Dynamics and Control, 6, 131–142 (1996)

Performance Analysis of Controlled Uncertain Systems

Y. H. CHEN yehwa.chen@me.gatech.edu
The George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA

Editor: W. Schmitendorf

Received May 5, 1994; Revised November 14, 1994

Abstract. Robust control which is designed via the Lyapunov approach has been shown to be effective for nonlinear uncertain systems. The performance of the controlled systems is studied by the Lyapunov argument. We propose to use the comparison principle which is based on the differential inequality to further explore the performance of controlled uncertain systems.

1. Introduction

The quest for robust control that is able to suppress (possibly fast) time-varying uncertainty in nonlinear systems has been one major theme in control society for the past two decades. One approach by which the control is designed and the resulting performance system is analyzed is often referred to as the *Lyapunov approach.* Two recent surveys on works in this area and relevant historical background are [7] and [15].

The major controlled system performance that has been proven so far, which includes practical stability and others, is *deterministic* in its nature in the sense that one is able to prescribe the performance regardless of the true value of uncertainty. The proof has been mainly based on the Lyapunov argument.

In this article, we propose to adopt the *comparison principle* which is based on *differential inequalities* to further explore deterministic properties of the controlled systems. This can be used to supplement the Lyapunov argument. To specifically demonstrate the use of the comparison principle, we work out a few examples which essentially cover all past work in this area.

The Lyapunov approach used for control design can lead to a differential inequality. Its solution is often a valid "metric" of the system's (worst case) performance. However, there is no available tool to obtain the solution directly. The comparison principle intends to use the solution of a differential equation to provide an upper bound for the solution of this differential inequality. If the initial value problem associated with the differential equation has a *unique* solution, then it can indeed serve as this upper bound. This in turn means that through the use of this comparison principle, one is able to obtain a good estimate of the upper bound of the system performance and hence *further* explore the deterministic properties of the controlled system.

Two remarks are in order. First, the comparison principle should be a supplement rather than a replacement of the Lyapunov argument. This is since one is already able to draw the major system performance (such as practical stability) directly from the Lyapunov argument. On the other hand, there is an extra layer of technicality, namely, the need of the (preferably analytic) solution of a differential equation, involved in using the comparison principle.

Second, the comparison principle is only based on a differential inequality and the related differential equation. Hence there is no way to distinguish the performance of two different controlled systems if their differential inequalities are the same.

2. Uncertain Systems

Consider the following class of uncertain dynamical systems

$$\dot{x}(t) = F(x(t), u(t), t, \omega), \qquad x(t_0) = x_0, \tag{2.1}$$

where $t \in \mathbf{R}$ is the "time" (or more adequately: the independent variable), $x(t) \in \mathbf{R}^n$ is the state, and $u(t) \in \mathbf{R}^m$ is the control. The system uncertainty is represented by the lumped uncertain element $\omega \in \Omega$. The only information assumed about ω is the knowledge of a nonempty set Ω to which it belongs. The uncertain element ω can represent a number of possibilities [16]. It can be an element of $\mathbf{R}^q$ representing constant unknown parameter and input disturbance; it can be a function from $\mathbf{R}$ to $\mathbf{R}^q$ representing unknown time-varying parameter and input disturbance; it can be a function from $\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}$ into $\mathbf{R}^m$ representing nonlinear elements which are difficult to characterize exactly; it can be merely an index; it can also be a combination of all. The function $F : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}^q \rightarrow \mathbf{R}^n$ is continuous.

The control task is to design u, which is possibly state or output dependent, that renders the state $x(t)$ certain desirable performance. A typical example of such performance is the *practical stability* [5] which includes existence and continuation of solutions, uniform boundedness, uniform ultimate boundedness, and uniform stability. The performance is *deterministic* in the sense that it is guaranteed regardless of the true value of the uncertain element ω.

The *Lyapunov approach* has been a major vehicle for control design for systems characterized by (2.1). For a review of the past work in this area, see [7] and [15]. We note that although the control design method in some work involves other tools such as the Riccati equation [19], the argument that leads to the system performance is still made via the Lyapunov "tongue." Thus it is considered appropriate to label this approach this way.

