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Surface-Induced Finite-Size Effects for First-Order Phase Transitions

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We consider classical lattice models describing first-order phase transitions, and study the finite-size scaling of the magnetization and susceptibility. In order to model the effects of an actual surface in systems such as small magnetic clusters, we consider models with free boundary conditions. For a field-driven transition with two coexisting phases at the infinite-volume transition point $h = h_t$, we prove that the low-temperature, finite-volume magnetization $m_{\text{free}}(L, h)$ per site in a cubic volume of size L^d behaves like

$$m_{\text{free}}(L, h) = \frac{m_+ + m_-}{2} + \frac{m_+ - m_-}{2} \tanh\left[\frac{m_+ - m_-}{2} L^d(h - h_\chi(L))\right] + O\left(\frac{1}{L}\right)$$

where $h_\chi(L)$ is the position of the maximum of the (finite-volume) susceptibility and $m_\pm$ are the infinite-volume magnetizations at $h = h_t + 0$ and $h = h_t - 0$, respectively. We show that $h_\chi(L)$ is shifted by an amount proportional to $1/L$ with respect to the infinite-volume transition point h_t provided the surface free energies of the two phases at the transition point are different. This should be compared with the shift for periodic boundary conditions, which for an asymmetric transition with two coexisting phases is proportional only to $1/L^{2d}$. One can consider also other definitions of finite-volume transition points, for example, the position $h_U(L)$ of the maximum of the so-called Binder cumulant $U_{\text{free}}(L, h)$. While $h_U(L)$ is again shifted by an amount proportional to $1/L$ with respect to the infinite-volume transition point h_t, its shift with respect to $h_\chi(L)$ is of the much smaller order $1/L^{2d}$. We give explicit formulas for the proportionality factors, and show that, in the leading $1/L^{2d}$ term, the relative shift is the same as that for periodic boundary conditions.

KEY WORDS: First-order phase transitions; finite-size scaling; free boundary conditions; surface effects.

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1. INTRODUCTION

In the last 20 years, the study of finite-size (FS) effects near first- and second-order phase transitions has gained increasing interest. While the study of FS effects for second-order phase transitions goes back to the work of Fisher and coworkers in the early 1970s,[16, 18, 15] finite-size effects for first-order phase transitions were first considered by Imry[21] and then by Fisher and Berker,[17] Blöte and Nightingale,[10] Binder and coworkers,[1, 9, 13] Privman and Fisher,[22] and others.

Recently, these studies have been systematized in a rigorous framework by Borgs and Kotecký[6] (see also refs. 7 and 8) and by Borgs and Imbrie.[4, 5, 2] Their results cover both finite-size effects in cubic volumes and long cylinders, and both field- and temperature-driven transitions, but were always limited to periodic boundary conditions. While the periodic boundary conditions are natural for the description of computer experiments that are used to study the bulk properties of a system (note that periodic boundary conditions are used in these computer experiments because they minimize the unwanted finite-size effects), they do not allow for the description of FS effects in actual physical systems, e.g., small magnetic clusters, where surface effects are of major importance.

In this paper we start a rigorous study of such surface effects. We consider spin systems in a finite box $\Lambda = \{1,..., L\}^d$, imposing free or so-called "weak" boundary conditions (see Section 2) instead of the periodic boundary conditions used in our previous work.

In order to explain our main ideas, let us first review the FSS for a system in a periodic box.[6–8] For a system describing the coexistence of two phases, say an Ising magnet at low temperatures, the partition function with periodic boundary conditions can be approximated by

$$Z_{\text{per}}(L, h) \cong Z_+(L, h) + Z_-(L, h) \tag{1.1}$$

where $Z_\pm$ contain small perturbations of the ground-state configurations $\sigma_\Lambda \equiv +1$ and $\sigma_\Lambda \equiv -1$, respectively. The error terms coming from the tunneling configurations can be bounded by $O(L^d e^{-L/L_0}) e^{-f(h)}$, where $f(h)$ is the free energy of the system and L_0 is a constant of the order of the infinite-volume correlation length.

In the asymptotic (large-volume) behavior of $\log Z_\pm$ there should appear, in principle, volume, surface,..., and corner terms. A periodic box, however, has neither surface,.., nor edges or corners, and one obtains

$$\begin{aligned} Z_{\text{per}}(L, h) &\cong e^{-f_+(h) L^d} + e^{-f_-(h) L^d} \\ &= 2 \cosh\left(\frac{f_+(h) - f_-(h)}{2} L^d\right) e^{-[f_+(h) + f_-(h)] L^d/2} \end{aligned} \tag{1.2}$$

where $f_+(h)$ and $f_-(h)$ are the (metastable) free energies of the plus and minus phases, respectively. Taylor expanding $f_\pm(h)$ around the transition point h_t, and introducing the spontaneous magnetizations $m_\pm$ of the plus and minus phases at h_t, one obtains the FSS of the magnetization

$$m_{\text{per}}(L, h) = L^{-d} d \log Z_{\text{per}}(L, h)/dh$$

in the form

$$m_{\text{per}}(L, h) \cong \frac{m_+ + m_-}{2} + \frac{m_+ - m_-}{2} \tanh\left[\frac{m_+ - m_-}{2}(h - h_t) L^d\right] \tag{1.3}$$

It describes the rounding of the infinite-volume transition in a region of width

$$\Delta h \sim L^{-d} \tag{1.4}$$

with a shift $h_t(L) - h_t$ that vanishes in the approximation (1.3). A more accurate calculation shows that, in fact, for a system describing the coexistence of two low-temperature phases at the infinite-volume transition point h_t and with infinite-volume susceptibilities $\chi_\pm$, one has

$$h_\chi(L) - h_t = \frac{6(\chi_+ - \chi_-)}{(m_+ - m_-)^3} L^{-2d} + O(L^{-3d}) \tag{1.5}$$

if $h_\chi(L)$ is defined as the position of the maximum of the susceptibility in the volume L^d.

Turning to free boundary conditions, we again expand $\log Z_\pm(L, h)$ into volume-surface-...-corner terms. This time, however, the volume Λ has a boundary, and the expansion yields

$$-\log Z_\pm(L, h) = f_\pm^{(d)}(h) L^d + f_\pm^{(d-1)}(h) 2dL^{d-1} + O(L^{d-2}) \tag{1.6}$$

where $f_\pm^{(d)}(h) = f_\pm(h)$ are the (metastable) bulk free energies, while $f_\pm^{(d-1)}(h)$ are the (metastable) surface free energies of the plus and minus phases, respectively. As a consequence, (1.2) gets replaced by

$$Z_{\text{free}}(L, h) \cong \exp\left(-\frac{f_+(h) + f_-(h)}{2} L^d - \frac{f_+^{(d-1)}(h) + f_-^{(d-1)}(h)}{2} 2dL^{d-1}\right)$$
$$\times 2 \cosh\left(\frac{f_+(h) - f_-(h)}{2} L^d + \frac{f_+^{(d-1)}(h) - f_-^{(d-1)}(h)}{2} 2dL^{d-1}\right) \tag{1.7}$$

At this point, one major difference with respect to (1.2) appears: while the free energies f_+ and f_- are equal at the transition point h_t, the surface free energies are typically different at h_t [obviously, there are systems for which $\tau_+ := f_+^{(d-1)}(h_t)$ and $\tau_- := f_-^{(d-1)}(h_t)$ are equal, such as in the symmetric Ising model where $\tau_+ = \tau_-$ by symmetry, but for asymmetric first-order transitions, this is typically not the case]. The leading terms in the expansion around h_t then lead to the formula

$$m_{\text{free}}(L, h) \cong \frac{m_+ + m_-}{2} + \frac{m_+ - m_-}{2} \tanh\left\{\frac{m_+ - m_-}{2}\,[(h - h_\chi(L)]\,L^d\right\} \quad (1.8)$$

Here

$$h_\chi(L) = h_t + \frac{\tau_+ - \tau_-}{m+ - m_-}\,\frac{2d}{L} + O\left(\frac{1}{L^2}\right) \quad (1.9)$$

which, for $\tau_- \neq \tau_+$, is now proportional to $1/L$, while the width Δh of the transition is still proportional to L^{-d}.

In fact, a formula of the form (1.8) has already been given in ref. 23, with heuristic arguments very similar to those presented above. Here, our goal is twofold: first, we want to make the arguments leading to (1.8) rigorous, deriving at the same time precise error bounds on the subleading terms [in fact, our method allows one to calculate in a systematic way the corrections to (1.8) in terms of an infinite asymptotic series in powers of $1/L$]. Second, we want to generalize these results to a wider class of situations, including, in particular, the finite-size scaling of expectation values of arbitrary local observables.

It will turn out that the more precise analysis of the subleading terms reveals an interesting fact: if one considers other standard definitions of the finite-volume transition points, e.g., the position $h_U(L)$ of the maximum of the so-called Binder cumulant $U_{\text{free}}(L, h)$, one finds that all of them are shifted, with respect to the infinite-volume transition point h_t, by an amount proportional to $1/L$. Their mutual shifts, however, are of the much smaller order $1/L^{2d}$, with proportionality factors that are the same as those for the corresponding shifts with periodic boundary conditions; see Section 2 for the precise statements.

The finite-size scaling of local observables, on the other hand, will lead to the construction of certain "metastable" states $\langle\cdot\rangle_\pm^h$ and their finite-volume analogs $\langle\cdot\rangle_\pm^{L,h}$, such that

$$\langle A\rangle_{\text{free}}^{L,h} \cong \frac{A_+(L) + A_-(L)}{2} + \frac{A_+(L) - A_-(L)}{2} \times \tanh\left\{\frac{m_+ - m_-}{2}\,[h - h_\chi(L)]\,L^d\right\} \quad (1.10)$$

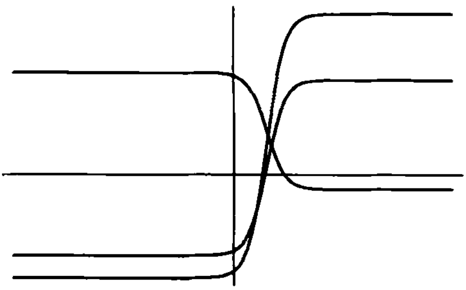

Fig. 1. Finite-size scaling of three different observables.

Here $A_\pm(L) = \langle A\rangle^{L,h_t}_\pm$ differ from the corresponding infinite-volume expectation values $A_\pm = \langle A\rangle^{h_t}_\pm$ by an amount which is exponentially small in the distance dist(supp A, $\partial\Lambda$); see Theorem 3.2 in Section 3.4 for the precise statement in the more general context of N-phase coexistence. Note that the argument of the hyperbolic tangent in (1.10) is the same as in (1.9), and is independent of the particular choice for A. Thus the finite-size scaling of all local observables is synchronized in the sense that, after subtracting the "offset" $[A_+(L) - A_-(L)]/2$, the functions $\langle A\rangle^{L,h}_{\text{free}}$ asymptotically only differ by a constant factor; see Fig. 1.

The organization of the paper is as follows: in the next section we present, in Theorem A, our main results for the finite-size scaling of the magnetization and susceptibility in the context of a field-driven transition with two coexisting phases. Section 3 is devoted to the contour representation of the models considered in Section 2, together with our main assumptions and results for a more abstract class of models describing the coexistence of N phases. We state two main theorems concerning the finite-size scaling: Theorem 3.1 on partition functions and other thermodynamic quantities, and Theorem 3.2 on the finite-size scaling of local observables. In Section 4 we construct suitable metastable free energies and prove Theorem 3.1, deferring the technical details to the appendices. In Section 5 we construct metastable states and prove the corresponding theorem, Theorem 3.2. In Section 6 we prove the results stated in Section 2, using the abstract results formulated in Section 3.

2. FIELD-DRIVEN TRANSITIONS

2.1. Definition of the Model

In order to explain our main ideas, we consider an *asymmetric* version of the *Ising model.* Working on a finite lattice $\Lambda = \{1,..., L\}^d$, $d \geq 2$, we consider configurations $\sigma_\Lambda : i \mapsto \sigma_i \in \{-1, 1\}$ and the reduced Hamiltonian

$$H(\sigma_\Lambda) = \frac{J}{4} \sum_{\langle ij \rangle \subset \Lambda} |\sigma_i - \sigma_j|^2 - h \sum_{i \in \Lambda} \sigma_i + \sum_{A \subset \Lambda} \kappa_A \prod_{i \in A} \sigma_i \tag{2.1}$$

where J is the reduced coupling (containing a factor $\beta = 1/k_B T$), the first sum goes over nearest neighbor pairs $\langle ij \rangle$, while the third one is a finite-range (i.e., $\kappa_A = 0$ for diam $A < R$, where $R < \infty$) perturbation with translation-invariant coupling constants $\kappa_A \in \mathbb{R}$. While the first two terms in (2.1) describe the standard Ising model, the third term is a perturbation that may break the $+/-$ symmetry of the Ising model. We will assume that it is small in the sense that

$$\|\kappa\| = \sum_{A:\, 0 \in A} \frac{|\kappa_A|}{|A|} \leqslant b_0 J$$

where $b_0 > 0$ is a constant to be specified in Theorem A below.

The partition function with *free boundary conditions* is

$$Z_{\text{free}}(L, h) = \sum_{\sigma_\Lambda} e^{-H(\sigma_\Lambda)} \tag{2.2}$$

The derivatives of its logarithm define the corresponding *magnetization*

$$m_{\text{free}}(L, h) = L^{-d} \frac{d}{dh} \log Z_{\text{free}}(L, h) \tag{2.3}$$

and the *susceptibility*

$$\chi_{\text{free}}(L, h) = \frac{d}{dh} m_{\text{free}}(L, h) \tag{2.4}$$

The *Binder cumulant* $U_{\text{free}}(L, h)$ is given as

$$U_{\text{free}}(L, h) = -\frac{\langle M^4 \rangle_c}{3\langle M^2 \rangle^2} = \frac{3\langle (M - \langle M \rangle)^2 \rangle^2 - \langle (M - \langle M \rangle)^4 \rangle}{3\langle (M - \langle M \rangle)^2 \rangle^2} \tag{2.5}$$

where $\langle \cdot \rangle$ denotes expectations with respect to the Gibbs measure corresponding to (2.1), $\langle \cdot \rangle_c$ denotes the corresponding truncated expectation values, and $M = \sum_{i \in \Lambda} \sigma_i$. Note that $U_{\text{free}}(L, h) \leqslant 2/3$ by the inequality $\langle F^2 \rangle \geqslant \langle F \rangle^2$ [applied to $F = (M - \langle M \rangle)^2$].

2.2. Heuristic Background, Main Ideas

For low temperatures (i.e., large J), the leading contributions to the partition function come from the constant ground-state configurations $\sigma_\Lambda \equiv -1$ and $\sigma_\Lambda \equiv +1$. In this approximation,

$$Z_{\text{free}}(L, h) \cong e^{-E_+(L, h)} + e^{-E_-(L, h)} \tag{2.6}$$

where

$$E_{\pm}(L, h) = \sum_{i \in \Lambda} e_{\pm}(i) \tag{2.7}$$

with the position-dependent "ground-state energies"

$$e_{\alpha}(i) = \sum_{A \subset \Lambda:\, i \in A} \kappa_A \frac{\alpha^{|A|}}{|A|} - h\alpha, \qquad \alpha = \pm 1 \tag{2.8}$$

In the same approximation, the magnetization $m_{\text{free}}(L, h)$ and susceptibility $\chi_{\text{free}}(L, h)$ are given by

$$m_{\text{free}}(L, h) \cong \tanh\left(\frac{E_{-}(L, h) - E_{+}(L, h)}{2}\right) \tag{2.9}$$

and

$$\chi_{\text{free}}(L, h) \cong L^d \cosh^{-2}\left(\frac{E_{-}(L, h) - E_{+}(L, h)}{2}\right) \tag{2.10}$$

Observing that $e_{\alpha}(i)$ differs from the bulk value e_{α} if i is in the vicinity of $\partial\Lambda$, we expand $E_{\pm}(L, h)$ into a bulk term $e_{\pm} L^d$ plus boundary terms,

$$E_{\pm}(L, h) = e_{\pm}(h) L^d + e_{\pm}^{(d-1)}(h) 2dL^{d-1} + O(L^{d-2}) \tag{2.11}$$

While, still within the approximation by ground states, the bulk transition point h_0 is the value of h at which $e_{+}(h) = e_{-}(h)$, the finite-volume transition point $h_0(L)$ corresponds to the equality of $E_{+}(L, h)$ and $E_{-}(L, h)$. By (2.11), this leads to a shift

$$h_0(L) - h_0 = O(1/L) \tag{2.12}$$

Notice that for periodic boundary conditions we get $h_0(L) = h_0$ for zero temperature and, for nonvanishing temperatures, a shift $h_0(L) - h_0$ proportional to $1/L^{2d}$ for periodic b.c.[6, 7]

In order to make the above considerations rigorous, one has to take into account the excitations around the two ground states $\sigma_{\Lambda} \equiv \pm 1$. This is done in Sections 3 and 4 and leads to a representation

$$Z_{\text{free}}(L, h) = (e^{-F_{+}(L, h)} + e^{-F_{-}(L, h)})[1 + O(L^d e^{-L/L_0})] \tag{2.13}$$

where L_0 is a constant of the order of the infinite-volume correlation length and $F_{\pm}(L, h)$ have an asymptotic expansion similar to (2.11), namely

$$F_{\pm}(L, h) = f_{\pm}(h) L^d + f_{\pm}^{(d-1)}(h) 2dL^{d-1} + O(L^{d-2}) \tag{2.14}$$

where $f_{\pm}(h)$ are metastable free energies and $f_{\pm}^{(d-1)}(h)$ are (metastable) surface free energies. Once these results (see Theorem 3.1 in Section 3 for the precise statements) are proven, we obtain the desired finite-size scaling results by a rigorous version of the method presented in the introduction.

2.3. Statements of Results

In order to state our results in the form of a theorem, we introduce, for $h \neq h_t$, the *free energy*

$$f(h) \equiv f^{(d)}(h) = -\lim_{L \to \infty} L^{-d} \log Z_{\text{free}}(L, h) \tag{2.15a}$$

the *surface free energy*

$$f^{(d-1)}(h) = -\lim_{L \to \infty} \frac{1}{2dL^{d-1}} \left[\log Z_{\text{free}}(L, h) + L^d f(h)\right] \tag{2.15b}$$

..., the *corner free energy*

$$f^{(0)}(h) = -\lim_{L \to \infty} \frac{1}{2^d} \left[\log Z_{\text{free}}(L, h) + L^d f(h) + \cdots + 2^{d-1} dL f^{(1)}(h)\right] \tag{2.15c}$$

as well as single-phase magnetizations $m_\pm$ and surface free energies $\tau_\pm$ at the transition point h_t,

$$m_\pm = -\frac{d}{dh} f(h)\bigg|_{h_t \pm 0} \tag{2.16}$$

$$\tau_\pm = f^{(d-1)}(h_t \pm 0) \tag{2.17}$$

We also recall that $\|\kappa\|$ was defined as

$$\|\kappa\| = \sum_{A:\, 0 \in A} \frac{|\kappa_A|}{|A|}$$

Theorem A. Finite-Size Scaling of *m* and χ. Consider a perturbed Ising model with a perturbation of the form (2.1), with translation-invariant coupling constants κ_A with range $R < \infty$. Then there are constants $J_0 < \infty$ and $b_0 > 0$ such that, for $\|\kappa\| < b_0 J$ and $J > J_0$, the following statements are true. Let

$$\begin{aligned} \Delta F(L) = {} & f^{(d-1)}(h_t + 0) 2dL^{d-1} + \cdots + f^{(0)}(h_t + 0) 2^d \\ & - f^{(d-1)}(h_t - 0) 2dL^{d-1} - \cdots - f^{(0)}(h_t - 0) 2^d \end{aligned} \tag{2.18}$$

and define $h_\chi(L)$ and $h_U(L)$ as the points where the susceptibility $\chi_{\text{free}}(L, h)$ and the Binder cumulant $U_{\text{free}}(L, h)$ are maximal. Then[3]

$$m_{\text{free}}(L, h) = \frac{m_+ + m_-}{2} + \frac{m_+ - m_-}{2} \tanh\left\{\frac{m_+ - m_-}{2}[h - h_\chi(L)]L^d\right\} + O\left(\frac{(1 + \|\kappa\|)}{L}\right) \tag{2.19}$$

and

$$\chi_{\text{free}}(L, h) = \left(\frac{m_+ - m_-}{2}\right)^2 \cosh^{-2}\left\{\frac{m_+ - m_-}{2}[h - h_\chi(L)]L^d\right\} L^d + O((1 + \|\kappa\|)L^{d-1}) \tag{2.20}$$

provided $|h - h_\chi(L)| \leqslant O((1 + \|\kappa\|)L^{-1})$.

In addition, for $\Delta F(L) \neq 0$, the shift $h_\chi(L)$ obeys the bound

$$h_\chi(L) = h_t + \frac{\Delta F(L)}{m_+ - m_-}\frac{1}{L^d}\left[1 + O\left(\frac{1}{L}\right)\right] \tag{2.21a}$$

In the leading order, the shift of the point $h_U(L)$ with respect to h_t is the same,

$$h_U(L) = h_t + \frac{\Delta F(L)}{m_+ - m_-}\frac{1}{L^d}\left[1 + O\left(\frac{1}{L}\right)\right] \tag{2.21b}$$

Remarks. (i) If $\tau_+ \neq \tau_-$, Eq. (2.21a) [and similarly for (2.21b)] can be simplified to

$$h_\chi(L) = h_t + \frac{\tau_+ - \tau_-}{m_+ - m_-}\frac{2d}{L}\left[1 + O\left(\frac{1}{L}\right)\right]$$

yielding a shift $\sim 1/L$ which is much larger than the width of the rounding, which, according to (2.19) and (2.20), is of the order $1/L^d$.

(ii) If it is interesting to consider the mutual shift $h_\chi(L) - h_U(L)$. While both $h_\chi(L) - h_t$ and $h_U(L) - h_t$ are of the order $1/L$, their mutual shift is actually much smaller, namely

$$h_\chi(L) - h_U(L) = 2\frac{\chi_+ - \chi_-}{(m_+ - m_-)^3}\frac{1}{L^{2d}} + O\left(\frac{1}{L^{2d+1}}\right) \tag{2.22}$$

[3] Here and in the following, $O(L^\alpha)$ stands for an error term which can be bounded by KL^α, with a constant K that does not depend on h, J, and κ, as long as $J > J_0$ and $\|\kappa\| < b_0 J$.

It is interesting to notice that, in the leading order $1/L^{2d}$, this mutual shift is exactly the same as the corresponding shift for periodic boundary conditions.

(iii) We stress that the condition $|h-h_\chi(L)| \leqslant O((1+\|\kappa\|)L^{-1})$ is not a very serious restriction in our context, because the width of the transition in the volume L^d is only proportional to L^{-d}. In fact, in Section 6 we will close the gap left in Theorem A by showing that for

$$|h-h_\chi(L)| > \frac{4d}{m_+ - m_-}(1+\|\kappa\|)L^{-1}$$

one has

$$|m_{\text{free}}(L,h)-m(h)| \leqslant O(1/L) \tag{2.23}$$

and

$$|\chi_{\text{free}}(L,h)-\chi(h)| \leqslant O(1/L) \tag{2.24}$$

where $m(h)$ and $\chi(h)$ are the infinite-volume magnetization and susceptibility, respectively, of the model (2.1).

(iv) Notice that, for periodic boundary conditions, it is possible to define finite-size transition points $h_t(L)$ with exponentially small shift, for example, the point where $m_{\text{per}}(L,h) = m_{\text{per}}(2L,h)$. Here, all these definitions lead to a shift $\sim 1/L$ yielding no qualitative improvement with respect to the point $h_\chi(L)$ or $h_U(L)$.

(v) In principle, the coefficients $m_\pm$, $\tau_\pm$,..., can be calculated up to arbitrary precision using standard series expansions, provided the microscopic Hamiltonian is known. On the other hand, the scaling (2.19)–(2.21) would allow one, in principle, to obtain the coefficients m_+, m_-, and the difference $\tau_+ - \tau_-$ from experimental measurements.

(vi) The general context considered in Section 3 allows one to analyze the finite-size scaling with more general boundary conditions than the free boundary conditions considered here, including, in particular, small applied boundary fields favoring one of the two phases near the boundary. In order to apply the techniques developed in this paper, it is necessary, however, to exclude boundary conditions which strongly favor one of the two phases. Such a condition is needed to ensure that the main contributions to the partition functions do in fact come from small perturbations of the two ground states $\sigma_\Lambda \equiv \pm 1$. For large boundary fields, the boundary may strongly favor one of the two phases. The leading contributions to the partition function then would include configurations which are in one phase near the boundary, and in the other one for the bulk. In such situations, wetting and roughening effects of the contour separating the boundary phase from the bulk phase would be important physical effects. We are not attempting to study these effects in the present paper.

