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ON SYSTEMS WITH SEPARATRIX CONTOUR CONTAINING TWO SADDLE-FOCI*

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1. Introduction

Let us consider a two-parameter family of C^r-smooth ($r \geq 3$) dynamical systems $X_\mu(x)$, $x \in \mathbf{R}^3$, $\mu \in \mathbf{R}^2$, which have, for $\mu = 0$, two isolated equilibrium states O_1 and O_2 of a saddle type. We assume that a system linearized at O_k has the spectrum $\sigma_k = \{\alpha_k \pm i\omega_k, \gamma_k\}$, where $\alpha_k\gamma_k < 0$, $\alpha_1\alpha_2 < 0$, $\omega_k \neq 0$, $k = 1, 2$, i.e., O_1 and O_2 are saddle-foci of different topological types.

Let Γ_i be a one-dimensional separatrix of the saddle-focus O_i, and let $W^s(O_1)$ and $W^u(O_2)$ be two-dimensional integral manifolds of the saddle-foci O_1 and O_2 respectively.

The class of systems under consideration is defined by the following additional conditions.

A. The system $X_0(x)$ has a structurally stable heteroclinic orbit Γ_0, which belongs to the transversal intersection of $W^s(O_1)$ with $W^u(O_2)$.

B. For $\mu = 0$, the one-dimensional separatrices Γ_1 and Γ_2 coincide and form a structurally unstable heteroclinic orbit Γ^0.

C. The quantity $\tau = \nu_2\omega_1/(\nu_1\omega_2) \neq 1$, where $\nu_i \equiv -\alpha_i/\gamma_i$ are saddle indices. The condition

$$G \equiv \nu_1^2/\omega_1^2 + \nu_2^2/\omega_2^2 \mp 2\nu_1\nu_2/(\omega_1\omega_2)\varepsilon + 1 - \varepsilon^2 \neq 0$$

is satisfied, where $\varepsilon > 1$ is a quantity that can be determined by solving the variational equation when integration is along the orbit Γ^0. It will be refined below.

By virtue of conditions A and B, systems with the properties described above have a separatrix contour $\mathcal{L} = O_1 \cup O_2 \cup \Gamma_0 \cup \Gamma^0$ in the phase space, and, by virtue of condition B, form a bifurcation set H^0 of codimension 2 in the space of dynamical systems with the C^3-topology.

On the basis of condition C, H^0 is divided into two connection components H_1^0 and H_2^0, defined by the conditions $G > 0$ and $G < 0$ respectively. It is known [6, 8] that already in the case of the homoclinic orbit Γ of the saddle-focus the set of orbits that lie entirely in the neighborhood of Γ has a very nontrivial structure if the saddle index of the saddle-focus is smaller than 1. Moreover, for systems with a separatrix loop the saddle index is a topological invariant [2]. However, in the case of a separatrix contour that contains a saddle and a saddle-focus, the complicated structure of the set of orbits that lie entirely in the neighborhood of this contour is revealed irrespective of the values of saddle indices [3, 9]. It is established in this paper that for the systems $X_0(x) \in H^0$ in the neighborhood $\mathcal{U}(\mathcal{L})$ of the contour $\mathcal{L}$, in addition to the orbits Γ_0 there also exists a countable set $\{\Gamma_{0i}\}$ of heteroclinic orbits Γ_{0i} along which $W^s(O_1)$ intersects $W^u(O_2)$ transversally. Moreover, H_2^0 systems with a structurally unstable heteroclinic orbit Γ_{0*} along which W_1^s touches W_2^u are everywhere dense. In addition to studying the indicated heteroclinic orbits, we deal with the simplest bifurcations connected with the splitting of the structurally unstable heteroclinic orbit Γ^0 into one-dimensional separatrices Γ_1 and Γ_2. In this way, the existence of two bifurcation curves l_1 and l_2 corresponding to the separatrices loops of the saddle-foci O_1 and O_2, respectively, and intersecting in the countable set $\{\mu^k\}$ of

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points μ^k is established. From this fact and the results of [6, 5, 10], it follows that for the saddle indices $\nu_1 < 1$ and $\nu_2 < 1$ and opposite signs of the divergence of the vector field at the saddle-foci (and this is equivalent to the fact that the saddle indices ν_i of the saddle-foci O_i simultaneously satisfy the same inequalities, namely, $1/2 < \nu_i < 1$ or $0 < \nu_i < 1/2$, $i = 1,2$), the values of parameters μ are everywhere dense in the neighborhood of every one of the points μ^k. For these parameters, there exist an attractor and a repeller in the system $X_\mu(x)$, which have a nonempty intersection of closures. One of them contains a countable set of periodic motions and the other contains a countable set of completely unstable periodic motions. It should be pointed out that the existence of a structurally unstable heteroclinic orbit Γ_{0*} along with the orbit Γ^0 in $\mathcal{U}(\mathcal{L})$ means the generation of codimension larger than 2, and this implies the impossibility of a complete investigation within the framework of a two-parameter family.