By using this Lyapunov approach, there have been numerous controls proposed for uncertain systems under different structural and uncertainty conditions. Besides their differences at technical levels, generally speaking, the Lyapunov approach starts with the choice of a legitimate Lyapunov function candidate $V(x, t)$. Very often it is sufficient if there are **KR**-functions of $\|x\|$ that bound $V(x, t)$ from below and above.[1] A control function is then selected. For any given trajectory $x(\cdot)$ of the controlled system, let $v(t) = V(x(t), t)$. The time derivative of $v(t)$ is then analyzed. The control function should contribute a negative term to the Lyapunov derivative. It can be further "fine-tuned" to address a number of issues/design criteria. These may include, for example, the actuator's bandwidth, the prescribed control bound, the desirable ultimate boundedness region, and the specific convergence rate. In general, the derivative of $v(t)$ is upper bounded by the following expression

$$\dot{v}(t) \leq \gamma(v(t), t), \qquad v(t_0) = v_0, \tag{2.2}$$

where $\gamma(\cdot)$ is continuous. Ineq. (2.2) is a *differential inequality* [13]. All previous work in this area falls into this expression.

Remark. In some early work [12], the Lyapunov approach is further labeled as the *minimax* or *min-max* approach. The reason was that the control design was often carried out in two steps. First, one *maximized* the worst case effect of the uncertainty with respect to the Lyapunov derivative. Second, one *minimized* this maximum effect by an appropriate choice of the control function. It was however later found that a *mixed* (rather than sequential) analysis of the uncertainty and control in the Lyapunov derivative might lead to a less conservative and hence more general result. This is especially true when there is input matrix uncertainty. As a result, one was able to relax the assumption from [2]

$$\max_{\omega \in \Omega} \|E(\omega, x, t)\| < 1 \tag{2.3}$$

to [15]

$$\min_{\omega \in \Omega} \lambda_{\min} \frac{1}{2}[E(\omega, x, t) + E^T(\omega, x, t)] =: \rho_E(x, t) > -1. \tag{2.4}$$

Here $E(\omega, x, t)$ stands for the input matrix uncertainty. Thus (2.4) enables one to tolerate more input matrix uncertainty.

Remark. The (often non-unique) solution $v(t)$ of (2.2), although a good "metric" of system performance, is however usually not available. There are no existing (systematic) methods to solve for $v(t)$ of (2.2).

Example 1. In the work of Leitmann [14], Chen [4], and Garofalo and Leitmann [10], it was shown that for linear uncertain systems under a few different classes of controls,

$$\dot{v}(t) \leq -\alpha_2 \|x(t)\|^2 + \alpha_1 \|x(t)\| + \alpha_0 \tag{2.5}$$

where $\alpha_{0,1,2} > 0$ are constants. Furthermore, there are constants $\beta_{1,2} > 0$ such that $\beta_1 \|x(t)\|^2 \leq v(t) \leq \beta_2 \|x(t)\|^2$. Thus $\|x(t)\|^2 \geq \beta_2^{-1} v(t)$ or $-\|x(t)\|^2 \leq -\beta_2^{-1} v(t)$. Besides, $\|x(t)\| \leq \beta_1^{-1} v^{\frac{1}{2}}(t)$. Therefore

$$\begin{aligned} \dot{v}(t) &\leq -\alpha_2 \beta_2^{-1} v(t) + \alpha_1 \beta_1^{-1} v^{\frac{1}{2}}(t) + \alpha_0 \\ &=: \gamma(v(t), t). \end{aligned} \tag{2.6}$$

The practical stability of the controlled system has been proven based on (2.5) [14]. Garofalo and Leitmann [10] also studied the upper bound of the finite time it takes for the state to be settled within the uniform ultimate boundedness region. This bound of the finite time is much less conservative than the one first studied in [14].

Example 2. In the work of Corless and Leitmann [8], Barmish *et al.* [2], Ambrosino *et al.* [1], and Pandey *et al.* [18], it was shown that, under various controls and uncertainty conditions, one could reach

$$\dot{v}(t) \leq -\gamma_3(\|x(t)\|) + k \tag{2.7}$$

where $\gamma_3(\cdot)$ is a **KR**-function, $k > 0$ is a constant. There are also **KR**-functions $\gamma_{1,2}(\cdot)$ such that $\gamma_1(\|x(t)\|) \leq v(t) \leq \gamma_2(\|x(t)\|)$. Thus $\gamma_2^{-1}(v(t)) \leq \|x(t)\|$ and $-\gamma_3(\gamma_2^{-1}(v(t)) \geq -\gamma_3(\|x(t)\|)$. We then have

$$\begin{aligned} \dot{v}(t) &\leq -\gamma_3(\gamma_2^{-1}(v(t))) + k \\ &=: \gamma(v(t), t). \end{aligned} \tag{2.8}$$