3. GENERAL SETTING AND MAIN THEOREM

3.1. Contour Representation of the Ising Model

In this section we review the contour representation for the model (2.1). To make this subsection as simple as possible, and to have a concrete example at hand, we use for illustration the simplest symmetry-breaking term, namely a perturbation of the form

$$\kappa \sum_{\langle ijk \rangle \subset \Lambda} \sigma_i \sigma_j \sigma_k \tag{2.1'}$$

where the sum goes over all triangles $\langle ijk \rangle$ made out of two nearest neighbor bonds $\langle ij \rangle$ and $\langle jk \rangle$. See refs. 24 and 25 for the contour representation for the more general model (2.1). It will be convenient to introduce, in addition to the finite lattice $\Lambda = \{1,..., L\}^d$, the subset $V = [\frac{1}{2}, L + \frac{1}{2}]^d$ of $\mathbb{R}^d$ which is obtained from Λ as the union of all closed unit cubes c_i with centers $i \in \Lambda$. For a given configuration $\sigma_\Lambda \subset \{-1, 1\}^\Lambda$, we then introduce the set ∂ as the boundary between the region $V_+ \subset V$ where $\sigma_i = +1$ and the region $V_- \subset V$ where $\sigma_i = -1$, and the contours $Y_1,..., Y_n$ corresponding to σ_Λ as the connected components of ∂.

To be more precise, we define an *elementary cube* as a closed unit cube with a center in Λ (we sometimes use the symbol c_i to denote an elementary cube with center $i \in \Lambda$), and introduce $\bar{V}_\pm$ as the union of all closed elementary cubes c_i for which $\sigma_i = \pm 1$, respectively. The set ∂ is then defined as $\bar{V}_+ \cap \bar{V}_-$, and the "ground-state regions" $V_\pm$ are defined as $\bar{V}_\pm \setminus \partial$. With these definitions, the partition function with Hamiltonian (2.1') can be rewritten in the form

$$Z_{\text{free}}(L, h) = \sum_{\partial} \sum_{\sigma_\Lambda}' e^{-H(\sigma_\Lambda)}$$

where the second sum is over all configurations consistent with ∂.

In order to specify the configuration σ_Λ, one has to decide which component of $V \setminus \partial$ corresponds to $\sigma_i = +1$ and which one to $\sigma_i = -1$. To this end, we introduce contours with labels. Given a configuration σ_Λ, the contours corresponding to σ_Λ are defined as pairs $Y = (\text{supp } Y, \alpha(\cdot))$, where supp Y is a connected component of ∂, while α is an assignment of a label $\alpha(c) \in \{-1, +1\}$ to each elementary cube that touches supp Y.[4] It is chosen in such a way that $\alpha(c_i) = \sigma_i$. Note that the labels of contours corresponding to a configuration σ_Λ are matching in the sense that the labels $\alpha(c)$ are constants on every component of $V \backslash \partial$.

In fact, a set of contours $\{Y_1, ..., Y_n\}$ corresponds to a configuration σ_Λ if and only if:

(i) supp $Y_i \cap$ supp $Y_j = \varnothing$ for $i \neq j$ and

(ii) the labels of $Y_1, ..., Y_n$ are matching.

We call a set of contours obeying (i) a *set of nonoverlapping contours* and a set of contours obeying (i) and (ii) a *set of nonoverlapping contours with matching labels*, or sometimes just a *set of matching contours.*

In order to rewrite $Z_{\text{free}}(L, h)$ in terms of contours, we assign a weight $\rho(Y)$ to each contour. This is done in such a way that

$$c^{-H(\sigma_\Lambda)} = e^{-E_+(V_+)} e^{-E_-(V_-)} \prod_{k=1}^{n} \rho(Y_k) \tag{3.1}$$

Here $H(\sigma_\Lambda)$ is the Hamiltonian (2.1′), $Y_1, ..., Y_n$ are the contours corresponding to σ_Λ, and

$$E_\pm(V_\pm) = \sum_{i \in \Lambda \cap V_\pm} e_\pm(i) \tag{3.2}$$

For the standard Ising model, $\rho(Y) = e^{-J|Y|}$, where $|Y|$ is the number of elementary $(d-1)$-dimensional faces in supp Y. The third term in (2.1′), however, introduces corrections yielding a weight of the form $\rho(Y) = e^{-J|Y| + O(\kappa|Y|)}$. As a consequence,

$$|\rho(Y)| \leqslant e^{-\tau|Y|} \qquad \text{with} \quad \tau = J - O(\kappa) \tag{3.3}$$

Similar bounds hold for the derivatives $|d^k \rho(Y)/dh^k|$.

With the help of (3.1), we rewrite the partition function $Z_{\text{free}}(L, h)$ as

$$Z_{\text{free}}(L, h) = \sum_{\{Y_1, ..., Y_n\}} e^{-E_+(V_+)} e^{-E_-(V_-)} \prod_{k=1}^{h} \rho(Y_k) \tag{3.4}$$

where the sum goes over all sets of matching contours in V.

[4] In the language of ref. 19, supp Y is called a (geometric) contour, while Y is called a labeled contour.

3.2. Assumptions for the General Model

In Section 3.3 below, we will state our main theorem, Theorem 3.1, from which we infer Theorem A of the preceding section. The setting of Theorem 3.1 is actually more general than what is needed for Theorem A and will include more general models. On one hand, we introduce contours in such a way that the notion of contours covers the Ising contours introduced above as well as thick Pirogov–Sinai contours[(24–26)] constructed as unions of elementary cubes.[5] On the other hand, we also consider the situation of general N-phase coexistence.

As before, we consider the finite lattice $\Lambda \subset \mathbb{Z}^d$, $d \geqslant 2$, and the corresponding volume $V \subset \mathbb{R}^d$. We introduce the set $\mathscr{C}$ of *elementary cells* as the set of all elementary cubes in V, all closed $(d-1)$-dimensional faces of these cubes,..., and all closed edges of these cubes. As usual, we define the boundary ∂W of a set $W \subset V$ as the set of all points x which have distance zero from both W and W^c and $\bar{W}$ as $W \cup \partial W$.

A contour in V is then a pair $Y = (\text{supp } Y, \alpha(\cdot))$ where supp Y is a connected union of elementary cells and $\alpha(\cdot)$ is an assignment of a label $\alpha(c)$ from a finite set $\{1,..., N\}$ to each elementary cube c in $\overline{V \backslash \text{supp } Y}$ which touches Y [by touching we mean that $c \cap \text{supp } Y \neq \emptyset$, while $(c \backslash \partial c) \cap \text{supp } Y = \emptyset$]. As before, we require that α is constant on each component C of $V\backslash\text{supp } Y$, and say that a set $\{Y_1,..., Y_n\}$ of contours is a *set of matching contours* (or, more explicitly, *a set of nonoverlapping contours with matching labels*) iff:

(i) $\text{supp } Y_i \cap \text{supp } Y_j \neq \emptyset$ for $i \neq j$ and

(ii) the labels of $Y_1,..., Y_n$ are matching in the sense that they are constant on components of $V\backslash(\text{supp } Y_1 \cup \cdots \cup \text{supp } Y_n)$.

In this way, each component C of $V\backslash(\text{supp } Y_1 \cup \cdots \cup \text{supp } Y_n)$ has constant boundary conditions on $\partial C \backslash \partial V$. The partition function of a statistical model with "weak" boundary conditions is then rewritten in terms of contours as

$$Z(V, h) = \sum_{\{Y_1,..., Y_n\}} \prod_{k=1}^{n} \rho(Y_k) \prod_{m=1}^{N} e^{-E_m(V_m)} \tag{3.5}$$

where the sum goes over sets of matching contours in V (including the empty set of contours), and V_m is the union of all components of $V\backslash(\text{supp } Y_1 \cup \cdots \cup \text{supp } Y_n)$ that have boundary condition m, and

$$E_m(V_m) = \sum_{c \subset \bar{V}_m} e_m(c) \tag{3.6}$$

[5] The contours are introduced in such a way that the more general cases considered in refs. 11 and 19 are covered as well.

We point out that the sum in (3.6) goes over all elementary cubes in the closure $\bar{V}_m$ of V_m, a convention which was chosen to ensure that all elementary cubes c with center in V_m are taken into account.[6] Note that by our definition of V as a closed subset of $\mathbb{R}^d$, the sum (3.5) contains contours that touch ∂V (in the sequel, we call these contours *boundary contours*) and contours that do not touch ∂V (*ordinary contours*). The contribution of the collection of empty contours to (3.5) is actually a sum of N terms, $\sum_m e^{-E_m(V)}$.

In the equalities (3.5) and (3.6) we have introduced "contour weights" $\rho(Y)\in\mathbb{R}$ and "ground-state energies" $e_m(c)\in\mathbb{R}$ that depend on a vector parameter $h\in\mathscr{U}$, where $\mathscr{U}$ is an open subset of $\mathbb{R}^\nu$. We assume that $\rho(Y)$ and $e_m(c)$ are translation invariant as long as Y and c do not touch the boundary of V. More generally, we assume translation invariance along a $(d-k)$-dimensional face in ∂V as long as Y (or c) does not touch the $[(d-k)-1]$-dimensional boundary of this face.

As usual, we have to assume a *Peierls condition*, together with several assumptions on the ground-state energies $e_m(c)$. Here, we assume that $e_m(c)$ and $\rho(Y)$ are C^6 functions of h obeying the following bounds:

$$|\rho(Y)|\leqslant e^{-\tau|Y|-E_0(Y)} \tag{3.7}$$

$$\left|\frac{d^k\rho(Y)}{dh^k}\right|\leqslant |k|!\,(C_0\,|Y|)^{|k|}\,e^{-\tau|Y|-E_0(Y)} \tag{3.8}$$

and

$$\left|\frac{d^k e_m(c)}{dh^k}\right|\leqslant C_0^{|k|} \tag{3.9}$$

Here $\tau>0$ is a sufficiently large constant, $|Y|$ denotes the number of elementary cells in[7] supp Y,

$$E_0(Y)=\sum_{c\subset \operatorname{supp} Y} e_0(c) \qquad \text{with} \quad e_0(c)=\min_m e_m(c) \tag{3.10}$$

k is a multi-index $k=(k_\alpha)_{\alpha=1,\dots,\nu}$ with $1\leqslant|k|\leqslant 6$, $|k|=\sum k_\alpha$, and C_0 is a constant independent of h and τ. In addition, we assume that the difference between $e_m(c)$ and the bulk term e_m is bounded,

$$|e_m(c)-e_m|\leqslant\gamma\tau \tag{3.11}$$

[6] A sum over elementary cubes $c\subset V_m$ would exclude those elementary cubes $c\subset\bar{V}_m$ which touch one of the contours $Y_1,\dots,Y_n$.

[7] Here, a k-dimensional cell c in supp Y is only counted if there is no $(k+1)$-dimensional cell c' in supp Y with $c\subset c'$.

with a constant $0 < \gamma < 1$ to be specified later. This condition is introduced to avoid a situation where free b.c. strongly favor cetain phases $n \in \{1,..., N\}$. Note that

$$|e_m(c) - e_m| \leqslant \|\kappa\|$$

for the asymmetric Ising model (2.1). For this model, the condition (3.11) is therefore satisfied once $\|\kappa\| \leqslant b_0 J$ for a suitable constant $0 < b_0 < \infty$.

3.3. Main Theorem

In this section we state our main result for the general model introduced in the last section. It actually generalizes Theorem A presented in Section 2 to a large class of models describing the coexistence of N phases. As in Section 2, the leading contribution to the partition function $Z(V, h)$ is the sum

$$\sum_{m=1}^{N} \exp\left\{-\sum_{c \subset V} e_m(c)\right\} \tag{3.12}$$

Introducing $|\partial_k V|$ as the joint k-dimensional area of all k-dimensional faces of V and $e_m^{(k)}$ as solutions of equations

$$\sum_{n=k}^{d} \binom{d-k}{n-k} e_m^{(n)} = e_m(c), \qquad k = d-1,..., 0 \tag{3.13}$$

whenever c is touching a k-dimensional face of V and not touching its $(k-1)$-dimensional boundary,[8] we rewrite

$$\sum_{c \subset V} e_m(c) = e_m\,|V| + e_m^{(d-1)}\,|\partial_{d-1} V| + \cdots + e_m^{(0)}\,|\partial_0 V| \tag{3.14}$$

To see that (3.13) implies (3.14), just notice that a hypercube c touching a k-dimensional face of V and not touching its $(k-1)$-dimensional boundary is touching $\binom{d-k}{n-k}$ different n-dimensional faces of ∂V. Each of these faces is specified by choosing $n-k$ directions among $d-k$ directions orthogonal to the concerned k-dimensional face.

As usual we define the bulk free energy $f(h)$ by

$$f(h) = -\lim_{V \to \mathbb{R}^d} \frac{1}{|V|} \log Z(V, h) \tag{3.15}$$

[8] Note that due to the translation invariance properties of $e_m(c)$, the right-hand side of this equation is constant for all such elementary cubes c.

and the magnetization $m(V, h) = (m_\alpha(V, h))_{\alpha=1,\dots,\nu}$ by

$$m(V, h) = \frac{1}{|V|} \frac{d}{dh} \log Z(V, h) \tag{3.16}$$

Theorem 3.1. There exist constants $b > 0$, $\gamma_0 > 0$, and $\tau_0 < \infty$ [where b and γ_0 depend on d and τ_0 depends on d, N, and the constant C_0 introduced in (3.8) and (3.9)], as well as metastable free energies $f_m(h)$, surface free energies $f_m^{(d-1)}(h),\dots$, edge free energies $f_m^{(1)}(h)$, and corner free energies $f_m^{(0)}(h)$, such that the following statements are true provided the effective decay constant $\tilde{\tau}$,

$$\tilde{\tau} := \tau(1 - \gamma/\gamma_0) - \tau_0 > 0 \tag{3.17}$$

[for the definition of τ and γ see (3.7), (3.8), and (3.11)]:

(i) $f(h) = \min_m f_m(h)$.

(ii) f_m and $f_m^{(l)}$, $l = d-1,\dots, 0$, are six-times differentiable functions of h.

(iii) If $|k| \leqslant 6$, then

$$\left| \frac{d^k}{dh^k} (f_m - e_m) \right| \leqslant e^{-b\tilde{\tau}} \qquad \text{and} \qquad \left| \frac{d^k}{dh^k} (f_m^{(l)} - e_m^{(l)}) \right| \leqslant e^{-b\tilde{\tau}}$$

where $l = d-1,\dots, 0$.

(iv) Let

$$F_m(V, h) = f_m(h)\,|V| + f_m^{(d-1)}(h)\,|\partial_{d-1} V| + \cdots + f_m^{(0)}(h)\,|\partial_0 V| \tag{3.18}$$

Then

$$\left| \frac{d^k}{dh^k} \left[Z(V, h) - \sum_{m=1}^{N} e^{-F_m(V, h)} \right] \right| \leqslant |V|^{|k|+1}\, O(e^{-b\tilde{\tau}L}) \max_m e^{-F_m(V, h)} \tag{3.19}$$

provided $0 \leqslant |k| \leqslant 6$.

(v) Let $0 \leqslant |k| \leqslant 5$ and define P_q as

$$P_q = \left[\sum_{m=1}^{N} e^{-F_m(V, h)} \right]^{-1} e^{-F_q(V, h)} \tag{3.20}$$

Then

$$\left| \frac{d^k}{dh^k} \left[m_\alpha(V, h) - \sum_{q=1}^{N} \frac{1}{|V|} \left(-\frac{dF_q(V, h)}{dh_\alpha} \right) P_q \right] \right| \leqslant |V|^{|k|}\, O(e^{-b\tilde{\tau}L}) \tag{3.21}$$

Here, as in the rest of this paper, $O(x)$ stands for a bound const $\cdot\, x$, where the constant depends only on d, N, and the constant C_0 introduced in (3.8) and (3.9).

Theorem 3.1 is the main theorem of this paper. Its proof has three major parts: the geometric analysis of contours touching the boundary, a decomposition of $Z(V, h)$ into pure phase partition functions, and the construction of metastable contour models allowing one to prove the bounds (3.19) and (3.21). Deferring the technical details to the appendices, we present the main steps of this proof in Section 4.

3.4. FSS for Local Observables

In addition to the FSS of thermodynamic quantities such as the magnetization or susceptibility, we want to study the FSS of local observables. In order to state our results in the context of the general models considered in Section 3.2, we introduce the following notation. An observable A is a function which associates to each configuration contributing to (3.5) a real number $A(Y_1,..., Y_n)$. Its expectation value in the volume V is defined as

$$\langle A \rangle_V^h = \frac{1}{Z(V, h)} Z(A \mid V, h) \tag{3.22}$$

where

$$Z(A \mid V, h) = \sum_{\{Y_1,..., Y_n\}} A(Y_1,..., Y_n) \prod_{k=1}^{n} \rho(Y_k) \prod_{m=1}^{N} e^{-E_m(V_m)} \tag{3.23}$$

As in (3.5), the sum in (3.23) goes over sets of matching contours in V, and V_m is the union of all components of $V \backslash (\text{supp } Y_1 \cup \cdots \cup \text{supp } Y_n)$ that have the boundary condition m.

An observable A is called a local observable if there is a finite set of elementary cubes, denoted supp A in the sequel, such that $A(Y_1,..., Y_n)$ does not depend on those contours Y_i for which $\text{supp } A \cap (\text{supp } Y_i \cup \text{Int } Y_i) = \varnothing$, where Int Y_i is the interior of Y_i (for the precise definition of Int Y_i see Section 4.1 below).

In most applications, local observables will be bounded, in the sense that the norm

$$\|A\| = \sup_{\{Y_1,..., Y_n\}} |A(Y_1,..., Y_n)| \tag{3.24}$$

is finite. In addition, the observable will either not depend on the vector parameter h at all, or obey bounds of the form

$$\left|\frac{d^k}{dh^k} A(Y_1, \ldots, Y_n)\right| \leqslant |k|!\, C_A(C_0\, |\text{supp}\, A|)^{|k|} \tag{3.25a}$$

where C_0 is the constant introduced in (3.8), C_A is a constant, and k is a multi-index of order $0 \leqslant |k| \leqslant 6$.

Here, we will allow for a slightly more general situation, requiring only that

$$\left|\frac{d^k}{dh^k}\left[A(Y_1, \ldots, Y_n) \prod_{j=1}^{n} \rho(Y_j)\right]\right|$$
$$\leqslant |k|!\, C_A(C_0\, |\overline{\text{supp}\, Y_A}|)^{|k|} \prod_{j=1}^{n} e^{-\tau |Y_j| - E_0(Y_j)} \tag{3.25b}$$

where $\overline{\text{supp}\, Y_A}$ stands for the set $\text{supp}\, A \cup \text{supp}\, Y_1 \cup \cdots \cup \text{supp}\, Y_n$, k is a multi-index of the order $0 \leqslant |k| \leqslant 6$, C_0 is the constant introduced in (3.8), and C_A is a constant that is finite[9] for all h and τ. Assuming this condition[10] and the conditions introduced in Section 3.2, we will be able to prove the following theorem.

Theorem 3.2. There are "metastable expectation functionals" $\langle \cdot \rangle^h_{V,q}$, $q = 1, \ldots, N$, such that the following statements are true provided the effective decay constant $\tilde{\tau} := \tau(1 - \gamma/\gamma_0) - \tau_0$ defined in Theorem 3.1 is positive and $0 \leqslant |k| \leqslant 6$:

(i) For each local observable obeying the bounds (3.25a) or (3.25b), one has

$$\left|\frac{d^k}{dh^k}\left[\langle A \rangle^h_V - \sum_{q=1}^{N} \langle A \rangle^h_{V,q} P_q\right]\right| \leqslant C_A e^{O(\varepsilon)\, |\text{supp}\, A|} O(e^{-b\tilde{\tau}L}) \tag{3.26}$$

where the probabilities P_q and the constant b are as in Theorem 3.1 and $\varepsilon = e^{-\tilde{\tau}/2}$.

(ii) For each local observable obeying the bounds (3.25a) or (3.25b), the limits

$$\langle A \rangle^h_q = \lim_{V \to \mathbb{R}^d} \langle A \rangle^h_{V,q} \tag{3.27}$$

[9] While we assumed that the constant C_0 is independent of h and τ, we do not require that C_A is independent of h and τ.

[10] Note that (3.7), (3.8), and (3.25a) imply the bound (3.25b).

exist as C^6 functions of h, and obey the bounds

$$\left|\frac{d^k}{dh^k} \langle A \rangle_q^h\right| \leqslant O(1)\, C_A\, |\text{supp } A|^{|k|}\, e^{O(\varepsilon)\, |\text{supp } A|} \tag{3.28}$$

where $\varepsilon = e^{-\bar{\tau}/2}$.

(iii) For each local observable obeying the bounds (3.25a) or (3.25b), one has

$$\left|\frac{d^k}{dh^k} [\langle A \rangle_q^h - \langle A \rangle_{V,q}^h]\right|$$
$$\leqslant C_A\, |\text{supp } A|^{|k|}\, e^{O(\varepsilon)\, |\text{supp } A|} O(e^{-b\bar{\tau}\, \text{dist}(\text{supp } A,\, \partial V)}) \tag{3.29}$$

where $\varepsilon = e^{-\bar{\tau}/2}$.

Proof. The proof of Theorem 3.2 is given in Section 5.

4. PROOF OF THEOREM 3.1

The proof of Theorem 3.1 has three major parts: the geometric analysis of contours, in particular a bound of the form

$$N_{\partial V}(\text{Int } Y) \leqslant \text{const } |Y|$$

where $N_{\partial V}(\text{Int } Y)$ denotes the number of elementary cubes in[11] $\overline{\text{Int } Y}$ that touch the boundary ∂V of V, the decomposition of $Z(V, h)$ into pure phase partition functions $Z_1(V, h),..., Z_N(V, h)$, and the construction of suitable metastable free energies $f_1,..., f_n$. Deferring the technical details to the appendices, we present the main steps in the following subsections.

4.1. The Geometry of Contours

An important notion in the Pirogov–Sinai theory of contour models is the notion of the interior and exterior of a contour. For ordinary contours $Y = (\text{supp } Y, \alpha(\cdot))$, one defines Int Y as the union of all finite components of $\mathbb{R}^d \backslash \text{supp } Y$ and $\text{Int}_m\, Y$ as the union of all components of Int Y which have the boundary condition m. Since ordinary contours do not touch the boundary ∂V of V, the set Ext $Y = V \backslash (\text{supp } Y \cup \text{Int } Y)$ is a connected set and $\alpha(c)$ is constant for all cubes c in $\overline{\text{Ext } Y}$ which touch supp Y. We say that Y is an m-contour if $\alpha(c) = m$ for these cubes.

[11] We recall that we use the symbol $\bar{W}$ to denote the closure of a set W.

We now generalize these notions to boundary contours. To this end, we first introduce, for each corner k of the box V, an "octant" $K(k)$. Namely, if k has components $k_1,..., k_d$, with $k_i = 1/2$ for $i \in I_-$ and $k_i = L + 1/2$ for $i \in I_+$, then

$$K(k) := \{x \in \mathbb{R}^d \mid x_i \geqslant 1/2 \text{ for } i \in I_-, \, x_i \leqslant L + 1/2 \text{ for } i \in I_+\}$$

We then say: a contour Y is *short* iff there is a corner k such that $\operatorname{supp} Y \cap \partial V \subset \partial K(k)$. Otherwise Y is called *long*. Note that short contours may be ordinary contours or boundary contours, while long contours are always boundary contours.

For a short contour Y, we then define Int Y as the union of all finite components of $K(k) \backslash \operatorname{supp} Y$, $\operatorname{Int}_m Y$ as the union of all components of Int Y which have the boundary condition m, Ext Y as $V \backslash (\operatorname{supp} Y \cup \operatorname{Int} Y)$, and $V(Y)$ as $\operatorname{supp} Y \cup \operatorname{Int} Y$. As before, Ext Y is a connected set, and the notion of an m-contour is defined by the condition that $\alpha(c) = m$ for all cubes c in $\overline{\operatorname{Ext} Y}$ that touch supp Y. Note that these definitions are equivalent to the previous ones if the short contour Y is in fact an ordinary contour. Note also that the above definitions do not depend on the choice of the corner k if there are several corners k for which $\operatorname{supp} Y \cap \partial V \subset K(k)$.

For long contours, there is *a priori* no natural notion of an exterior or interior. We choose a convention that ensures that the volume of a component C_i of Int Y cannot exceed the value $L^d/2$ if Y is a long contour. Namely, if Y is a long boundary contour, and $C_1,..., C_n$ are the components of $V \backslash \operatorname{supp} Y$, then the component C_i with the largest volume is called the exterior Ext Y. If there are several such components $C_{i_1},..., C_{i_l}$, we choose the first one in some arbitrary fixed order (for example, the lexicographic order) as Ext Y. We then define $\operatorname{Int} Y = V \backslash (\operatorname{supp} Y \cup \operatorname{Ext} Y)$, $V(Y) = \operatorname{supp} Y \cup \operatorname{Int} Y$, $\operatorname{Int}_m Y$ as the union of all components of Int Y which have the boundary condition m, and an m-contour Y as a contour for which $\alpha(c) = m$ on all cubes c in $\overline{\operatorname{Ext} Y}$ that touch supp Y.