2. Constructing Successor Maps

A system of differential equations in the neighborhood of the equilibrium state of the saddle-focus type can be written as

$$\begin{aligned} \dot{x} &= -\nu x - \omega y + f_1 x + f_2 y, \\ \dot{y} &= \omega x - \nu y + g_1 x + g_2 y, \\ \dot{z} &= z, \end{aligned} \tag{2.1}$$

where $f_i, g_i \in C^{r-1}$ and $f_i(0,0,0) = g_i(0,0,0) = 0$, $i = 1,2$.

Lemma 2.1 [6]. *There exist changes of coordinates and time after which the functions f_i and g_i in system* (2.1) *satisfy the conditions*

$$f_i(x,y,0) = g_i(x,y,0) = f_i(0,0,z) = g_i(0,0,z) \equiv 0. \tag{2.2}$$

Let us now pass to cylindrical coordinates

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z.$$

Then system (2.1) assumes the form

$$\begin{aligned} \dot{\rho} &= -\nu\rho + F_1(\rho,\varphi,z), \\ \dot{\varphi} &= (\omega + F_2(\rho,\varphi,z))z, \\ \dot{z} &= z, \end{aligned}$$

where, by virtue of (2.2) and [1, 10], $F_1, F_2 \in C^{r-1}$ and $\lim\limits_{\rho\to 0} F_i(\rho,\varphi,z) = 0$, $F_i(\rho,\varphi,0) \equiv 0$. Since $\omega \neq 0$, it follows that in a sufficiently small neighborhood of O we can change the time $d\tau = (\omega + F_2)dt$ and pass, in this way, to the system

$$\begin{aligned} \dot{\rho} &= -\nu\rho/\omega + R(\rho,\varphi,z), \\ \dot{z} &= \omega^{-1}(1 + \Phi(\rho,\varphi,z))z, \\ \dot{\varphi} &= 1, \end{aligned} \tag{2.3}$$

where

$$\lim_{\rho\to 0} R(\rho,\varphi,z) = \lim_{\rho\to 0}\Phi(\rho,\varphi,z) \equiv 0, \quad R(\rho,\varphi,0) = \Phi(\rho,\varphi,0) \equiv 0, \quad \lim_{\rho\to 0}\frac{\partial R(\rho,\varphi,z)}{\partial\rho} \equiv 0. \tag{2.4}$$

Following [7], we shall consider the system of integral equations

$$\begin{aligned} \rho(t) &= \rho_0 e^{-\nu/\omega t} + \textstyle\int_0^t e^{-\nu(t-\tau)/\omega} R(\rho(\tau),\varphi(\tau),z(\tau))d\tau, \\ z(t) &= z_1 e^{(t-t_0)/\omega} + \textstyle\int_{t_0}^t e^{(t-t_0)/\omega}\Phi(\rho(\tau),\varphi(\tau),z(\tau))d\tau, \end{aligned} \tag{2.5}$$

where $t_0 > 0$, and by virtue of the last equation of system (2.4), $\varphi(t) = \varphi_0 + t$. We can show [7] that for $0 \le t \le t_0$ and for all sufficiently small ρ_0 and z_1 the system of integral equations (2.5) has the unique solution

$$\rho(t) = \tilde{\rho}(t,t_0,\rho_0,\varphi_0,z_1), \quad z(t) = \tilde{z}(t,t_0,\rho_0,\varphi_0,z_1),$$

which, at the same time, is the solution of system (2.3) and which passes, for $t = 0$, through the point $M_0(\rho_0, z_0, \varphi_0)$, where $z_0 = \tilde{z}(0, t_0, \rho_0, \varphi_0, z_1)$ and, for $t = t_0$, through the point $M_1(\rho_1, z_1, \varphi_1)$, where $\rho_1 = \tilde{\rho}(0, t_0, \rho_0, \varphi_0, z_1)$, $\varphi = \varphi_0 + t_0$. Using the method of successive approximations and properties (2.4), we can, just as is done in [6], make the form of the functions $\tilde{\rho}$ and $\tilde{z}$ more precise.

Lemma 2.2. *The solution of the boundary value problem* (2.5) *can be represented as*

$$\begin{aligned} \rho(t) &= \rho_0 e^{-\nu t/\omega} + \rho_\alpha(t, t_0, \rho_0, \varphi_0, z_1), \\ z(t) &= z_1 e^{(t-t_0)/\omega} + z_\alpha(t, t_0, \rho_0, \varphi_0, z_1), \end{aligned} \tag{2.6}$$

where, for $t = t_0$, we have the estimates

$$\|\rho_\alpha\| + \left\|\frac{\partial^k \rho_\alpha}{\partial \rho_0^k}\right\| + \left\|\frac{\partial^k \rho_\alpha}{\partial t^k}\right\| + \left\|\frac{\partial^k \rho_\alpha}{\partial t_0^k}\right\| \le M e^{-(\varkappa-\varepsilon)t_0}, \quad k = 1, 2, \tag{2.7}$$

and, for $t = 0$, the estimates

$$\|z_\alpha\| + \left\|\frac{\partial^k z_\alpha}{\partial \rho_0^k}\right\| + \left\|\frac{\partial^k z_\alpha}{\partial t^k}\right\| + \left\|\frac{\partial^k z_\alpha}{\partial t_0^k}\right\| \le M e^{-(\varkappa-\varepsilon)t_0}, \quad k = 1, 2, \tag{2.8}$$

Here $\varkappa = \min(\nu/\omega, 1/\omega)$ and ε is a certain sufficiently small number.

