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Semantics for Connexive Logics. I

According to the third view in the ancient debate on the nature of conditionals, a sound conditional requires a connexion between antecedent and consequent. Both material implication (the first ancient view) and strict implication (the second ancient view) were rejected by the third view as satisfactory accounts of the conditional relation because they deliver conditionals such as those of the paradoxes of implication which are unsound, presumably on the ground that they fail to meet the requirement of connexion. We do not know what conditionals were said to meet this requirement, other than Identity, $A \to A$ (a principle that was explicitly rejected under the fourth ancient view).

The third ancient view has reappeared in the modern debate as to the nature of entailment, implication and conditionality, where the connexion requirement is commonly imposed as a requirement of meaning or content connexion between antecedent and consequent of valid implications. This requirement coincides with the broad requirement of relevance: for if antecedent and consequent enjoy a meaning connexion then they are relevant in meaning to one another, and if they are relevant in meaning to one another then they have through the relevance relation a connexion in meaning. Thus the general classes of connexive and relevant logics are one and the same. And all these logics — provided only that they contain a quite minimal negation — can conform to a characterization of implication in terms of incompatibility, i.e. of $A \to B$ as $A / \sim B$ where / is the Chrysippus-Sheffer stroke, and thus can satisfy the only further piece of information we now retain as to the third ancient view, namely that a conditional is sound when the contradictory of its consequent is incompatible with its antecedent (see [5], p. 129 ff.) Since material and strict accounts can also satisfy this condition, it was presumably intended that the incompatibility relation concerned be genuinely two-place and not reducible to a modal operator applying to truth functions. In fact very many connexive logics will meet this irreducibility requirement (cf. [4], p. 462 ff.)

As a great variety of logics fall under the general head of *connexive (or relevant) logic*, even when suitably rigorous connexion and irreducibility requirements are imposed, various rudimentary classifications of these logics have been attempted, for example along the route pursued in [3] in terms of the way the logics resolve — as they are bound to do — Lewis's "independent" arguments for paradoxes of (strict) implication. For instance, relevant logics in the narrower sense are at bottom connexive logics which solve the paradoxes through the (criticism and) rejection of the principles of Disjunctive Syllogism, Antilogism, and the like. (Of course, to characterise such relevant logics more fully some inclusions as well as exclusions should be specified: this may be achieved as in ([3], 2.9), by requiring that a relevant logic, narrowly construed, is also a conservative extension of distributive lattice logic, DLL.) Connexive logics, in the modern narrow sense, resolve the standard paradoxes by qualifying or rejesting Simplification ($A \bullet B \to A$, $A \bullet B \to B$) and, usually, its dual Addition ($A \to A \vee B$, $B \to A \vee B$).

But what typifies *modern connexivism* is not merely — or even so much — the qualification of Simplification, as the perhaps surprising acceptance of certain non-classical but traditional principles such as Aristotle, $\sim(A \to \sim A)$ and some of its strengthenings. As McCall, to whom we owe the modern use of 'connexive', asserts ([4], p. 438) 'the idee maîtresse of connexive logic ... is that no proposition should imply or be implied by its own negation'. But in fact these two features, which we take to characterise connexive logics in the narrow sense, namely (1) rejection or qualification of Simplification and its equivalents and (2) acceptance of Aristotle's thesis along with Identity, are intuitively related: both may be explained, and in this way rendered not at all surprising, by a traditional and rather appealing intuitive theory of incompatibility and negation.

Specifically connexive principles of importance other than Aristotle and its mate $\sim(\sim A \to A)$ are:

$$\sim[(A \to B) \bullet (A \to \sim B)] \qquad \text{(Strawson), and}$$
$$A \to B \to . \sim(A \to \sim B); \quad A \to \sim B \to . \sim(A \to B) \quad \text{(Boethius)}$$

These principles not only have a venerable history; they have received substantial support in recent literature (see [3], 2.3). Aristotle long ago gave the following argument for Strawson:

$$(A \to B) \bullet (A \to \sim B) \to (A \to B) \bullet (B \to \sim A) \\ \to (A \to \sim A).$$

Hence as $\sim(A \to \sim A)$, $\sim[(A \to B) \bullet (A \to \sim B)]$[1]. And Boethius is affirmed by Boethius (see [5], p. 191)[2]. Modern connexivism, with connexive logic as a formal sentential logic, begins with Nelson [9], and has since

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[1] This argument is discussed in Łukasiewicz [6]. Łukasiewicz, I think correctly, takes Aristotle to be asserting the principle $\sim(A \to \sim A)$ quite generally. But there remains room for argument, because it could be claimed that Aristotle's (rather primitive) variables are restricted to contingent statements.

[2] An affirmation the Kneales far too hastily write off as a mistake. But the Kneales procedure does rather nicely illustrate the way history gets coloured by currently-received perceptions of correctness.

received more semantically tractable formulation in the work of Angell (see especially [1]) and McCall (e.g. [2]). A semantical analysis of McCall's system of [2] and some related systems has been presented previously, in [10]; but the semantics proposed was not adequate to deal with more satisfactory connexive systems, such as Angell's system (of [1]) and systems in its vicinity. The present paper closes that gap and provides semantics for a very extensive class of connexive logics.

Why is it worth looking for semantics for these logics? Partly because to float a bold hypothesis, connexivism seems to underlie much of traditional Aristotelian logic, and semantics for connexive logics could help to clarify substantially the underlying thinking. That is, much traditional logical and semantical thinking is connexivist in character. (In this respect, however, the semantical analysis so far obtained falls seriously short, as will be indicated below, in § 6). More weakly and less speculatively, connexivism does represent an important, but neglected, traditional position in logic, and one of much modern interest. Firstly, unlike most alternatives to classical logic, it is not a mere sublogic of classical logic, but contains distinctive non-classical principles of its own. In this respect it stands, as McCall has remarked ([4], p. 435) 'to classical logic rather as Reimannian and Lobatchevskian geometries stand to Euclidean.' Secondly, it offers what is often regarded as a very appealing line on the paradoxes of implication, attributing these paradoxes not to suppression features, e.g. of strict implication, but to the addition in the first place of irrelevant components. And the position has a simple and direct explanation, in terms of its quite intuitive subtraction, i.e. cancellation, account of negation as to why the very first steps in Lewis' hard paradox argument, namely $A \bullet {\sim}A \rightarrow A$ and $A \bullet {\sim}A \rightarrow {\sim}A$, fail. For ${\sim}A$ cancels out A, so that the conjoined content, of $A \bullet {\sim}A$, is less than that of A and of that of ${\sim}A$. But implication requires content inclusion, so these (degenerate) examples of Simplification fail. This explains, in a sketchy way, the character of the connexivist argument against Simplification.[3] The same argument explains why $A \bullet {\sim}A$ does not imply ${\sim}(A \bullet {\sim}A)$, and provides a basis for an argument for Aristotle, along these lines: — No contingent or necessary statement implies its own negation, i.e. if B is possible then ${\sim}(B \rightarrow {\sim}B)$. But if B is impossible then B is equivalent to some statement A & ${\sim}A$; hence, by replacement of equivalents in ${\sim}((A \bullet {\sim}A) \rightarrow {\sim}(A \bullet {\sim}A))$, ${\sim}(B \rightarrow {\sim}B)$ when B is impossible. Hence Aristotle, ${\sim}(B \rightarrow {\sim}B)$, holds for every B.