The following three lemmas state that the sets Ext Y and Int Y are defined in such a way that they have the main properties of an exterior and interior of the set supp Y. They are proven in Appendix B.

The first of them expresses the fact that for two contours Y_1 and Y_2 which do not touch each other, Y_1 together with its interior is necessarily contained in one of the components of $\operatorname{Ext} Y_2 \cup \operatorname{Int} Y_2$.

Lemma 4.1. Let Y_1, Y_2 be nonoverlapping contours. Then the following statements are true:

(i) If $\operatorname{supp} Y_2 \subset \operatorname{Ext} Y_1$ and $\operatorname{supp} Y_1 \subset \operatorname{Ext} Y_2$, then $V(Y_2) \subset \operatorname{Ext} Y_1$ and $V(Y_1) \subset \operatorname{Ext} Y_2$.

(ii) If supp $Y_1 \subset C_2$, where C_2 is a component of Int Y_2, then $V(Y_1) \subset C_2$.

(iii) If supp $Y_1 \subset \text{Int } Y_2$, then $V(Y_1) \subset \text{Int } Y_2$.

The next lemma expresses the fact that it is not possible that two contours which do not touch are both included in the interior of each other.

Lemma 4.2. Let Y_1 and Y_2 be nonoverlapping contours. Then one and only one of the following three cases is true:

(i) supp $Y_2 \subset \text{Ext } Y_1$ and supp $Y_1 \subset \text{Ext } Y_2$.

(ii) supp $Y_2 \subset \text{Ext } Y_1$ and supp $Y_1 \subset \text{Int } Y_2$.

(iii) supp $Y_2 \subset \text{Int } Y_1$ and supp $Y_1 \subset \text{Ext } Y_2$.

Definition 4.3. Let $\{Y_1,..., Y_n\}$ be a set of nonoverlapping contours. Then $Y_k \in \{Y_1,..., Y_n\}$ is called an *internal contour* iff there exists a contour $Y_i \in \{Y_1,..., Y_n\}$ with supp $Y_k \subset \text{Int } Y_i$. Otherwise Y_k is called an *external contour*. Finally, $\{Y_1,..., Y_n\}$ is called a set of *mutually external contours* if all contours in $\{Y_1,..., Y_n\}$ are external.

The next lemma will be used in Section 4.2 to conclude that all external contours of a given configuration contributing to (3.5) have the same external label. This observation will be an important ingredient in the decomposition of $Z(V, h)$ into single-phase partition functions $Z_m(V, h)$, and therefore in the proof of Theorem 3.1.

Lemma 4.4. Let $\{Y_1,..., Y_n\}$ be a set of nonoverlapping contours in V, and let

$$\text{Ext} = V \Big\backslash \bigcup_{i=1}^{n} (\text{Int } Y_i \cup \text{supp } Y_i) \tag{4.1}$$

Then Ext is a connected component of $V \backslash \bigcup_{i=1}^{n} \text{supp } Y_i$.

Remark. Let Y_0 be a contour, and let W_0 be one of the components of Int Y_0. Then Lemma 4.4 remains valid if V is replaced by W_0, as can be seen immediately from the proof in Appendix B.

While the preceding three lemmas, even though tedious to prove, just express our intuitive notions about exteriors and interiors (in fact, our definitions were chosen in such a way that they do), the next lemma is less obvious. In order to explain the need for it, we recall that the ground-state energies $e_m(c)$ may be different from the corresponding bulk term e_m. As a consequence, the boundary may favor an otherwise unstable phase. In the expansion about the leading contribution $e^{-E_m(V)}$ to the single-phase partition functions $Z_m(V, h)$, this will have the tendency to increase the weight of boundary contours which describe transitions into one of these "boundary-favored" phases. In order to control the contributions coming from such contours (using the exponential decay $e^{-\tau |Y|}$), we need a bound of the form

$$N_{\partial V}(\text{Int } Y) \leqslant \text{const} \cdot |Y|$$

where $N_{\partial V}(\text{Int } Y)$ denotes the number of elementary cubes in $\overline{\text{supp } Y}$ that touch the boundary ∂V of V. This is the main statement of the next lemma.

Lemma 4.5. Let Y be a contour in V, and let $W_1,..., W_n$ be the components of Int Y. Then

$$N_{\partial V}(\text{Int } Y) \leqslant C_1 |Y| \tag{4.2}$$

$$\sum_{i=1}^{n} |\partial W_i| \leqslant C_2 |Y| \tag{4.3}$$

and

$$|\partial V(Y)| \leqslant C_3 |Y| \tag{4.4}$$

where $C_1 = 2d(2^{1/d} + 1)/(2^{1/d} - 1)$, $C_2 = C_1 + 2d$, and $C_3 = C_2 + 2d$.

The proof of this lemma relies on a lattice version of the isoperimetric inequality and is given in Appendix B. The proof of the required isoperimetric inequality is given in Appendix A.

4.2. Decomposition of $Z(V, h)$ into Pure Partition Functions

The first step in the proof of Theorem 3.1 is the decomposition of $Z(V, h)$ into N terms $Z_q(V, h)$, $q = 1,..., N$, which are obtained as perturbations of the leading terms $e^{-E_q(V)}$. We start with the observation that all external contours contributing to (3.5) touch the set Ext introduced in (4.1). Given that these contours are matching, we conclude that all external contours of a given configuration contributing to (3.5) have the same label. Therefore

$$Z(V, h) = \sum_{q=1}^{N} Z_q(V, h) \tag{4.5}$$

with

$$Z_q(V, h) = \sum_{\{Y_1,..., Y_n\}} \prod_{k=1}^{n} \rho(Y_k) \prod_{m=1}^{N} e^{-E_m(V_m)} \tag{4.6}$$

where the sum goes over sets of matching contours in V for which all external contours are q-contours. As before, V_m is the union of all components of $V\backslash(\text{supp } Y_1 \cup \cdots \cup \text{supp } Y_n)$ that have boundary condition m, and $E_m(V_m)$ is defined in (3.6).

More generally, let W be a component of the interior Int Y_0 of some contour Y_0 in V, a set of the form (4.1), or a set obtained from a component W_0 of an interior Int Y_0 by a similar construction,

$$W = W_0 \Big\backslash \bigcup_{i=1}^{n} (\text{Int } Y_i \cup \text{supp } Y_i) \tag{4.7a}$$

where $\{Y_1, ..., Y_n\}$ is a set of nonoverlapping contours in W_0. We then define $Z_q(W, h)$ as

$$Z_q(W, h) = \sum_{\{Y_1, ..., Y_n\}} \prod_{k=1}^{n} \rho(Y_k) \prod_{m=1}^{N} e^{-E_m(V_m)} \tag{4.7b}$$

where the sum goes over sets of matching contours in V for which all external contours are q-contours with $V(Y) = (\text{supp } Y \cup \text{Int } Y) \subset W$. Here, V_m is now defined as the union of all components of $W\backslash(\text{supp } Y_1 \cup \cdots \cup \text{supp } Y_n)$ that have boundary condition m. Note that the sum in (4.7b) contains no contours which surround the holes in W. Finally, given a volume W which is a disjoint union of volumes $W_1, ..., W_n$ of the form (4.7a), we define $Z_q(W, h)$ as the product of the partition functions $Z_q(W_i, h)$, $i = 1, ..., n$.

Returning to (4.6), we derive a second expression for $Z_q(V, h)$, which eliminates the matching condition for the labels of $Y_1, ..., Y_n$. To this end we first sum over all sets $\{Y_1, ..., Y_n\}$ with a fixed collection of external contours. For each external contour Y this resummation produces a factor $\prod_{m=1}^{N} Z_m(\text{Int}_m\ Y, h)$. This yields the expression

$$Z_q(V, h) = \sum_{\{Y_1, ..., Y_n\}_{\text{ext}}} e^{-E_q(\text{Ext})} \prod_{k=1}^{n} \left[\rho(Y_k) \prod_{m=1}^{N} Z_m(\text{Int}_m\ Y_k, h) \right] \tag{4.8}$$

where the sum runs over sets $\{Y_1, ..., Y_n\}_{\text{ext}}$ of mutually external q-contours in V and Ext is the set defined in (4.1). Assuming that $Z_q(\text{Int}_m\ Y_k, h) \neq 0$, we divide each Z_m by the corresponding Z_q and multiply it back again in the form (4.7b.). Iterating the same procedure on the terms $Z_q(\text{Int}_m\ Y_k, h)$, we eventually get

$$Z_q(V, h) = e^{-E_q(V)} \sum_{\{Y_1, ..., Y_n\}} \prod_{k=1}^{n} K_q(Y_k) \tag{4.9}$$

where the sum goes over sets of nonoverlapping contours which are all q-contours, while

$$K_q(Y) := \rho(Y)\, e^{E_q(Y)} \prod_{m=1}^{N} \frac{Z_m(\mathrm{Int}_m\, Y, h)}{Z_q(\mathrm{Int}_m\, Y, h)} \tag{4.10}$$

The equality (4.9) is the desired alternative expression for $Z_q(V, h)$ which contains no matching condition on contours. Assuming that the new contour activities $K_q(Y)$ are sufficiently small (for h in the transition region, this is actually the case, see Section 4.4), it also expresses the fact that $e^{-E_q(V)}$ is the leading contribution to $Z_q(V, h)$.

Obviously, (4.9) can be generalized to volumes W of the form considered in (4.7). One obtains

$$Z_q(W, h) = e^{-E_q(W)} \sum_{\{Y_1,\dots,Y_n\}} \prod_{k=1}^{n} K_q(Y_k) \tag{4.11}$$

where the sum goes over sets of nonoverlapping q-contours $Y_1,\dots, Y_n$ with $V(Y_i) \subset W$.

4.3. Truncated Contour Models

Given the decomposition (4.5) of $Z(V, h)$ and the representation (4.9) for $Z_q(V, h)$, one might try to obtain the FSS of $Z(V, h)$ by a cluster expansion analysis of the partition functions $Z_q(V, h)$. For such an analysis, one would need a bound of the form $|K_q(Y)| \leqslant \varepsilon^{|Y|}$ with a sufficiently small constant $\varepsilon > 0$. While it turns out that such a bound can be proven for stable phases q, it is false for unstable phases.

In order to overcome this problem, we will construct truncated contour activities $K'_q(Y)$ and the corresponding partition functions

$$Z'_q(W, h) = e^{-E_q(W)} \sum_{\{Y_1,\dots,Y_n\}} \prod_{k=1}^{n} K'_q(Y_k) \tag{4.12}$$

in such a way that:

(i) The truncated contour activities $K'_q(Y)$ obey a bound

$$|K'_q(Y)| \leqslant \varepsilon^{|Y|} \tag{4.13}$$

for some small $\varepsilon > 0$.

(ii) $Z'_q(W, h) = Z_q(W, h)$ if the corresponding (infinite-volume) free energy $f_q = f_q(h)$ is equal to $f \equiv \min_{m \in Q} f_m$, so that the truncated model is identical to the original model if $f_q = f$ (following refs. 26 and 29, we call these q "stable").

(iii) The truncated contour activities and the corresponding free energies are smooth functions of the external fields h.

Heuristically, the truncated model will be a model where contours corresponding to supercritical droplets in the corresponding droplet model are suppressed with the help of a smoothed characteristic function. In the case of a two-phase model, this idea could be implemented by defining

$$K'_+(Y) = K_+(Y)\,\chi(\alpha\,|Y| - (f_+ - f_-)\,|V(Y)|)$$
$$K'_-(Y) = K_-(Y)\,\chi(\alpha\,|Y| - (f_- - f_+)\,|V(Y)|)$$

where χ is a smoothed characteristic function and α is a constant of the order of τ, for example, $\alpha = \tau/2$. While the presence of the characteristic function would not affect the stable phases since $f_+ - f_- \geqslant 0$ if the + phase is stable (and $f_- - f_+ \geqslant 0$ if the − phase is stable), it would suppress contours immersed into an unstable + phase as soon as the volume term $(f_+ - f_-)\,|V(Y)|$ is bigger then the decay term proportional to $|Y|$. As a consequence, all contours contributing to the "metastable" partition function Z'_q obey a bound of the form (4.13) as desired.

Unfortunately, the above definition of $K'_q(Y)$ is circular because it uses free energies f_q that are defined as free energies of a model with activities $K'_q(Y)$. To overcome this problem, we will use the following inductive procedure.

Assume that $K'_q(Y)$ has already been defined for all q and all contours Y with $|V(Y)| < n$, $n \in \mathbb{N}$, and that it obeys a bound of the form (4.13). Introduce $f_q^{(n-1)}$ as the free energy of a contour model with activities

$$K^{(n-1)}(Y^q) = \begin{cases} K'(Y^q) & \text{if } |V(Y^q)| \leqslant n-1 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

Consider then a contour Y with $|V(Y)| = n$. Since $|V(\tilde{Y})| < n$ for all contours $\tilde{Y}$ in Int Y, the truncated partition functions $Z'_q(\mathrm{Int}_m\, Y, h)$ are well defined for all q and m. Their logarithm can be controlled by a convergent cluster expansion, and $Z'_q(\mathrm{Int}_m\, Y, h) \neq 0$ for all q and m. We therefore may define $K'_q(Y)$ for a q-contour Y with $|V(Y)| = n$ by

$$K'_q(Y) = \chi'_q(Y)\,\rho(Y)\,e^{E_q(Y)} \prod_m \frac{Z_m(\mathrm{Int}_m\, Y, h)}{Z'_q(\mathrm{Int}_m\, Y, h)} \tag{4.15a}$$

with

$$\chi'_q(Y) = \prod_{m \neq q} \chi(\alpha\,|Y| - (f_q^{(n-1)} - f_m^{(n-1)})\,|V(Y)|) \tag{4.15b}$$

Here α is a constant that will be chosen later and χ is a smoothed characteristic function. We assume that χ has been defined in such a way that χ is a C^6 function that obeys the conditions

$$0 \leqslant \chi(x) \leqslant 1 \tag{4.16a}$$

$$\chi(x) = 0 \text{ if } x \leqslant -1 \qquad \text{and} \qquad \chi(x) = 1 \text{ if } x \geqslant 1 \tag{4.16b}$$

$$0 \leqslant \frac{d}{dx}\chi(x) \leqslant 1 \tag{4.16c}$$

$$\left|\frac{d^k}{dx^k}\chi(x)\right| \leqslant \tilde{C}_0 \qquad \text{for all} \quad k \leqslant 6 \tag{4.16d}$$

for some constant $\tilde{C}_0$.

As the final element of the construction of K'_q, we have to establish the bound (4.13) for contours Y with $|V(Y)| = n$. We defer the proof, together with the proof of the following Lemma 4.6, to Appendix C.

We use $f_q = f_q(h)$ to denote the free energy corresponding to the partition function $Z'_q(V, h)$,

$$f_q = -\lim_{V \to \mathbb{R}^d} \frac{1}{|V|} \log Z'_q(V, h) \tag{4.17}$$

and introduce $f = f(h)$ and $a_q = a_q(h)$ as

$$f = \min_m f_m \tag{4.18a}$$

$$a_q = f_q - f \tag{4.18b}$$

Finally, we recall that for a volume W of the form (4.7a), $|W|$ denotes the Euclidean volume of W, while for a contour Y or for the boundary ∂W of a volume W, $|Y|$ and $|\partial W|$ are used to denote the number of elementary cells, i.e., the number of elementary cubes, plaquettes,..., and bonds in Y and ∂W, respectively.

Lemma 4.6. Assume that $\rho(\cdot)$ and $e_q(\cdot)$ obey the conditions (3.7) and (3.11), and let

$$\varepsilon = e^{2+\alpha} e^{-\tau(1-(1+2C_1))\gamma} \tag{4.19a}$$

$$\bar{\alpha} = \frac{2d}{C_3}(\alpha - 2) \tag{4.19b}$$

Then there exists a constant $\varepsilon_0 > 0$ (depending only on d and N) such that the following statements hold provided $\varepsilon < \varepsilon_0$ and $\bar{\alpha} \geqslant 1$:

(i) The contour activities $K'_q(Y)$ are well defined for all Y and obey (4.13).

(ii) If $a_q\,|V(Y)|^{1/d} \leqslant \bar{\alpha}$, then $\chi_q(Y)=1$ and $K_q(Y)=K'_q(Y)$.

(iii) If $a_q\,|W|^{1/d} \leqslant \bar{\alpha}$, then $Z_q(W, h)=Z'_q(W, h)$.

(iv) For all volumes W of the form (4.7a), one has

$$|Z_q(W, h)| \leqslant e^{-f\,|W| + O(\varepsilon)\,|\partial W| + \gamma\tau N_{\partial V}(W)} \tag{4.20}$$

where $N_{\partial V}(W)$ is the number of elementary cubes in $\bar{W}$ which touch ∂V.

(v) For $W=V$ the bound (4.20) can be sharpened to

$$|Z_q(V, h)| \leqslant e^{-f\,|V|} e^{(1+\gamma\tau)\,|\partial V|} \max\{e^{-a_q\,|V|/4}, e^{-(4C_3)^{-1}\tau\,|\partial V|}\} \tag{4.21}$$

where $C_3=C_3(d)$ is the constant defined in Lemma 4.5.

Remarks. (i) Due to the bound (4.13), the partition function $Z'_q(V, h)$ can be analyzed by a convergent cluster expansion. As a consequence, one can prove the usual volume, surface,..., corner asymptotics for its logarithm. Namely, using $f_q^{(d)}, f_q^{(d-1)},..., f_q^{(0)}$ to denote the bulk, surface,..., corner free energies corresponding to $Z'_q(V, h)$, and introducing $F_q(W)$ as

$$F_q(W)=\sum_{c\in\bar{W}} f_q(c) \tag{4.22}$$

where $f_q(c)=f_q$ if c does not touch the boundary ∂V of V, and, in analogy to (3.13),

$$f_q(c)=\sum_{n=k}^{d} \binom{d-k}{n-k} f_m^{(n)}, \qquad k=d-1,..., 0 \tag{4.23}$$

if c is touching a k-dimensional face of V and not touching its $(k-1)$-dimensional boundary, we get

$$|\log Z'_q(V, h)+F_q(V)| \leqslant |V|\; O((K\varepsilon)^L) \tag{4.24}$$

for some $K<\infty$ depending only on N and d.

(ii) It is interesting to present a heuristic derivation of the bound (4.21) in the approximation of the droplet model. To this end, we recall that the diameter of a critical droplet is proportional to τ/a_q. Assume now that $a_q L/\tau$ is small. Then the size of a critical droplet is larger then the system size, and $Z_q(V, h)$ is a partition function describing small perturbations around a metastable ground state, with the weight

$$Z_q(V, h) \sim e^{-f_q\,|V| + O(|\partial V|)} = e^{-f\,|V| + O(|\partial V|)} e^{-a_q\,|V|} \tag{4.25a}$$

For large values of $a_q L/\tau$, on the other hand, supercritical droplets do fit into the volume V. As a consequence, the leading configuration contributing to $Z_q(V, h)$ contains a big contour (with an interior that is essentially all of V) describing a transition from the unstable boundary condition q to a stable phase $\bar{q}$ with $f_{\bar{q}} = f$. We conclude that

$$Z_q(V, h) \sim e^{-f|V| + O(|\partial V|)} e^{-O(\tau)|\partial V|} \tag{4.25b}$$

if $a_q L/\tau$ is large. Except for the numerical value of the involved constants, the bound (4.21) exactly describes this behavior.

(iii) The fact that $\chi_q(Y)$ suppresses supercritical droplets manifests itself in the fact that

$$\chi_q(Y) = 0 \qquad \text{unless} \quad a_q |V(Y)| \leqslant [\alpha + 1 + O(\varepsilon)] |Y| \tag{4.26}$$

See Appendix C for the proof of (4.26).

4.4. Bounds on Derivatives

We finally turn to the continuity properties of Z_q and Z'_q. As a finite sum of C^6 functions, $Z_q(V, h)$ is a C^6 function of h. The following lemma yields a bound on the derivatives of $Z_q(V, h)$.

Lemma 4.7. There is a constant K [depending on d, N, and the constants introduced in (3.8), (3.9), and (4.16)] such that the following statements are true provided $\varepsilon < \varepsilon_0$ and $\bar{\alpha} \geqslant 1$:

(i) $Z_q(W, h)$ is a C^6 function of h and

$$\left| \frac{d^k}{dh^k} Z_q(W, h) \right| \leqslant |k|! \, \{ [C_0 + O(\varepsilon)] \, |W| \}^{|k|} \, e^{-f|V|} e^{O(\varepsilon)|\partial W|} e^{\gamma \tau N_{\partial V}(W)} \tag{4.27}$$

for all multi-indices k of order $1 \leqslant |k| \leqslant 6$.

(ii) $K'_q(Y)$ is a C^6 functions of h, and

$$\left| \frac{d^k}{dh^k} K'_q(Y) \right| \leqslant (K\varepsilon)^{|Y|} \tag{4.28}$$

for all multi-indices k of order $1 \leqslant |k| \leqslant 6$.

(iii) $\log Z'_q(W, h)$ is a C^6 functions of h, and

$$\left| \frac{d^k}{dh^k} \log Z'_q(W, h) \right| \leqslant [C_0^{|k|} + O(\varepsilon)] \, |W| \tag{4.29}$$

for all multi-indices k of order $1 \leqslant |k| \leqslant 6$.

(iv) For $W = V$ (and $1 \leqslant |k| \leqslant 6$), the bound (4.27) can be sharpened to

$$\left| \frac{d^k}{dh^k} Z_q(V, h) \right| \leqslant |k|! \, \{ [C_0 + O(\varepsilon)] \, |V| \}^{|k|} \, e^{-f|V|} e^{(1+\gamma\tau)|\partial V|} \times \max\{ e^{-a_q |V|/4}, e^{-(4C_3)^{-1} \tau |\partial V|} \} \tag{4.30}$$

Proof. The proof of this lemma is given in Appendix D.

Remarks. (i) For many models, including the perturbed Ising model introduced in Section 2, it is possible to prove a *degeneracy-removing condition.* In the context of a model with N ground states and a driving parameter $h \in \mathbb{R}^{N-1}$ ($N = 2$ for the perturbed Ising model), one considers the matrix

$$\mathbb{E} = \left(\frac{d}{dh_i} (e_q - e_N) \right)_{q,\, i = 1, \ldots, N-1} \tag{4.31}$$

and its inverse $\mathbb{E}^{-1}$. One then proves that for some value h_0 of h, all ground-state energies are equal, and that $\mathbb{E}^{-1}$ obeys a bound

$$\|\mathbb{E}^{-1}\|_\infty = \max_i \sum_q |(\mathbb{E}^{-1})_{iq}| \leqslant \text{const} \tag{4.32}$$

in a neighborhood $\mathscr{U}$ of h_0, which does not depend on τ.

On the other hand, $s_q = f_q - e_q$ is a C^6 function of h with

$$|f_q - e_q| \leqslant O(\varepsilon) \tag{4.33}$$

and

$$\left| \frac{d}{dh_i} (f_q - e_q) \right| \leqslant O(\varepsilon) \tag{4.34}$$

by Lemmas 4.6 and 4.7. As a consequence, the inverse of the matrix

$$\mathbb{F} = \left(\frac{d}{dh_i} (f_q - f_N) \right)_{q,\, i = 1, \ldots, N-1} \tag{4.35}$$

obeys a bound of the same form as $\mathbb{E}^{-1}$, with a slightly larger constant on the right-hand side; combined with the inverse function theorem and the fact that $f_q(h_0) - f_N(h_0) \leqslant O(\varepsilon)$, one immediately obtains the existence of a point $h_t \in \mathscr{U}$, $|h_t - h_0| \leqslant O(\varepsilon)$, for which all a_q are zero, i.e., all phases are stable. More generally, one may construct differentiable curves $h_q(t)$, starting at h_t, on which only the phase q is unstable, surfaces $h_{q\bar{q}}(t, s)$ on which phases q and $\bar{q}$ are unstable, etc. A possible parametrization of these curves, surfaces, etc., is given by

$$a_m(h_q(t)) = \delta_{mq} t, \qquad a_m(h_{q\bar{q}}(t, s)) = \delta_{mq} t + \delta_{m\bar{q}} s, \ldots,$$

(ii) Due to Lemma 4.7(ii), the bound (4.24) can be generalized to the first six derivatives of $\log Z'_q(V, h)$. Namely,

$$\left| \frac{d^k}{dh^k} [\log Z'_q(V, h) + F_q(V)] \right| \leqslant |V| \, O((K\varepsilon)^L) \tag{4.36}$$

for all multi-indices k of order $1 \leqslant |k| \leqslant 6$.

