and classical dynamics seem to share the same characteristics with respect to global stability. To determine the absence of any global diffusion in the quantum model, however, more detailed studies are necessary.

To this end, we introduce the *extended Hilbert space* $\mathscr{H} := L^2([0, \mathscr{T}], dt) \otimes L^2([0, 1], dx)$ fitted with the canonical scalar product. Note that $\mathscr{H}$ is the quantum analog to the extended phase space in classical mechanics. So-called *Floquet Hamiltonians* $K = -i\partial_t + H(t)$ defined on $\mathscr{H}$, with $\{H(t)\}$ a periodic family, represent the particular quantum system plus external perturbation; see ref. 18, for instance.

Hence, we shall be interested in self-adjoint realizations on $\mathscr{H}$ of

$$\mathscr{K}(\alpha, \beta = 0) = -i\partial/\partial t - \partial^2/\partial x^2 + \alpha(t)\,\delta(x - 1) \tag{12}$$

where the *Dirac delta distribution* formally indicates the nontrivial logarithmic derivative at $x=1$ of $\psi \in \mathscr{D}(-\partial_x^2(\alpha(t), \beta=0))$.

To furnish (12) with a rigorous meaning, we introduce several operators on $\mathscr{H}$:

(i) "Free" oscillations between two fixed walls are sketched on $\mathscr{H}$ by the self-adjoint operator $K_0 := \dot{K}_0$

$$\dot{K}_0 = \overline{D \otimes \mathbb{I}} + \overline{\mathbb{I} \otimes (-\Delta_N)} \tag{13}$$

where the self-adjoint $D := -i\partial/\partial t$ is defined on $\mathscr{T}$-periodic functions and $-\Delta_N$ is the Neumann Laplacian.

(ii) Time-dependent boundary conditions enter via the Floquet Hamiltonian $\dot{K}_\alpha$, which is given by

$$\begin{aligned} \dot{K}_\alpha &= \overline{D \otimes \mathbb{I}} - \int_{[0,\mathscr{T}]}^{\oplus} dt\, \partial_x^2(\alpha(t), \beta=0) \\ \mathscr{D}(\dot{K}_\alpha) &= \{\Psi \in \mathscr{H}\colon \dot{K}_\alpha \Psi \in \mathscr{H} \text{ with } \Psi(0,\cdot) = \Psi(\mathscr{T},\cdot),\ \Psi_x(\cdot, x=0) = 0 \\ &\qquad \text{and } \Psi_x(\cdot, x=1) = \alpha(\cdot)\, \Psi(\cdot, x=1) \text{ a.e.}\} \end{aligned} \tag{14}$$

Thus, from a formal point of view, both operators act as the partial derivative $k := -i\partial/\partial t - \partial^2/\partial x^2$, i.e., we have to deal with different self-adjoint extensions of a common (maximal) symmetric operator $\dot{\mathscr{K}}$, which is defined as

$$\begin{aligned} \mathscr{D}(\dot{\mathscr{K}}) &= \{f \in \mathscr{H}\colon \partial f/\partial t \in \mathscr{H},\ \partial^2 f/\partial x^2 \in \mathscr{H} \text{ with} \\ &\qquad f(0,\cdot) = f(\mathscr{T},\cdot),\ f_x(\cdot, x=0) = 0 \\ &\qquad \text{and supp } f(\cdot) \subset [0, m_f],\ m_f < 1 \text{ a.e.}\} \\ \dot{\mathscr{K}} f &= [-i\partial/\partial t - \partial^2/\partial x^2] f \qquad \forall f \in \mathscr{D}(\dot{\mathscr{K}}) \end{aligned}$$

Proposition 2. The closed symmetric operator $\dot{K} = \dot{\mathscr{K}}^{**}$ has deficiency indices equal to infinity and the defect space $\ker(\dot{K}^* - z)$ is the closed linear hull $\overline{\mathbb{L}(\mathscr{S}(z))}$ of the set

$$\mathscr{S}(z) := \{\Phi_j(z; t, x) = c_j(z) \exp(2\pi i j t/\mathscr{T}) \cos[(z - 2\pi j/\mathscr{T})^{1/2} x]\}$$

with $c_j(z) := \|\Phi_j(z)\|_{\mathscr{H}}^{-1}$.

We omit the proof of Proposition 2 since it is identical to the proof of Lemma 2.1 of ref. 19. Proposition 2 allows the determination of the self-adjoint Floquet Hamiltonian corresponding to the simplified quantum Fermi accelerator.