Given Aristotle, Boethius is derivable in stronger logical systems. But in less strong, and more satisfactory systems, Boethius requires in-

[3] A fuller development, and assesment, of the argument may be found in [3]. I owe several of the informal arguments for connexive assumptions to V. Routley.

dependent argument. McCall's argument for Boethius is premissed on the plausibility of the principle, Compatibility, $A \to B \to . A \circ B$, that what a statement implies it is compatible with. Boethius, $A \to B \to \sim(A \to \sim B)$, then follows using the definition of compatibility, $A \circ B =_{Df} \sim(A \to \sim B)$, and connexively acceptable negation principles (which are included in basic system *CB*). Compatibility, which is indeed a principle of wide appeal – for example, Lewis and Langford ([7], p. 157; my italics) in presenting the orthodox strict position, say that 'the principle $p \prec q \prec$ $\prec . p \circ q$ *which might be expected to hold*, does not, in fact, hold without exceptions' – may be defended using traditional accounts of implication. For example, if A implies B then B is (contentwise) part of A; but if B is part of A, it must (surely?) be compatible with A, for something cannot have a part incompatible with it. A slightly deeper argument for Compatibility, which gets closer to the traditional ideas underlying connexivism, goes as follows: – if A is incompatible with B then $A \bullet B$ says less than A and than B because B has negated and so cancelled part of A; thus $A \bullet B$ does not imply A and does not imply B, so A does not imply B. In short, the cancellation account of negation, in combination with a content account of implication, especially one which connects $A \to B$ with $A \bullet B \leftrightarrow A$, vindicates Boethius.

§1. Connexive systems and others. The systems – many of which are not connexive systems in the narrower sense explained – are formulated (as in Angell [1]) with connective set $\{\to, \sim, \bullet\}$ and with $\vee$ defined: $A \vee B =_{Df} \sim(\sim A \bullet \sim B)$, $/$ defined: $A / B =_{Df} A \to \sim B$, $\circ$ defined: $A \circ B =_{Df} \sim(A \to \sim B)$, and $\leftrightarrow$ defined: $A \leftrightarrow B =_{Df} (A \to B) \bullet (B \to A)$. Orthodox notational conventions are adopted without further elaboration in what follows: they are essentially those in [3] and [4]. The basic system CB has the following postulate: –

A1. $A \to A$ (Identity) A2. $\sim\sim A \to A$ (Double Negation)
R1. $A, A \to B \to B$; read: where A and $A \to B$ are theorems so is B (Modus Ponens)
R2. $A \to B, C \to D \to B \to C \to . A \to D$ (Affixing)
R3. $A \to C, B \to D \to A \bullet B \to C \bullet D$ (Rule Praeclarum)
R4. $A \to \sim B \to B \to \sim A$ (Rule Contraposition)

CB can be alternatively, and perhaps more neatly, axiomatised in terms different primitive, e.g. with $/$ or $\circ$ replacing $\to$. Some of the theorems and derived rules of *CB* used in what follows are these:

i. $A \to \sim\sim A$; from A1 and B4.

Note that the resources of *CB* do not permit the integration of both forms of Double Negation to $A \leftrightarrow \sim\sim A$, i.e. $(A \to \sim\sim A) \bullet (\sim\sim A \to A)$, since the system lacks an Adjunction rule. Nor would it permit the adoption of $A \leftrightarrow \sim\sim A$ as a workable axiom since the basic system does not sanction even Rule Simplification. These are weaknesses stronger connexive type systems will do something to repair.

ii. $A \to B \dashrightarrow B \to C \to . A \to C$ (Rule Suffixing)
iii. $A \to B \dashrightarrow C \to A \to . C \to B$ (Rule Prefixing)
iv. $A \to B, B \to C \dashrightarrow A \to C$ (Rule Syllogism)
v. $A \to B \dashrightarrow \sim B \to \sim A$

And, more generally, all forms of Rule Contraposition are forthcoming, using the postulates derived.

Additional postulates drawn from the following lists may be added to the basic system *CB* singly or combination to yield a wealth of stronger systems (some of the postulate labels are taken over from [3]): —

B3. $A \to B \to . B \to C \to . A \to C$ (Suffixing)
B4. $A \to B \to . C \to A \to . C \to B$ (Prefixing)
B5. $A \to (A \to B) \to . A \to B$ (Contraction)
B6. $A \to . A \to B \to B$ (Assertion)
B7. $A \to (B \to C) \to . B \to . A \to C$ (Commutation)
B8. $A \to (B \to C) \to . A \to B \to . A \to C$ (Self-Distribution)
B10. $A \to . B \to B$ (S4 Paradox)

C1. $A \bullet (A \to B) \to B$ (Conjunctive Assertion)
C2. $(A \to B) . (B \to C) \to . A \to C$ (Conjunctive Syllogism)
C3. $A \to (B \to C) \to . (A \bullet B) \to C$ (Importation)
C4. $A \bullet B \to B \bullet A$ (Commutation)
C5. $A \bullet (B \bullet C) \to (A \bullet B) \bullet C$ (Associativity)
C6. $A \to B \to . (C \bullet A) \to (C \bullet B)$ (Factor)
C7. $A \to . B \to A \bullet B$ (Adjunction)
C8. $A \to A \bullet A$ (Tautology)
C9. $A \to B \bullet C \to D \to . A \bullet C \to B \bullet D$ (Praeclarum)

D1. $A \bullet B \to C \to . (A \bullet \sim C) \to \sim B$ (Antilogism)
D2. $\sim (A \bullet \sim A)$ (Non-contradiction)
D3. $A \to \sim A \to . \sim A$ (Reduction)
D4. $A \to \sim B \to . B \to \sim A$ (Contraposition)
D5. $A \bullet \sim (A \bullet \sim B) \to B$ (Disjunctive Syllogism)
D6. $A \bullet (B \vee C) \to . A \bullet B \vee A \bullet C$ (Distribution)
D7. $A \bullet B \to \sim (A \to \sim B)$ (i.e. $A \to B \to \sim (A \bullet \sim B)$) (Ackermann, Counterexample)
D8. $\sim (A \to \sim B) \to A \bullet B$ (Sheffer)
D9. $A \bullet B \to A \vee B$

DC1. $\sim (A \to \sim A)$ (Aristotle)
DC2. $\sim ((A \to B) \bullet (A \to \sim B))$ (Strawson)
DC3. $A \to B \to . \sim (A \to \sim B)$ (Boethius)

DR1. $A \to (A \to B) \to B$ (Necessitation)
DR2. $A, B \to A \bullet B$ (Rule Adjunction)
DR3. $A \to B \to \sim(A \bullet \sim B)$ (Rule Counterexample)
DR4. $A \to (B \to C) \to A \bullet B \to C$ (Rule Importation)
DR5. $A \bullet B \to C \to A \to .B \to C$ (Rule Exportation)

Several of the rules are deribitionally equivalent to axioms, e.g. DR3 is equivalent to $\sim(A \bullet \sim A)$. The rule yields the principle using Identity. Conversely,

$$\begin{aligned} & A \to B \to \sim B \to \sim A && \text{by R4} \\ & \to A \bullet \sim B \to A \bullet \sim A && \text{by R3} \\ & \to \sim(A \bullet \sim A) \to \sim(A \bullet \sim B) && \text{by R4} \\ & \to \sim(A \bullet \sim B) && \text{using R1.} \end{aligned}$$

DR4 is equivalent to: $(A \to B) \bullet A \to B$. The rule yields the thesis immediately using Identity. Conversely,

$$\begin{aligned} & A \to (B \to C) \to A \bullet B \to (B \to C) \bullet B && \text{by R3} \\ & \to A \bullet B \to C && \text{using R2.} \end{aligned}$$

DR5 is equivalent to $A \to .B \to A \bullet B$. The thesis results using Identity, and the rule results by Affixing.