4.5. Proof of Theorem 3.1

In order to prove Theorem 3.1, we introduce the sets

$$Q = \{1, \ldots, N\} \qquad \text{and} \qquad S = \{q \in Q \mid a_q L < \bar{\alpha}\} \tag{4.37}$$

Using the decomposition (4.5) together with Lemma 4.6(iii), we bound

$$\begin{aligned} &\left| \frac{d^k}{dh^k} \left[Z(V, h) - \sum_{q=1}^{N} e^{-F_q(V)} \right] \right| \\ &\qquad \leqslant \sum_{q \in S} \left| \frac{d^k}{dh^k} [Z'_q(V, h) - e^{-F_q(V)}] \right| \\ &\qquad \quad + \sum_{q \notin S} \left| \frac{d^k}{dh^k} Z_q(V, h) \right| + \sum_{q \notin S} \left| \frac{d^k}{dh^k} e^{-F_q(V)} \right| \end{aligned} \tag{4.38}$$

where k is an arbitrary multi-index of order $0 \leqslant |k| \leqslant 6$.

Next, we observe that for $1 \leqslant |k| \leqslant 6$,

$$\left| \frac{d^k}{dh^k} F_q(V) \right| \leqslant O(1) \, |V| \tag{4.39}$$

by the assumption (3.9) and the fact that $F_q(V) - E_q(V)$ can be analyzed by a convergent expansion using Lemmas 4.6 and 4.7.

For $q \in S$, we then rewrite

$$[Z'_q(V, h) - e^{-F_q(V)}] = -e^{-F_q(V)}[1 - e^{F_q(V) + \log Z'_q(V, h)}]$$

Using the bounds (4.24), (4.36), and (4.39), we obtain the following bound on the first sum on the r.h.s. of (4.38):

$$\sum_{q \in S} \left| \frac{d^k}{dh^k} [Z'_q(V, h) - e^{-F_q(V)}] \right|$$

$$\begin{aligned} &\leqslant O((K\varepsilon)^L) \, |V|^{|k|+1} \sum_{q \in S} e^{-F_q(V)} \\ &\leqslant O((K\varepsilon)^L) \, |V|^{|k|+1} \max_{q \in S} e^{-F_q(V)} \\ &\leqslant O((K\varepsilon)^L) \, |V|^{|k|+1} \max_{q \in Q} e^{-F_q(V)} \end{aligned} \tag{4.40}$$

In order to bound the last sum in (4.38), we observe that for $q \notin S$ one has

$$\begin{aligned} \left| \frac{d^k}{dh^k} e^{-F_q(V)} \right| &\leqslant O(1) \, |V|^{|k|} \, e^{-F_q(V)} \\ &\leqslant O(1) \, |V|^{|k|} \, e^{[\gamma\tau + O(\varepsilon)] \, |\partial V|} e^{-f_q \, |V|} \\ &= O(1) \, |V|^{|k|} \, e^{[\gamma\tau + O(\varepsilon)] \, |\partial V|} e^{-a_q \, |V|} e^{-f \, |V|} \\ &\leqslant O(1) \, |V|^{|k|} \, e^{[2\gamma\tau + O(\varepsilon)] \, |\partial V|} e^{-a_q \, |V|} \max_{q \in Q} e^{-F_q(V)} \\ &\leqslant O(1) \, |V|^{|k|} \, e^{-[\bar{\alpha}/2d - 2\gamma\tau - O(\varepsilon)] \, |\partial V|} \max_{q \in Q} e^{-F_q(V)} \end{aligned} \tag{4.41}$$

where we used the definition (4.37) of S and, in the last step, the fact that $L^{-1} \, |V| = (1/2d) \, |\partial V|$.

Finally, again for $q \notin S$, we have

$$\begin{aligned} \left| \frac{d^k}{dh^k} Z_q(V, h) \right| &\leqslant O(1) \, |V|^{|k|} \, e^{(1+\gamma\tau) \, |\partial V|} \max\{e^{-\bar{\alpha}/4 \, |V|}, e^{-\tau/4C_3 \, |\partial V|}\} \, e^{-f \, |V|} \\ &\leqslant O(1) \, |V|^{|k|} \, e^{(1 + 2\gamma\tau - \min\{\bar{\alpha}/8d, \, \tau/4C_3\}) \, |\partial V|} \max_{q \in Q} e^{-F_q(V)} \end{aligned} \tag{4.42}$$

by (4.21) and (4.30).

Inserting the bounds (4.40)–(4.42) into (4.38) and observing that $|\partial V| \geqslant 2dL$ for all $d \geqslant 2$, we finally obtain the bound

$$\left| \frac{d^k}{dh^k} \left[Z(V, h) - \sum_{q=1}^{N} e^{-F_q(V)} \right] \right| \leqslant O(e^{-L/L_0}) \, |V|^{|k|+1} \max_{q \in Q} e^{-F_q(V)} \tag{4.43}$$

where

$$\frac{1}{L_0} := \min\left\{-\log(K\varepsilon), \bar{\alpha} - 4d\gamma\tau - O(\varepsilon), \frac{\bar{\alpha}}{4} - 2d - 4d\gamma\tau, \frac{2d}{4C_3}\tau - 2d - 4d\gamma\tau\right\}$$

$$= \min\left\{-\log(K\varepsilon), \frac{\bar{\alpha}}{4} - 2d - 4d\gamma\tau, \frac{2d}{4C_3}\tau - 2d - 4d\gamma\tau\right\} \tag{4.44}$$

Recalling the definitions (4.19) of $\bar{\alpha}$ and ε, together with the fact that $C_3 = C_1 + 4d$, we now rewrite

$$-\log(K\varepsilon) = \tau - \alpha - (1 + 2C_1)\gamma\tau - O(1) \tag{4.45a}$$

$$\frac{\bar{\alpha}}{4} - 2d - 4d\gamma\tau = \frac{2d}{4C_3}[\alpha - (32d + 8C_1)\gamma\tau] - O(1) \tag{4.45b}$$

and

$$\frac{2d}{4C_3}\tau - 2d - 4d\gamma\tau = \frac{2d}{4C_3}[\tau - (32d + 8C_1)\gamma\tau] - O(1)$$

$$= 2\frac{2d}{4C_3}\left[\frac{\tau}{2} - (16d + 4C_1)\gamma\tau\right] - O(1) \tag{4.45c}$$

Choosing $\alpha = \tau/2 + (16d + 3C_1 - 1/2)\gamma\tau$, we obtain

$$-\log(K\varepsilon) = \frac{\tau}{2} - \left(\frac{1}{2} + 5C_1 + 16d\right)\gamma\tau - O(1) \tag{4.46a}$$

$$\frac{\bar{\alpha}}{4} - 2d - 4d\gamma\tau = \frac{2d}{4C_3}\left[\frac{\tau}{2} - \left(\frac{1}{2} + 16d + 5C_1\right)\gamma\tau\right] - O(1) \tag{4.46b}$$

and

$$\frac{1}{L_0} = \frac{2d}{4C_3}\left[\frac{\tau}{2} - \left(\frac{1}{2} + 16d + 5C_1\right)\gamma\tau\right] - O(1)$$

$$= \frac{d}{4C_3}[\tau - (1 + 32d + 10C_1)\gamma\tau - \tau_0] \tag{4.46c}$$

where τ_0 is a constant that depends on N, d, and the constants introduced in (3.8), (3.9), and (4.16).

Defining

$$b = b(d) = \frac{d}{4C_3} \tag{4.47a}$$

$$\gamma_0 = \gamma_0(d) = \frac{1}{1 + 32d + 10C_1} \tag{4.47b}$$

and

$$\tilde{\tau} = \tau(1 - \gamma/\gamma_0) - \tau_0 \tag{4.47c}$$

we obtain $1/L_0 = b\tilde{\tau}$ and hence the bound (3.19) of Theorem 3.1.

Observing that

$$\varepsilon = e^{2 - (\tau/2)(1 - (1 + 16d + 10C_1)\gamma)} = e^2 e^{-(\tau/2)(1 - \gamma/\gamma_0)} \tag{4.48}$$

we note that the condition $\tilde{\tau} > 0$ implies the inequality $\varepsilon < \varepsilon_0$ provided τ_0 is chosen large enough. The condition $\bar{\alpha} \geqslant 1$, on the other hand, is trivial, since

$$\bar{\alpha} = \frac{2d}{C_3}(\alpha - 2) \geqslant \frac{d}{C_3}\tau - O(1)$$

It remains to prove statements (iii) and (v). While (v) is a direct consequence of (iv), statement (iii) follows from the fact that $(f_m - e_m)$ and $(f_m^{(l)} - e_m^{(l)})$ can be analyzed by a convergent cluster expansion involving the decay constant ε. Observing that $O(\varepsilon) \leqslant O(e^{-\tilde{\tau}})$ can be bounded by $e^{-b\tilde{\tau}}$, this proves (iii). ▮

5. PROOF OF THEOREM 3.2

5.1. Decomposition of $Z(A \mid V, h)$ into Pure Phase Partition Functions

The first step in the proof of Theorem 3.2 is the same as the first step in the proof of Theorem 3.1. Namely, we decompose $Z(A \mid V, h)$ as

$$Z(A \mid V, h) = \sum_{q=1}^{N} Z_q(A \mid V, h) \tag{5.1}$$

with

$$Z_q(A \mid V, h) = \sum_{\{Y_1,\ldots,Y_n\}} A(Y_1,\ldots,Y_n) \prod_{k=1}^{n} \rho(Y_k) \prod_{m=1}^{N} e^{-E_m(V_m)} \tag{5.2}$$

Here the sum goes over sets of matching contours in V for which all external contours are q-contours.

Next, we group all contours Y_i for which $V(Y_i) \cap \operatorname{supp} A \neq \varnothing$ into a new contour $\check{Y}_A$, and introduce the sets

$$\operatorname{supp} Y_A = \bigcup_{Y \in Y_A} \operatorname{supp} Y, \qquad V(Y_A) = \bigcup_{Y \in Y_A} V(Y)$$

$$\operatorname{Int} Y_A = V(Y_A) \backslash \operatorname{supp} Y_A, \qquad \operatorname{Ext} Y_A = V \backslash V(Y_A)$$

as well as

$$\overline{\text{supp } Y_A} = \text{supp } Y_A \cup \text{supp } A, \qquad \overline{V(Y_A)} = V(Y_A) \cup \text{supp } A$$
$$\text{Int}^{(0)}\, Y_A = \text{Int } Y_A \backslash \text{supp } A, \qquad \text{Ext}^{(0)}\, Y_A = \text{Ext } Y_A \backslash \text{supp } A$$

As usual, $\text{Int}_m\, Y_A$ is the union of all components of Int Y_A which have boundary condition m, $\text{Int}_m\, Y_A = \text{Int } Y_A \cap V_m$, while $\text{Int}_m^{(0)}\, Y_A = \text{Int}^{(0)}\, Y_A \cap V_m$.

Recalling that A only depends on those contours for which $V(Y) \cap \text{supp } A \neq \emptyset$, we then define

$$\rho(Y_A) = A(Y_1', \ldots, Y_{n'}') \prod_{k=1}^{n'} \rho(Y_k') \prod_{m=1}^{N} e^{-E_m(V_m \cap (\text{supp } A \backslash \text{supp } Y_A))} \tag{5.3}$$

where $Y_1', \ldots, Y_{n'}'$ are the contours in Y_A. Fixing now, for a moment, all contours Y_i in (5.2) for which $V(Y_i) \cap \text{supp } A \neq \emptyset$, and resumming the rest, we obtain

$$Z_q(A \mid V, h) = \sum_{Y_A} \rho(Y_A)\, Z_q(\text{Ext}^{(0)}\, Y_A, h) \prod_{m=1}^{N} Z_m(\text{Int}_m^{(0)}\, Y_A, h) \tag{5.4}$$

Introducing

$$K_q(Y_A) = \rho(Y_A)\{\exp[E_q(\overline{\text{supp } Y_A})]\} \prod_{m=1}^{N} \frac{Z_m(\text{Int}_m^{(0)}\, Y_A, h)}{Z_q(\text{Int}_m^{(0)}\, Y_A, h)} \tag{5.5}$$

we further rewrite (5.4) as

$$Z_q(A \mid V, h) = \sum_{Y_A} K(Y_A)\{\exp[-E_q(\overline{\text{supp } Y_A})]\}\, Z_q(\text{Ext}^{(0)}\, Y_A, h) \times Z_q(\text{Int}^{(0)}\, Y_A, h) \tag{5.6}$$

Using finally the representation (4.11) for $Z_q(\text{Ext}^{(0)}\, Y_A, h)$ and $Z_q(\text{Int}^{(0)}\, Y_A, h)$, we get

$$Z_q(A \mid V, h) = e^{-E_q(V)} \sum_{Y_A} K_q(Y_A) \sum_{\{Y_1, \ldots, Y_n\}} \prod_{k=1}^{n} K_q(Y_k) \tag{5.7}$$

Here the second sum goes over the set of nonoverlapping q contours $Y_1, \ldots, Y_n$, such that for all contours Y_i, the set $V(Y_i)$ does not intersect the set $\overline{\text{supp } Y_A}$.

In order to make the connection to the standard Mayer expansion for polymer systems, we then introduce $G(Y_A, Y_1, \ldots, Y_n)$ as the graph on the vertex set $\{0, 1, \ldots, n\}$ which has an edge between two vertices $i \geqslant 1$ and

$j \geqslant 1$, $i \neq j$, whenever $\operatorname{supp} Y_i \cap \operatorname{supp} Y_j \neq \varnothing$, and an edge between the vertex 0 and a vertex $i \neq 0$ whenever $V(Y_i) \cap \overline{\operatorname{supp} Y_A} \neq \varnothing$. Implementing the nonoverlap constraint in (5.7) by a characteristic function $\phi(Y_A, Y_1, ..., Y_n)$ which is zero whenever the graph G has less than $n+1$ components, we find that the standard Mayer expansion for polymer systems (see, for example, ref. 27) then yields

$$\frac{Z_q(A \mid V, h)}{Z_q(V, h)} = \sum_{Y_A} K_q(Y_A) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{Y_1,..., Y_n\}} \left[\prod_{k=1}^{n} K_q(Y_k) \right] \phi_c(Y_A, Y_1,..., Y_n) \tag{5.8}$$

Here $\phi_c(Y_A, Y_1,..., Y_n)$ is a combinatoric factor defined in term of the connectivity properties of the graph $G(Y_A, Y_1,..., Y_n)$.[27] It vanishes if $G(Y_A, Y_1,..., Y_n)$ has more than one component.

5.2. Truncated Expectation Values

In the context of Section 5.1, the expansion (5.8) is a formal power series in the activities $K(Y_i)$. In order to use this expansion, one has to prove its convergence. As in Section 4, it is useful to introduce truncated models.

For a contour Y with $V(Y) \cap \operatorname{supp} A = \varnothing$, we define $K'_q(Y)$ as before [see (4.15a)], while for $Y_A = \{Y_1,..., Y_n\}$, where $\{Y_1,..., Y_n\}$ is a set of contours with $V(Y) \cap \operatorname{supp} A \neq \varnothing$ for all $Y \in Y_A$, we define

$$K'_q(Y_A) = \rho(Y_A)\{\exp[E_q(\overline{\operatorname{supp} Y_A})]\} \prod_{m=1}^{N} \frac{Z_m(\operatorname{Int}_m^{(0)} Y_A, h)}{Z'_q(\operatorname{Int}_m^{(0)} Y_A, h)} \prod_{Y \in Y_A} \chi_q(Y) \tag{5.9}$$

with $\chi_q(Y)$ as in (4.15b). Given this definition, we introduce

$$Z'_q(A \mid V, h) = e^{-E_q(V)} \sum_{Y_A} K'_q(Y_A) \sum_{\{Y_1,..., Y_n\}} \prod_{k=1}^{n} K'_q(Y_k) \tag{5.10}$$

and

$$\langle A \rangle_{V,q}^{h} = \frac{Z'_q(A \mid V, h)}{Z'_q(V, h)} \tag{5.11}$$

which can again be expanded as

$$\langle A \rangle_{V,q}^{h} = \sum_{Y_A} K'_q(Y_A) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{Y_1,..., Y_n\}} \left[\prod_{k=1}^{n} K'_q(Y_k) \right] \phi_c(Y_A, Y_1,..., Y_n) \tag{5.12}$$

The following lemma will allow us to prove absolute convergence of the expansion (5.12), which immediately yields the statements (ii)–(iv) of Theorem 3.2.

Lemma 5.1. Let ε, ε_0, and $\bar{\alpha}$ be as defined in Lemma 4.6, and assume that $\varepsilon < \varepsilon_0$ and $\bar{\alpha} \geqslant 1$. Then the following statements are true:

(i) If

$$a_q \max_{Y \in Y_A} |V(Y)|^{1/d} \leqslant \bar{\alpha} \tag{5.13}$$

then $K_q(Y_A) = K'_q(Y_A)$.

(ii) Let

$$|Y_A| = \sum_{Y \in Y_A} |\text{supp } Y| \tag{5.14}$$

Then

$$|K'_q(Y_A)| \leqslant C_A e^{O(\varepsilon)\, |\text{supp } A|} \varepsilon^{|Y_A|} \tag{5.15}$$

(iii) Let k be a multi-index of order $1 \leqslant |k| \leqslant 6$. Then

$$\left| \frac{d^k}{dh^k} K'_q(Y_A) \right| \leqslant |\text{supp } A|^{|k|}\, C_A e^{O(\varepsilon)\, |\text{supp } A|} (K\varepsilon)^{|Y_A|} \tag{5.16}$$

where K is a constant that depends only on d, N, and the constants introduced in (3.8), (3.9), and (4.16).

Proof. The proof of Lemma 5.1 is given in Appendix E.

Using standard estimates for polymer expansions (see, for example, ref. 27), we see that the bounds of Lemmas 4.6 and 5.1 immediately imply the absolute convergence of the expansion (5.12),

$$\begin{aligned} |\langle A \rangle^h_{V,q}| &\leqslant \sum_{Y_A} |K'_q(Y_A)| \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{Y_1,\dots,Y_n\}} \prod_{k=1}^{n} |K'_q(Y_k)| \cdot |\phi_c(Y_A, Y_1,\dots, Y_n)| \\ &\leqslant O(1)\, C_A e^{O(\varepsilon)\, |\text{supp } A|} \end{aligned} \tag{5.16a}$$

and similar bounds for the derivatives, in particular,

$$\left| \frac{d^k}{dh^k} \langle A \rangle^h_{V,q} \right| \leqslant O(1)\, |\text{supp } A|^{|k|}\, C_A e^{O(\varepsilon)\, |\text{supp } A|} \tag{5.16b}$$

Theorem 3.2(ii)–(iv) then follows using standard arguments.

5.3. Bounds on $Z_q(A \mid V, h)$

In conjunction with Lemma 4.6, Lemma 5.1 allows one to analyze $Z_q(A \mid V, h)/Z_q(V, h)$ provided $a_q L \leqslant \bar{\alpha}$. In order to prove Theorem 3.2 in the case where $a_q L > \bar{\alpha}$ for some of the phases q, we need an analog of the bounds (3.21) and (4.30) for $Z_q(A \mid V, h)$.

Lemma 5.2. Let ε, ε_0, and $\bar{\alpha}$ be as defined in Lemma 4.6, let $\tilde{\varepsilon} = \max\{\varepsilon, e^{-3\tau/4}\}$, and assume that $\tilde{\varepsilon} < \varepsilon_0$ and $\bar{\alpha} \geqslant 1$. Then the following statements are true:

(i) $$|Z_q(A \mid V, h)| \leqslant C_A e^{O(\tilde{\varepsilon})\,|\text{supp}\, A|} e^{[\gamma\tau + O(\tilde{\varepsilon})]\,|\partial V|} e^{-f\,|V|} \times \max\{e^{-(a_q/4)\,|V|}, e^{-(\tau/4C_3)\,|\partial V|}\}$$

(ii) Let k be a multi-index of order $1 \leqslant |k| \leqslant 6$. Then

$$\left|\frac{d^k}{dh^k} Z_q(A \mid V, h)\right| \leqslant |k|!\, \{[C_0 + O(\varepsilon)]\, |V|\}^{|k|}\, C_A e^{O(\tilde{\varepsilon})\,|\text{supp}\, A|} \times e^{[\gamma\tau + O(\tilde{\varepsilon})]\,|\partial V|} e^{-f\,|V|} \max\{e^{-(a_q/4)\,|V|}, e^{-(\tau/4C_3)\,|\partial V|}\}$$

Proof. The proof of Lemma 5.2 is given in Appendix E.

5.4. Proof of Theorem 3.2

As pointed out before, the absolute convergence of the cluster expansion (5.12) immediately implies the statements (ii)–(iv). In order to prove Theorem 3.2(i), we proceed as in the proof of Theorem 3.1, using the decomposition (5.1), Lemma 5.1, and Lemma 5.2 instead of the decomposition (4.5), Theorem 4.6, and Theorem 4.7. Defining S as in Section 4.5, and observing $Z_q(A \mid V, h) = Z'_q(A \mid V, h)$ if $q \in S$, we bound

$$\begin{aligned}
&\left|\frac{d^k}{dh^k}\left[Z(A \mid V, h) - \sum_{q=1}^{N} e^{-F_q(V)} \langle A \rangle_{V,q}^{h}\right]\right| \\
&\quad \leqslant \sum_{q \in S} \left|\frac{d^k}{dh^k}\{\langle A \rangle_{V,q}^{h} [Z'_q(V, h) - e^{-F_q(V)}]\}\right| \\
&\quad\quad + \sum_{q \notin S} \left|\frac{d^k}{dh^k} Z_q(A \mid V, h)\right| + \sum_{q \notin S} \left|\frac{d^k}{dh^k}[\langle A \rangle_{V,q}^{h} e^{-F_q(V)}]\right| \qquad (5.17)
\end{aligned}$$

where k is an arbitrary multi-index of order $0 \leqslant |k| \leqslant 6$.