Example 3. In Corless and Leitmann [9], it was shown that if the bound of uncertainty was known and hence no adaptation was needed, then

$$\dot{v}(t) \leq -\gamma_3(\|x(t)\|) + \epsilon_0 e^{-l(t-t_0)} \tag{2.9}$$

where $\epsilon_0 > 0$, $l > 0$ are both constants. There are also **KR**-functions $\gamma_{1,2}(\cdot)$ that lower and upper bound $v(t)$. Therefore

$$\begin{aligned} \dot{v}(t) &\leq -\gamma_3(\gamma_2^{-1}(v(t))) + \epsilon_0 e^{-l(t-t_0)} \\ &=: \gamma(v(t), t). \end{aligned} \tag{2.10}$$

Example 4. In Chen and Leitmann [5] and Han and Chen [11], it was shown that either under mismatched uncertainty and/or polynomial-type controls,

$$\dot{v}(t) \leq -\gamma_3(\|x(t)\|) + \gamma_4(\|x(t)\|) + \gamma_5 \tag{2.11}$$

where $\gamma_{3,4}(\cdot)$ are of class **KR** and γ_5 is a constant. (Han and Chen [11] specifically considered that $\gamma_{3,4}(\cdot)$ were both of polynomial types). There are also **KR** functions $\gamma_{1,2}(\cdot)$ that bound $v(t)$. Then

$$\begin{aligned} \dot{v}(t) &\leq -\gamma_3(\gamma_2^{-1}(v(t))) + \gamma_4(\gamma_1^{-1}(v(t))) + \gamma_5 \\ &=: \gamma(v(t), t). \end{aligned} \tag{2.12}$$

3. Differential Inequality and Comparison Principle

We now study the differential inequality (2.2) to further explore the controlled system performance.

Definition 1. [13] If $w(\psi, t)$ is a scalar function of the scalars ψ, t in some open connected set $\mathcal{D}$, we say a function $\psi(t)$, $t_0 \leq t \leq \bar{t}$, $\bar{t} > t_0$ is a solution of the differential inequality

$$\dot{\psi}(t) \leq w(\psi(t), t) \tag{3.1}$$

on $[t_0, \bar{t})$ if $\psi(t)$ is continuous on $[t_0, \bar{t})$ and its derivative on $[t_0, \bar{t})$ satisfies (3.1).

THEOREM 1 [13] *Let $w(\phi, t)$ be continuous on an open connected set $\mathcal{D} \in \mathbf{R}^2$ and such that the initial value problem for the scalar equation*

$$\dot{\phi}(t) = w(\phi(t), t), \qquad \phi(t_0) = \phi_0 \tag{3.2}$$

has a unique solution. If $\phi(t)$ is a solution of (3.2) on $t_0 \leq t \leq \bar{t}$ and $\psi(t)$ is a solution of (3.1) on $t_0 \leq t < \bar{t}$ with $\psi(t_0) \leq \phi(t_0)$, then $\psi(t) \leq \phi(t)$ for $t_0 \leq t \leq \bar{t}$.

Instead of exploring the solution of the differential inequality (3.1), which is often not feasible, Theorem 1 suggests to study the upper bound of this solution instead. The theorem can be applied directly to study (2.2).

THEOREM 2 *Consider the differential inequality (2.2). Consider also the scalar equation*

$$\dot{r}(t) = \gamma(r(t), t), \qquad r(t_0) = v_0. \tag{3.3}$$

Suppose that $\gamma(\cdot)$ is continuous on an open connected set $\mathcal{D} \in \mathbf{R}^2$. Suppose also that for some constant $L > 0$, the function $\gamma(\cdot)$ satisfies a Lipschitz condition

$$|\gamma(v_1, t) - \gamma(v_2, t)| \leq L|v_1 - v_2| \tag{3.4}$$

for all points (v_1, t), $(v_2, t) \in \mathcal{D}$. Then any function $v(t)$ that satisfies the differential inequality (2.2) for $t_0 \leq t < \bar{t}$ satisfies also the inequality

$$v(t) \leq r(t) \tag{3.5}$$

for $t_0 \leq t \leq \bar{t}$.

Proof: Since the function $\gamma(\cdot)$ satisfies a Lipschitz condition, the initial value problem of $\dot{r} = \gamma(r, t)$, $r(t_0) = v_0$ has a unique solution. The result follows by using Theorem 1. ■

Remark. The main result of this theorem, which can be described as the *comparison principle* [3, 20] since it invokes the comparison of two solutions, can be used to supplement the Lyapunov analysis used in other literature (see, e.g., [8]) for analyzing the system performance. Since (3.3) is a first-order ordinary differential equation, it is often possible to solve it analytically. This is considered advantageous to study the performance of the controlled system, which is related to $v(t)$.