§2. Connexive modellings. Connexive modellings do not differ from relevant modellings as to the implication connective: thus the pure entailment theories are the same. It is only when negation and conjunction are introduced that marked differences begin to emerge. Both negation, where the crucial distinguishing formulae are located, and conjunction can be added separated to the implicational base, but here they will be treated together: their separate treatment can easily be isolated, and will be drawn attention to subsequently.

As in [3] chapter 3, both affixing and reduced models are elaborated at once. The reduced models, which provide neater modellings for connexive logics in the literature, will be assembled in a later section. *A CB model* $\mathcal{M}$ – a basic model for connexive logics – is a structure $\mathcal{M} = \langle T, O, K, R, S, ^*, G, I \rangle$, where O and K are sets with $O \subseteq K$, T is an element of O, R and S are three-place relations on K, $*$ is an operation on K, G is a relation on *Wff* and worlds, i.e. on $wff \times K$, and I is a two-valued interpretation function from $Wff \times K$ to $\text{II} = \{1, 0\}$, such that generally:

p1. $a \leqslant a$, where $b \leqslant c =_{Df} (Px \in O) Rxbc$.
p2. if $a \leqslant d$ and $Rdbd$ then $Rabc$.
p3. $a = a^{**}$.
p4. if $a \leqslant b$ then $b^* \leqslant a^*$.
p5. if $a \leqslant b$ and $Scda$ then $Scdb$.

i1. if $I(P, a) = 1$ and $a \leqslant b$ then $I(P, b) = 1$, for every sentential parameter P.

i2. $I(B \to C, a) = 1$ iff for every b and c such that $Rabc$ if $I(B, b) = 1$ then $I(C, c) = 1$.
i3. $I(B \bullet C, a) = 1$ iff for some b and c such that $Sbca$ $I(B, b) = 1$ and $I(C, c) = 1$.
i4. $I(\sim A, a) = 1$ iff $I(A, a^*) \neq 1$
i5. if AGb then $I(A, b) = 1$

What is new in connexive modelling, as distinct from relevant logic modelling, is, first, the evaluation of conjunction through a 3-place relation S, and second, the introduction of a relation G of generation. The first has a good precedent in relevant logic, in the semantical treatment of fusion (or intensional conjunction) $\circ$ (see [3], chapter 4) which is evaluated in a quite analogous way to $\bullet$ but using relation R. (Since $\bullet$ is not however linked with $\to$ by Portation rules as $\circ$ is, a different relation than R has to be chosen to assess intensional conjunction generally). The evaluation rule for $\bullet$ reveals at once that the connective is indeed intensional: its assessment in a situation involves consideration of what happens in other (not merely operationally linked) situations. And there is a fair sense, as will emerge, in which connexive conjunction is irreducibly intensional. The second innovation, the incorporation of G in the model, appears to be without precedents, though the notion of generation used is well-known from algebra and is regularly used in completeness arguments for relevant logics. 'AGb' reads: A generates (situation) b – by which is meant that everything that holds in situation b is implied by A. Relation G plays an important role in the modelling conditions for distinctively connexive logical principles. Condition i5, though not entirely desirable since it is not inductively defined, is of a type now familiar from accepted semantical analyses (e.g. those for conditional logics and for da Costa's C systems).

A *reduced* CB model $\mathscr{M}$ is a structure $\langle T, K, R, S, {}^*, G, I\rangle$, i.e. with O elided, which otherwise differs from a CB model only in defining $b \leqslant c$ as $RTbc$. In reduced models certain modelling conditions simplify; but only (excessively) powerful systems have reduced modellings. A sufficient condition for reduced modelling can be given through the following: a *strong* connexive logic L is an extension of CB which contains at least the following principles B3, B4, C1, C9, D4, DR2. Essentially a strong logic converts the rules of CB to implications. The minimal strong logic under this characterisation is the system SB: –

$A \to B \to . B \to C \to . A \to C$	$B \to C \to . A \to B \to . A \to C$
$A \to A$	$\sim\sim A \to A$
$A \to \sim B \to . B \to \sim A$	$A \to B \bullet C \to D \to . A \bullet C \to B \bullet D$
$A \bullet (A \to B) \to B$	
$A, A \to B \to B$	$A, B \to A \bullet B$

SB so formulated contains some redundancies. Both $A \to A$ and $A \to$

$\to B\to.B\to C\to.A\to C$ can be deleted without altering the class of theorems (see [4], pp. 140-1). SB also has some rather conspicuous deficiencies for a "strong" logic, e.g. such properties as associativity of conjunction are not guaranteed.

Semantical notions are defined in the usual fashion. In particular a wff A is *true in* $\mathscr{M}$ iff $I(A, T) = 1$, and A is *CB valid* iff A is true in every CB model. These notions are extended to apply to each extension L of CB considered. Furthermore A is *reduced L valid* iff A is true in every reduced L model. Modelling conditions for the extensions are as follows, with q_i corresponding to postulate B_i, t_i to C_i, s_i to D_i, ds_i to DC_i, and db_i to rule DRi. These definitions are used: $R^2abcd = (Px)(Rabx \,\&\, Rxcd)$; $R^2a(bc)d =_{Df} (Px)(Raxd \,\&\, Rbcx)$; $R^3ab(cd)e =_{Df} (Px)(R^2abxe \,\&\, Rcdx)$; $U/Va(bc)d =_{Df} (Px)(Uaxd \,\&\, Vbcx)$; $U/Vabcd = (Px)(Uabx \,\&\, Vxcd)$; $U^2 = U/U$. (Here U and V range over three-place relations such as R and S.)

q3. if R^2abcd then $R^2b(ac)d$
q4. if R^2abcd then $R^2a(bc)d$.
q5. if $Rabc$ then R^2abbc
q6. if $Rabc$ then $Rbac$
q7. if R^2abcd then R^2acbd
q8. if R^2abcd then $R^3ac(bc)d$
q10. if $Rabc$ then $b \leqslant c$

r1. if $Sbca$ then $Rcba$
r2. if $S/Rdebc$ then $R^2e(db)c$
r3. if $R/Sa(de)c$ then R^2adec
r4. if $Sbca$ then $Scba$
r5. if $S^2b(de)a$ then S^2bdea
r6. if $R/Sa(de)c$ then $S/Rd(ce)c$
r7. if $Rabc$ then $Sabc$
r8. $Saaa$
r9. if $S/(R/S)fg(de)c$ then $R/(S/R)fd(ge)c$, i.e. if $R/Sa(de)c$ and $Sfga$ then for some x $Rfdx$ and $S/Rx(ge)c$.