Theorem 3. Let $\alpha \in \mathscr{C}^1((0, \mathscr{T}))$ such that $\alpha(t=0)=\alpha(t=\mathscr{T})=0$. Then there is a self-adjoint Floquet Hamiltonian $K(\alpha)$ given by the operator closure of $\dot{K}_\alpha$ defined in (14). Furthermore,

$$\{\exp[i\mathscr{T}K(\alpha)]\Psi\}(\mathscr{T}) = \mathscr{U}_\alpha(\mathscr{T}, 0)\,\Psi(0) \qquad \forall \Psi \in \mathscr{D}(\dot{K}_\alpha)$$

Proof. From Proposition 2 we infer $\dot{K}_\alpha^* \subset \mathscr{H}^*$. Yet, integration by parts assures us that $\mathbb{L}(\mathscr{S}(z))$ is not contained in $\mathscr{D}(\dot{K}_\alpha^*)$. Therefore the operator $\dot{K}_\alpha$ has deficiency indices $(0, 0)$. Owing to $\alpha \in \mathscr{C}^1((0, \mathscr{T}))$, the boundary conditions are compatible.

Hence, $K(\alpha)$ generates the one-parameter unitary group $\{\exp[i\tau K(\alpha)]\}_{\tau \in \mathbb{R}}$ and the Trotter product formula applies (ref. 20, for instance):

$$\exp[i\tau K(\alpha)] = \text{s-lim}_{n \to \infty} \left\{ \exp[i(\tau/n)\,\overline{D \otimes \mathbb{I}}] \times \exp\left[-i(\tau/n) \int_{[0, \mathscr{T}]}^{\oplus} dt\, \partial_x^2(\alpha(t), \beta=0) \right] \right\}^n \tag{15}$$

Using the shift properties of $\{\exp(i\tau D)\}$ and

$$\exp\left[-i\tau \int_{[0, \mathscr{T}]}^{\oplus} dt\, \partial_x^2(\alpha(t), \beta=0) \right] \Psi(t) = \exp[-i\tau \partial_x^2(\alpha(t), \beta=0)]\,\Psi(t)$$

we can explicitly compute the nth term on the RHS of (15). For instance, with $n=1$ we obtain

$$\exp(i\tau\overline{D \otimes \mathbb{I}}) \exp\left[-i\tau \int_{[0, \mathscr{T}]}^{\oplus} dt\, \partial_x^2(\alpha(t), \beta=0) \right] \Psi(t)$$
$$= \sum_{k \in \mathbb{N}} \exp[i\tau E_{k,\alpha}(t-\tau)] \langle \psi_{k,\alpha}(t-\tau), \Psi(t-\tau) \rangle_h\, \psi_{k,\alpha}(t-\tau)$$

with $E_{k,\alpha}(t)$ and $\psi_{k,\alpha}(t)$ defined in Theorem 1. Iteration of (15) for arbitrary $n \in \mathbb{N}$ shows that in every step the object of evolution Ψ appears as $\Psi(t-\tau)$. Hence, from the definition of the norm on $\mathscr{H}$ and (15) we infer that there exists a unitary transformation $\mathscr{U}_\alpha(t, t-\tau)$ on $h = L^2([0, 1], dx)$. Finally, strong differentiation on $\mathscr{D}(\dot{K}_\alpha)$ proves that $\mathscr{U}_\alpha(t, t-\tau)$ solves

$$-i\partial_t \psi = \partial_x^2(\alpha(t), \beta=0)\psi \quad \blacksquare$$

Henceforth, there indeed exists a spectral equivalence between $K(\alpha)$ and $\mathscr{U}_\alpha(\mathscr{T}, 0)$. In particular,

$$\exp(i\mathscr{T}\lambda)\psi_\lambda = \mathscr{U}_\alpha(\mathscr{T}, 0)\psi_\lambda$$

for $\lambda \in \sigma_{pp}(K(\alpha))$ with corresponding eigenfunction(s) ψ_λ. Theorem 3 permits the investigation of $\sigma(K(\alpha))$ instead of $\sigma(\mathcal{U}_\alpha(\mathcal{T}, 0))$. However, to relate $K(\alpha)$ to the classical simplified accelerator, the "explicit" form of α has to be determined!

As we are unaware of any physical argument providing a detailed knowledge of α, we shall use the approximations to the classical map (5) discussed in Section 1 as a guide.

In particular, we note that the quantization of the standard map is unambiguous. In ref. 21 we introduce a quantum standard map and relate a certain "linearization" of it to the classical map linearized around its period-one fixed points. The physical idea behind this procedure is *Ehrenfest's theorem*,[22] which connects expectations of linear quantum systems to the classical dynamics. (For a related investigation see ref. 23.) Henceforth, using the deviation via the (quantum) standard map, we obtain a quantum model which corresponds to the classical Fermi map in vicinities of its (stable) period-one fixed points.