It follows from the form of the right-hand sides of system (2.3) and from the properties (2.4) of the functions R and Φ that there exist sufficiently small δ, $\bar{r}$, and $\bar{d}$ such that the area elements

$$S_0 = \{(\rho, z, \varphi) \mid \varphi = 0, |\rho - \rho^*| < \delta, |z| < \delta\},$$

$$S^0 = \{(\rho, z, \varphi) \mid |\rho| < \delta, |z| = d\},$$

where $0 < \rho^* < \bar{r}$, $0 < d < \bar{d}$, are local secants for the orbits of system (2.3). Therefore, setting $\varphi_0 = 0$, $z_1 = d$ in relations (2.6), and, in addition, $t = t_0$ in the first relation and $t = 0$ in the second, we obtain, with due account of the fact that $\varphi = \varphi_0 + t_0$, the parametric representation of the mapping T_0: $S_0 \to S^0$ written in the form

$$\begin{aligned} \rho &= \rho_0 e^{-\nu t/\omega}(1 + \chi_1(\rho_0, \varphi)), \\ z_0 &= d e^{-\varphi/\omega}(1 + \chi_2(\rho_0, \varphi)), \end{aligned} \tag{2.9}$$

where, on the basis of (2.7), (2.8), we have

$$\|\chi_i\| + \left\|\partial^j \chi_i / \partial^j \varphi^j\right\| + \left\|\partial^j \chi_i / \partial \rho_0^j\right\| \le M e^{(\varkappa-\varepsilon)t_0}, \quad i, j = 1, 2. \tag{2.10}$$

We shall consider on S^0 the polar coordinate system (ρ, θ) induced by the cylindrical system, introduced earlier, in the neighborhood of O. In this case, every value of φ from the interval $[2\pi n, 2\pi(n+1))$ can be identified with the value of the polar angle $\theta = \varphi - 2\pi n$. We introduce on S_0 the Cartesian coordinates (u, ξ), where $\xi = z_0$ and $u = \rho_0 - \rho^*$. Let S_i^0 be a surface that is transversal to Γ_i in the neighborhood of the saddle-focus O_i and S_{01} (S_{02}) be a surface transversal, also in the neighborhood of O_1 (O_2), to the two-dimensional manifold $W^s(O_1)$ ($W^u(O_2)$). Let $M_i^0 = \Gamma_i \cap S_i^0$, $M_{0i} = \Gamma_0 \cap S_{0i}$, $\mathcal{U}(M_i^0)$, and $\mathcal{U}(M_{0i})$ be neighborhoods of these points. We introduce in $\mathcal{U}(M_{0i})$ the coordinates (ξ_i, u_i). The successor maps T_{0i}: $S_{0i} \to S_i^0$, T_1: $S_1^0 \to S_2^0$, and T_2: $S_{02} \to S_{01}$ are defined along the orbits of the system in the neighborhood of every saddle-focus O_i, $i = 1, 2$, and in the neighborhood of the segments of the orbits Γ_0 and Γ^0 bounded by the points M_1^0, M_2^0 and M_{02}, M_{01} respectively. It follows from the general theorems of differential equations that the mapping T_2 can be written as

$$T_2: \begin{aligned} u_1 &= F_1(u_2, \xi_2, \mu), \\ \xi_1 &= F_2(u_2, \xi_2, \mu), \end{aligned} \tag{2.11}$$

where $\frac{\partial(F_1,F_2)}{\partial(u_2,\xi_2)}|_{u_2=\xi_2=0} = \Delta \neq 0$ and, by virtue of condition A, $\frac{\partial F_2(0,0,0)}{\partial u_2} \neq 0$. Therefore we can solve the second equation in (2.11) for u_2 and obtain

$$\begin{aligned} u_1 &= a_1(\xi_1,\xi_2,\mu), \\ u_2 &= a_2(\xi_1,\xi_2,\mu), \end{aligned} \tag{2.12}$$

where $a_i(\xi_1,\xi_2,\mu)$ are sufficiently smooth functions and $\frac{\partial a_i}{\partial \xi_j}|_{(0,0,\mu)} \neq 0$, $i = 1,2$, $j = i+1 \pmod 2$. In order to obtain relations that will define the mapping T_{0i}, it is necessary to replace ρ_0 and z_0 in (2.9) by $\rho^*_{0i} + u_i$ and ξ_i, respectively, where ρ^*_{0i} is the distance from O_i to one of the intersection points of the orbit Γ_0 on the area element $\varphi_0 = 0$. Replacing then the coordinate u_i by the corresponding expression from (2.12) and solving the second equation obtained for ξ_i, we shall have for the mappings $(\xi_1,\xi_2) \to (\rho_i,\varphi_i)$, which we shall denote by T_{0i} as before, the relations

$$\begin{aligned} \rho_i &= \rho^*_{0i}(1 + a_i(\xi_1,\xi_2,\mu))e^{-\nu_i\varphi_i}(1 + \chi_{1i}(\xi_1,\xi_2,\varphi_i,\mu)), \\ \xi_i &= de^{-\varphi_i/\omega_i}(1 + \chi_{i2}(\xi_j,\varphi_i,\mu)), \end{aligned} \tag{2.13}$$