s1. if $R/Sa(de)c$ then $R/Sa(dc^*)e^*$
s2. if $Sbcx^*$ then $b \leqslant c^*$, for $x \in 0$; or in reduced form if $SbcT^*$ then $b \leqslant c^*$
s3. Raa^*a
s4. if $Rabc$ then Rac^*b^*
s5. if $Sbca$ then Sba^*c^*
s6. if $Sbca$ and $Sdea^*$ then, for some x and y, $Sbxd^*$ and Sby^* and $Sxyc^*$
s7. if $Sbca$ then Ra^*bc^*
s8. if Ra^*bc^* then $Sbca$
s9. if $Sbca$ and $Sdea^*$ then either $b \leqslant d^*$ or $c \leqslant e^*$

ds1. $(Py)(Rx^*yy^* \,\&\, AGy)$, for every wff A and every x in O

ds2. if $Sbcx^*$ then $(Px, y)(Rbyz \,\&\, Rcyz^* \,\&\, AGy)$, for $x \in O$
ds3. $(Px, y, z)(Raxy \,\&\, Ra^*xz \,\&\, AGx \,\&\, y \leqslant z^*)$
dr1. $(Px \in O)$ $Raxa$; or in reduced modellings $RaTa$.
dr2. $(Px \in O)$ $Sxxx$; or in reduced form $STTT$
dr3. if $Sbcx^*$ then $b \leqslant c^*$ for $x \in O$ (cf. s2)
dr4. if $Sabc$ then $Rabc$ (cf. r1)
dr5. if Rabc then Sabc (cf. r7)

§3. Soundness theorems. Proofs of soundness follows the lines of [3], chapter 3, and begin with lemmas which simplify verification procedures.

HEREDITARINESS LEMMA. *If* $a \leqslant b$ *and* $I(A, a) = 1$ *then* $I(B, b) = 1$.

PROOF is by induction from the given basis. The cases for $\rightarrow$ and $\sim$ are as for relevant semantics.

ad$\bullet$. Suppose $a \leqslant b$ and $I(B \bullet C, a) = 1$. Then, for some c and d for which $Scda$, $I(B, c) = 1 = I(C, d)$. But by p5 $Scdb$, so $I(B \bullet C, b) = 1$.

Where L is any of the logics under examination, i.e. any of the extensions of CB considered, and A and B any wff of L, A *L-implies* B in L model $\mathscr{M}$ iff, for every a in K, if $I(A, a) = 1$ then $I(B, a) = 1$, i.e. iff $[A] \subseteq [B]$; and A *L-implies* B iff A L-implies B in every L model.

SOUNDNESS LEMMAS. (1) *If A L-implies B in $\mathscr{M}$ then $A \rightarrow B$ is true in $\mathscr{M}$, and, where $\mathscr{M}$ is a reduced L model, A L-implies B iff $A \rightarrow B$ is true in $\mathscr{M}$.*

(2) *A L-implies B iff $A \rightarrow B$ is L-valid.*

PROOF is like that for lemma 3.2 of [3].

SOUNDNESS THEOREM for CB. *If A is a theorem of CB then A is CB valid.*

PROOF is by the usual induction over proofs. That the axioms of CB are valid follows directly using the soundness lemma (2). Since $[A] \subseteq [A]$ always, A1 is CB valid. As to A2, for every L model $[\sim\sim A] = \{a \in K: I(\sim\sim A, a) = 1\} = \{a: I(A, a^{**}) = 1\} = \{a: I(A, a) = 1\} = [A]$, by p3 and i4, so $\sim\sim A \rightarrow A$ is valid by (2) above. That the rules preserve validity is proved as in [3], theorem 3.9; and only R3 is not treated there.

ad R3. Suppose $A \bullet B \rightarrow C \bullet D$ is not valid. Then for some model $\mathscr{M}$ and some $a \in K$, $I(A \bullet B, a) = 1 \neq I(C \bullet D, a)$. By the first, for some b and c, $Sbca$ and $I(A, b) = 1 = I(B, c)$; and so by the second, as $Sbca$, either $I(C, b) \neq 1$ or $I(D, c) \neq 1$. Hence either A does not L-imply C or B does not L-imply D; that is not both $A \rightarrow C$ and $B \rightarrow D$ are valid. Thus, by contraposition, R3 preserves validity.

SOUNDNESS THEOREMS for extensions L of CB. *For such logic L, if A is a theorem of L then A is L valid, and so reduced L valid.*

PROOF. It suffices to show that where a modelling condition holds the corresponding axiom or rule is appropriately valid. The arguments for the purely implicational axioms and rules and for the implication-negation axioms are exactly as in [3]. Most of the new cases, involving conjunction, follow very similar lines, so only a few illustrative cases are presented.

ad C1. Suppose $I(A \bullet (A \to B), a) = 1$. To show, using r1, that $I(B, a) = 1$; for the validity of C1 then follows using (2). By i3, for some b, c, $Sbca$ and $I(A, b) = 1 = I(A \to B, c)$. By r1, $Rcba$, so by i2, if $I(A, b) = 1$ then $I(B, a) = 1$, whence the result.

ad C2. Suppose $I(A \to B \bullet B \to C, a) = 1 \neq I(A \to C, a)$. Then for some b, c, d, e in K, $Rabc$, $Sdea$, $I(A, b) = 1 \neq I(C, c)$ and $I(A \to B, d) = 1 = I(B \to C, e)$. By r2, for some $x \in K$, $Rdbx$ and $Rexc$, so as $I(A, b) = 1$, $I(B, x) = 1$ and hence $I(C, c) = 1$, which is impossible.

ad D2. By s2, $SbcT^* \supset b \leqslant c^*$. Suppose $I(\sim(A \bullet \sim A), T) \neq 1$. $I(A \bullet \sim A, T^*) = 1$, so for some b and c $I(A, b) = 1$ and $I(A, c^*) \neq 1$ and $SbcT^*$. But then $I(A, c^*) = 1$ also, which is impossible.

ad DC1. Suppose $I(\sim(A \to \sim A), T) \neq 1$ for some L model $\mathscr{M}$ with base T. $I(A \to \sim A, T^*) = 1$. Hence, by ds1, $I(A, y) = 1 \supset I(\sim A, y) = 1$ and AGy. By i5, $I(A, y) = 1$, so $I(\sim A, y^*) = 1$; that is, $I(A, y) \neq 1$, which is impossible.

ad DC3. Suppose $I(A \to B, a) = 1 \neq I(\sim(A \to \sim B), a)$. Since by ds3, for some x, y and z, $Raxy$ and Ra^*xz and AGx, $I(A, x) = 1$. Thus $I(B, y) = 1 = I(\sim B, z)$. So $I(B, z^*) \neq 1$; but $y \leqslant z^*$, so $I(B, z^*) = 1$, which is impossible.

§4. Completeness theorems.

Proofs again follow the lines of [3], chapter 3, but the proofs are somewhat simpler than corresponding relevant proofs because primeness of theories is nowhere required. Where, as before, L is any of the connexive logics under examination, an L-theory a is a set of wff closed under provable L-implication, i.e. whenever $A \in a$ and $\vdash_L A \to B$ then $B \in a$. Note that closure under adjunction is not required. An L-theory a is *regular* when all theorems of L are in a, i.e. iff when $\vdash_L A$, $A \in a$. Where T is a regular L-theory closed under adjunction (i.e. when $A, B \in T$ then $A \bullet B \in T$) a *T-L-theory* a is a set of wff of L which is closed under T-implication, i.e. whenever $A \to B \in B$ (also written $\vdash_T A \to B$) and $A \in a$ then $B \in a$.