To be more detailed, remark that (7) basically is a mapping in ($-$external energy, time) variables (the minus sign refers to Newton's third law: *actio est reactio*). In the extended Hilbert space approach the external energy E is represented by $i\partial/\partial t$ on $L^2([0, \mathcal{T}], dt)$ and in case of the Fermi problem the operator $-D$ is appropriate (see ref. 19 for the form of the external energy operator in the kicked rotor model).

Therefore, quantization of (7) yields the change in the external energy due to the periodic δ-kick-like interactions with the "quantum ball" as

$$D_+ = D_- - \kappa[\dot{a}(\omega t)]_- \tag{16}$$

with $\kappa \in \mathbb{R}$ related to K from (7) and the plus and minus signs refer to the auxiliary time τ measuring the number of interactions.[21] The brackets around $\dot{a}$ express the fact that the "time operator" conjugate to $-D$ has to be some $\mathcal{T}$-periodic multiplication. {We refer to ref. 21 for details on the τ evolution of $[\dot{a}(\omega t)]_-$.} On the other hand, there is always global conservation of energy:

$$dE/d\tau = -\partial H/\partial t = \partial\{\partial_x^2(\alpha(t), \beta = 0)\}/\partial t \tag{17}$$

Thus, combining the quantization of (7), i.e., (16) as one part of it, and (17), we conclude that by adopting $\alpha_0(t) \cong \mathfrak{F}(\omega t)$ with $\mathfrak{F}$ defined in (5), we obtain a quantum counterpart to the simplified classical Fermi accelerator in the interval of intermediate u values, i.e., in the region of local stability where the standard map (7) is a meaningful approximation to the full mapping (5).

The actual study of spectral properties of $K(\alpha_0)$ is rather involved.[24]

We therefore present the final result for the special case of $\alpha_0(\kappa, t) = k \sin 2\pi t/\mathscr{T}$ as a proposition without proof. Mathematical details can be found in ref. 19, where an analogous machinery is developed to investigate the dynamics of the kicked rotor.

Proposition 4. Define $K(\alpha_0)$ as in Theorem 3 with $\alpha_0(\kappa, t) = \kappa \sin 2\pi t/\mathscr{T}$, $|\kappa| \in \mathbb{R}^+$, sufficiently small and $\mathscr{T} = 2/\pi\nu$, ν typically Diophantine. Then $\sigma(K(\alpha_0)) = \overline{\sigma_{pp}(K(\alpha_0))}$ and

$$\sigma_{pp}(K(\alpha_0)) \subset \{j\nu\pi^2 + \gamma_n^2(\pm\kappa)\}_{(j,n)\in\mathbb{Z}\times\mathbb{N}}$$

Here $\gamma_n(\kappa)$ is the root of the equation $\gamma_n(\kappa) = -\kappa \operatorname{ctg} \gamma_n(\kappa)$ and $\gamma_n^2(\pm\kappa) = (n\pi)^2 \pm 2\kappa + \mathscr{O}(n^{-1})$ for $n \in \mathbb{N}$ sufficiently large.

The content of Proposition 4 highlights the relationship between the quasienergy eigenvalues and the period-one fixed points of (5), since the roots $\gamma_n^2(\pm\kappa)$ correspond to $\hat{\alpha}_0^{\pm} = \pm\kappa$. Hence, both sets are determined by the extremal values of the external perturbation. In physical terms, both the quantum eigenvalues and the period-one fixed points are related to the (hypothetical) turning points of the wall oscillation. (A similar observation is made in the kicked rotor model.[19, 21]) Thus, for small enough coupling (and sufficiently short periods in time[21]), the quantum system defined above ($\triangleq$"*partially linearized quantum Fermi accelerator*") and the classical map (7) share similiar dynamics.

However, as there are technical limitations on the method of proof to be applied to Proposition 4, we are unable to follow the *route to chaos* in the actual accelerator, i.e., increase the coupling κ to move toward the stochastic sea; cf. Section 1. Yet, Proposition 4 indicates that the choice of α_0 is reasonable in a limited approximation of the simplified Fermi accelerator problem.