$j = i+1 (\mathrm{mod}\, 2)$, $i = 1,2$, where, by virtue of (2.12) and the relation $\rho_{0i} = \rho^*_i + u_i$, instead of (2.10) we have the estimates

$$\begin{aligned} &\|\chi_{1i}\| + \|\tfrac{\partial^k \chi_{1i}}{\partial \xi_1^k}\| + \|\tfrac{\partial^k \chi_{1i}}{\partial \xi_2^k}\| + \|\tfrac{\partial^k \chi_{1i}}{\partial \varphi_i^k}\| \leq Me^{-(\varkappa_i-\varepsilon)\varphi_i}, \\ &\|\chi_{2i}\| + \|\tfrac{\partial^k \chi_{2i}}{\partial \xi_j^k}\| + \|\tfrac{\partial^k \chi_{2i}}{\partial \varphi_i^k}\| \leq Me^{-(\varkappa_i-\varepsilon)\varphi_i}, \quad i = 1,2\; j = i+1 \pmod 2,\; k = 1,2. \end{aligned} \tag{2.14}$$

It follows from the expressions obtained that the mapping T_{0i} is defined on the set $\sigma_0 = I_1 \times I_2$, where $I_i = (0 < \xi_i \leq \delta)$, $i = 1,2$, and the set $\sigma_{1i} = T_{0i}\sigma_0$ is the range. Hence, the mapping $T_{21} = T_{02}^{-1} \circ T_{01}$: $S_{02} \to S_{01}$ is defined on σ_{12} and maps σ_{12} onto σ_{11}. Each of these domains is a set bounded, by virtue of (2.13), by two spirals that twist at the point M_{0i}. One of the boundary curves σ_{11} (σ_{12}) is a connection component $w_1(O_2) \in W^u(O_2) \cap S^0_1$ ($w_2(O_1) \in W^s(O_1) \cap S^0_2$) that consists of points such that none of the semiorbits, beginning at $w_1(O_2)$ ($w_2(O_1)$) as $t \to -\infty$ ($t \to \infty$), have any other points of intersection with S^0_1 (S^0_2). Geometrically, the mapping T_{0i} acts as follows. Any segment $I_{\xi_i}(\xi_j) = \{(\xi_1,\xi_2) \mid \xi_j \in I_j, \xi_i = \mathrm{const}\}$, $i = 1,2$, $j = i+1 \pmod 2$, passes on S^0_j, $j = i+1 \pmod 2$, into a spiral under the action of T_{0j} and, under the action of T_{0i} into a segment of the ray whose origin and terminus are on the boundary of σ_{1i} (Fig. 1). Thus we can suppose, formally, that σ_{1i} is fibered into segments of rays $\sigma_{1i} = \bigcup\limits_{0\leq\xi_i\leq\delta} T_{0i}I_{\xi_i}(\xi_j)$ or spirals $\sigma_{1i} = \bigcup\limits_{0\leq\xi_j\leq\delta} T_{0i}I_{\xi_j}(\xi_i)$ so that, under the action of the mapping T_{21}, every spiral on S^0_2 passes into the corresponding segment of the ray on S^0_1 and the segment of the ray passes into a spiral.

Representing $[\bar{\varphi}_i,\infty)$ as $\bigcup\limits_n^{\infty} I_{ik}$, $i = 1,2$, where $I_{ik} = [\pi(k-1/2), \pi(k+1/2))$, we find, by virtue of the second relation of (2.12), that σ_0 can be represented as the union of an infinite number of strips σ^k_{01} or σ^k_{02} that accumulate at the straight line $\xi_2 = 0$ or $\xi_1 = 0$ respectively. Each of these strips is the image of the set $I_j \times I_{ik}$ under the action of the mapping

$$T_i: \xi_j = \xi_j, \quad \xi_i = de^{-\varphi_i/\omega_i}(1 + \chi_{i2}(\xi_j,\varphi_i,\mu)), \quad j = i+1 \pmod 2, \quad i = 1,2.$$

The image of the boundaries of σ^k_{0i} under the action of T_{0i} are spirals belonging to int σ_{1i} and under the action of T_{0j} segments of rays whose endpoints lie on the boundary spirals σ_{1j} (Fig. 2).

Along with the polar coordinates on S^0_i we shall use Cartesian coordinates, namely, (x,y) on S^0_1 and (u,v) on S^0_2. Then, with due account of condition B, the mapping T_1: $\mathcal{U}(M^0_1) \to \mathcal{U}(M^0_2)$ can be represented in the Cartesian coordinates as

$$\begin{pmatrix} u \\ v \end{pmatrix} = E(\mu) + D(x,y,\mu)\begin{pmatrix} x \\ y \end{pmatrix}, \tag{2.15}$$