The *canonical CB model* $\mathscr{M}_c$ is the structure $\mathscr{M}_c = \langle L, O, K, R, S, {}^*, G, I\rangle$ defined as follows: — the base L is the class of theorems of L, or more generally *some* regular L-theory; O is the class of regular L-theories; and K the class of L-theories. Where a, b, c, d are L-theories, $Rabc$ iff for every wff A and B if $A \to B \in a$ and $A \in b$ then $B \in c$; $Sabc$ iff for every A

and B if $A \in a$ and $B \in b$ then $A \bullet B \in c$; $a^* = \{A: \sim A \notin a\}$; AGb iff $b = \{D: \vdash_L A \to D\}$; and $I(A, a) = 1$ iff $A \in a$. Where T is a regular L-theory, the *T-canonical CB model $\mathcal{M}_c$ on* base T is the similar structure $\langle T, [O], K, R, S, *, G, I\rangle$, but T-L-theories replace L-theories throughout definitions. O is bracketed because it will prove eliminable.

CANONICAL LEMMA for CB. (I) *The canonical CB model $\mathcal{M}_c$ is a CB model.*

(II) *The T-canonical CB model $\mathcal{M}_c$ on base T is a CB model.*

PROOF. Most details are simplifications of corresponding details for basic relevant system B in [3], chapter 3. But some adjustments emerge from the different characterisation of K. Case (II) will only be dealt with where details diverge from those (I).

It is immediate that $L \in O$, $O \subseteq K$, and that R and S are appropriate three-place relations. By Rule Contraposition a^* is an operation on K. The set $b = \{D: \vdash_L A \to D\}$ is an L-theory (by Affixing and Modus Ponens, so G is well-defined. Similarly where $b = \{D: \vdash_T A \to D\}$, b is a T-L-theory. For suppose $\vdash_T B \to C$ and $B \in b$. Then $\vdash_T A \to B$. By B3 (or, but slightly differently, B4) and the fact that T is an L-theory, $\vdash_T B \to C \to . A \to C$: Hence, by adjunction closure and B1, $\vdash_T A \to C$, i.e. $C \in b$. The p-postulates are readily established given the connection

a) $\qquad a \leqslant b$ iff $a \subseteq b$

Suppose $a \subseteq b$ and, to show $RLab$, that $A \to B \in L$ and $A \in a$. By inclusion $A \in b$ and since b is an L-theory $B \in b$. For the converse suppose $a \leqslant b$ and $A \in a$. Then for some regular x, $Rxab$. By regularity $A \to A \in a$, so as $A \in a$ and $Rxab$, $A \in b$. p1 and p2 are immediate from a), p3 from the definition of * and Double Negation, and p4 from a) definition of * and Rule Contraposition.

ad p5. Suppose $a \leqslant b$ and $Scda$ and also $C \in c$ and $D \in d$; to show $C \bullet D \in b$. By the assumptions $C \bullet D \in a$, so, using a), $C \bullet D \in b$.

ad i1. $a \leqslant b \;\&\; P \in a \supset P \in b$, by a). In fact, quite generally, $a \leqslant b \;\&\; \& \; I(A, a) = 1 \supset . I(A, b) = 1$.

ad i2. What has to be shown is: $B \to C \in a$ iff $(b, c)(Rabc \;\&\; B \in b \supset C \in c)$, where R has its canonical definition. One half is immediate using definition of R. For the converse suppose $B \to C \notin a$. Define $b = \{D: \vdash_L B \to D\}$, $c = \overline{\{E: \vdash_L E \to C\}}$. Then $B \in b$, $C \notin c$ and b and c are L-theories. As to the last, suppose on the contrary that for some wff E and F, $E \in c$, $\vdash_L E \to F$, but $F \notin c$. Then $\sim \vdash_L E \to C$ and $\vdash_L F \to C$, so $\vdash_L E \to C$, which is impossible. (In case (II) B3 or B4 and adjunction closure and B1 are again invoked.) It remains to show $Rabc$. Suppose otherwise that for some E and F, $E \to F \in a$ and $E \in b$ but $F \notin c$. Then $\vdash_L B \to E$ and $\vdash_L F \to C$, so by Affixing $\vdash_L E \to F \to . B \to C$, whence $B \to C \in a$, contradicting assumptions (In case (II) use B3 and B4).

ad i3. Suppose $Sbca$ & $B \in b$ & $C \in c$. Then, by canonical definition of S, $B \bullet C \in a$. For the converse suppose $B \bullet C \in a$. Define $b = \{D \vdash_L B \to D\}$ and $c = \{E \vdash_L C \to E\}$. Then, by Identity, $B \in b$ and $C \in c$. To show $Sbca$, suppose $D \in b$ and $E \in c$. Then $\vdash_L B \to D$ and $\vdash_L C \to E$, so by Praeclarum $\vdash_L B \bullet C \to D \bullet E$, whence $D \bullet E \in a$, as required. (In case (II) adjunction closure and the Praeclarum axiom are used).

ad i4, i.e. $\sim A \in a$ iff $A \notin a^*$. By definition of $*$.

ad i5, i.e. if $b = \{D: \vdash_L A \to D\}$ then $A \in b$. From Identity.

CANONICAL LEMMAS for extensions L of CB. (I) *The canonical* L *model* $\mathscr{M}_c$ *is an* L *model.*

(II) *The T-canonical* L *model* $\mathscr{M}_c$ *is an* L *model.*

PROOF. It suffices given the previous lemma, to prove that where an axiom or rule is valid the corresponding modelling condition holds generally. Many of the cases are simplifications of analogous cases in [3], and very many of the new cases conform to a common pattern; so once again only a few illustrative cases are set out.

ad r2. Set $x = \{D: (PE)(E \to D \in d$ & $E \in b\}$. Then x is an L theory and $Rdbx$. To show, what remains to be proved, that $Rexc$, suppose $A \to B \in e$ and $A \in x$. It is enough to show $B \in c$. As $A \in x$, for some E, $E \to A \in d$ and $E \in b$. As $A \to B \in e$ and, by hypothesis of r2, $Sdea$, $E \to A \bullet A \to B \in b$. Hence by C2, $E \to B \in a$. Thus as $Rabc$, by hypothesis of r2 again, and $E \in b$, $B \in c$.

ad r5. Suppose $Sbca$ and $Sdec$, and set $x = \{D: (PB, C)(\vdash B \bullet C \to D$ & $B \in b$ & $C \in d)\}$. Then x is an L theory and $Sbdx$. To show, what remains, $Sxea$, suppose $D \in x$ and $E \in e$. As $Sdec$, $C \bullet E \in c$, so as $Sbca$, $B \bullet (C \bullet E) \in a$. Hence by C5, $(B \bullet C) \bullet E \in a$; but $\vdash (B \bullet C) \bullet E \to D \bullet E$ by R3, whence $D \bullet E \in a$ as required.