Combining the bounds (4.40) and (5.16), and bounding terms of the form $|\text{supp}\, A|^{|k|}$ and $|V|^{|k|+1}$ by $e^{O(1)L}$, we get an estimate for the first sum on the right-hand side of (5.17) by

$$\sum_{q\in S}\left|\frac{d^k}{dh^k}\left[Z'_q(A\mid V,h)-\langle A\rangle^h_{V,q}\,e^{-F_q(V)}\right]\right| \leqslant C_A e^{O(\varepsilon)\,|\text{supp}\,A|}(K\varepsilon)^L \max_{q\in Q} e^{-F_q(V)} \tag{5.18}$$

Here K is a constant that depends only on N, d, and the constants introduced in (3.8), (3.9), and (4.16). The terms for $q\notin S$ are bound in a similar way, leading to

$$\left|\frac{d^k}{dh^k}\left[\langle A\rangle^h_{V,q}\,e^{-F_q(V)}\right]\right| \leqslant C_A e^{O(\varepsilon)\,|\text{supp}\,A|} e^{-[\bar{\alpha}/2d-2\gamma\tau-O(1)]\,|\partial V|} \max_{q\in Q} e^{-F_q(V)} \tag{5.19}$$

and

$$\left|\frac{d^k}{dh^k}Z_q(A\mid V,h)\right| \leqslant C_A e^{O(\bar{\varepsilon})\,|\text{supp}\,A|} e^{(2\gamma\tau+O(1)-\min\{\bar{\alpha}/8d,\,\tau/4C_3\})\,|\partial V|} \max_{q\in Q} e^{-F_q(V)} \tag{5.20}$$

Inserting the bounds (5.18)–(5.20) into (5.17), and choosing α as in Section 4.5, we get $\tilde{\varepsilon}=\varepsilon$ and

$$\left|\frac{d^k}{dh^k}\left[Z(A\mid V,h)-\sum_{q=1}^{N} e^{-F_q(V)}\langle A\rangle^h_{V,q}\right]\right| \leqslant C_A e^{O(\varepsilon)\,|\text{supp}\,A|} O(e^{-b\tilde{\tau}L}) \max_{q\in Q} e^{-F_q(V)} \tag{5.21}$$

where b and $\tilde{\tau}$ are the constants introduced in (4.47). Together with Theorem 3.1 and the observation that a prefactor $|V|^{|k|+1}$ can be absorbed into the exponential decay term $e^{-b\tilde{\tau}L}$, the bound (5.21) implies Theorem 3.2(i). ▮

6. PROOF OF THEOREM A

Even though the statement of this section have a generalization (sometimes a very straightfoward one) to the case of several phases, we will restrict ourselves to the situation where only two phases, plus and minus, come into play and the driving parameter is an external field h. However, we do not restrict ourselves to the model (2.1) (for which Theorem A is stated), but consider the two-phase case in the general setting of Section 3. In particular, we have two ground-state energies $e_\pm$ satisfying, for h in an interval $\mathscr{U}$ [containing the point h_0 for which $e_+(h)=e_-(h)$], the (non-degeneracy) bounds

$$0<\bar{a}\leqslant\frac{d}{dh}[e_-(h)-e_+(h)]\leqslant\bar{A} \tag{6.1}$$

which imply the bounds

$$0<a\leqslant\frac{d}{dh}[f_-(h)-f_+(h)]\leqslant A \tag{6.2}$$

on the free energies $f_\pm(h)$ from Theorem 3.1 [cf. also (4.17)]. Actually, $\bar{A}=2C_0$ according to the assumption (3.9). In the situation of Theorem A we have $(d/dh)[e_-(h)-e_+(h)]=2$. Considering now the free energies[12]

$$F_\pm(L,h)=\sum_{k=0}^{d} f_\pm^{(k)}(h)\,|\partial_k V| \tag{6.3}$$

[cf. (3.18)] and their derivatives

$$M_\pm(L,h)=\sum_{k=0}^{d} m_\pm^{(k)}(h)\,|\partial_k V| \tag{6.4}$$

where $m_\pm^{(k)}(h)=-df_\pm^{(k)}(h)/dh$, and introducing

$$\Delta F(L,h)=F_+(L,h)-F_-(L,h) \tag{6.5a}$$

$$\Delta M(L,h)=M_+(L,h)-M_-(L,h) \tag{6.5b}$$

$$F_0(L,h)=\frac{F_+(L,h)+F_-(L,h)}{2} \tag{6.5c}$$

$$M_0(L,h)=\frac{M_+(L,h)+M_-(L,h)}{2} \tag{6.5d}$$

we reformulate the bounds (3.21) of Theorem 3.1 for the two-phase case as

$$\left|\frac{d^k}{dh^k}\left\{m(L,h)-\left[\frac{1}{L^d}M_0(L,h)+\frac{1}{L^d}\frac{\Delta M(L,h)}{2}\tanh\left(-\frac{\Delta F(L,h)}{2}\right)\right]\right\}\right| \leqslant e^{-b\bar{\tau}L} \tag{6.6}$$

[12] We take here $\partial_d V\equiv V$.

with $0 \leqslant k \leqslant 5$. For the magnetization $m(h, L)$ and its derivative, the susceptibility $\chi(h, L) = dm(h, L)/dh$, these bounds yield

$$m(L, h) = \frac{1}{L^d} M_0(L, h) + \frac{1}{L^d} \frac{\Delta M(L, h)}{2} \tanh\left(-\frac{\Delta F(L, h)}{2}\right) + O(e^{-b\bar{\tau}L}) \tag{6.7}$$

and

$$\chi(L, h) = \frac{1}{L^d} \chi_0(L, h) + \frac{1}{L^d} \frac{\Delta\chi(L, h)}{2} \tanh\left(-\frac{\Delta F(L, h)}{2}\right) + \frac{1}{L^d}\left(\frac{\Delta M(L, h)}{2}\right)^2 \cosh^{-2}\left(-\frac{\Delta F(L, h)}{2}\right) + O(e^{-b\bar{\tau}L}) \tag{6.8}$$

Here $\chi_0(L, h) = dM_0(L, h)/dh$ and $\Delta\chi(L, h) = d\Delta M(L, h)/dh$.

In order to obtain Theorem A, and more generally the corrections to it in terms of an asymptotic power series in $1/L$, we proceed in several steps:

(i) We expand the funtions $\Delta F(L, h)$, $M_0(L, h)$, $\Delta M(L, h)$, $\chi_0(L, h)$, and $\Delta\chi(L, h)$ around the point $h_t(L)$ where $\Delta F(L, h) = 0$, obtaining a power series in $(h - h_t(L))$ with coefficients that are derivatives of $\Delta F(L, h)$ and $F_0(L, h)$ at the point $h_t(L)$.

(ii) We Taylor expand the coefficients in (i) into a power series in $(h_t(L) - h_t)$. Combined with the volume, surface,..., corner expansion for the derivatives of $F_\pm(L, h)$ and the fact that $h_t(L) - h_t$ can be represented as an asymptotic expansion in powers of $1/L$, we obtain the coefficients of (i) as power series in $1/L$, with coefficients that are derivatives of the infinite-volume free energies $f_\pm(h)$, surface free energies $f_\pm^{(d-1)}(h)$,..., and corner free energies $f_\pm^{(0)}(h)$ at the infinite-volume transition point h_t.

(iii) At h_t, the derivatives of $f_\pm^{(k)}(h)$ are identified with the one-sided derivatives of the free energies $f^{(k)}(h)$ defined by (2.15).

(iv) We use Lemma 6.1 below to replace the argument of the hyperbolic functions in (6.7) and (6.8) by few expansion terms with an additive error.

(v) In a final step, we use Lemma 6.3 to replace $h - h_t(L)$ by $h - h_\chi(L)$, where $h_\chi(L)$ is the position of the susceptibility maximum.

Lemma 6.1. Let x and y be two nonzero real numbers which have the same sign. Then

$$|\tanh x - \tanh y| \leqslant \min\left(\frac{\tanh x}{x}, \frac{\tanh y}{y}\right) |x - y| \tag{6.9a}$$

and

$$|\cosh x - \mathrm{cohh}\, y| \leqslant 2 \min\left(\frac{\tanh x}{x}, \frac{\tanh y}{y}\right) |x-y| \tag{6.9b}$$

Lemma 6.2. For large L there exists a unique point $h_t(L) \in \mathscr{U}$ for which $F_+(L, h) = F_-(L, h)$. This point satisfies the bound

$$h_t(L) = h_t + \frac{\Delta F(L, h_t)}{m_+ - m_-} \frac{1}{L^d}\left[1 + O\left(\frac{1}{L}\right)\right] \tag{6.10}$$

Lemma 6.3. For large L there exists a unique point $h_\chi(L) \in \mathscr{U}$ as well as a unique point $h_U(L) \in \mathscr{U}$ for which the susceptibility $\chi(L, h)$ and the Binder cumulant $U(L, h)$, respectively, attain its maximum. To the leading order in $1/L$, their shift with respect to the point $h_t(L)$ is given by

$$h_\chi(L) = h_t(L) + 6\frac{\chi_+ - \chi_-}{(m_+ - m_-)^3} \frac{1}{L^{2d}} + O\left(\frac{1}{L^{2d+1}}\right) \tag{6.11}$$

and

$$h_U(L) = h_t(L) + 4\frac{\chi_+ - \chi_-}{(m_+ - m_-)^3} \frac{1}{L^{2d}} + O\left(\frac{1}{L^{2d+1}}\right) \tag{6.12}$$

Proof of Theorem A. Let us begin with the identification (iii). Introducing

$$m_\pm^{(k)}(h) = -\frac{df_\pm^{(k)}(h)}{dh}, \qquad \chi_\pm^{(k)}(h) = -\frac{d^2 f_\pm^{(k)}(h)}{dh^2}, \qquad k = d, \dots, 0 \tag{6.13}$$

we get

$$f_\pm^{(k)}(h_t) = \lim_{h \to h_t \pm 0} f^{(k)}(h) \tag{6.14}$$

and

$$m_\pm^{(k)}(h_t) = -\left.\frac{df^{(k)}(h)}{dh}\right|_{h_t \pm 0}, \qquad \chi_\pm^{(k)}(h_t) = -\left.\frac{d^2 f^{(k)}(h)}{dh^2}\right|_{h_t \pm 0} \tag{6.15}$$

for $k = d, \dots, 0$. In particular, the one-sided derivatives (2.16) as well as the limits (2.17) and (2.18) are expressed in terms of derivatives and limits of the corresponding differentiable function $f_\pm^{(k)}$:

$$m_\pm = -\left.\frac{d}{dh} f_\pm(h)\right|_{h = h_t} \tag{6.16}$$

and

$$\tau_{\pm} = f_{\pm}^{(d-1)}(h_t) \tag{6.17}$$

Also

$$\Delta F(L) = \Delta F(L, h_t) \tag{6.18}$$

with $\Delta F(L, h) = (F_+(L, h) - F_+(L, h))$. To show all this, we first notice that, for the two-phase case, the bound (3.19) from Theorem 3.1 reads

$$\left| \frac{d^k}{dh^k} [Z(L, h) - e^{-F_+(L,h)} - e^{-F_-(L,h)}] \right|$$
$$\leqslant |V|^{k+1} \max(e^{-F_+(L,h)}, e^{-F_-(L,h)})\, O(e^{-b\bar{\tau}L}) \tag{6.19}$$

Taking into account that $F_{\pm}(L, h)$ are asymptotically dominated by $f_{\pm}(h)\, L^d$, the bound (6.19) implies that for $h > h_t$ the free energies $f^{(k)}(h)$, $k = d,..., 0$, defined by (2.15) actually equal the corresponding free energies $f_+^{(k)}(h)$, $k = d,..., 0$, from Theorem 3.1 [we have chosen the notation for which $\min(f_+(h), f_-(h)) = f_+(h)$ for $h > h_t$]. Similarly, $f^{(k)}(h) = f_-^{(k)}(h)$, $k = d,..., 0$, for every $h < h_t$.[13] This identification immediately implies the equalities (6.14)–(6.18).

Notice also that by (3.9) and Theorem 3.1(iii) one has

$$\left| \frac{d^k f_{\pm}^{(l)}}{dh^k} \right| \leqslant C_0^k + e^{-b\bar{\tau}} \tag{6.20}$$

and thus also

$$m_{\pm}(h_t(L)) = m_{\pm} + O(|h_t(L) - h_t|) = m_{\pm} + O\left(\frac{1 + \|\kappa\|}{L}\right) \tag{6.21}$$

according to Lemma 6.2, where we evaluate $\Delta F(L)$ with the help of (2.8) and Theorem 3.1(iii).

Expanding now $M_{\pm}(L, h)$ and $F_{\pm}(L, h)$, we have

$$\begin{aligned} M_{\pm}(L, h) &= M_{\pm}(L, h_t(L)) + O([h - h_t(L)]\, L^d) \\ &= m_{\pm}(h_t(L))\, L^d + O([h - h_t(L)]\, L^d) + O(L^{d-1}) \\ &= m_{\pm} L^d + O([h - h_t(L)]\, L^d) + O((1 + \|\kappa\|)\, L^{d-1}) \end{aligned} \tag{6.22}$$

[13] For $h = h_t$, the asymptotic behavior will be determined by the first $k = d - 1,..., 0$, for which $f_+^{(k)}(h_t) \neq f_-^{(k)}(h_t)$. For example, if $f_+^{(d-1)}(h_t) > f_-^{(d-1)}(h_t)$, then $f^{(k)}(h_t) = f_-^{(k)}(h_t)$ for all $k = d,..., 0$ [of course, $f_+^{(d)}(h_t) = f_-^{(d)}(h_t)$].

and

$$-\Delta F(L,h) = [M_+(L,h_t(L)) - M_-(L,h_t(L)][h-h_t(L)] + O([h-h_t(L)]^2 L^d)$$
$$= (m_+ - m_-)[h-h_t(L)]L^d + O([h-h_t(L)]^2 L^d) + O([1+\|\kappa\|][h-h_t(L)]\,L^{d-1})$$
$$= 2x\{1+O([h-h_t(L)]\} + O([1+\|\kappa\|]\,L^{-1}) \qquad (6.23)$$

Here

$$x = \frac{m_+ - m_-}{2}[h-h_t(L)]\,L^d \qquad (6.24)$$

Using Lemma 6.1 to replace the argument of the hyperbolic functions in (6.7) and (6.8) by x, we get

$$m(L,h) = \frac{m_+ + m_-}{2} + \frac{m_+ - m_-}{2}\tanh(x) + O([1+\|\kappa\|]\,L^{-1}) + O([h-h_t(L)] \qquad (6.25)$$

and

$$\chi(L,h) = \left(\frac{m_+ - m_-}{2}\right)^2 \cosh^{-2}(x) L^d + O([1+\|\kappa\|]\,L^{d-1}) + O([h-h_t(L)]L^d) \qquad (6.26)$$

In order to replace further the argument x of the hyperbolic functions by the argument

$$\tilde{x} = \frac{m_+ - m_-}{2}[h-h_\chi(L)]L^d \qquad (6.27)$$

used in (2.19) and (2.20), we finally use Lemma 6.3 to bound

$$|\tanh x - \tanh \tilde{x}| \leqslant |x-\tilde{x}| \leqslant O(L^{-d}) \qquad (6.28a)$$

and

$$|\cosh^{-2} x - \cosh^{-2}\tilde{x}| \leqslant |x-\tilde{x}| \leqslant O(L^{-d}) \qquad (6.28b)$$

Combining the bounds (6.24) and (6.28) with the assumption $|h-h_\chi(L)| \leqslant O([1+\|\kappa\|]\,L^{-1})$, we get the bounds (2.19) and (2.20).

The shifts (2.21a) and (2.21b) as well as the bound (2.22) on the mutual shift $h_U(L)-h_\chi(L)$ follow from Lemmas 6.2 and 6.3. ∎

Proof of Lemma 6.1. Without loss of generality, we may assume that $x>y>0$. Since $(\tanh t)/t$ is a decreasing function of t, we have $(\tanh y)/y>(\tanh x)/x$ and thus

$$\left|\frac{\tanh x-\tanh y}{\tanh x}\right|=1-\frac{\tanh y}{\tanh x}\leqslant 1-\frac{y}{x}=\frac{|x-y|}{|x|} \tag{6.29}$$

This concludes the proof of (6.9a). In order to prove (6.9b), we bound

$$\begin{aligned}|\cosh^{-2} x-\cosh^{-2} y| &= 2\left|\int_x^y \frac{\sin t}{\cosh^3 t}\,dt\right| \\ &\leqslant 2\left|\int_x^y \cosh^{-2} t\,dt\right| \\ &= 2\,|\tanh x-\tanh y| \end{aligned}\tag{6.30}$$

and use (6.9a). ∎

Proof of Lemma 6.2. Using the bounds (6.2), we get, for sufficiently large L, the bound

$$\frac{a}{2}L^d\leqslant -\frac{d\Delta F(L,h)}{dh}\leqslant 2AL^d \tag{6.31}$$

Since $\Delta F(L)=\Delta F(L,h_t)=O(L^{d-1})$, we get the existence of a unique $h_t(L)$ for which $\Delta F(L,h)=0$. Moreover, $h_t(L)\in(h_t-\bar{B}/L,\ h_t+\bar{B}/L)$ for some $\bar{B}$. For h in this interval, the Taylor expansion of $\Delta F(L,h)$ around h_t yields

$$\Delta F(L,h)=\Delta F(L)-(m_+-m_-)\,L^d(h-h_t)+(h-h_t)\,O(L^{d-1}) \tag{6.32}$$

This implies (6.10) [valid also for $\Delta F(L)=0$ when $h_t(L)=h_t$]. ∎

Proof of Lemma 6.3. To get (6.11), we can actually follow the proof of Theorem (3.3) in ref. 6, replacing only h_t by $h_t(L)$. Thus we use first (6.8) combined with the bound

$$\left|\frac{d^kF_\pm(L,h)}{dh^k}\right|\leqslant \bar{C}_kL^d \tag{6.33}$$

[here $\bar{C}_k$ may be chosen as $\bar{C}_k = (C_0)^k + O(e^{-b\bar{\tau}})$ by (3.9) and Theorem 3.1(iii)] to get

$$\left| \chi(L, h) - \frac{1}{4L^d} [\Delta M(L, h)]^2 \cosh^{-2}\left(\frac{\Delta F(L, h)}{2}\right) \right| \leqslant 1 + 2\bar{C}_2 \qquad (6.34)$$

Hence

$$\chi(L, h_t(L)) - \chi(L, h) \geqslant \frac{1}{4L^d} [\Delta M(L, h_t(L))]^2 - \frac{1}{4L^d} [\Delta M(L, h)]^2 \cosh^{-2}\left(\frac{\Delta F(L, h)}{2}\right) - 2 - 4\bar{C}_2 \qquad (6.35)$$

Next, we use (6.31) and (6.33) to bound $|(d/dh)[\Delta M(L, h)]^2|$ by $8A\bar{C}_2 L^{2d}$. As a consequence,

$$|[\Delta M(L, h)]^2 - [\Delta M(L, h_t(L))]^2| \leqslant 8A\bar{C}_2 L^{2d} \, |h - h_t(L)| \qquad (6.36)$$

and

$$\chi(L, h_t(L)) - \chi(L, h) \geqslant \frac{1}{4L^d} [\Delta M(L, h_t(L))]^2 \left[1 - \cosh^{-2}\left(\frac{\Delta F(L, h)}{2}\right)\right] - 2A\bar{C}_2 \, |h - h_t(L)| \, L^d \cosh^{-2}\left(\frac{\Delta F(L, h)}{2}\right) - 2 - 4\bar{C}_2 \qquad (6.37)$$

On the other hand,

$$\frac{a}{4} \, |h - h_t(L)| \, L^d \leqslant \left| \frac{\Delta F(L, h)}{2} \right| \leqslant A \, |h - h_t(L)| \, L^d \qquad (6.38)$$

by (6.31) and the fact that $\Delta F(L, h_t(L)) = 0$. Using the lower bound

$$\cosh^2 \alpha \geqslant \left(1 + \frac{\alpha^2}{2}\right)^2 \geqslant 1 + \alpha^2 \geqslant 2 \, |\alpha| \qquad (6.39)$$

valid for any α, we imply that

$$\cosh^{-2}\left(\frac{\Delta F(L, h)}{2}\right) \leqslant \frac{2}{a \, |h - h_t(L)| \, L^d} \qquad (6.40)$$

Thus, using once more (6.31) and (6.38), we get

$$\chi(L, h_t(L)) - \chi(L, h) \geqslant \frac{1}{4L^d}\left(\frac{a}{2}L^d\right)^2\left[1 - \cosh^{-2}\left(\frac{a}{4}B\right)\right] - \frac{4A\bar{C}_2}{a} - 2 - 4\bar{C}_2 \tag{6.41}$$

whenever we suppose that $|h - h_t(L)| \geqslant BL^{-d}$, where $B > 0$ will be chosen later. Observing that

$$\cosh^{-2}\left(\frac{a}{4}B\right) < 1 \tag{6.42}$$

it is clear that the right-hand side of (6.41) is positive, once L is sufficiently large (how large depends on the choice of B).

Thus, it remains to consider the case $|h - h_t(L)| < BL^{-d}$. Taking into account that $\Delta F(L, h_t(L)) = 0$ we get

$$\begin{aligned}\left.\frac{d\chi(L,h)}{dh}\right|_{h_t(L)} &= -\frac{1}{L^d}\left.\frac{d^2M_0(L,h)}{dh^2}\right|_{h_t(L)} \\ &\quad + \frac{3}{4L^d}\left.\frac{d\Delta M(L,h)}{dh}\right|_{h_t(L)} \Delta M(L, h_t(L)) + O(e^{-b\bar{\tau}L})\end{aligned} \tag{6.43}$$

Using the bound (6.33), we get

$$\left.\frac{d\chi(L,h)}{dh}\right|_{h_t(L)} = \frac{3}{4}(\chi_+ - \chi_-)(m_+ - m_-)L^d + O(L^{d-1}) \tag{6.44}$$

Applying once more the bound (6.6), this time for $k = 3$, and using (6.31), we get, for $|h - h_t(L)| < BL^{-d}$, the bound

$$\frac{d^2\chi(L,h)}{dh^2} = -2\left[\frac{1}{2}\Delta M(L,h)\right]^4 L^{-d}\frac{1 - 3\tanh^2[\Delta F(L,h)/2]}{\cosh^2[\Delta F(L,h)/2]} + O(L^{2d}) \tag{6.45}$$

Taking into account that according to (6.38) one has $|\Delta F(L, h)| \leqslant 2A\,|h - h_t(L)|\,L^d$ and choosing $B > 0$ so that

$$\varepsilon := \frac{1 - 3\tanh^2(AB)}{\cosh^2(AB)} > 0 \tag{6.46}$$

we get

$$\frac{d^2\chi(L,h)}{dh^2} \leqslant -\frac{1}{16}(m_+ - m_-)^4 L^{3d}\varepsilon \tag{6.47}$$

for $|h-h_t(L)|<BL^{-d}$ and L large enough. The bound (6.47) together with the bound (6.44) implies that there exists a unique zero $h_\chi(L)$ of $d\chi(L,h)/dh$ in the interval $(h_t(L)-B/L^d,\ h_t(L)+B/L^d)$ and that $h_\chi(L)-h_t(L)=O(L^{-2d})$.

For $h-h_t(L)=O(1/L^{2d})$ we have $\Delta F(L,h)=O(1/L^d)$ by (6.38) and thus, using (6.45) and (6.36), we get

$$\frac{d^2\chi(L,h)}{dh^2}=-2\left[\frac{1}{2}(m_+-m_-)\right]^4 L^{3d}+O(L^{3d-1}) \tag{6.48}$$

Taking into account (6.44), we conclude the bound (6.11).

To prove the bound (6.12), we first notice, by a straightforward computation, that

$$U(h_t(L))=\frac{2}{3}\left[1-4\frac{\chi_++\chi_-}{(m_+-m_-)^2}\frac{1}{L^d}+O\left(\frac{1}{L^{d+1}}\right)\right] \tag{6.49}$$

Using the fact that, for L large,

$$\begin{aligned}\frac{d^2\chi}{dh^2}=&-\frac{1}{L^d}\frac{d^3M_0(L,h)}{dh^3}+\frac{1}{2L^d}\frac{d^3\Delta M(L,h)}{dh^3}\tanh\left(\frac{\Delta F(L,h)}{2}\right)\\&+\frac{1}{L^d}\frac{d^2\Delta M(L,h)}{dh^2}\frac{\Delta M(L,h)}{\cosh^2(\Delta F(L,h)/2)}\\&+\frac{3}{4L^d}\left(\frac{d\Delta M(L,h)}{dh}\right)^2\frac{1}{\cosh^2(\Delta F(L,h)/2)}\\&-\frac{3}{2L^d}\frac{d\Delta M(L,h)}{dh}[\Delta M(L,h)]^2\frac{\tanh(\Delta F(L,h)/2)}{\cosh^2(\Delta F(L,h)/2)}\\&-\frac{1}{8L^d}[\Delta M(L,h)]^4\frac{1-3\tanh^2(\Delta F(L,h)/2)}{\cosh^2(\Delta F(L,h)/2)}\\\leqslant\ & O(L^{2d})-L^{3d}\frac{1}{8}\left(\frac{a}{2}\right)^4\frac{1-3\tanh^2(\Delta F(L,h)/2)}{\cosh^2(\Delta F(L,h)/2)}\end{aligned} \tag{6.50}$$

we find that $\dot{U}(L,h)$ is negative [and thus smaller than $U(h_t(L))$] whenever

$$\frac{1-3\tanh^2(\Delta F(L,h)/2)}{\cosh^2(\Delta F(L,h)/2)}\leqslant-\varepsilon \tag{6.51}$$

for some positive ε. To meet this condition, it suffices to take $|h-h_t(L)|>B/L^d$ with B such that

$$\cosh^2\left(\frac{aB}{4}\right)\geqslant\frac{3}{2}(1+\tilde{\varepsilon}) \tag{6.52}$$

for some $\tilde{\varepsilon}>0$. Indeed, using (6.38), we get $|\Delta F(L,h)|\geqslant\frac{1}{2}aL^d|h-h_t(L)|$ and thus

$$\cosh^2\left(\frac{\Delta F(L,h)}{2}\right)>\frac{3}{2}(1-\tilde{\varepsilon})$$

which implies (6.51) with $\varepsilon=2\tilde{\varepsilon}/(1+\varepsilon)$.