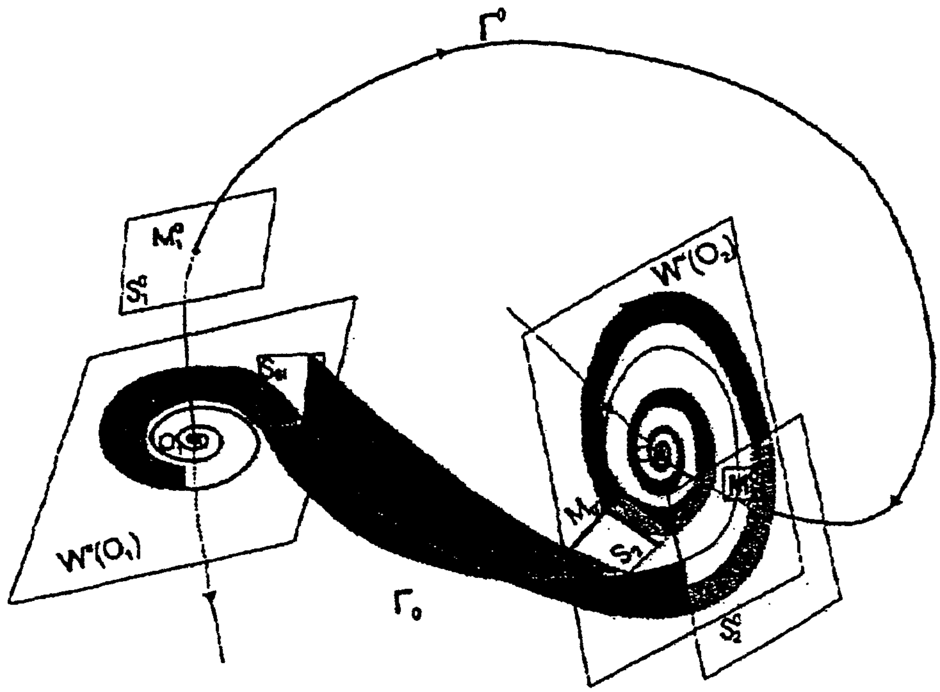

Fig. 1

where $\mathrm{Det}\,(D) = |D| \neq 0$, $E(\mu) = (\mu_1, \mu_2)$.

The action of the linear part of the mapping T_1 is equivalent to the composition of transpositions, i.e., the rotation of the axes of coordinates and a change of scales. Therefore, after a change the reference point for the angular coordinates and a change of the scale, we can assume, without loss of generality, that $D(x, y, \mu)$ in (2.3) can be represented as

$$D(x,y,\mu) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \pm\lambda \end{pmatrix} + \dots, \tag{2.16}$$

where the dots mean smooth functions which tend to zero as $x, y \to 0$. We denote $(D(x,y) - D(0,\mu))\begin{pmatrix} x \\ y \end{pmatrix}$ by $F(x,y,\mu)$. Obviously, $F(0,\mu) \equiv 0$, $\frac{\partial F(0,0,\mu)}{\partial x} = \frac{\partial F(0,0,\mu)}{\partial y} \equiv 0$. We make the change of coordinates $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + D(0,\mu)^{-1}F(\eta_{01},\mu)$ and return to the old notations. Then the mapping T_1 becomes linear. It is easy to verify that this change of coordinates does not change the principal terms of the mapping T_{01} in (2.12) and preserves the degree of its smoothness. Introducing a new notation for the polar coordinates on S_1^0 and S_2^0 and setting $\rho = \rho_1$, $\varphi_1 = \varphi + \pi(n - 1/2)$, and $r = \rho_2$, $\varphi_2 = \psi + \pi(m - 1/2)$, respectively, on the basis of (2.14), (2.16), we find that in polar coordinates, for $\mu = 0$, the mapping T_1: $(\rho, \varphi) \to (r, \varphi)$ can be represented as

$$\begin{aligned} r &= \rho \cdot \sqrt{(\lambda^{-2} \cdot \cos^2(\varphi) + \lambda^2 \sin^2(\varphi)}; \\ \psi &= \arctan\,(\lambda^2 \cdot \tan\,(\varphi)). \end{aligned} \tag{2.17}$$