ad s1. Suppose $Rabc$ and $Sdeb$, and define $x^* = \{D: (PA)(\sim D \to A \in a$ & $\sim A \in e\}$. Then x^* is an L theory; for suppose $D \in x^*$ and $\vdash D \to E$. For some A, $\sim D \to A \in a$ and $\sim A \in e$. Since $\vdash \sim E \to \sim D$, $\vdash \sim D \to A \to . \to . \sim E \to A$, so $\sim E \to A \in a$ and $E \in x^*$. Also $Raxe^*$. For suppose $C \to D \in a$ and $C \in x$ but $D \notin e$ for some C and D. Then $\sim D \in e$ and $\sim C \notin x^*$, i.e for every A, when $\sim\sim C \to A \in a$ then $\sim A \in e$, so $\sim\sim C \in a$. As to Sdc^*x, suppose otherwise $A \in d$, $B \in c^*$ but $A \bullet B \notin x$. Then $\sim(A \bullet B) \in x^*$, so for some C, $\sim\sim(A \bullet B) \to C \in a$ and $\sim C \in e$. Since $Sdeb$ however, $A \bullet \sim C \in b$, whence, as $Rabc$, $\sim B \in c$, contradicting $B \in c^*$.

ad s2. Suppose $Sbcx^*$ and $B \in b$, for $x \in O$. Then as $\sim(B$ & $\sim B) \in x$, B & $\sim B \notin x^*$, so $\sim B \notin c$, and $B \in c^*$.

ad ds1. Let $x \in O$ and define $b = \{D: \vdash_L A \to D\}$. Then AGb. To show Rx^*bb^*, suppose otherwise, for some B and C, $B \to C \in x^*$, $B \in b$ and $C \notin b^*$. Then $\vdash A \to B$ and, as $\sim C \in b$, $\vdash A \to \sim C$, whence $\vdash C \to \sim A$. So by Affixing, $\vdash B \to C \to . A \to \sim A$, and Contraposing $\vdash \sim(A \to \sim A)$

$\to\sim(B\to C)$. Since $x \in O$, $\sim(A\to\sim A) \in x$ by DC1, so $\sim(B\to C) \in x$, contradicting $(B\to C) \in x^*$.

ad ds3. Define x as the situation generated by A, $y = \{F\colon (PE)(E\to F \in a \;\&\; E \in x)\}$ and $z = \{C\colon (PB)(B\to C) \in a^* \;\&\; B \in x)\}$. Then AGx, $Raxy$- and Ra^*xz. To show $y \leqslant z^*$, suppose otherwise that for some D, $D \in y$, and $D \notin z^*$. Then $\sim D \in z$. As $D \in y$, for some E $E\to D \in a$ where $\vdash A\to E$. Hence as $\vdash E\to D\to.A\to D$, $A\to D \in a$, so by DC3, $\sim(A\to\sim D) \in a$. As $\sim D \in z$, for some B, $B\to\sim D \in a^*$ where $\vdash A\to B$. Hence as $\vdash B\to\sim D \to.A\to\sim D$, $A\to\sim D \in a^*$, and so $\sim(A\to\sim D) \notin a$, contradicting $\sim(A \to\sim D) \in a$.

COMPLETENESS THEOREMS for extensions L of CB.

I. *For each such logic L considered, if A is L valid then A is a theorem of L.*

II. *For each such strong logic L, if A is reduced L valid, then A is a theorem of L.*

PROOF. In both cases suppose A is not a theorem of L. Form the canonical L model $\mathcal{M}_c$ with base consisting of the class T of theorems of L. Then, by the lemma, $\mathcal{M}_c$ is an L model; and since $A \notin T$, $I(A, T) \neq 1$. Thus A is not true in an (the canonical) L model, so A is not L valid. In case (II) it can be verified that a strong logic provides all the properties required for reduced canonical models to be models (some of the details are recorded in the proof of the canonical lemma).

COROLLARY (*Implicational Adequacy*). $\vdash_L A\to B$ iff A L-implies B.

The semantical apparatus developed will also deliver stronger completeness theorems and therewith further information, such as compactness results. An [*adjunctive*] *L-derivation* of A from set S of wff of A, written $S \vdash_L A$, is a finite sequence of wff $A_1, \ldots, A_n$, with $A_n = A$, such that each member of the sequence either belong to S or is obtainable from predecessors in the sequence by a provable L-implication [or by Rule Adjunction]. A is [*adjunctively*] *L-derivable* from S iff there is an [adjunctive] L-derivation of A from S.

STRONGER COMPLETENESS THEOREMS. *Where A is not [adjunctively] L-derivable from regular set S, there is a [reduced] L-model $\mathcal{M}$ under which every member of S is true but A is not true.*

PROOF. Let T be the L-theory closure of S, i.e. $T = \{D\colon\; S \vdash_L D\}$ Then $A \notin T$ and T is a regular (adjunctive) L-theory. Form the (reduced) canonical model $\mathcal{M}_c$ with base T, and then proceed as in the previous theorem.

At various points the completeness arguments conspicuously fail to use all the information now available. For example, in the strong completeness result T could be inflated to a maximal L-theory using Zorn's lemma, but properties of maximality are not drawn upon by the modelling. Similarly an important extension lemma ([3], lemma 3.3), obtained using Lindenbaum methods, holds for connexive logics which conform to Rule Adjunction and Distribution, but the properties yielded by the lemma are not exploited by the modellings given. All this suggests that for significant classes of connexive logics there are improved and more informative modellings yet to be found.

§5. Alternative semantics, and augmented semantics for Simplification principles. Firstly the semantics given may be represented in operational form with the relations R and S on K replaced by operations on K. The operational evaluation rule for implication may be written in either of the forms.

i2′ $I(B \to C, a) = 1$ iff, for every b, if $I(B, b) = 1$ then $I(C, a \oplus b) = 1$,
or
i2″ $I(B \to C, a) = 1$ iff, for every b, if $I(B, a \odot b = 1$ then $I(C, b) = 1$,

depending on whether the suffixing or prefixing rule is favoured in the completeness argument (see [8]). Here the more familiar i2′ is chosen. Correspondingly the rule for conjunction may be written in two forms, of which the following is chosen:

i3′ $I(B \bullet C, a) = 1$ iff, for some b, $I(B, b) = 1 = I(C, a \odot b)$.

An L *operational model* differs from an L model primarily in replacing R by operation $\oplus$ and S by operation $\odot$ and adding an ordering relation $\leqslant$. But modellings conditions have also to be reexpressed in operational form. To illustrate the conditions are set down in the case of basic system CB: operational conditions for many of the postulates listed are given in [3]. In a CB operational model conditions p1 - p5 are replaced by the following conditions.

p1′. $a \leqslant x \oplus a$, for $x \in O$
p2′. if $a \leqslant b$ then $a \oplus c \leqslant b \oplus c$
p3 and p4 are as before.
p5′. if $a \leqslant b$ then $a \odot c \leqslant b \odot c$.

Semantical notions are extended in the expected way; in particular a wff A is L *o* (perationally) *valid* iff it is true in all L operational models, i.e. $I(A, T) = 1$ for every L *o* model.

Operational Adequacy Theorem for CB. *A is a theorem of CB iff A is CB o valid.*

PROOF of soundness is like that for the relational semantics. The main new feature of the completeness proof lies in the canonical definitions of the operations. In the canonical o model define $a \oplus b = \{C\colon (PB)(B \to C \in a \ \& \ B \in b)\}$ and $a \odot b = \{C\colon (D)(D \in b \supset D \bullet C \in a\}$. Then $a \oplus b$ and $a \odot b$ are L-theories where a and b are. Consider, to illustrate the new features, the verification of i3′ in the canonical model. Suppose, firstly, $B \in b$ and $C \in a \odot b$; to show $B \bullet C \in a$. But this is immediate using the definition of $\odot$. For the converse, suppose $B \bullet C \in a$. Define $b = \{D\colon \vdash B \to D\}$. b is a CB-theory and $B \in b$. To show $C \in a \odot b$, suppose $D \in b$ arbitrary D; to show $D \bullet C \in a$. Since $\vdash B \to D$, $\vdash B \bullet C \to D \bullet C$ by R3 (or Rule Factor), whence $D \bullet C \in a$.