In the interval $|h-h_t(L)|\leqslant B/L^d$ we consider the leading terms to $dU(L,h)/dh$ and $d^2U(L,h)/dh^2$. Namely,

$$\begin{aligned}\frac{dU(L,h)}{dh}\sim -\frac{8}{3}L^d\cosh\left(\frac{\Delta F(L,h)}{2}\right)\bigg[&-\frac{\Delta M(L,h)}{2}\sinh\left(\frac{\Delta F(L,h)}{2}\right)\\&-\frac{d\Delta M(L,h)/dh}{\Delta M(L,h)}\cosh^3\left(\frac{\Delta F(L,h)}{2}\right)\\&-3\,\frac{d\Delta M_0(L,h)/dh}{\Delta M(L,h)}\sinh\left(\frac{\Delta F(L,h)}{2}\right)\bigg]\end{aligned} \tag{6.53}$$

yielding

$$\left.\frac{dU(L,h)}{dh}\right|_{h_t(L)}\sim\frac{8}{3}L^d\frac{\chi_+-\chi_-}{m_+-m_-} \tag{6.54}$$

and

$$\begin{aligned}\frac{d^2U(L,h)}{dh^2}&\sim -\frac{2}{3}L^{3d}(m_+-m_-)^2\left[1+2\sinh^2\left(\frac{\Delta F(L,h)}{2}\right)\right]+O(L^{3d-1})\\&\leqslant -\frac{1}{3}L^{3d}(m_+-m_-)^2\end{aligned} \tag{6.55}$$

Thus, there exists a unique root $h_U(L)$ of the equation $dU(L,h)/dh=0$ in the interval $|h-h_t(L)|\leqslant B/L^d$ and $h_U(L)-h_t(L)=O(L^{-2d})$. Moreover, for $h-h_t(L)=O(L^{-2d})$ we have $\Delta F(L,h)=O(L^{-d})$ and thus

$$\frac{d^2U(L,h)}{dh^2}=-\frac{2}{3}L^{3d}(m_+-m_-)^2+O(L^{3d-1})$$

Hence,

$$\frac{dU(L,h)}{dh} = \frac{dU(L,h)}{dh}\bigg|_{h_t(L)} + \frac{d^2U(L,h)}{dh^2}\bigg|_{h_t(L)+\xi(h-h_t(L))} |h-h_t(L)| = 0 \qquad (6.56)$$

yields the shift (6.12) claimed in the lemma. ∎

We are left with the proof of (2.23) and (2.24) for

$$|h-h_\chi(L)| > \frac{4d(1+\|\kappa\|)}{(m_+ - m_-)L} \qquad (6.57)$$

Using (6.2), (6.11), and the bound (6.38), we first note that the condition (6.57) implies

$$\frac{\Delta F(L,h)}{2} > \frac{a4d}{4AL} L^d + O(L^{-d}) \geqslant \frac{aL}{A} + O(L^{-d}) \qquad (6.58)$$

Combined with (6.7) and (6.8), we obtain

$$m(L,h) = L^{-d}M_+(L,h) + O(e^{-aL/A})$$

$$\chi(L,h) = L^{-d}\frac{dM_+(L,h)}{dh} + O(e^{-aL/A})$$

if $h > h_\chi(L) + 4d(1+\|\kappa\|)/(m_+ - m_-)L$ and

$$m(L,h) = L^{-d}M_-(L,h) + O(e^{-aL/A})$$

$$\chi(L,h) = L^{-d}\frac{dM_-(L,h)}{dh} + O(e^{-aL/A})$$

if $h < h_\chi(L) - 4d(1+\|\kappa\|)/(m_+ - m_-)L$. Expanding $M_\pm(L,h)$ and its derivative into volume, surface,..., corner terms, this leads to

$$m(L,h) = m_+(h) + O(1/L)$$

$$\chi(L,h) = \chi_+(h) + O(1/L)$$

if $h > h_\chi(L) + 4d(1+\|\kappa\|)/(m_+ - m_-)L$ and

$$m(L,h) = m_-(h) + O(1/L)$$

$$\chi(L,h) = \chi_-(h) + O(1/L)$$

if $h < h_\chi(L) + 4d(1+\|\kappa\|)/(m_+ - m_-)L$.

Next, we recall $|e_{\pm}(c) - e_{\pm}| \leqslant \|\kappa\|$ for the asymmetric Ising model (2.1). As a consequence, $|\Delta F(L)| \leqslant 2dL^{d-1}[2\,\|\kappa\| + O(e^{-b\bar{\tau}})]$. Combined with Lemmas 6.2 and 6.3, we conclude that

$$|h_\chi(L) - h_t| \leqslant \frac{4d(\|\kappa\| + 1)}{(|m_+ - m_-|)L} \tag{6.59}$$

As a consequence, $h < h_\chi(L) + 4d(1 + \|\kappa\|)/(m_+ - m_-)L$ implies $h < h_t$ and hence $m(h) = m_-(h)$ and $\chi(h) = \chi_-(h)$, while $h > h_\chi(L) + 4d(1 + \|\kappa\|)/(m_+ - m_-)L$ implies $h > h_t$ and hence $m(h) = m_+(h)$ and $\chi(h) = \chi_+(h)$. The condition (6.57) therefore implies the bounds (2.23) and (2.24).

APPENDIX A. STRONG ISOPERIMETRIC INEQUALITY

Using the standard isoperimetric inequality,

$$|\partial W| \geqslant \frac{\sqrt{\pi}}{\Gamma(d/2+1)^{1/d}}\, d\, |W|^{(d-1)/d} \tag{A.1}$$

in the proof of Lemma B.3 below, we would get, for $d \geqslant 4$, a negative factor on the right-hand side of the bound (B.2). We strengthen (A.1) with the help of additional information—the fact that the considered sets W are finite unions of closed elementary cubes.

Lemma A.1. Let W be a union of closed elementary cubes. Then

$$|\partial W| \geqslant 2d\, |W|^{(d-1)/d} \tag{A.2}$$

Proof. The proof is just a particularly simple case of the proof of optimality of the Wulff shape.[28] Namely,

$$|\partial W| = \lim_{\varepsilon \to 0} \frac{|W + \varepsilon C| - |W|}{\varepsilon} \tag{A.3}$$

where εC is the rescaling, by the factor ε, of the (hyper)cube C of side 2 with the center at the origin, and

$$W + \varepsilon C = \{x + y\colon x \in W, y \in \varepsilon C\} \tag{A.4}$$

is the ε-neighborhood of W in the maximum metric. The Brunn–Minkowski inequality (valid also for nonconvex W; see, for example, ref. 14) yields

$$|W + \varepsilon C|^{1/d} \geqslant |W|^{1/d} + |\varepsilon C|^{1/d} \tag{A.5}$$

Thus

$$\lim_{\varepsilon\to 0}\frac{|W+\varepsilon C|-|W|}{\varepsilon}\geqslant \lim_{\varepsilon\to 0}\frac{(|W|^{1/d}+|\varepsilon C|^{1/d})^d-|W|}{\varepsilon}$$
$$=d\,|C|^{1/d}\,|W|^{(d-1)/d}=2d\,|W|^{(d-1)/d}\quad\blacksquare \qquad \text{(A.6)}$$

APPENDIX B. PROOF OF LEMMAS 4.1–4.5

We start with two Lemmas B.1 and B.2 that are an important technical ingredient to prove Lemmas 4.1, 4.2, and 4.4.

Lemma B.1. Let Y be a short contour with supp $Y\cap\partial V\subset\partial K(k)$. Then:

(i) supp $Y\cap\partial V=$ supp $Y\cap\partial K(k)=$ supp $Y\cap(\partial V\cap\partial K(k))$.

(ii) ∂ Int $Y\subset\partial K(k)\cup\partial$ supp Y.

(iii) Int $Y\cap\partial V=$ Int $Y\cap\partial K(k)=$ Int $Y\cap(\partial V\cap\partial K(k))$.

Proof.

(i) supp $Y\cap\partial K(k)=$ supp $Y\cap\partial K(k)\cap V=$ supp $Y\cap\partial K(k)\cap\partial V$ since $V\cap\partial K(k)=\partial V\cap\partial K(k)$ and supp $Y\subset V$. On the other hand, supp $Y\cap\partial V\subset\partial K(k)$ implies supp $Y\cap\partial V\subset\partial K(k)\cap\partial V$ and hence supp $Y\cap\partial V\subset$ supp $Y\cap\partial K(k)\cap\partial V$. Combining this with supp $Y\cap\partial K(k)\cap\partial V\subset$ supp $Y\cap\partial V$, we obtain (i).

(ii) Follows from the fact that all components of Int Y are components of $K(k)\setminus$supp Y.

(iii) In order to prove (iii), we first prove Int $Y\cap\partial V\subset\partial$ Int $Y\cap\partial V$. This can be proven as follows: the inclusion Int $Y\subset V$ implies $V^c\subset(\text{Int } Y)^c$ implies $\text{dist}(x, V^c)\geqslant\text{dist}(x,(\text{Int } Y)^c)$. Therefore $\text{dist}(x,(\text{Int } Y)^c)=0$ for all $x\in\partial V$ and hence for all $x\in$ Int $Y\cap\partial V$. Since $x\in$ Int Y and $\text{dist}(x,(\text{Int } Y)^c)=0$ implies $x\in\partial$ Int Y, this proves Int $Y\cap\partial V\subset\partial$ Int $Y\cap\partial V$.

Using (ii), the (just proven) fact that Int $Y\cap\partial V\subset\partial$ Int $Y\cap\partial V$, and the fact that ∂ supp $Y\cap\partial V\subset\partial K(k)$, one proves that Int $Y\cap\partial V\subset\partial K(k)$. Intersecting both sides with Int Y and observing that Int $Y\subset V$ while $V\cap\partial K(k)=\partial K(k)\cap\partial V$, one concludes that

$$\text{Int } Y\cap\partial V\subset\text{Int } Y\cap\partial K(k)=\text{Int } Y\cap\partial K(k)\cap\partial V$$

This combined with the fact that

$$\text{Int } Y\cap\partial K(k)\cap\partial V\subset\text{Int } Y\cap\partial V$$

proves (iii). $\blacksquare$

Lemma B.2. Let Y_1 and Y_2 be two nonoverlapping contours with $\operatorname{supp} Y_1 \subset \operatorname{Int} Y_2$. Assume that Y_2 is a short contour with $\partial V \cap \operatorname{supp} Y_2 \subset \partial K(k)$ for some corner k. Then Y_1 is a short contour with $\partial V \cap \operatorname{supp} Y_1 \subset \partial K(k)$ as well. In addition, $\operatorname{supp} Y_2 \cap \operatorname{Int} Y_1 = \emptyset$.

Proof. By Lemma B.1(iii) and the fact that $\operatorname{supp} Y_1 \subset \operatorname{Int} Y_2$, $\operatorname{supp} Y_1 \cap \partial V \subset \operatorname{Int} Y_2 \cap \partial K(k) \subset \partial K(k)$. Let now $x_0 \in \operatorname{supp} Y_1 \subset \operatorname{Int} Y_2$. By the definition of $\operatorname{Int} Y_2$ and $\operatorname{Ext} Y_2$, and by the fact that $\operatorname{supp} Y_2$ is connected, we may construct a path ω in $K(k)$ which connects $x_0 = \omega(0)$ to infinity such that

$$\begin{aligned} &\omega(t) \in \operatorname{Int} Y_2 && \text{for} \quad t \in [0, 1) \\ &\omega(t) \in \operatorname{supp} Y_2 && \text{for} \quad t \in [1, 2) \\ &\omega(t) \in \operatorname{Ext} Y_2 && \text{for} \quad t \in [2, 3) \\ &\omega(t) \in K(k) \backslash V && \text{for} \quad t \in [3, \infty) \end{aligned}$$

Assume now that $\operatorname{supp} Y_2 \cap \operatorname{Int} Y_1 \neq \emptyset$. Since $\operatorname{supp} Y_2$ is a connected set which does not intersect $\operatorname{supp} Y_1$, this implies that $\operatorname{supp} Y_2 \subset \operatorname{Int} Y_1$. As a consequence, $x_2 = \omega(2) \in \operatorname{Int} Y_1$ and $\omega|_{[2,\infty)}$ is a path in $K(k)$ which connects $x_2 \in \operatorname{Int} Y_1$ to infinity. But this implies that $\omega|_{[2,\infty)}$ must intersect the set $\operatorname{supp} Y_1$, and hence, by the assumption that $\operatorname{supp} Y_1 \subset \operatorname{Int} Y_2$, the set $\operatorname{Int} Y_2$. This is a contradiction, because ω was constructed in such a way that $\omega(t) \notin \operatorname{Int} Y_2$ for $t \geqslant 1$. ∎

Proof of Lemma 4.1. (i) Since $\operatorname{supp} Y_2 \subset \operatorname{Ext} Y_1$, $\operatorname{supp} Y_2 \cap \operatorname{Int} Y_1 = \emptyset$. It follows that each point in $\operatorname{Int} Y_1$ can be connected to $\partial \operatorname{Int} Y_1$ (and therefore to $\operatorname{supp} Y_1$) by a path ω which does not intersect $\operatorname{supp} Y_2$. As a consequence, all points in $\operatorname{Int} Y_1$ lay in the same connectivity component of $V \backslash \operatorname{supp} Y_2$ as $\operatorname{supp} Y_1$. Since $\operatorname{supp} Y_1 \subset \operatorname{Ext} Y_2$, we conclude that $\operatorname{Int} Y_1 \subset \operatorname{Ext} Y_2$, and hence $\operatorname{Int} Y_1 \cup \operatorname{supp} Y_1 \subset \operatorname{Ext} Y_2$. In a simillar way, $\operatorname{Int} Y_2 \cup \operatorname{supp} Y_2 \subset \operatorname{Ext} Y_1$.

(ii) Let us first assume that Y_2 is a short contour. Then Y_1 is a short contour as well and $\operatorname{supp} Y_2 \cap \operatorname{Int} Y_1 = \emptyset$ by Lemma B.2. Continuing as in the proof of (i), we obtain that $\operatorname{Int} Y_1 \cup \operatorname{supp} Y_1 \subset C_2$.

Next, consider the case where Y_1 is short while Y_2 is long, and assume that $\operatorname{supp} Y_2 \cap \operatorname{Int} Y_1 \neq \emptyset$. Since $\operatorname{supp} Y_2$ is connected, this would imply that $\operatorname{supp} Y_2 \subset \operatorname{Int} Y_1$. By Lemma B.2, this would imply that Y_2 is short as well. Therefore $\operatorname{supp} Y_2 \cap \operatorname{Int} Y_1$ must be empty. Again, this implies $\operatorname{Int} Y_1 \cup \operatorname{supp} Y_1 \subset C_2$.

As the last case, assume now that both Y_1 and Y_2 are long. Since $C_2 \subset \operatorname{Int} Y_2$, $|C_2| \leqslant L^d/2$ by the definition of the exterior for long contours. Since $\operatorname{supp} Y_2$ is a connected set, while C_2 is a connected component of

$V \backslash \text{supp } Y_2$, both C_2 and $V \backslash C_2$ are connected sets. Observing that $\text{supp } Y_1 \subset C_2$ implies $V \backslash \text{supp } Y_1 \supset V \backslash C_2$, we then introduce the component C_1 of $V \backslash \text{supp } Y_1$ which contains $V \backslash C_2$. A moment of reflection shows that $|C_1| > |V \backslash C_2|$, which, by the fact that $|V \backslash C_2| \geqslant L^d/2$, implies that $C_1 = \text{Ext } Y_1$. As a consequence,

$$\text{Int } Y_1 \subset V \backslash \text{Ext } Y_1 \subset C_2$$

which concludes the proof of (ii).

(iii) Follows from (ii). ∎

Proof of Lemma 4.2. We only have to show that $\text{supp } Y_1 \subset \text{Int } Y_2$ and $\text{supp } Y_2 \subset \text{Int } Y_1$ leads to a contradiction. In fact, $\text{supp } Y_1 \subset \text{Int } Y_2$ implies that $\text{supp } Y_1 \cup \text{Int } Y_1 \subset \text{Int } Y_2$ by Lemma 4.1. As a consequence, $\text{Ext } Y_1 \supset \text{supp } Y_2 \cup \text{Int } Y_2$, which implies that $\text{supp } Y_2 \subset \text{Ext } Y_1$. But this is incompatible with $\text{supp } Y_2 \subset \text{Int } Y_1$. ∎

Proof of Lemma 4.4. Let Y_k be an internal contour. Due to Lemma 4.2, this implies that $(\text{Int } Y_k \cup \text{supp } Y_k) \subset \text{Int } Y_j$ for some $j \neq k$ and hence $\text{Ext} = V \backslash \bigcup_{i \neq k} (\text{Int } Y_i \cup \text{supp } Y_i)$. Iterating this argument, we get that the set Ext is given by

$$\text{Ext} = V \Big\backslash \bigcup_{i=1}^{\tilde{n}} (\text{Int } Y_i^e \cup \text{supp } Y_i^e) \tag{B.1}$$

where $\{Y_1^e, ..., Y_{\tilde{n}}^e\}$ are the external contours in $\{Y_1, ..., Y_n\}$. Obviously, Ext is separated from the rest of V by the support of the contours $Y_1^e, ..., Y_{\tilde{n}}^e$. We therefore only have to show that Ext is connected.

Let

$$E_1 = V \backslash (\text{Int } Y_1^e \cup \text{supp } Y_1^e) = \text{Ext } Y_1^e$$

and

$$E_k = E_{k-1} \backslash (\text{Int } Y_k^e \cup \text{supp } Y_k^e).$$

Assume by induction that E_{k-1} is connected. Let $x, y \in E_k$. We have to show that x and y can be connected by a path ω_k in E_k. Using the inductive assumption, we can connect x and y by a path ω_{k-1} in E_{k-1}. Assume without loss of generality that ω_{k-1} intersects the set $W = \text{Int } Y_k^e \cup \text{supp } Y_k^e$, and let x_1 be the first and y_1 the last intersection point of ω_{k-1} with W. Since both W and $V \backslash W$ are connected, the boundary

$$\partial_V W = \{x \in V \mid \text{dist}(x, W) = \text{dist}(x, V \backslash W) = 0\}$$

is connected, and x_1 and y_1 can be connected by a path ω in $\partial_V W$. Using the path ω_{k-1} from x to x_1, the path ω from x_1 to y_1, and again the path ω_{k-1} from y_1 to y, we obtain a path $\tilde{\omega}_k$ in $E_k \cup \partial_V W$ which connects x to y. The desired path ω_k is obtained by a small deformation of $\tilde{\omega}_k$ which ensures that ω_k is a path in E_k. ▌

In order to prove Lemma 4.5, we need the following lemma, which is based on the strong isoperimetric inequality proven in Appendix A.

Lemma B.3. Let W be a union of elementary cubes in V, with $|W| \leqslant L^d/2$. Then

$$|\partial W \cap \partial V| \leqslant \frac{2^{1/d}+1}{2^{1/d}-1} |\partial W \backslash \partial V| \tag{B.2}$$

$$|\partial W| \leqslant \left(1 + \frac{2^{1/d}+1}{2^{1/d}-1}\right) |\partial W \backslash \partial V| \tag{B.3}$$

Proof. We introduce the $(d-1)$-dimensional faces

$$F_i = \{x \in \mathbb{R}^d \mid x_i = 1/2,\ 1/2 \leqslant x_k \leqslant L + 1/2,\ k \neq i\}, \qquad i = 1,..., d$$

$$F_{d+i} = \{x \in \mathbb{R}^d \mid x_i = L + 1/2,\ 1/2 \leqslant x_k \leqslant L + 1/2,\ k \neq i\}, \qquad i = 1,..., d$$

together with the projections π_i: $V \to F_i$, π_{d+i}: $V \to F_{d+i}$, where $x' = \pi_i(x)$ has coordinates $x'_k = x_k$ for $k \neq i$ and $x'_i = 1/2$, while $x'_i = L + 1/2$ for $x' = \pi_{d+i}(x)$. Finally, for each elementary $(d-1)$-cell $p \in \mathscr{C}$, we define $\pi_-(p)$ as the projection $\pi_i(p)$ onto the face F_i which is parallel to p, and $\pi_+(p)$ as $\pi_{d+i}(p)$.

Let $G_i = F_i \cap \partial W$, $i = 1,..., 2d$, and consider an elementary $(d-1)$-cell $p \in G_i$, together with the line l that links the center of $\pi_-(p)$ to the center of $\pi_+(p)$. Then l must intersect ∂W an even number of time. Define

$$H_i = \{p \in G_i \mid \text{there does not exist } p' \in \partial W \backslash \partial V \text{ with } \pi_-(p') = \pi_-(p)\}$$

and consider an elementary $(d-1)$-cell $p \in G_i \backslash H_i$, $i = 1,..., 2d$. Then either both $\pi_-(p)$ and $\pi_+(p)$ lie in $\bigcup_{j=1}^d G_j \backslash H_j$, in which case there are at least two elementary $(d-1)$-cells $p' \in \partial W \backslash \partial V$ with $\pi_-(p') = \pi_-(p)$, or only one of $\pi_-(p)$ and $\pi_+(p)$ lies in $\bigcup_{j=1}^d G_j \backslash H_j$, in which case there is at least one elementary $(d-1)$-cell $p' \in \partial W \backslash \partial V$ with $\pi_-(p') = \pi_-(p)$. As a consequence,

$$\sum_{i=1}^{2d} |G_i \backslash H_i| \leqslant |\partial W \backslash \partial V| \tag{B.4}$$

On the other hand,

$$|H_i| \leqslant |W|\, L^{-1} \leqslant (\tfrac{1}{2})^{1/d}\, |W|^{1-1/d}$$

by the fact that $|W| \leqslant L^d/2$. Using the strong isoperimetric inequality (see Appendix A), we obtain that

$$|H_i| \leqslant \frac{1}{2d} 2^{-1/d} |\partial W| \tag{B.5}$$

Combining (B.4) and (B.5), we get

$$\begin{aligned} |\partial W \cap \partial V| &= \sum_{i=1}^{2d} |G_i| = \sum_{i=1}^{2d} |G_i \backslash H_i| + \sum_{i=1}^{2d} |H_i| \\ &\leqslant |\partial W \backslash \partial V| + 2^{-1/d} |\partial W| \\ &= (1 + 2^{-1/d})\, |\partial W \backslash \partial V| + 2^{-1/d} |\partial W \cap \partial V| \end{aligned}$$

and hence

$$|\partial W \cap \partial V| \leqslant \frac{1 + 2^{-1/d}}{1 - 2^{-1/d}} |\partial W \backslash \partial V|$$

which implies (B.2). The bound (B.3) follows from (B.2). ∎

Proof of Lemma 4.5. We start with the observation that

$$|Y| = |Y|_d + |Y|_{d-1} + \cdots + |Y|_1 \tag{B.6a}$$

where $|Y|_k$ denotes the number of k-dimensional elementary cells in[14] Y, and similarly for the boundary ∂W_i of a component W_i of Int Y,

$$|\partial W_i| = |\partial W_i|_{d-1} + \cdots + |\partial W_i|_1 \tag{B.6b}$$

Using the fact that $|\partial W_i \cap \partial V|_{d-1} = |\partial \bar{W}_i \cap \partial V|_{d-1}$, we then decompose $|\partial W_i|$ as

$$|\partial W_i| = |\partial W_i \backslash \partial V|_{d-1} + |\partial \bar{W}_i \cap \partial V|_{d-1} + \sum_{k=1}^{d-2} |\partial W_i|_k \tag{B.7}$$

For a long contour Y, we use Lemma B.3 applied to the set $\bar{W}_i$, together with the fact that $\partial \bar{W}_i \subset \partial W_i$ to bound

$$|\partial \bar{W}_i \cap \partial V|_{d-1} \leqslant \left(\frac{2^{1/d}+1}{2^{1/d}-1}\right) |\partial \bar{W}_i \backslash \partial V|_{d-1} \leqslant \left(\frac{2^{1/d}+1}{2^{1/d}-1}\right) |\partial W_i \backslash \partial V|_{d-1} \tag{B.8}$$

[14] As in Section 3, a k-dimensional cell c in supp Y is only counted if there is no $(k+1)$-dimensional cell c' in supp Y with $c \subset c'$.

For short contours Y, on the other hand, $|\partial \bar{W}_i \cap \partial V| \leqslant |\partial \bar{W}_i \backslash \partial V|$, which implies (B.8) with a better constant. Therefore (B.8) is valid for both long and short contours.

As a last step we observe that each cube c in supp Y can be shared by at most $2d$ elementary faces in $(\partial W_1 \backslash V) \cup \cdots \cup (\partial W_n \backslash V)$, while each $(d-1)$-dimensional elementary face in Y may be shared by the boundary of at most two different components of Int $Y \cup$ Ext Y. Since all lower-dimensional elementary cells in Y belong to a unique component of Int $Y \cup$ Ext Y, we get that

$$\sum_{i=1}^{n} |\partial W_i \backslash \partial V|_{d-1} + \sum_{i=1}^{n} \sum_{k=1}^{d-2} |\partial W_i|_k \leqslant 2d\,|Y| \tag{B.9}$$

Combining (B.7) with (B.8) and (B.9) and the fact that

$$N_{\partial V}(\text{Int } Y) \leqslant \sum_{i=1}^{n} |\partial \bar{W}_i \cap \partial V| \tag{B.10}$$

we obtain the first two bounds of the lemma.

In order to prove (4.4), we observe that $V(Y) = \text{supp } Y \cup W_1 \cup \cdots \cup W_n$, which in turn implies $\partial V(Y) \subset \partial \text{ supp } Y \cup \partial W_1 \cup \cdots \cup \partial W_n$ and hence

$$|\partial V(Y)| \leqslant |\partial \text{ supp } Y| + |\partial W_1| + \cdots + |\partial W_n|$$

Combined with the bound $|\partial \text{ supp } Y| \leqslant 2d\,|Y|$, we obtain the remaining bound of Lemma 4.5. ∎

APPENDIX C. INDUCTIVE PROOF OF LEMMA 4.6

In this appendix, we prove Lemma 4.6. Actually, we first prove the following Lemma C.1. In order to state the lemma, we recall the definition of $f_q^{(n)}$ as the free energy of an auxiliary contour model with activities

$$K^{(n)}(Y^q) = \begin{cases} K'(Y^q) & \text{if } \ |V(Y^q)| \leqslant n \\ 0 & \text{otherwise} \end{cases} \tag{C.1}$$

and define

$$f^{(n)} = \min_q f_q^{(n)} \tag{C.2}$$

$$a_q^{(n)} = f_q^{(n)} - f \tag{C.3}$$

We also assign a number $v(W)$ to each volume of the form (4.7a),

$$v(W) = \max_{Y \text{ in } W} |V(Y)| \tag{C.4}$$

where the maximum goes over all contours Y with supp $Y \subset W$. Obviously, $v(\text{Int } Y) \leqslant |V(Y)|$ for all contours Y. In fact,

$$v(\text{Int } Y) < |V(Y)| \tag{C.5}$$

due to the fact that dist$(\tilde{Y}, Y) \geqslant 1$ if $\tilde{Y}$ is a contour in Int Y.