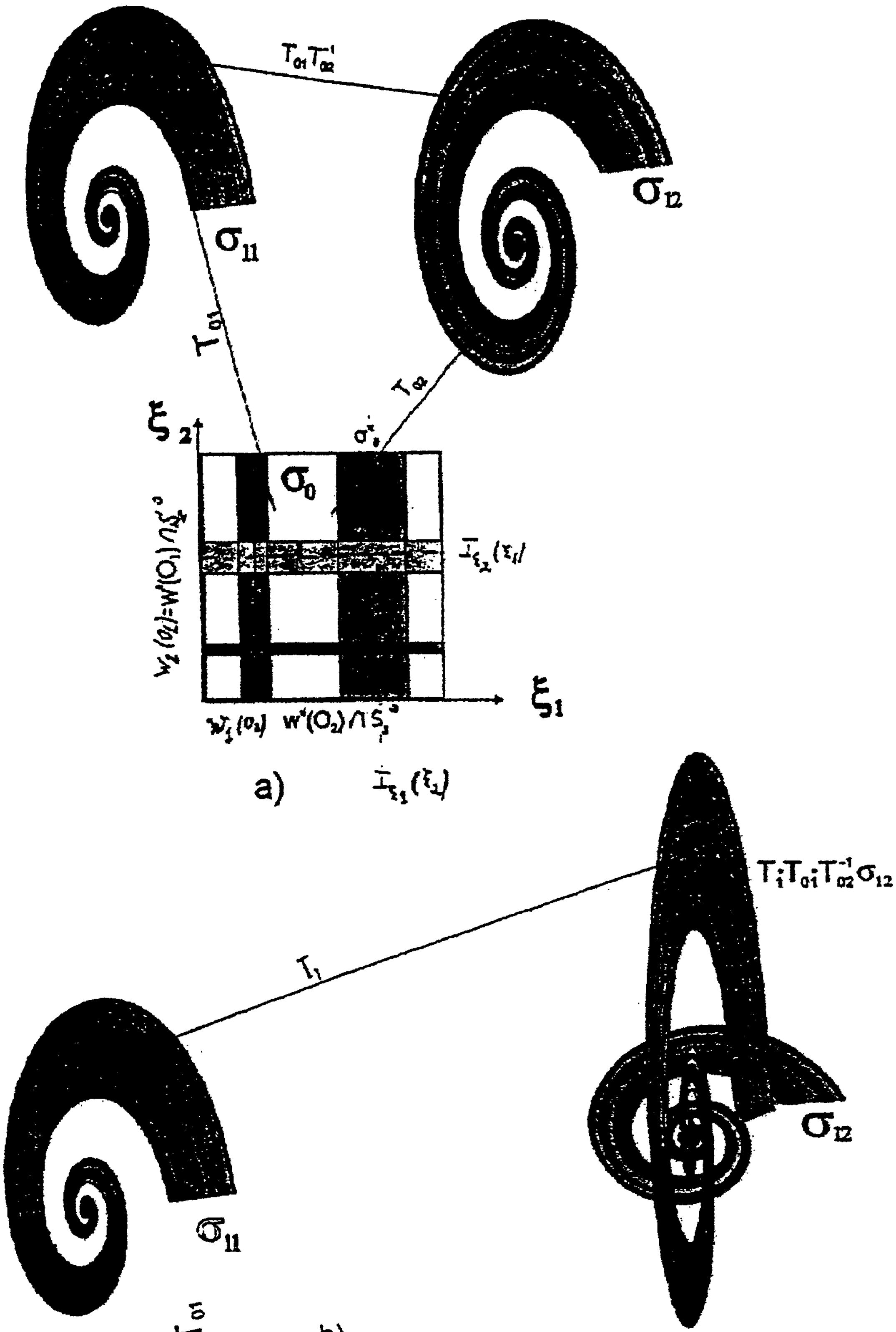

Fig.2.

3. Heteroclinic Orbits of Saddle-Foci

The question concerning the structure of a set of orbits that lie entirely in $\mathcal{U}(\mathcal{L})$ is closely connected with the behavior in $\mathcal{U}(\mathcal{L})$ of two-dimensional manifolds $W^s(O_1)$ and $W^u(O_2)$ of the saddle-foci O_1 and O_2 respectively. Let us consider the possibility of the existence of other orbits, different from Γ_0, belonging to the intersection of the two-dimensional manifolds $W^s(O_1)$ and $W^u(O_2)$. We shall restrict our consideration to orbits that belong to $W^s(O_1) \cap W^u(O_2)$, each of which has one point of intersection with S_{0i}, $i = 1, 2$.

Lemma 3.1. *For $\mu = 0$, along with the orbit Γ_0, there also exists a countable set $\{\Gamma_{0i}\}$ of orbits $\Gamma_{0i} \in W^s(O_1) \cap W^u(O_2)$, $i = 1, 2, \ldots$, along each of which $W^s(O_1)$ intersects $W^u(O_2)$ transversally if the quantity G defined by condition D exceeds zero, and the systems in H_2^0 that have, in addition to the set $\{\Gamma_{0i}\}$ of structurally stable orbits Γ_{0i}, an orbit Γ_{0*} along which $W^s(O_1)$ touches $W^u(O_2)$ are everywhere dense if $G < 0$.*

Proof. Since $w_1(O_2)$ ($w_2(O_1)$) is the image of the segment $\xi_2 = 0$ ($\xi_1 = 0$) under the action of mapping T_{01} (T_{02}), it follows, by virtue of (2.13), that in polar coordinates $w_1(O_2)$ and $w_2(O_1)$ can be described by the equations

$$
\begin{aligned}
\rho &= c_{01}(1 + a_1(\xi_1, 0, 0))e^{-\nu_1(\pi(n-1/2)+\varphi)/\omega_1}(1 + \chi_{11}^n(\xi_1, 0, \varphi)),\\
\xi_1 &= de^{-(\pi(n-1/2)+\varphi)/\omega_1}(1 + \chi_{21}^n(\xi_1, 0, \varphi)),\\
r &= c_{02}(1 + a_2(0, \xi_2, 0))e^{-\nu_2(\pi(n-1)/2+\psi)/\omega_2}(1 + \chi_{21}^m(0, \xi_2, \psi)),\\
\xi_2 &= de^{-(\pi(m-1/2)+\psi)/\omega_2}(1 + \chi_{22}^m(0, \xi_2, \psi)),
\end{aligned}
\tag{3.1}
$$

where

$$
\chi_{1j}^n(\xi_1, 0, \varphi) = \chi_{1j}(\xi_1, 0, \varphi + \pi(n - 1/2), 0),
$$
$$
\chi_{2j}^m(0, \xi_2, \psi) = \chi_{2j}(0, \xi_2, \psi + \pi(m - 1/2), 0), \quad c_{0i} = \rho + 0{*}e^{-\nu_i\theta_{0i}^*},
$$

and θ_{0i}^* is the angle of rotation of the coordinate system relative to the cylindrical system introduced earlier.