Remark. The differential equation (3.3) can be directly constructed by the differential inequality (2.2) which is the direct consequence of the Lyapunov analysis. Hence all one needs to use the comparison principle is the upper bound of the Lyapunov derivative. However, we stress that the uniqueness requirement should be observed for otherwise there is no existing proof which shows $v(t) \leq r(t)$. That is, it is possible that $v(t) > r(t)$ if the solution of (3.3) is non-unique.

Remark. Suppose the solution r is given by, in the general form,

$$r = r(t; v_0, t_0). \tag{3.6}$$

Suppose also that $\gamma_1(\|x(t)\|) \leq v(t)$ where $\gamma_1(\cdot)$ is of class **KR**, then $\|x(t)\| \leq \gamma_1^{-1}(v(t))$ and by $v(t) \leq r(t)$, we have

$$\|x(t)\| \leq \gamma_1^{-1}(r(t; v_0, t_0)). \tag{3.7}$$

This shows the upper bound of $\|x(t)\|$. The controlled system performance can be estimated by using this bound.

4. Case Study—I

Consider Example 1. The differential equation

$$\dot{r}(t) = -\alpha_2\beta_2^{-1}r(t) + \alpha_1\beta_1^{-1}r^{\frac{1}{2}}(t) + \alpha_0, \quad r(t_0) = v_0, \tag{4.1}$$

belongs to the second type of the Abel equation [21]. The right-hand side satisfies the Lipschitz condition

$$\begin{aligned}
&\left|(-\alpha_2\beta_2^{-1}r_1 + \alpha_1\beta_1^{-1}r_1^{1/2} + \alpha_0) - (-\alpha_2\beta_2^{-1}r_2 + \alpha_1\beta_1^{-1}r_2^{1/2} + \alpha_0)\right| \\
&\quad \leq \max_{r \geq \underline{r} > 0} \left|\alpha_2\beta_2^{-1} + \frac{\alpha_1\beta_1^{-1}}{2}r^{-1/2}\right| |r_1 - r_2| \\
&\quad =: L|r_1 - r_2|
\end{aligned} \tag{4.2}$$

where $L < \infty$ and $\underline{r} > 0$ is any constant. We now only consider $v_0 > \underline{r}$ and $r(t)$ for all $t \in \mathcal{T}$ where $\mathcal{T}$ is the time internal within which $r(t) \geq \underline{r}$. The solution of (4.1) is given by [17]

$$r(t) = \tilde{r}^2(t), \tag{4.3}$$

where $\tilde{r}(t)$ is such that

$$(\tilde{r}(t) - r_a)^{r_a} = C(\tilde{r}(t) - r_b)^{r_b} \exp[(r_b - r_a)(t - t_0)], \tag{4.4}$$

$$r_a := \frac{\alpha_1\beta_1^{-1} + \sqrt{\alpha_1^2\beta_1^{-2} + 4\alpha_2\beta_2^{-1}\alpha_0}}{2\alpha_2\beta_2^{-1}}, \tag{4.5}$$

$$r_b := \frac{\alpha_1\beta_1^{-1} - \sqrt{\alpha_1^2\beta_1^{-2} + 4\alpha_2\beta_2^{-1}\alpha_0}}{2\alpha_2\beta_2^{-1}}, \tag{4.6}$$

$$C := \frac{(v_0^{1/2} - r_a)^{r_a}}{(v_0^{1/2} - r_b)^{r_b}}. \tag{4.7}$$

For any $\xi > 0$, $\tilde{r}(t) - r_a < \xi$ for all $t \geq t_0 + T(v_0, \xi)$ where

$$T(v_0, \xi) = \frac{1}{r_a - r_b}\ln\frac{C(r_a - r_b + \xi)^{r_b}}{\xi^{r_a}}. \tag{4.8}$$

This in turn will provide an alternative estimation of the finite entering time for uniform ultimate boundedness (cf. [14, 10]).

5. Case Study—II

Consider the case in Example 2 with

$$\gamma_1(\|x\|) = \hat{\gamma}_1\|x\|^p,\ \gamma_2(\|x\|) = \hat{\gamma}_2\|x\|^p,\ \gamma_3(\|x\|) = \hat{\gamma}_3\|x\|^p, \tag{5.1}$$

where $\hat{\gamma}_{1,2,3} > 0$, $p > 0$ are constants. A similar problem is also considered in other literature (see, e.g., [6]). The differential equation used in the comparison principle is given by

$$\dot{r}(t) = -\frac{\hat{\gamma}_3}{\hat{\gamma}_2}r(t) + k, \quad r(t_0) = v_0. \tag{5.2}$$