Each of the L modelling conditions given maybe reexpressed in operational form. Sometimes the operational conditions are very attractive combinatorically and algebraically, sometimes they merely mirror the corresponding axioms just two examples:

B3.	$A \to B \to . B \to C \to . A \to C$	q3′.	$b \oplus (a \oplus c) \leqslant (a \oplus b) \oplus c$
DC3.	$A \to B \to \sim(A \to \sim B)$	ds3′.	$(Px)(AGx \ \& \ a \oplus x \leqslant (a^* \oplus x)^*)$

OPERATIONAL ADEQUACY THEOREM for extension, L of CB. *For each extension considered, theoremhood and validity coincide.*

The operational semantics lead directly to alternative formulations of connexive logics, to subscripted semantic tableaux, subscripted natural deduction formulations, and subscripted Gentzen formulation for each system L (details are as in [3]). The semantics also lead, using filtration methods, to decidability results for several of the logics considered, but not — so far at least — in the case of strong logics of interest.

The semantics may also be reformulated functionally, in a way that eliminates all reference to worlds, by rewriting $I(A, a)$ as $a(A)$ and reexpressing modelling conditions as conditions on functions, e.g. $\leqslant$ is replaced by a relation $\subseteq$ of functional extension.

In order to model Conjunctive Simplification, $A \bullet B \to A$, and its special case, $A \bullet A \to A$, the models given have to be augmented in the first case by a property C of situational occupation, and in the second case by an operation $\cap$ of situational intersection.

(Both semantical devices have been previously exploited, in forerunners to [3].) The notions are subject to these interpretational conditions:

i6. If, for some B, $I(B, a) = 1$, Ca;
i7. If $I(A, a) = 1 = I(A, b)$, $I(A, a \cap b) = 1$.

The modelling conditions are these:

E1. $A \bullet A \to A$ t1. if $Sbca$ then $b \cap c \leqslant a$

E2. $A \bullet B \to A$ t2. if $Sbca$ and Cc then $b \leqslant a$
E3. $B \bullet A \to A$ t3. if $Sbca$ and Cb then $c \leqslant a$
E4. $A \bullet B \to A$ t4. if $Sbcx$ and Cc then $b \leqslant x$ for x in O.

ENLARGEMENT OF ADEQUACY THEOREMS to cover E1-E4.

PROOF of adequacy is by cases.

ad E1. Suppose $I(A \bullet A, a) = 1$. Then for some b, c, $Sbca$ and $I(A, b) = 1 = I(A, c)$. Hence, by i7, $I(A, b \cap c) = 1$, so by E1, $I(A, a) = 1$.

ad t1, and associated requirements. Define $b \cap c = \{A\colon A \in b \ \& \ A \in c\}$. Then $\cap$ is an operation on K. For suppose, to show $b \cap c$ is an L theory when b and c are, $A \in b \cap c$ and $\vdash A \to B$. Then as $A \in b$ and $A \in c$, $B \in b$ and $B \in c$, i.e. $B \in b \cap c$. Also i7 follows. To establish t1, suppose $Sbca$ and $A \in b \cap c$. Then $A \in b$ and $A \in c$, so by $Sbca$, $A \bullet A \in a$; and thus, by E1, $A \in a$.

ad E2. Suppose $I(A \bullet B, a) = 1$. Then for some b, c, $Sbca$ and $I(A, b) = 1 = I(B, c)$. By i6, Cc, so by t2, $b \leqslant a$. Hence $I(A, a) = 1$.

ad t2, and associated requirements. Define Ca iff, for some B, $B \in a$. So i6 holds. As to t2, suppose $Sbca$ and Cc, and $A \in b$. Let D be a wff in c. By $Sbca$, $A \bullet D \in a$, whence, by E2, $A \in a$. The further cases are similar.

Some may have reservations, by this stage, at the multiplication of semantical apparatus. But really nothing has been introduced that is not reflected in commonplace semantical thinking. The notion of situational occupation, for example, is on a par with the notion of non-nullness of situational domains, yet no objection is generally made to the requirement that some domain be non-null or to the explicit statement of this in a semantics. And the idea of situational-intersection, of the common part of two situations, is an almost everyday one.

The enlargement, to include Simplification principles, means that the analysis given comprehends all the usual relevant affixing systems as well as the modern connexivist systems, which however reject or qualify Simplification. What the enlargement fails to do is to reflect the underlying semantical reasons for connexivists' rejection of Simplification or to cater directly for all the qualified forms they would allow. But these are problems for later more sophisticated and less brutal days.

§6. Semantics for Angell's system P_{AI}. Angell's 'propositional logic with subjunctive conditionals', P_{AI} of [1], has, when reformulated with axiom schemes, the following postulates:

A_1 (= B4). $B \to C \to . A \to B \to . A \to C$
A_2 (= C6, C4). $A \to B \to . C \bullet A \to B \bullet C$
A_3 (= D1). $A \to \sim(B \bullet C) \to . B \bullet A \to \sim C$
A_4 (= C5). $A \bullet (B \bullet C) \to B \bullet (A \bullet C)$

A_5 $(= D4)$. $A \to \sim B \to . B \to \sim A$
A_6 $(= A2)$. $\sim\sim A \to A$
A_7 $(= D7)$. $A \to B \to \sim(A \bullet \sim B)$
A_8 (cf. E2). $\sim((A \bullet B) \bullet \sim A)$
A_9 (cf. C8). $\sim(A \bullet \sim(A \bullet A))$
A_{10} $(= DC3)$. $A \to B \to \sim(A \to \sim B)$
R_{12} $(= R1)$. $A, A \to B \to B$ $R_2(= DR2)$. $A, B \to A \bullet B$

Since $\boldsymbol{P_{A1}}$ includes system SB (see the derivations in [1], especially *24, *43 and those following *52), a reduced modelling can be adopted. A $\boldsymbol{P_{A1}}$ model is then a reduced CB model $\langle T, K, R, S, {}^*G, C, I\rangle$, with situational occupancy C, such that

q4. if R^2abcd then $R^2a(bc)d$
r6′. if $R/Sa(de)c$ then $R/Saedc$
s1′. if $R/Sa(de)c$ then $R/Sd(ae)c$
r5′. if $S^2b(de)a$ then $S^2d(be)a$
s4. if $Rabc$ then Rac^*b^*
s7. if $Sbca$ then Ra^*bc^*
e5. if S^2decT^* and Ce then $d \leqslant c^*$
s10. if $SbcT^*$ then for some x $Sxxc^*$ and $b \leqslant x$
ds3. $(Px, y, z)(Raxy \mathbin{\&} Ra^*xz \mathbin{\&} AGx \mathbin{\&} y \leqslant z^*$
dr2. $STTT$

Adequacy Theorem for $\boldsymbol{P_{A1}}$. *Wff A is a theorem of $\boldsymbol{P_{A1}}$ iff A is $\boldsymbol{P_{A1}}$ valid.*