It is obvious that the points of intersection of the orbit $\Gamma_{0i} \in W^s(O_1) \cap W^u(O_2)$ with S_2^0 belong to $w_1(O_2) \cap T_1 w_2(O_1)$, and therefore the coordinates of each of these points are defined by the equation resulting from (2.17) upon replacement of ρ and r by their expressions from (3.1) in the first equation and the subsequent replacement of the angular coordinate ψ in the resulting relation by its expression from the second relation in (2.17). Hence we have

$$
\begin{aligned}
&c_{02}(1 + a_2(0, \xi_2, 0))e^{-\nu_2(\pi(m-1/2)+\arctan(\lambda^2\tan\varphi)/\omega_2}(1 + \ldots),\\
&c_{01}(1 + a_1(\xi_1, 0, 0))e^{-\nu_1(\pi(n-1/2)+\varphi)/\omega_1}(1 + \ldots)\sqrt{\lambda^{-2}\cos^2\varphi + \lambda^2\sin^2\varphi},\\
&\xi_1 = de^{-(\varphi+\pi(n-1/2))/\omega_1}(1 + \ldots),\\
&\xi_2 = de^{(\arctan(\lambda^2\tan\varphi+\pi(m-1/2))/\omega_1}(1 + \ldots),
\end{aligned}
\tag{3.2}
$$

where the dots mean terms which smoothly tend to zero as $n, m \to \infty$. After taking logarithms of both sides of Eq. (3.2), we get

$$
n - (\nu_2/\nu_1)\cdot(\omega_2/\omega_1)\cdot m = (\omega_1/(\nu_1\pi)C - \theta/\pi + f(\theta), \tag{3.3}
$$

where $C = \ln(c_{02}/c_{01})$,

$$
f(\theta) = (\nu_2/\nu_1)\cdot(\omega - 2/\omega_1)\arctan(\lambda^2\tan\theta) + \ln((\lambda^{-2})\cos^2\theta\lambda^2\sin^2\theta + \ldots), \tag{3.4}
$$

and the dots mean terms, smooth with respect to θ, which tend to zero as $n, m \to \infty$. After the differentiation of $f(\theta)$ and simple transformations, we obtain the equation

$$
\frac{\nu_1(\lambda^4 + \lambda^2)}{\omega_1} - 2\frac{\nu_2\lambda^2}{\omega_2} = \frac{\nu_1(\lambda^4 - 1)}{\omega_1}\cos(2\theta) + \frac{\lambda^4 - \lambda^2}{\omega_1}\sin(2\theta) + \ldots \tag{3.5}
$$

for determining the values θ corresponding to the extrema of the function $f(\theta) - \theta/\pi$. After introducing the quantity $\varepsilon = (\lambda^2 + 1/\lambda^2)/2$, we find that for $G > 0$ Eq. (3.5) has no roots and, consequently, the right-hand side of Eq. (3.3) is a monotonic function of θ, whereas for $G < 0$ Eq. (3.5) has two roots θ_1^0 and θ_2^0 for which the function $-\theta/\pi + f(\theta)$, which appears on the right-hand side of Eq. (3.3), has two extrema.

It is easy to see that the difference of values of the right-hand side at the endpoints of the interval $[-\pi/2, \pi/2]$ is equal to $\tau - 1 + \ldots$, where $\tau = (\nu_1/\nu_2) \cdot (\omega_1/\omega_2)$. Then, if the quantity $\tau \neq 1$ is an integer, then the left-hand side of $\mathcal{F}(n, m, \tau) \equiv n - m\tau$ assumes all integer values, and, for every integer k, there exists a countable set of pairs $(n_i(k), m_i(k))$ such that $\mathcal{F}(n_i(k), m_i(k), \tau) = k$, $i = 1, 2, \ldots$. Hence there exist at least $k - 1$, and, for $G > 0$, exactly $k - 1$, values $\theta_1, \theta - 2, \ldots, \theta_{k-1}$, each of which is a solution of Eq. (3.3) for the countable set of pairs of integers $(m_i^j(k), n_i^j(k))$, $i \in \mathbf{Z}$, $j = 1, 2, \ldots, k - 1$. If $\tau = p/q$ is rational, then, by virtue of the fact that for any integer l and for coprime p and q the relation $qi - pj = l$ is satisfied for an infinite set of pairs (i, j), the set $\{l/q, l \in \mathbf{Z}\}$ is the range of the function $\mathcal{F}(n, m, \tau)$. Taking into account that $f(\pi/2) - f(-\pi/2) = \tau - 1$, $\tau = k + p'/q$, where k is the integer part of τ and p' is the numerator of the fractional part, we obtain the existence of $(k-1)q + p'$ roots θ_j for $k > 0$ and $q - p'$ roots for $k = 0$, each of which is associated with a countable set of pairs (m_i, n_i) of integers. Finally, for an irrational τ we get an everywhere dense set on the interval $[-\pi/2, \pi/2]$ of the values of θ that are roots of Eq. (3.3). Thus, for any values τ there exists a countable set of pairs (m_i, n_i), for each of which Eq. (3.3) has a solution θ_i^*. In this case, for $G > 0$ every root θ_i is simple and every simple root θ_i^* is associated with the heteroclinic orbit Γ_{0i} belonging to the transversal intersection of $W^s(O_1)$ with $W^u(O_2)$. Thus, if $G > 0$, then the first part of the lemma is proved.