Finally, we recall that for a volume W of the form (4.7a), $|W|$ is used to denote the Euclidean volume of W, while for a contour Y and the boundary ∂W of a volume W, $|Y|$ and $|\partial W|$ are used to denote the number of elementary cells in Y and ∂W, respectively [see Eqs. (B.6a) and (B.6b)].

Lemma C.1. Let

$$\varepsilon = e^{-\tau(1-(2C_1+1)\gamma)} e^{\alpha+2} \qquad \text{and} \qquad \bar{\alpha} = \frac{(\alpha - 2)\, 2d}{C_3} \tag{C.6}$$

Then there is a constant ε_0, depending only on d and N, such that the following statements are true for all $\varepsilon < \varepsilon_0$ and all $n \geqslant 0$, provided $|V(Y)| \leqslant n$, $v(W) \leqslant n$, and $\bar{\alpha} \geqslant 1$:

(i) $|K'_q(Y)| \leqslant e^{|Y|}$.

(ii) If $a_q^{(n)} |V(Y)|^{1/d} \leqslant \bar{\alpha}$, then $\chi'_q(Y) = 1$.

(iii) If $a_q^{(n)} |V(Y)|^{1/d} \leqslant \bar{\alpha}$, then $K'_q(Y) = K_q(Y)$.

(iv) If $a_q^{(n)} |W|^{1/d} \leqslant \bar{\alpha}$, then $Z_q(W, h) = Z'_q(W, h)$.

(v) $|Z_q(W, h)| \leqslant e^{-f^{(n)}|W|} e^{O(\varepsilon)|\partial W|} e^{\gamma\tau N_{\partial V}(W)}$.

Proof. We proceed by induction on n, first proving the lemma for $n = 0$ and then for any given $n \in \mathbb{N}$, assuming that it has been already proven for all integers smaller than n.

Proof of Lemma C.1 for $n = 0$. For $|V(Y)| = 0$ we have $\chi'_q(Y) = 1$ and thus $K'_q(Y) = K_q(Y) = \rho(Y)$. This makes (i)–(iii) trivial statements. Using (iii) for $|V(Y)| = 0$, we then conclude that $Z_q(W, h) = Z'_q(W, h)$ for $v(W) = 0$. By (i), $Z_q(W, h) = Z'_q(W, h)$ and thus the partition function can be analyzed by a convergent expansion yielding

$$|Z_q(W, h)| \leqslant e^{-f_q^{(0)}|W|} e^{O(\varepsilon)|\partial W|} e^{(e_q|W| - E_q(W))} \leqslant e^{-f_q^{(0)}|W|} e^{O(\varepsilon)|\partial W|} e^{\gamma\tau N_{\partial V}(W)}$$

Observing that $f_q^{(0)} \geqslant f^{(0)}$, this concludes the proof of Lemma C.1 for $n = 0$.

Proof of Lemma C.1(i) for $|V(Y)|=n$. Due to (C.5), $v(\text{Int } Y)<n$, and all contours $\tilde{Y}$ contributing to $Z'_q(\text{Int}_m Y, h)$ obey the condition $|V(\tilde{Y})|<n$. This implies that $|K'_q(\tilde{Y})| \leqslant \varepsilon^{|\tilde{Y}|}$ by the inductive assumption (i). As a consequence, the logarithm of $Z'_q(\text{Int}_m Y, h)$ can be analyzed by a convergent expansion, and

$$|\log Z'_q(\text{Int}_m Y, h) + f_q^{(n-1)} |\text{Int}_m Y|| \leqslant O(\varepsilon) |\partial \text{ Int}_m| + \gamma\tau N_{\partial V}(\text{Int}_m Y) \tag{C.7}$$

Combining (C.7) with the induction assumption (v), we get

$$\begin{aligned}\prod_m \left|\frac{Z_m(\text{Int}_m Y, h)}{Z'_q(\text{Int}_m Y, h)}\right| &\leqslant e^{a_q^{(n-1)} |\text{Int } Y|} e^{2\gamma\tau N_{\partial V}(\text{Int } Y)} e^{O(\varepsilon) \sum_m |\partial \text{ Int}_m Y|} \\ &\leqslant e^{a_q^{(n-1)} |\text{Int } Y|} e^{(2C_1 \gamma\tau + O(\varepsilon)) |Y|} \end{aligned} \tag{C.8}$$

where we have used Lemma 4.5 in the last step. Observing that

$$|e_m - f_m^{(n-1)}| \leqslant O(\varepsilon) \tag{C.9a}$$

which implies the bound

$$|(e_q - e_0) - a_q^{(n-1)}| \leqslant O(\varepsilon) \tag{6.9b}$$

we use the assumptions (3.7) and (3.11) to bound

$$\begin{aligned}|\rho(Y) e^{E_q(Y)}| &\leqslant {}^{-\tau |Y|} e^{\gamma\tau B_{\partial V}(\text{supp } Y)} e^{(e_q - e_0) |Y|_d} \\ &\leqslant e^{-(\tau - \gamma\tau - O(\varepsilon)) |Y|} e^{a_q^{(n-1)} |Y|_d}\end{aligned} \tag{C.10}$$

Here $|Y|_d$ is defined as the number of d-cells in Y and thus $|V(Y)| = |\text{Int } Y| + |Y|_d$. Combining now (C.10) with (C.8), we obtain

$$|K'_q(Y)| \leqslant \chi'_q(Y) e^{a_q^{(n-1)} |V(Y)|} e^{-(\tau - O(\varepsilon) - (1 + 2C_1) \gamma\tau) |Y|} \tag{C.11}$$

Without loss of generality, we may now assume that $\chi'_q(Y) > 0$ [otherwise $K'_q(Y) = 0$ and the statement (i) is trivial]. By the definition of $\chi'_q(Y)$, this implies

$$(f_q^{(n-1)} - f_m^{(n-1)}) |V(Y)| \leqslant 1 + \alpha |Y| \leqslant (1 + \alpha) |Y|$$

for all $m \neq q$. As a consequence,

$$a_q^{(n-1)} |V(Y)| \leqslant (1 + \alpha) |Y| \tag{C.12}$$

provided $\chi_q'(Y) \neq 0$. Combined with (C.11) and the fact that $\chi_q'(Y) \leqslant 1$, this implies that

$$|K_q'(Y)| \leqslant e^{-[\tau - 1 - O(\varepsilon) - \alpha - (1+2C_1)\gamma\tau]\,|Y|} \tag{C.13}$$

which yields the desired bound (i) for $|V(Y)| = n$.

Proof of Lemma C.1(ii) for $k = |V(Y)| \leqslant n$ and $a_q^{(n)}\,|V(Y)|^{1/d} \leqslant \bar{\alpha}$. We just have proved that (i) us true for all contours Y with $|V(Y)| \leqslant n$. As a consequence, both $f_m^{(k)}$ and $f_m^{(n)}$ may be analyzed by a convergent cluster expansion. On the other hand,

$$|V(Y)|^{(d-1)/d} \leqslant \frac{1}{2d}\,|\partial V(Y)| \leqslant \frac{C_3}{2d}\,|Y| \tag{C.14}$$

by the isoperimetric inequality and Lemma 4.5. Using this bound and the definition of $f_m^{(n)}$, one may easily see that all contours Y contributing to the cluster expansion of the difference $f_m^{(k)} - f_m^{(n)}$ obey the bound

$$|Y| \geqslant \frac{2d}{C_3}(k+1)^{(d-1)/d} \geqslant \frac{d}{C_3}\,k^{1/d} =: n_0$$

As a consequence,

$$|f_m^{(k)} - f_m^{(n)}| \leqslant (K\varepsilon)^{n_0}$$

where K is a constant depending only on the dimension d and the number of phases N. Using the bound (C.14) for the second time and recalling that $|V(Y)| = k$, we get

$$|f_m^{(k)} - f_m^{(n)}| \cdot |V(Y)| \leqslant (K\varepsilon)^{n_0}\,|V(Y)|^{1/d}\,\frac{C_3}{2d}\,|Y| = O(1)\,n_0(K\varepsilon)^{n_0}\,|Y| \leqslant O(\varepsilon)\,|Y| \tag{C.15}$$

Combining (C.15) with the assumption $a_q^{(n)}\,|V(Y)|^{1/d} \leqslant \bar{\alpha}$ and the bound (C.14), we obtain the lower bound

$$\begin{aligned}\alpha\,|Y| - [f_q^{(k)} - f_m^{(k)}]\,|V(Y)| &\geqslant \alpha\,|Y| - a_q^{(n)}\,|V(Y)| - O(\varepsilon)\,|Y| \\ &\geqslant \left(\alpha - \bar{\alpha}\,\frac{C_3}{2d} - O(\varepsilon)\right)|Y| \\ &= (2 - O(\varepsilon))\,|Y| \geqslant 2 - O(\varepsilon)\end{aligned}$$

where, in the next to the last step, we used the definition of $\bar{\alpha}$ [see (C.6)]. Combined with (4.16b) we obtain the equality $\chi_q'(Y) = 1$.

Proof of Lemma C.1(iii) and (iv). Given (ii) and the definitions of $K'_q(Y)$ and $Z'_q(W, h)$, the statement is obvious. See ref. 6 for a formal proof using induction on the subvolumes of W and Int Y.

Proof of Lemma C.1(v). We say a contour Y is *small* if $a_q^{(n)} |V(Y)|^{1/d} \leqslant \bar{\alpha}$, while it is *large* if $a_q^{(n)} |V(Y)|^{1/d} > \bar{\alpha}$. We then use the relation (4.8) to rewrite $Z_q(W, h)$ by splitting the set $\{Y_1,..., Y_k\}_{\text{ext}}$ of external contours into $\{X_1,..., X_{k'}\} \cup \{Z_1,..., Z_{k''}\}$, where $Z_1,..., Z_{k''}$ are the small contours in $\{Y_1,..., Y_k\}_{\text{ext}}$ and $X_1,..., X_{k'}$ are the large contours in $\{Y_1,..., Y_k\}_{\text{ext}}$. Note that for a fixed set $\{X_1,..., X_{k'}\}$, the sum over $\{Z_1,..., Z_{k''}\}$ runs over sets of mutually external small q-contours in $\text{Ext} = W \backslash \bigcup_{i=1}^{k'} V(X_i)$. Resumming the small contours, we thus obtain

$$Z_q(W, h) = \sum_{\{X_1,..., X_{k'}\}_{\text{ext}}} Z_q^{\text{small}}(\text{Ext}, h) \prod_{i=1}^{k'} \left[\rho(X_i) \prod_m Z_m(\text{Int}_m X_i, h \right] \tag{C.16}$$

Here the sum goes over sets of mutually external large contours in W and $Z_q^{\text{small}}(\text{Ext}, h)$ is obtained from $Z_q(\text{Ext}, h)$ by dropping all large external q-contours.

Due to the inductive assumption (iii), $K_q(Y) = K'_q(Y)$ if Y is small. Since $|K'_q(Y)| \leqslant \varepsilon^{|Y|}$ by (i), $Z_q^{\text{small}}(\text{Ext}, h)$ can be controlled by a convergent cluster expansion, and

$$|Z_q^{\text{small}}(\text{Ext}, h)| \leqslant e^{-f_q^{\text{small}} |\text{Ext}|} e^{O(\varepsilon) |\partial \text{Ext}|} e^{\gamma\tau N_{\partial V}(\text{Ext})} \tag{C.17}$$

where f_q^{small} is the free energy of the contour model with activities

$$K_q^{\text{small}}(Y) = \begin{cases} K'_q(Y) & \text{if } |V(Y)| \leqslant n \text{ and } Y \text{ is small} \\ 0 & \text{otherwise} \end{cases} \tag{C.18}$$

On the other hand,

$$\prod_m |Z_m(\text{Int}_m X_i, h)| \leqslant e^{-f^{(n-1)} |\text{Int } X_i|} e^{O(\varepsilon) |\partial \text{ Int } X_i|} e^{\gamma\tau N_{\partial V}(\text{Int } X_i)}$$

by the induction assumption (v). Observing that the small contours contributing to the difference of $f_m^{(n)}$ and $f_m^{(n-1)}$ obey the bound

$$|Y| \geqslant \frac{d}{C_3} n^{(d-1)/d} \geqslant \frac{2d}{C_3} n^{1/d} =: n_0$$

while $|V(X_i)| \leqslant n$, we may continue as in the proof of (C.15) to bound

$$|f^{(n-1)} - f^{(n)}| \cdot |\text{Int } X_i| \leqslant |f^{(n-1)} - f^{(n)}| \cdot |V(X_i)| \leqslant O(1)\, n_0 (K\varepsilon)^{n_0} \leqslant O(\varepsilon)$$

Since $|\text{Int } X_i| \leqslant O(1) \, |X_i|$ by Lemma 4.5, we conclude that

$$\prod_m |Z_m(\text{Int } X_i, h)| \leqslant e^{-f^{(n)} |\text{Int } X_i|} e^{O(\varepsilon) |X_i|} e^{\gamma\tau N_{\partial V}(\text{Int } X_i)} \tag{C.19}$$

Combining (C.17) and (C.19) with the bounds

$$\begin{aligned} |\rho(X_i)| &\leqslant e^{-\tau |X_i| - e_0 |X_i|_d} e^{\gamma\tau N_{\partial V}(\text{supp } X_i)} \\ &\leqslant e^{-[\tau - O(\varepsilon)] |X_i|} e^{-f^{(n)} |X_i|_d} e^{\gamma\tau N_{\partial V}(\text{supp } X_i)} \end{aligned} \tag{C.20}$$

and

$$|\partial \text{ Ext}| \leqslant |\partial W| + \sum_{i=1}^{k'} |\partial V(X_i)| \leqslant |\partial W| + C_3 \sum_{i=1}^{k'} |X_i| \tag{C.21}$$

and the equality

$$N_{\partial V}(\text{Ext}) + \sum_{i=1}^{k'} [N_{\partial V}(\text{supp } X_i) + N_{\partial V}(\text{Int } X_i)] = N_{\partial V}(W)$$

we conclude that

$$\begin{aligned} |Z_q(W, h)| &\leqslant e^{O(\varepsilon) |\partial W|} e^{\gamma\tau N_{\partial V}(W)} e^{-f^{(n)} |W|} \\ &\quad \times \sum_{\{X_1,\dots,X_{k'}\}_{\text{ext}}} e^{-[f_q^{\text{small}} - f^{(n)}] |\text{Ext}|} \prod_{i=1}^{k'} e^{-[\tau - O(\varepsilon)] |X_i|} \end{aligned} \tag{C.22}$$

Next, we bound the difference $f_q^{\text{small}} - f_q^{(n)}$. In a first step, we use the isoperimetric inequality together with Lemma 4.5 and the definition of large contours to bound

$$|X| \geqslant \frac{1}{C_3} |\partial V(X)| \geqslant \frac{d}{C_3} |V(X)|^{(d-1)/d} \geqslant \frac{d}{C_3} |V(X)|^{1/d} \geqslant l_0 := \frac{2d\bar{\alpha}}{C_3} \frac{1}{a_q^{(n)}} \tag{C.23}$$

for all large contours X. Next, we observe that

$$|f_q^{(n)} - f_q^{\text{small}}| \leqslant (K\varepsilon)^{l_0} \leqslant \frac{1}{-l_0 \log(K\varepsilon)} \tag{C.24}$$

where K is a constant depending only on d and N. Recalling the condition $\bar{\alpha} \geqslant 1$, we get

$$|f_q^{(n)} - f_q^{\text{small}}| \leqslant \tfrac{1}{2} a_q^{(n)} \tag{C.25}$$

provided ε is chosen small enough. Combining (C.22) with (C.25), we finally obtain

$$|Z_q(W, h)| \leqslant e^{O(\varepsilon)\,|\partial W|}\, e^{\gamma\tau N_{\partial V}(W)}\, e^{-f^{(n)}\,|W|} \sum_{\{X_1,\dots,X_{k'}\}_{\rm ext}} e^{-(a_q^{(n)}/2)\,|{\rm Ext}|} \prod_{i=1}^{k'} e^{-\tilde{\tau}\,|X_i|} \tag{C.26}$$

with

$$\tilde{\tau} = (\tau - 1) \tag{C.27}$$

At this point we need the following Lemma C.2, which is a variant of a lemma first proven in ref. 29; see also ref. 3.

Lemma C.2. Consider an arbitrary contour functional $\tilde{K}_q(Y) \geqslant 0$, and let $\tilde{Z}_q$ be the partition function

$$\tilde{Z}_q(W) = \sum_{\{Y_1,\dots,Y_n\}} \prod_{i=1}^{h} (\tilde{K}_q(Y_i)\, e^{|Y_i|}) \tag{C.28}$$

Let $\tilde{s}_q$ be the corresponding free energy, and assume that $\tilde{K}_q(Y) \leqslant \tilde{\varepsilon}^{|Y|}$, where $\tilde{\varepsilon}$ is small (depending on N and d). Then for any $a \geqslant -\tilde{s}_q$ the following bound is true:

$$\sum_{\{Y_1,\dots,Y_k\}_{\rm ext}} e^{a\,|{\rm Ext}|} \prod_i \tilde{K}_q(Y_i) \leqslant e^{O(\tilde{\varepsilon})\,|\partial W|} \tag{C.29}$$

where the sum goes over sets of mutually external q-contours in W.

In order to apply the lemma, we define $\tilde{K}_q(Y) = e^{-\tilde{\tau}\,|Y|}$ if Y is a large q-contour, and $\tilde{K}_q(Y) = 0$ otherwise. With this choice,

$$0 \leqslant -\tilde{s}_q \leqslant (K\varepsilon)^{l_0} \leqslant \frac{1}{-l_0 \log(K\varepsilon)} \tag{C.30}$$

where l_0 is the constant from (C.23). As a consequence,

$$-\tilde{s}_q \leqslant \tilde{a} := \frac{a_q^{(n)}}{2} \tag{C.31}$$

provided ε is small enough. Applying Lemma C.2 to the right-hand side of (C.26), and observing that $\tilde{\varepsilon} := e^{-\tilde{\tau}} \leqslant \varepsilon$, we finally obtained the desired inequality

$$|Z_q(W, h)| \leqslant e^{O(\varepsilon)\,|\partial W|}\, e^{\gamma\tau N_{\partial V}(W)}\, e^{-f_0^{(n)}\,|W|}$$

This concludes the inductive proof of Lemma C.1. ∎

Proof of Lemma C.2. The partition function $\tilde{Z}_q$ is defined in terms of the polymer model with activities $K^*(Y) = \tilde{K}_q(Y)\, e^{|Y|}$. For $\tilde{\varepsilon}$ small enough, $\tilde{Z}_q$ can be controlled by a convergent cluster expansion and

$$|\log \tilde{Z}_q(\text{Int } Y) + \tilde{s}_q\, |\text{Int } Y|\,| \leqslant O(\tilde{\varepsilon})\, |\partial \text{ Int } Y| \leqslant O(\tilde{\varepsilon})\, |Y|$$

On the other hand, $|W| = |\text{Ext}| + \sum_i (|\text{Int } Y_i| + |Y_i|_d)$ if $\{Y_1, ..., Y_k\}_{\text{ext}}$ is a set of mutually external contours in W. Combined with the fact that $-\tilde{a} \leqslant \tilde{s}_q = O(\tilde{\varepsilon})$, we obtain

$$\begin{aligned}
&\sum_{\{Y_1, ..., Y_k\}_{\text{ext}}} e^{-\tilde{a}\,|\text{Ext}|} \prod_{i=1}^{k} \tilde{K}_q(Y_i) \\
&\leqslant e^{\tilde{s}_q |W|} \sum_{\{Y_1, ..., Y_k\}_{\text{ext}}} \prod_{i=1}^{k} \tilde{K}_q(Y_i)\, e^{-\tilde{s}_q(|\text{Int } Y_i| + |Y_i|_d)} \\
&\leqslant e^{\tilde{s}_q |W|} \sum_{\{Y_1, ..., Y_k\}_{\text{ext}}} \prod_{i=1}^{k} \tilde{K}_q(Y_i)\, \tilde{Z}_q(\text{Int } Y_i)\, e^{O(\tilde{\varepsilon})\,|Y_i| - \tilde{s}_q |Y_i|_d} \\
&\leqslant e^{\tilde{s}_q |W|} \sum_{\{Y_1, ..., Y_k\}_{\text{ext}}} \prod_{i=1}^{k} \tilde{K}_q(Y_i)\, e^{|Y_i|}\, \tilde{Z}_q(\text{Int } Y_i) \\
&= e^{\tilde{s}_q |W|} \tilde{Z}_q(W) \leqslant e^{O(\tilde{\varepsilon})\,|\partial W|} \quad \blacksquare
\end{aligned}$$

Proof of Lemma 4–6. Lemma 4.6(i)–(iv) follows from Lemma C.1 and the fact that $f = \lim_{n \to \infty} f^{(n)}$ and $a_q = \lim_{n \to \infty} a_q^{(n)}$.

In order to prove the statement (v), we extract the factor

$$\begin{aligned}
&\max_{\{X_1, ..., X_{k'}\}} e^{-(a_q^{(n)}/4)\,|\text{Ext}|}\, e^{-(\tau/4) \sum_i |X_i|} \\
&\leqslant \max_{\{X_1, ..., X_{k'}\}} e^{-(a_q^{(n)}/4)\,|\text{Ext}|}\, e^{-(\tau/4C_3) \sum_i |\partial V(X_i)|} \\
&\leqslant \max_{U \subset W} e^{-(a_q^{(n)}/4)\,|W \backslash U|}\, e^{-(\tau/4C_3)\,|\partial U|}
\end{aligned}$$

from the right-hand side of (C.26), and bound the remaining sum as before. Taking the limit $n \to \infty$ in the resulting bound, this yields

$$\begin{aligned}
|Z_q(W, h)| \leqslant\, & e^{\gamma\tau N_{\partial V}(W)}\, e^{[O(\varepsilon) + O(e^{-3\tau/4})]\,|\partial W|}\, e^{-f|W|} \\
& \times \max_{U \subset W} e^{-(a_q/4)\,|W \backslash U|}\, e^{-(\tau/4C_3)\,|\partial U|} \qquad \text{(C.32)}
\end{aligned}$$

We conclude, with the help of the isoperimetric inequality, that

$$|Z_q(W, h)| \leqslant e^{\gamma\tau N_{\partial V}(W)} e^{[O(\varepsilon)+O(e^{-3\tau/4})]\,|\partial W|} e^{-f|W|} \times \max_{U\subset W} e^{-(a_q/4)\,|W\setminus U| - (2d\tau/4C_3)\,|U|^{d/(d-1)}}$$
$$= e^{\gamma\tau N_{\partial V}(W)} e^{[O(\varepsilon)+O(e^{-3\tau/4})]\,|\partial W|} e^{-f|W|} \times \max\{e^{-(a_q/4)\,|W|}, e^{-(2d\tau/4C_3)\,|W|^{d/(d-1)}}\} \tag{C.33}$$

where we used the fact that the maximum over U is obtained for either $U = W$ or $U = \emptyset$.

Observing that $2d\,|V|^{d/(d-1)} = |\partial V|$ and that $N_{\partial V}(V)$ can be bounded by $|\partial V|$, we see that the bound (C.33) implies Lemma 4.6(v). ∎

Proof of the Bound (4.26). Due to the bound (C.12), we have $a_q^{(n-1)}\,|V(Y)| \leqslant (1+\alpha)\,|Y|$ if $\chi_q(Y) \neq 0$. Using the strategy which was used to prove (C.15), we replace $a_q^{(n-1)}$ by a_q, concluding that $\chi_q(Y) \neq 0$ implies $a_q\,|V(Y)| \leqslant [1+O(\varepsilon)+\alpha]\,|Y|$. ∎

APPENDIX D. PROOF OF LEMMA 4.7

We start with a combinatoric lemma that will be used throughout this appendix.