Finally, what about a quantum analog to the stochastic portion of the classical dynamics? Again we propose to follow the classical approximation: Equation (8) provides a probabilistic account of the change in external energy due to the ball-oscillating wall interaction. As relation (17) is still valid, we therefore propose that the *forcing function α in (14) has to carry some random characteristics*—presumably modeled along the classical situation described in (8). (After all, the external perturbation is *always* classical.) However, since the theory of random (unbounded) perturbations of Schrödinger operators is not as developed as its deterministic counterpart, we face severe technical challenges in even thinking about quasienergy spectra of "operators $K(\alpha(\Omega)) = \overline{D \otimes \mathbb{I} - \Delta(\alpha(\Omega))}$."

It is known that singular spectra are unstable under certain random

perturbations,[25] whereas on the other hand, various random Schrödinger operators on the half-line $\mathbb{R}^+$ tend to have pure point spectrum a.s.[26]

Thus, the existence of a singular continuous component in $\sigma(K(\alpha(\Omega)))$ cannot be excluded from the very beginning. Moreover, as seen from Proposition 4, the quasienergy eigenvalues of the linearized accelerator are related to the stable period-one points of the classical map. Hence, in the absence of the latter we may indeed hope to find traces of classical chaos in form of a singular continuous quasienergy spectrum.

Therefore, the following dynamical structure emerges in the simplified quantum Fermi accelerator model: *Global stability is achieved by purely singular quasienergy spectrum for* $\alpha \in \mathscr{C}^2$ *and in the approximation sketched above we observe pure point quasienergy spectrum of the approximate ("partially linearized") Floquet operator and that model is in correspondence to the intermediate phase space region of the classical accelerator.* Whether or not the stochastic region of the latter finds its quantum equivalent in $\sigma_{sc}(\mathscr{U}_\chi(\mathscr{T}, 0)) \neq \varnothing$ is open to speculation.

REFERENCES

1. E. Fermi, *Phys. Rev.* **15**:1169 (1949).
2. I. Percival and D. Richards, *Introduction to Dynamics* (Cambridge University, Cambridge, 1982).
3. S. Ulam, In *Proceedings 4th Berkeley Symposium on Mathematical Statistics and Probability* (University of California Press, 1961), Vol. 3, p. 315.
4. A. J. Lichtenberg and M. A. Lieberman, *Phys. Rev. A* **5**:1852 (1972).
5. A. J. Lichtenberg and M. A. Lieberman, *Physica* **1D**:291 (1980).
6. G. Bolz and T. Krüger, in G. Bolz, Ph.d. Thesis, Universität Bielefeld (1991).
7. A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer, New York, 1983).
8. B. V. Chirikov, *Phys. Rep.* **52**:263 (1979).
9. G. Karner, *Lett. Math. Phys.* **17**:329 (1989).
10. J. S. Howland, Spectrum of the Fermi accelerator, preprint, University of Virginia (1992).
11. P. Šeba, *Phys. Rev. A* **41**:2306 (1990).
12. G. Karner, *J. Math. Anal. Appl.* **164**:206 (1992).
13. J. S. Howland, *Ann. Inst. H. Poincaré A* **50**:309, 325 (1989).
14. G. Nenciu, *Ann. Inst. H. Poincaré A* **59**:91 (1993).
15. N. Dunford and J. T. Schwartz, *Linear Operators*, Part III (Interscience, New York, 1963).
16. M. S. Birman and M. G. Krein, *Dokl. Akad. Nauk SSSR* **144**:475 (1962).
17. J. M. Combes, Connections between quantum dynamics and spectral properties of time-evolution operators, in *Differential Equations with Applications to Mathematical Physics*, W. F. Ames, E. M. Harrell II, and J. V. Herod, eds. (Academic Press, New York, 1993).
18. H. Jauslin, Stability and chaos in classical and quantum Hamiltonian systems, in *Proceedings II Granada Seminar on Computational Physics*, P. Carrido and J. Marro, eds. (World Scientific, Singapore, 1993).
19. G. Karner, The dynamics of the quantum standard map, preprint, University of Virginia (1993).

20. M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. I (Academic Press, New York, 1978).
21. G. Karner, *Nonlinearity* 7:623 (1994).
22. P. Ehrenfest, *Z. Physik* **45**:455 (1927).
23. R. F. Fox and B. L. Lan, *Phys. Rev. A* **41**:2952 (1990).
24. G. Karner, Quantum Fermi maps revisited, in *Proceedings of the IXth Bielefeld Encounters in Mathematics and Physics*, S. Albeverio and L. Streit, eds. (World Scientific, Singapore, 1993).
25. J. S. Howland, Random perturbations of singular spectra, preprint, University of Virginia (1991).
26. S. Kotani, *Contemp. Math.* **50**:277 (1986).