Note that $\gamma(\cdot)$ is globally Lipschitz. The solution of (5.2) is give by

$$r(t) = \left(v_0 - \frac{\hat{\gamma}_2}{\hat{\gamma}_3}k\right)\exp\left[-\frac{\hat{\gamma}_2}{\hat{\gamma}_3}(t-t_0)\right] + \frac{\hat{\gamma}_2}{\hat{\gamma}_3}k. \tag{5.3}$$

Therefore,

$$\begin{aligned}\|x(t)\|^p &\le \hat{\gamma}_1^{-1}r(t)\\ &= \frac{1}{\hat{\gamma}_1}\left(v_0 - \frac{\hat{\gamma}_2}{\hat{\gamma}_3}k\right)\exp\left[-\frac{\hat{\gamma}_2}{\hat{\gamma}_3}(t-t_0)\right] + \frac{\hat{\gamma}_2}{\hat{\gamma}_1\hat{\gamma}_3}k\end{aligned} \tag{5.4}$$

or

$$\|x(t)\|^p - \frac{\hat{\gamma}_2}{\hat{\gamma}_1\hat{\gamma}_3}k \le \frac{1}{\hat{\gamma}_1}\left(v_0 - \frac{\hat{\gamma}_2}{\hat{\gamma}_3}k\right)\exp\left[-\frac{\hat{\gamma}_2}{\hat{\gamma}_3}(t-t_0)\right] \tag{5.5}$$

for all $t \ge t_0$. Let

$$\eta := \left(\frac{\hat{\gamma}_2}{\hat{\gamma}_1\hat{\gamma}_3}\right)^{1/p} \tag{5.6}$$

Then if $v_0 > (\hat{\gamma}_2/\hat{\gamma}_3)k$, one can guarantee that $\|x(t)\|^p \to \eta^p$ exponentially where the rate of convergence is $\hat{\gamma}_2/\hat{\gamma}_3$. Furthermore, for any $\kappa > 0$, there exists

$$T = \frac{\hat{\gamma}_3}{\hat{\gamma}_2}\ln\left(\frac{\frac{\hat{\gamma}_2}{\hat{\gamma}_1\hat{\gamma}_3}k + \kappa}{\frac{\hat{\gamma}_2}{\hat{\gamma}_1\hat{\gamma}_3}k}\right) \tag{5.7}$$

such that

$$\left(\frac{\hat{\gamma}_2}{\hat{\gamma}_1\hat{\gamma}_3}k + \kappa\right)\exp\left[-\frac{\hat{\gamma}_2}{\hat{\gamma}_3}(t-t_0)\right] \ge \frac{\hat{\gamma}_2}{\hat{\gamma}_1\hat{\gamma}_3}k \tag{5.8}$$

for all $t \in [t_0, t_0 + T]$. Thus if $v_0 > (\hat{\gamma}_2/\hat{\gamma}_3)k$, then

$$\begin{aligned}&\frac{1}{\hat{\gamma}_1}\left(v_0 - \frac{\hat{\gamma}_2}{\hat{\gamma}_3}k\right)\exp\left[-\frac{\hat{\gamma}_2}{\hat{\gamma}_3}(t-t_0)\right] + \left(\frac{\hat{\gamma}_2}{\hat{\gamma}_1\hat{\gamma}_3}k + \kappa\right)\exp\left[-\frac{\hat{\gamma}_2}{\hat{\gamma}_3}(t-t_0)\right]\\ &\quad\ge \frac{1}{\hat{\gamma}_1}\left(v_0 - \frac{\hat{\gamma}_2}{\hat{\gamma}_3}k\right)\exp\left[-\frac{\hat{\gamma}_2}{\hat{\gamma}_3}(t-t_0)\right] + \frac{\hat{\gamma}_2}{\hat{\gamma}_1\hat{\gamma}_3}k\end{aligned} \tag{5.9}$$

for all $t \in [t_0, t_0 + T]$. This is equivalent to saying that

$$\left[\frac{1}{\hat{\gamma}_1}\left(v_0 - \frac{\hat{\gamma}_2}{\hat{\gamma}_3}k\right) + \frac{\hat{\gamma}_2}{\hat{\gamma}_1\hat{\gamma}_3}k + \kappa\right]\exp\left[-\frac{\hat{\gamma}_2}{\hat{\gamma}_3}(t - t_0)\right] \geq \|x(t)\|^p \tag{5.10}$$

or

$$\left[\frac{1}{\hat{\gamma}_1}\left(v_0 - \frac{\hat{\gamma}_2}{\hat{\gamma}_3}k\right) + \frac{\hat{\gamma}_2}{\hat{\gamma}_1\hat{\gamma}_3}k + \kappa\right]^{1/p}\exp\left[-\frac{\hat{\gamma}_2}{p\hat{\gamma}_3}(t - t_0)\right] \geq \|x(t)\| \tag{5.11}$$

for all $t \in [t_0, t_0 + T]$. Therefore $x(t)$ exponentially converges to 0 for any $t \in [t_0, t_0 + T]$. The rate of convergence is $\hat{\gamma}_2/p\hat{\gamma}_3$. Since T is also the finite entering time for uniform ultimate boundedness, this implies that $x(t)$ enters the region exponentially. Note however there is no guarantee that $x(t) \to 0$ (either exponentially or not) for all $t \geq t_0$.