Proof is a variation of earlier results; the only fresh detail is that concerning e5 and s10.

ad s10. Suppose $SbcT^*$ and define $x = \{A\colon A \bullet A \in c^*\}$. Then $Sxxc^*$, Also x is a $\boldsymbol{P_{A1}}$ theory. For suppose $A \in x$ and $\vdash \boldsymbol{P_{A1}} A \to B$. Then, by Factor, $\vdash A \bullet A \to B \bullet A$ and $\vdash B \bullet A \to B \bullet B$, so $\vdash A \bullet A \to B \bullet B$ As $A \in x$, $A \bullet A \in c^*$; but c^* is a $\boldsymbol{P_{A1}}$ theory, so $B \bullet B \in c^*$, whence $B \in x$. To show $b \subseteq x$, suppose $A \in b$. By A_9, $A \bullet \sim(A \bullet A) \notin T^*$, so $A \notin b$ or $\sim(A \bullet A) \in c$. Thus $A \bullet A \in c^*$, i.e. $A \in x$ as required.

But the modelling, though formally adequate, is not intuitively very satisfying: as it stands the modelling is rather complex, with the modelling conditions exceeding in number the postulates they model, and basic connexive postulates like Boethius, instead of being validated in a natural way, have fairly intractable conditions.

§7. Two reductions of connexivism and further shortcomings of the semantics. One of the constant trials confronting exponents of non-classical logics is that of reduction: reduction of the espoused systems to something else, at worst to classical precepts. In the connexivists' case full classical reduction would be a fate worse than death, since total triviality would be the aftermath. But lesser reductions remain a worry, e.g. modal reductions, as with Meyer's reduction of McCall's first degree connexive logic strict implication (see [4]).

The semantics given suggests two reductions, of S to R or vice versa, both of which should also be resisted. The first — which would make connexive logic *a*, perhaps bizarre, branch of relevant logic — simply supposes that $Sabc$ iff $Rabc$. Such a requirement models exactly the two-way rule

$$A \bullet B \to C \leftrightarrow A \to . B \to C \qquad \text{(Portation)}$$

or equivalently the axioms: $A \to B \bullet A \to B$ and $A \to . B \to A \bullet B$. But all these principles are unacceptable, as [3] tries to explain. If the reduction were to succeed, connexive conjunction would be nothing but the fusion connective of relevant logics, and modern connexivism would be a study within the implication-fusion-negation part of relevant logics.

The second reduction supposes that generally $Sbca$ iff Ra^*bc^*. The axioms these conditions model are the questionable $A \bullet B \to {\sim}(A \to {\sim}B)$ and the unacceptable ${\sim}(A \,{\sim}B) \to A \bullet B$. That is, in a connexive logic with an adequate biconditional — many connexive logics are deficient in this department — Ra^*bc^* iff $Sbca$ corresponds exactly to ${\sim}(A \to {\sim}B) \leftrightarrow A \bullet B$, i.e. $A \bullet B \leftrightarrow A \circ B$ and $A \bullet B \leftrightarrow {\sim}(A/B)$. The last interconnection, together with $A \to B \leftrightarrow . A/{\sim}B$, suggests trying to refound this "classical" connexivism — it does assert $A \to B \leftrightarrow {\sim}(A \,{\sim}B)$, so that classical logic has the conditional correctly defined, even if it went a little astray in its extensional axiomatisation of conjunction — on incompatibility as main, or sole, primitive. Primitives have already been reduced to the pleasantly negative pair, incompatibility and negation: but attempts to reduce negation to incompatibility appear to founder on fundamental connexive assumptions. Consider Sheffer's proposal: ${\sim}A =_{Df} A/A$, negation is self-incompatibility. By Aristotle, no statement is self-incompatible, so Sheffer's proposal would trivialise connexivism, every statement would hold true.

Within the confines of incompatibility sentential logic it is hard — perhaps impossible — to find connexively plausible reductions of negation, but given sentential quantifiers or constants prospects looks brighter. For example, given a constant t, representing the True, it is tempting to define ${\sim}A$ as A/t. Then Aristotle becomes $(A/A)/t$, which need not lead to disaster. (Alternatively, when implication alone is taken as primitive, and conjunction, defined, negation can be defined in minimal logic style thus ${\sim}A =_{Df} A \to f$, with f a constant representing the False.) But the reduction, although it can provide an alternative semantics for "classical" connexive logics (and also for other connexive logics), does not ease the semantical problems negation causes in connexive logics, unless a new beginning is made, e.g. with a new evaluation rule for compatibility which somehow delivers Aristotle directly and without the need for devices such as the generation relation G. (But even a recasting of the compatibility rule as $I(A \circ B, a) = 1$ iff $(Pb, c)(Tabc \;\&\; I(A, b) = 1 = I(B, c))$ still appears to require use of G in modelling $A \circ A$: the requisite condition is $(Pb)(AGb \;\&\; Tabb)$.)

The chief reason for dissatisfaction with the semantics furnished for narrow connexive logics is not, however, that Aristotle and Boethius are modelled, so to speak, by specially invoked semantical trickery which too closely parallels the syntax, but that the semantics does not correspond to, or satisfactorily reflect, semantical ideas underlying connexivist thinking and much traditional logic. For example, it does not link with the traditional theory of positive and negative propositions, or with the idea underlying the connexivist rejection of $A \;\&\; {\sim}A \to A$ and $A \;\&\; {\sim}A \to {\sim}A$, that a negative proposition cancels out or deletes its positive, so that $A \;\&\; {\sim}A$, which says zero as regards A, says less than either A or ${\sim}A$ and so can not imply them — a point which explains, in a way the semantics does not, the connexive rejection of $A \;\&\; {\sim}A \to {\sim}(A \;\&\; {\sim}A)$, and so how simple counterexamples to Aristotle are dealt with.

Although the semantical analysis resolves many problems concerning connexive logics and provides new means of attacking other problems, it leaves many problems — perhaps too many — open. The matter of decidability of strong logics, already mentioned, is one. Other problems are those of consistency and non-triviality. Semantical modellings of the sort given provide no quarantee that the logics modelled are consistent or even other than trivial. The system $R + \{{\sim}(A \to {\sim}A)\}$ which was modelled is trivial, and very many of the systems which were modelled are inconsistent, e.g. $B + \{{\sim}(A \to {\sim}A)\}$. ($R$ and B are standard relevant logics, studied e.g. in [3].) The modellings given do simplify the task of establishing consistency or non-triviality where these properties hold; for example the modellings yield matrices which can then be routinely tested by computer. But it would be nice to obtain a somewhat more general and routine procedure for detecting inconsistency and triviality, e.g. special classes of models, which like the models of classical model theory, ensure one or other of these properties. This is just one respect in which the modellings given fail — to a perhaps greater extent that other modellings, e.g. those of relevant logics — to direct choice of system among connexive logics.

§8. Other Outstanding issues. The semantics furnished do not cater for Nelson's original connexive logic (of [9] and elsewhere) since Nelson's system lacks the basic conjunction rule, Praeclarum, and equivalents such as Rule Factor. In a sequel the connexive semantics will be enlarged to cope with systems like Nelson's. Also an attempt will be made to approximate more closely to the underlying intuitive semantics for certain of the connexive systems already treated.

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