Lemma D.1. Let k_0 be a positive integer and let $G(h)$ be a function which satisfies the bounds

$$\left|\frac{d^k}{dh^k} G(h)\right| \leqslant \lambda^{|k|}$$

for all multi-indices k with $1 \leqslant |k| \leqslant k_0$ and some $\lambda > 0$. Then

$$\left|\frac{d^k}{dh^k} e^{G(h)}\right| \leqslant |k|!\,\lambda^{|k|} e^{G(h)}$$

for all multi-indices k with $1 \leqslant |k| \leqslant k_0$.

Proof. Observing that

$$\frac{d^k}{dh^k} e^{G(h)} = H_k(h)\, e^{G(h)}$$

where $H_k(h)$ is a polynomial of degree $|k|$ in the derivatives of G, we immediately obtain the lemma by induction on $|k|$. ∎

Keeping the notation of Appendix C, we now prove the following lemma, which contains statements (i)–(iii) of Lemma 4.7.

Lemma D.2. There is a constant $K<\infty$, depending only on N, d, and the constants introduced in (3.8), (3.9), and (4.16), such that the following statements are true provided $\varepsilon<\varepsilon_0$, $\bar{\alpha}\geqslant 1$, and $n\geqslant 0$:

(i) For $|V(Y)|\leqslant n$ and $h_0\in\mathscr{U}$ one has

$$\left|\frac{d^k}{dh^k}K_q'(Y)\right|_{h=h_0}\leqslant (K\varepsilon)^{|Y|} \tag{D.1}$$

provided $1\leqslant |k|\leqslant 6$.

(ii) For $v(W)\leqslant n$ and $h_0\in\mathscr{U}$ one has

$$\left|\frac{d^k}{dh^k}\log Z_q'(W,h)\right|_{h=h_0}\leqslant [C_0^{|k|}+O(\varepsilon)]\,|W| \tag{D.2}$$

provided $1\leqslant |k|\leqslant 6$.

(iii) For $v(W)\leqslant n$ and $h_0\in\mathscr{U}$ one has

$$\left|\frac{d^k}{dh^k}Z_q(W,h)\right|_{h=h_0}\leqslant |k|!\,\{[C_0+O(\varepsilon)]\,|W|\}^{|k|}\,e^{-f|W|}\,e^{O(\varepsilon)\,|\partial W|}\,e^{\gamma\tau N_{\partial V}(W)} \tag{D.3}$$

provided $1\leqslant |k|\leqslant 6$.

Proof. As in the proof of Lemma C.1, we proceed by induction on n.

Proof of Lemma D.2 for $n=0$. For $|V(Y)|=0$, $K_q'(Y)=K_q(Y)=\rho(Y)$, which makes (i) a trivial statement. As a consequence, the left-hand side of (D.2) can be analyzed by a convergent cluster expansion, leading immediately to the bound (D.2) for $v(W)=0$. Bounding finally $[C_0^{|k|}+O(\varepsilon)]\,|W|$ by $\{[C_0+O(\varepsilon)]\,|W|\}^{|k|}$ and observing that $Z_q(W,h)=Z_q'(W,h)$ if $v(W)=0$, we obtain (iii) with the help of Lemma D.1.

Proof of Lemma D.2(i) for $|V(Y)|=n$. Using the assumptions (3.8) and (3.9) together with Lemma D.1, we can easily generalize the bound (C.10) to derivatives, giving

$$\left|\frac{d^k}{dh^k}[\rho(Y)\,e^{E_q(Y)}]\right|\leqslant |k|!\,(2C_0\,|Y|)^{|k|}\,e^{-[\tau-\gamma\tau-O(\varepsilon)]\,|Y|}\,e^{a_q\,|Y|_d} \tag{D.4}$$

In a similar way, the bound (C.8) can be generalized to derivatives, using the inductive assumptions (ii) and (iii) together with Lemma D.1 and Lemma 4.5. This gives

$$\left|\frac{d^k}{dh^k}\left[\prod_m \frac{Z_m(\mathrm{Int}_m\, Y, h)}{Z'_q(\mathrm{Int}_m\, Y, h)}\right]\right|$$
$$\leqslant |k|!\ \{[2C_0 + O(\varepsilon)]\ |\mathrm{Int}\ Y|\}^{|k|}\ e^{a_q\, |\mathrm{Int}\ Y|}\ e^{[2C_1\gamma\tau + O(\varepsilon)]\,|Y|} \quad \text{(D.5)}$$

Using finally the possibility to analyze the derivatives of $f_m^{(n-1)}(h)$ be a convergent expansion due to the inductive assumption (i), we bound

$$\left|\frac{d^k f_m^{(n-1)}(h)}{dh^k}\right| \leqslant C_0^{|k|} + O(\varepsilon) \leqslant [C_0 + O(\varepsilon)]^{|k|}$$

As a consequence,

$$\left|\frac{d^k}{dh^k}\chi'_q(Y)\right| \leqslant [\bar{C}_1\ |V(Y)|]^{|k|} \quad \text{(D.6)}$$

for all multi-indices of order $|k| \leqslant 6$. Here $\bar{C}_1$ is a constant that depends on N and the constants introduced in (3.8), (3.9), and (4.16). Combining (D.4)–(D.6) and bounding terms of the form $O(1)\ |V(Y)|$ and $O(1)\ |Y|$ by $e^{O(1)\,|Y|}$, we obtain the bound (D.1).

Proof of Lemma D.2(ii) for $v(W) = n$. We just have proved that (i) is true for all contours Y with $|V(Y)| \leqslant n$. As a consequence the deivatives of $\log Z'_q(W, h)$ can be analyzed by a convergent cluster expansion. The bound (D.2) immediately follows.

Proof of Lemma D.2(iii) for $v(W) = n$. We define: a contour Y is *small* if $a_q(h_0)\ |V(Y)|^{1/d} \leqslant \bar{\alpha}$, while a contour Y is called *large* if $a_q(h_0)\ |V(Y)|^{1/d} > \bar{\alpha}$. As in Appendix C, we then rewrite $Z_q(W, h)$ as

$$Z_q(W, h) = \sum_{\{X_1, \dots, X_n\}_{\mathrm{ext}}} Z_q^{\mathrm{small}}(\mathrm{Ext}, h) \prod_{i=1}^{n} \left[\rho(X_i) \prod_m Z_m(\mathrm{Int}_m\, X_i, h)\right] \quad \text{(D.7)}$$

where the sum goes over sets of mutually external large contours in W and $Z_q^{\mathrm{small}}(\mathrm{Ext}, h)$ is obtained from $Z_q(\mathrm{Ext}, h)$ by dropping all large external q-contours.

Due to Lemma 4.6, $K_q(Y) = K'_q(Y)$ if Y is small and $h = h_0$. Combining this with the bound (4.13) and the inductive assumption (D.1), we conclude that

$$\left|\frac{d^k}{dh^k} K'_q(Y)\right| \leqslant (2K\varepsilon)^{|Y|} \quad \text{(D.8)}$$

is a certain neighborhood $\mathscr{U}_0$ of h_0. As a consequence, the derivatives of $\log Z_q^{\text{small}}(\text{Ext}, h)$ can be controlled by a convergent cluster expansion, and

$$\left|\frac{d^k}{dh^k}\log Z_q^{\text{small}}(\text{Ext}, h)\right| \leqslant [C_0^{|k|} + O(\varepsilon)]\,|\text{Ext}| \leqslant \{[C_0 + O(\varepsilon)]\,|\text{Ext}|\}^{|k|} \tag{D.9}$$

provided $h \in \mathscr{U}_0$. Combining (D.9) with the bound

$$|Z_q^{\text{small}}(\text{Ext}, h_0)| \leqslant e^{-f_q^{\text{small}}|\text{Ext}|}\, e^{O(\varepsilon)\,|\partial\,\text{Ext}|}\, e^{\gamma\tau N_{\partial V}(\text{Ext})} \tag{D.10}$$

where f_q^{small} is the free energy of the contour model with activities

$$K_q^{\text{small}}(Y) = \begin{cases} K_q'(Y) & \text{if} \quad Y \text{ is small,} \\ 0 & \text{if} \quad Y \text{ is large} \end{cases} \tag{D.11}$$

we obtain

$$\left|\frac{d^k}{dh^k} Z_q^{\text{small}}(\text{Ext}, h)\right|_{h=h_0} \leqslant |k|!\,\{[C_0 + O(\varepsilon)]\,|\text{Ext}|\}^{|k|}\, e^{-f_q^{\text{small}}|\text{Ext}|}\, e^{O(\varepsilon)\,|\partial\,\text{Ext}|}\, e^{\gamma\tau N_{\partial V}(\text{Ext})} \tag{D.12}$$

On the other hand,

$$\begin{aligned}\left|\frac{d^k}{dh^k}\rho(X_i)\right| &\leqslant |k|!\,(C_0\,|X_i|)^{|k|}\, e^{-[\tau - O(\varepsilon)]\,|X_i|}\, e^{-f\,|X_i|_d + N_{\partial V}(\text{supp}\,X_i)} \\ &\leqslant |k|!\, C_0^{|k|}\, e^{-[\tau - |k|/e - O(\varepsilon)]\,|X_i|}\, e^{-f\,|X_i|_d + N_{\partial V}(\text{supp}\,X_i)}\end{aligned} \tag{D.13}$$

Combining (D.7) with the inductive assumption (D.3) and the bounds (D.12) and (D.13), we may continue as in Appendix C to get

$$\begin{aligned}\left|\frac{d^k}{dh^k} Z_q(W, h)\right|_{h=h_0} &\leqslant |k|!\,\{[C_0 + O(\varepsilon)]\,|W|\}^{|k|}\, e^{-f\,|W|}\, e^{\gamma\tau N_{\partial V}(W)}\, e^{O(\varepsilon)\,|\partial W|} \\ &\quad \times \sum_{\{X_1,\ldots,X_n\}_{\text{ext}}} e^{-(a_q/2)\,|\text{Ext}|} \prod_{i=1}^{n} e^{-\tilde{\tau}\,|X_i|}\end{aligned} \tag{D.14}$$

where now

$$\tilde{\tau} = \tau - 6/e - 1 \tag{D.15}$$

Note the extra term $6/e$ with respect to (C.27), which comes from the term $|k|/e$ in (C.13) (recall that we assumed $|k| \leqslant 6$).

Given the bound (D.14), the proof of (D.3) for $v(W)=n$ now follows using Lemma C.2 from Appendix C. This concludes the proof of Lemma D.2. ▮

Proof of Lemma 4.7(v). Starting from (D.14), statement (iv) of Lemma 4.7 is obtained in the same way as statement (v) of Lemma 4.6 was obtained in Appendix C. ▮

As a corollary of this proof, one obtains the analogs of (C.32) and (C.33) for derivatives, namely

$$\left|\frac{d^k}{dh^k} Z_q(W, h)\right|_{h=h_0} \leqslant |k|! \left\{[C_0 + O(\varepsilon)] \, |W|\right\}^{|k|} e^{\gamma\tau N_\partial v(W)} e^{[O(\varepsilon)+O(e^{-3\tau/4})] \, |\partial W|} e^{-f|W|} \times \max_{U \subset W} e^{-(a_q/4)\, |W\setminus U|} e^{-(\tau/4C_3)\, |\partial U|} \tag{D.16}$$

and

$$\left|\frac{d^k}{dh^k} Z_q(W, h)\right|_{h=h_0} \leqslant |k|! \left\{[C_0 + O(\varepsilon)] \, |W|\right\}^{|k|} e^{\gamma\tau N_\partial v(W)} e^{[O(\varepsilon)+O(e^{-3\tau/4})] \, |\partial W|} e^{-f|W|} \times \max\left\{e^{-(a_q/4)\, |W|}, e^{-(2d\tau/4C_3)\, |W|^{d/(d-1)}}\right\} \tag{D.17}$$

APPENDIX E. PROOF OF LEMMAS 5.1 AND 5.2

Proof of Lemma 5.1. Observing that all components W of $\mathrm{Int}^{(0)}\, Y_A$ obey the bound $|W| \leqslant \max_{Y \in Y_A} |V(Y)|$, we see that the statement (i) of Lemma 5.1 immediately follows from Lemma 4.6.

In order to prove (ii), we first note that for $Y_A = \{Y_1,..., Y_n\}$,

$$\rho(Y_A) \exp[E_q(\overline{\mathrm{supp}\; Y_A})] = A(Y_1,..., Y_n) \prod_{i=1}^{n} \rho(Y_i) \exp[E_q(Y_i)] \times \prod_{m \neq q} \exp[E_q(\mathrm{Int}_m\, Y_A \cap \mathrm{supp}\, A) - E_m(\mathrm{Int}_m\, Y_A \cap \mathrm{supp}\, A)] \tag{E.1}$$

which implies that

$$|\rho(Y_A) \exp[E_q(\overline{\mathrm{supp}\; Y_A})]| \leqslant C_A \exp[-(\tau - \gamma\tau)\, |Y_A|] \exp[2\gamma\tau N_\partial(\mathrm{Int}\; YA \cap \mathrm{supp}\, A)] \times \exp[(e_q - e_0)(|\mathrm{supp}\; Y_A|_d + |\mathrm{Int}\; Y_A \cap \mathrm{supp}\, A|_d)] \tag{E.2}$$

Next we use Lemma 4.6 to bound

$$\prod_{m=1}^{N}\left|\frac{Z_m(\mathrm{Int}_m^{(0)}\,Y_A, h)}{Z_q'(\mathrm{Int}_m^{(0)}\,Y_A, h)}\right| \leqslant e^{a_q\,|\mathrm{Int}^{(0)}\,Y_A| + O(\varepsilon)\,|\partial\,\mathrm{Int}^{(0)}\,Y_A|}\, e^{2\gamma\tau N_\partial(\mathrm{Int}^{(0)}\,Y_A)} \tag{E.3}$$

Bounding $e_q - e_0 \leqslant a_q + O(\varepsilon)$, observing that

$$|\mathrm{Int}^{(0)}\,Y_A| + |\mathrm{Int}\,Y_A \cap \mathrm{supp}\,A| = |\mathrm{Int}\,Y_A| \tag{E.4}$$

and bounding

$$|\partial\,\mathrm{Int}^{(0)}\,Y_A| \leqslant |\partial\,\mathrm{Int}\,Y_A| + |\partial\,\mathrm{supp}\,A| \leqslant |\partial\,\mathrm{Int}\,Y_A| + 2d\,|\mathrm{supp}\,A|$$

we get the bound

$$|K_q'(Y_A)| \leqslant C_A e^{-[\tau - \gamma\tau - O(\varepsilon)]\,|Y_A|}\, e^{2\gamma\tau N_\partial(\mathrm{Int}\,Y_A)}\, e^{a_q\,|V(Y_A)| + O(\varepsilon)\,|\mathrm{supp}\,A|} \tag{E.5}$$

Bounding now $N_\partial(\mathrm{Int}\,Y_A)$ by $C_1\,|Y_A|$, and observing that $\prod_{Y\in Y_A}\chi_q(Y) \neq 0$ implies that $a_q\,|V(Y_A)| \leqslant [\alpha + 1 + O(\varepsilon)]\,|Y_A|$ due to the bound (4.26), we finally get

$$|K_q'(Y_A)| \leqslant C_A e^{-\tau[1-(1+2C_1)\gamma]\,|Y_A|}\, e^{[1+\alpha+O(\varepsilon)]\,|Y_A|}\, e^{O(\varepsilon)\,|\mathrm{supp}\,A|} \tag{E.6}$$

which implies the bound (5.15).

We are left with the proof of (iii). By (E.1), Lemma D.1, and the assumptions (3.9) and (3.25b),

$$\begin{aligned}
&\left|\frac{d^k}{dh^k}\rho(Y_A)\exp[E_q(\overline{\mathrm{supp}\,Y_A})]\right| \\
&\qquad \leqslant |k!|\; C_A C_0^{|k|}(|\overline{\mathrm{supp}\,Y_A}| + |Y_A| + 2\,|\mathrm{Int}\,Y_A \cap \mathrm{supp}\,A|)^{|k|} \\
&\qquad\quad \times \exp[-(\tau - \gamma\tau)\,|Y_A|]\,\exp[2\gamma\tau N_\partial(\mathrm{Int}\,Y_A \cap \mathrm{supp}\,A)] \\
&\qquad\quad \times \exp[(e_q - e_0)(|\mathrm{supp}\,Y_A|_d + |\mathrm{Int}\,Y_A \cap \mathrm{supp}\,A|_d)]
\end{aligned} \tag{E.7}$$

On the other hand,

$$\begin{aligned}
&\left|\frac{d^k}{dh^k}\prod_{m=1}^{N}\frac{Z_m(\mathrm{Int}_m^{(0)}\,Y_A, h)}{Z_q'(\mathrm{Int}_m^{(0)}\,Y_A, h)}\right| \\
&\qquad \leqslant \{[2C_0 + O(\varepsilon)]\,|\mathrm{Int}^{(0)}\,Y_A|\}^{|k|}\, e^{a_q\,|\mathrm{Int}^{(0)}\,Y_A|} \\
&\qquad\quad \times e^{O(\varepsilon)\,|\partial\,\mathrm{Int}^{(0)}\,Y_A|}\, e^{2\gamma\tau N_\partial(\mathrm{Int}^{(0)}\,Y_A)}
\end{aligned} \tag{E.8}$$

by Lemma 4.7, while

$$\left|\frac{d^k}{dh^k}\prod_{Y\in Y_A}\chi_q(Y)\right|\leqslant\left[\bar{C}_1\sum_{Y\in Y_A}|V(Y)|\right]^{|k|} \tag{E.9}$$

by (D.6).

Combining the bounds (E.7)–(E.9), and bounding

$$\begin{aligned}
&C_0(|\overline{\operatorname{supp} Y_A}|+|Y_A|+2\,|\operatorname{Int} Y_A\cap\operatorname{supp} A|)\\
&\quad+[2C_0+O(\varepsilon)]\,|\operatorname{Int}^{(0)} Y_A|+\bar{C}_1\sum_{Y\in Y_A}|V(Y)|\\
&\qquad\leqslant C_0\,|\operatorname{supp} A|+[2C_0+O(\varepsilon)](|Y_A|+|\operatorname{Int} Y_A|)+\bar{C}_1\sum_{Y\in Y_A}|V(Y)|\\
&\qquad\leqslant|\operatorname{supp} A|\,e^{O(1)\,|Y_A|}
\end{aligned} \tag{E.10}$$

we may then continue as in the proof of (E.6) to obtain te bound (5.16). ∎

Proof of Lemma 5.2. We start from the representation (5.4) and use the assumptions (3.25) and (3.11) to bound

$$\begin{aligned}
|\rho(Y_A)|\leqslant{}& C_A\exp[-\tau\,|Y_A|-E_0(\operatorname{supp} Y_A)]\exp[-E_0(\operatorname{Int} Y_A\cap\operatorname{supp} A)]\\
&\times\exp[-E_q(\operatorname{Ext} Y_A\cap\operatorname{supp} A)]\\
\leqslant{}& C_A\exp[\gamma\tau N_\partial(\overline{\operatorname{supp} Y_A})]\exp\{-[\tau-O(\varepsilon)]\,|Y_A|\}\\
&\times\exp[O(\varepsilon)\,|\operatorname{supp} A|]\exp(-f\,|\overline{\operatorname{supp} Y_A}|)\\
&\times\exp(-a_q\,|\operatorname{Ext} Y_A\cap\operatorname{supp} A|)
\end{aligned} \tag{E.11}$$

Lemma 4.6 to bound

$$\begin{aligned}
&\left|\sum_{m=1}^{N} Z_m(\operatorname{Int}^{(0)}_m Y_A, h)\right|\\
&\qquad\leqslant e^{-f\,|\operatorname{Int}^{(0)} Y_A|}\,e^{O(\varepsilon)(|Y_A|+|\operatorname{supp} A|)}\,e^{\gamma\tau N_\partial(\operatorname{Int}^{(0)} Y_A)}
\end{aligned} \tag{E.12}$$

and the inequality (C.32) in conjunction with the estimate

$$|\partial\operatorname{Ext}^{(0)} Y_A|\leqslant|\partial\operatorname{supp} A|+|\partial V|+|\partial V(Y_A)|\leqslant 2d\,|\operatorname{supp} A|+|\partial V|+C_3\,|Y_A|$$

to bound

$$\begin{aligned}
|Z_q(\operatorname{Ext}^{(0)} Y_A, h)|\leqslant{}& e^{O(\bar{\varepsilon})(|\partial V|+|\operatorname{supp} A|)}\,e^{\gamma\tau N_\partial(\operatorname{Ext}^{(0)} Y_A)}\,e^{-f\,|\operatorname{Ext}^{(0)} Y_A|}\\
&\times e^{O(\bar{\varepsilon})\,|Y_A|}\max_{U\subset\operatorname{Ext}^{(0)} Y_A} e^{-(a_q/4)\,|\operatorname{Ext}^{(0)} Y_A\setminus U|}\,e^{-(\tau/4C_3)\,|\partial U|}
\end{aligned} \tag{E.13}$$

where $\tilde{\varepsilon}$ is the constant introduced in Lemma 5.2. Combining the bounds (E.11)–(E.13) with (5.4), we obtain

$$|Z_q(A \mid V, h)| \leqslant C_A e^{O(\tilde{\varepsilon})\,|\text{supp}\,A|}\, e^{[\gamma\tau + O(\tilde{\varepsilon})]\,|\partial V|}\, e^{-f\,|V|} \times \sum_{Y_A} e^{-(\tau-1)\,|Y_A|} \max_{U \subset \text{Ext}^{(0)}\,Y_A} e^{-(a_q/4)\,|\text{Ext}\,Y_A \backslash U|}\, e^{-(\tau/4C_3)\,|\partial U|} \tag{E.14}$$

where we used the bounds

$$a_q\,|\text{Ext} \cap \sup A| + (a_q/4)\,|\text{Ext}^{(0)}\,Y_A \backslash U| \leqslant (a_q/4)\,|\text{Ext}\,Y_A \backslash U|$$

and $N_\partial(V) \leqslant |\partial V|$. Extracting the factor

$$\begin{aligned}
&\max_{Y_A} e^{-(\tau/4)\,|Y_A|} \max_{U \subset \text{Ext}^{(0)}\,Y_A} e^{-(a_q/4)\,|\text{Ext}\,Y_A \backslash U|}\, e^{-(\tau/4C_3)\,|\partial U|} \\
&\quad\leqslant \max_{Y_A} e^{-(\tau/4C_3)\,|\partial V(Y_A)|} \max_{U \subset \text{Ext}^{(0)}\,Y_A} e^{-(a_q/4)\,|[V \backslash V(Y_A)] \backslash U|}\, e^{-(\tau/4C_3)\,|\partial U|} \\
&\quad\leqslant \max_{S \subset V} e^{-(a_q/4)\,|V \backslash S|}\, e^{-(\tau/4C_3)\,|\partial S|} \\
&\quad\leqslant \max\{e^{-(a_q/4)\,|V|}, e^{-(\tau/4C_3)\,|\partial V|}\}
\end{aligned}$$

from the right-hand side of (E.14), we are left with a sum $\sum_{Y_A} e^{-(3\tau/4-1)\,|Y_A|}$ which we bound as follows:

$$\sum_{Y_A} e^{-(3\tau/4-1)\,|Y_A|} \leqslant \sum_{n=0}^{\infty} \frac{1}{n!} \left[\sum_{Y:\, V(Y) \cap \text{supp}\,A \neq \varnothing} e^{-(3\tau/4)-1)\,|Y|} \right]^n \leqslant e^{O(\tilde{\varepsilon})\,|\text{supp}\,A|}$$

Putting everything together, we obtain the bound (i) of Lemma 5.2.

In order to prove (ii); we generalize (E.11)–(E.13) to derivatives. In (E.11), these derivatives produce an extra factor,

$$\begin{aligned}
&C_A\,|k|!\,(C_0\,|\overline{\text{supp}\,Y_A}| + C_0\,|\text{supp}\,A \backslash \text{supp}\,Y_A|)^{|k|} \\
&\quad\leqslant C_A\,|k|!\,(2C_0\,|\text{supp}\,A| + C_0\,|Y_A|)^{|k|} \\
&\quad\leqslant C_A\,|k|!\,(C_0\,|\text{supp}\,A|)^{|k|}\, e^{O(1)\,|Y_A|}
\end{aligned}$$

while in (E.12) and (E.13), they produce factors

$$|k|!\,\{[C_0 + O(\varepsilon)]\,|\text{Int}^{(0)}\,Y_A|\}^{|k|}$$

and

$$|k|!\,\{[C_0 + O(\varepsilon)]\,|\text{Ext}^{(0)}\,Y_A|\}^{|k|}$$

On the right-hand side of (E.14), this leads to an extra factor

$$C_A\, |k|!\, \{[C_0 + O(\varepsilon)]\, |V|\}^{|k|}\, e^{O(1)\,|Y_A|}$$

Observing that the sum $\sum_{Y_A} e^{-[3\tau/4 - O(1)]\,|Y_A|}$ can be bounded by $e^{O(\bar{\varepsilon})\,|\mathrm{supp}\, A|}$ as well, we obtain Lemma 5.2(ii). ∎

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