6. Case Study—III

Consider again the case in Example 2 with $\gamma_1(\|x\|) = \hat{\gamma}_1\|x\|^p$, $\gamma_2(\|x\|) = \hat{\gamma}_2\|x\|^p$, $\gamma_3(\|x\|) = \hat{\gamma}_3\|x\|^{2p}$, where $\hat{\gamma}_{1,2,3}$, $p > 0$. Therefore,

$$\gamma_3(\gamma_2^{-1}(v)) = \frac{\hat{\gamma}_3}{\hat{\gamma}_2^2}v^2. \tag{6.1}$$

The differential equation

$$\dot{r}(t) = -\frac{\hat{\gamma}_3}{\hat{\gamma}_2^2}r^2(t) + k, \qquad r(t_0) = v_0 \tag{6.2}$$

belongs to the scalar Riccati equation type [21]. Since the right-hand side of (6.2), which is continuous, is negative for any $r > \sqrt{\frac{k\hat{\gamma}_2^2}{\hat{\gamma}_3}}$, the solution remains finite. Let us denote the region in which r will remain by $\mathcal{R}$. The right-hand side then satisfies the Lipschitz condition

$$\left|\left(-\frac{\hat{\gamma}_3}{\hat{\gamma}_2^2}r_1^2 + k\right) - \left(-\frac{\hat{\gamma}_3}{\hat{\gamma}_2^2}r_2^2 + k\right)\right| \leq L|r_1 - r_2| \tag{6.3}$$

where

$$L := \max_{r_1, r_2 \in \mathcal{R}} \frac{\hat{\gamma}_3}{\hat{\gamma}_2^2}|r_1 + r_2| < \infty. \tag{6.4}$$

The solution of (6.2) is given by [17]

$$r(t) = \frac{k}{a}\frac{\exp\{2[\hat{C} + a(t - t_0)]\} - 1}{\exp\{2[\hat{C} + a(t - t_0)]\} + 1} \tag{6.5}$$

where

$$a = \sqrt{\frac{\hat{\gamma}_3 k}{\hat{\gamma}_2^2}}, \qquad \hat{C} = \frac{a v_0 \left(\frac{-\hat{\gamma}_2^2}{\hat{\gamma}_3}\right) + 1}{a v_0 \left(\frac{-\hat{\gamma}_2^2}{\hat{\gamma}_3}\right) - 1}. \tag{6.6}$$

The state $x(t)$ is then upper bounded by

$$\|x(t)\| \le \left(\frac{\hat{\gamma}_2\sqrt{k}}{\hat{\gamma}_1\sqrt{\hat{\gamma}_3}}\right)^{1/p} \left\{\frac{\exp\{2[\hat{C} + a(t-t_0)]\} - 1}{\exp\{2[\hat{C} + a(t-t_0)]\} + 1}\right\}^{1/p} \tag{6.7}$$

The right-hand side of (6.7) approaches $\left(\frac{\hat{\gamma}_2\sqrt{k}}{\hat{\gamma}_1\sqrt{\hat{\gamma}_3}}\right)^{1/p}$ as $t \to \infty$.

7. Case Study—IV

Consider the case in Example 3 with $\gamma_1(\|x\|) = \hat{\gamma}_1\|x\|^p$, $\gamma_2(\|x\|) = \hat{\gamma}_2\|x\|^p$, $\gamma_3(\|x\|) = \hat{\gamma}_3\|x\|^p$, $\hat{\gamma}_{1,2,3}$, $p > 0$. The solution of the differential equation (by letting $\delta = \hat{\gamma}_3/\hat{\gamma}_2$)

$$\dot{r}(t) = -\delta r(t) + \epsilon_0 e^{-l(t-t_0)}, \qquad r(t_0) = v_0, \tag{7.1}$$

which is unique since the right-hand side of (7.1) is globally Lipschitz, is given by

$$r(t) = \begin{cases} e^{-\delta(t-t_0)}\left(v_0 - \frac{\epsilon_0}{\delta - l}\right) + e^{-l(t-t_0)}\frac{\epsilon_0}{\delta - l}, & \text{if } \delta \ne l, \\ e^{-\delta(t-t_0)}v_0 + e^{-\delta t}e^{l t_0}\epsilon_0(t - t_0), & \text{if } \delta = l. \end{cases} \tag{7.2}$$

This in turn shows the exponential convergence of $x(t)$ to 0 for all $t \ge t_0$ if $\delta \ne l$. The rate of convergence is

$$\underline{\delta} = \frac{1}{p}\min\{\delta, l\}. \tag{7.3}$$

The above statement can be easily checked since $\|x(t)\| \le \hat{\gamma}_1^{-1/p} r^{1/p}(t)$ and hence

$$\begin{aligned} \|x(t)\| &\le \frac{1}{\hat{\gamma}_1^p} e^{-\underline{\delta}(t-t_0)} \left[e^{-(\frac{\delta}{p} - \underline{\delta})(t-t_0)} \left(v_0 - \frac{\epsilon_0}{\delta - l}\right)^{1/p} \right. \\ &\quad \left. + e^{-(\frac{l}{p} - \underline{\delta})(t-t_0)} \left(\frac{\epsilon_0}{\delta - l}\right)^{1/p} \right] \\ &\le \bar{L} e^{-\underline{\delta}(t-t_0)}, \end{aligned} \tag{7.4}$$

where $\bar{L} < \infty$ is a constant. When in most of the cases both p and δ are fixed, the maximum rate, that is, δ/p, is obtained by choosing $l > \delta$. We also note that the exponential convergence analysis does not hold for $l = \delta$.

Computer simulations were performed to demonstrate the behavior of $r(t)$. In (7.1), we choose $\epsilon_0 = 1$ and $l = 1.1$. All other coefficients in (4.1), (5.2), (6.2), and (7.1) are chosen to be 1. The initial condition $v_0 = 10$. The results are shown in Figure 1 (line 1 for (4.1), line 2 for (5.2), line 3 for (6.2), line 4 for (7.1)).

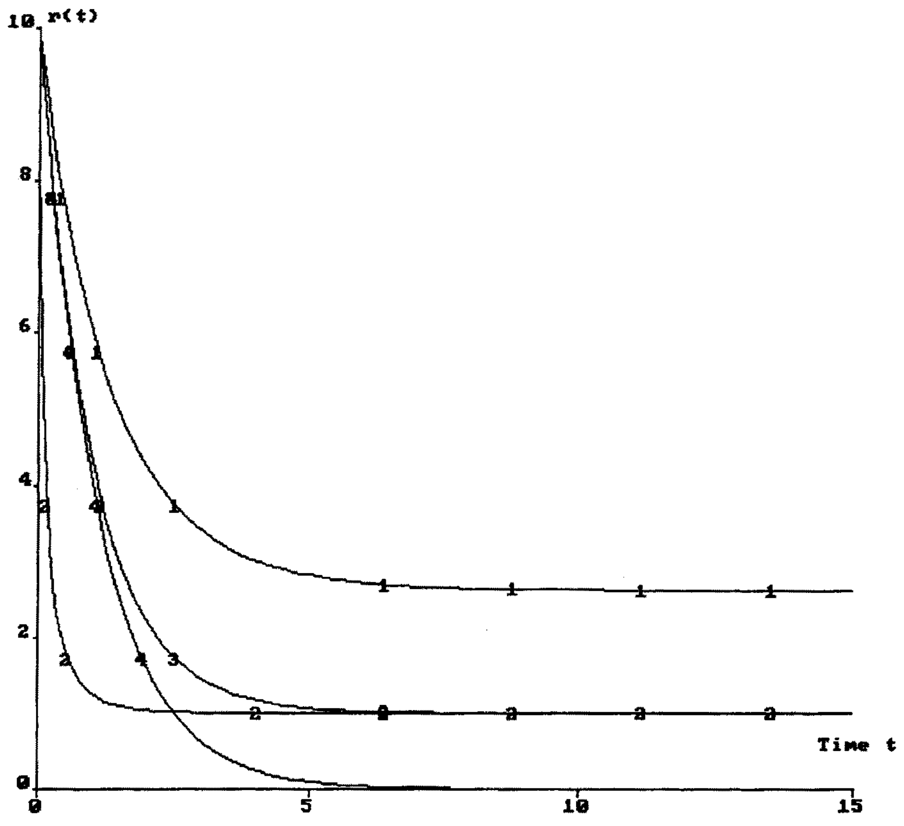

Figure 1. Behavior of $r(t)$.

8. Conclusions

The comparison principle is based on the differential inequality that is the result of the Lyapunov argument. Since the major performance result such as practical stability can be proven solely based on the Lyapunov argument, it seems reasonable to only anticipate that the comparison principle may supplement the Lyapunov argument. Furthermore, the use of the comparison principle specifically requires the initial value problem associated with the differential equation to have a *unique* solution, there may be domains of uncertain systems and controls that the comparison principle can not reach while the Lyapunov argument can. On the other hand, the technical level involved in using the comparison principle is usually a first-order ordinary differential equation whose closed-form solution is often available. The solution will help the designer to further explore the characteristics of the (especially transient) performance. Thus the current method is considered to be a valid addition to the Lyapunov approach.

Acknowledgments

The author is grateful to Professor George Leitmann for many helpful discussions and comments.

Notes

1. By a **KR**-function $\gamma : \mathbf{R}_+ \to \mathbf{R}_+$, we mean $\gamma(0) = 0$, $\gamma(\cdot)$ is strictly increasing, and $\lim_{r\to\infty} \gamma(r) = \infty$.

References

1. Ambrosino, G., Celentano, G., and Garofalo, F., "Robust model tracking for a class of nonlinear plants," *IEEE Transactions on Automatic Control*, vol. 30, pp. 275–279, 1985.
2. Barmish, B. R., Corless, M., and Leitmann, "A new class of stabilizing controllers for uncertain dynamical systems," *SIAM Journal of Control and Optimization*, vol. 21, pp. 246–255, 1983.
3. Birkhoff, G., and Rota, G-C., *Ordinary Differential Equations*, Blaisdell: Waltham, Massachusetts, 1962.
4. Chen, Y. H., "On the deterministic performance of uncertain dynamical systems," *International Journal of Control*, vol. 43, no. 5, pp. 1557–1579, 1986.
5. Chen, Y. H., and Leitmann, G., "Robustness of uncertain systems in the absence of matching assumptions," *International Journal of Control*, vol. 45, pp. 1527–1542, 1987.
6. Corless, M., "Guaranteed rates of exponential convergence for uncertain systems," *Journal of Optimization Theory and Applications*, vol. 76, pp. 471–484, 1990.
7. Corless, M., "Control of uncertain nonlinear systems," *Journal of Dynamic Systems, Measurement, and Control*, vol. 115, pp. 362–372, 1993.
8. Corless, M. J., and Leitmann, G., "Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems," *IEEE Transactions on Automatic Control*, vol. 26, pp. 1139–1144, 1981.
9. Corless, M., and Leitmann, G, "Adaptive control of systems containing uncertain functions and unknown functions with uncertain bounds," *Journal of Optimization Theory and Applications*, vol. 41, pp. 155–168, 1983.
10. Garofalo, F., and Leitmann, G., "Guaranteeing ultimate boundedness and exponential rate of convergence for a class of nominally linear uncertain systems," *Journal of Dynamic Systems, Measurement, and Control*, vol. 111, pp. 584–589, 1989.
11. Han, M. C., and Chen, Y. H., "Polynomial robust control design for uncertain systems," *Automatica*, vol. 28, pp. 809–814, 1992.
12. Gutman, S., "Uncertain dynamical systems: a Lyapunov min-max approach," *IEEE Transactions on Automatic Control*, vol. 24, pp. 437-443, 1979; correction, vol. 25, p. 613, 1980.
13. Hale, J. K., *Ordinary Differential Equations*, Krieger: Huntington, 1969.
14. Leitmann, G., "On the efficacy of nonlinear control in uncertain linear systems," *Journal of Dynamic Systems, Measurement, and Control*, vol. 103, pp. 95–102, 1981.
15. Leitmann, G., "On one approach to the control of uncertain systems," *Journal of Dynamic Systems, Measurement, and Control*, vol. 115, pp. 373–380, 1993.
16. Leitmann, G., "One approach to the control of uncertain dynamical systems," *Sixth Workshop of Dynamics and Control*, Vienna, Austria, 1993.
17. Murphy, G.M., *Ordinary Differential Equations and Their Solutions*, D. Van Nostrand Company, Inc.: Princeton, New Jersey. 1960.
18. Pandey, S., Leitmann, G., and Corless, M., "A deterministic controller for a new class of uncertain systems," *Proceedings of the IEEE Conference on Decision and Control*, pp. 2615–2617, 1991.
19. Petersen, I. R., and Hollot, C. V., "A Riccati equation approach to the stabilization of uncertain linear systems," *Automatica*, vol. 22, no. 4, pp. 397–411, 1986.

20. Siljak, D. D., 1978, *Large-Scale Dynamical Systems: Stability and Structure*, North-Holland: New York, 1978.

21. Zwillinger, D., *Handbook of Differential Equations*, Second Edition, Academic Press: San Diego, 1992.