Let us prove the second part of the statement of the lemma. It suffices to show that for any one of the values θ_i^0, $i = 1, 2$, and any $\tau = \nu_1\omega_2/(\nu_2\omega_1)$ there exist arbitrarily close $\tau' = \nu_1'\omega_2'/(\nu_2'\omega_1')$ and m and n such that the relation $n - \tau' m = (\omega_1'/(\nu_1'\pi)C - \theta_i^0/\pi + f(\theta_i^0)$ is satisfied. We take as τ' the rational number p/q that is arbitrarily close to the original τ and has a sufficiently large denominator q. Then, as was already noted, there exists a number l' such that the inequality $l'/q < (\omega_1'/(\nu_1'\pi)C - \theta_i^0/\pi + f(\theta_i^0) < (l'+1)/q$ is satisfied, and, for the infinite set $\{m_i\}$ of values, $m_i \to \infty$ as $i \to \infty$. From the fact that l'/q and $(l'+1)/q$ are values of the functions $\mathcal{F}^-(n_j, m_j, \tau')$ and $\mathcal{F}^+(n_j, m_j, \tau')$, $\partial\mathcal{F}/\partial\tau = -m$, and the derivative with respect to τ of the right-hand side of Eq. (3.3) is bounded, by virtue of (3.4), by a certain constant K, it follows that there exists $\tau'' \to \tau'$ as $m_i \to \infty$ for which $\mathcal{F}^+(n_j, m_j, \tau'') > (\omega_1'/(\nu_1'\pi)C - \theta_i^0/\pi + f(\theta_i^0)$. Consequently, there exists $\tau' < \tau^* < \tau''$, which is arbitrarily close to τ, for which one of the values θ_1^0 or θ_2^0 is also a root of Eq. (3.3). We have proved the lemma.

We say that the heteroclinic orbits Γ_{0i}, defined in the lemma given above, are "one-circuit" orbits because the arcs of these orbits intercepted by the secants S_{01} and S_{02} and lying in the neighborhood of the orbit Γ^0 consist of one connection component. Since Γ_{0i} belong to the transversal intersection of $W^s(O_1)$ with $W^u(O_2)$, we can define the separatrix contour $\mathcal{L}_i = O_1 \cup O_2 \cup \Gamma^0 \cup \Gamma_{0i}$, which satisfies all initial conditions A–C. Hence, the neighborhood $\mathcal{U}(\mathcal{L}_i) \subset \mathcal{U}(\mathcal{L})$ also contains the countable set $\{\Gamma_{0i}^j\}$ of "two-circuit" heteroclinic orbits Γ_{0i}^j belonging to the transversal intersection of $W^s(O_1)$ with $W^u(O_2)$. Every one of these orbits has, in the neighborhood of Γ^0, two disconnected components intercepted by the secants S_{01} and S_{02}. Reasoning by induction, we obtain the existence of a countable set $\{\Gamma^{j_1}\}$ of heteroclinic orbits $\Gamma^{j_1} \in W^s(O_1) \cap W^u(O_2)$ of any number of circuits k. In addition, the validity of the following statement can be established in the same way.

Systems, that have a structurally unstable heteroclinic orbit Γ^{j_1} of any number of circuits along which $W^s(O_1)$ touches $W^u(O_2)$ are everywhere dense in H_0^2.

4. Bifurcations of Homoclinic Orbits

Lemma 4.1. *In the plane of parameters $\mu = (\mu_1, \mu_2)$, there exists a bifurcation curve l_1 (l_2) such that for $\mu \in l_1$ ($\mu \in l_2$) there exists a one-circuit separatrix loop Γ_1 (Γ_2) of the saddle-focus O_1 (O_2). Every one of these curves is shaped as a spiral that twists to the point $(0,0)$.*

Proof. It is obvious that the existence of the loop of the saddle-focus O_1 (O_2) is equivalent to the fact that $T^{-1}w_2(O_1)$ ($T_1 w_1(O_2)$) passes through the origin on the secant S_1^0 (S_2^0) (recall that the point $(0,0)$ on S_1^0 (on S_2^0) is the first intersection point of Γ_1 with S_1^0 (Γ_2 with S_2^0)). By virtue of (2.15), the first point of intersection of Γ_1 with S_2^0 has Cartesian coordinates (μ_1, μ_2) and the equation of $w_2(O_1)$ is defined by the first relation of (2.13) for $\xi_1 = 0$. Therefore, replacing ξ_2 in the first relation of (2.13) by its expression from the second relation of (2.13) and substituting $\xi_1 = 0$, $\mu_1 = \rho_\mu \cos\psi$, $\mu_2 = \rho_\mu \sin\psi$, where $\rho_\mu = \sqrt{\mu_1^2 + \mu_2^2}$, and then solving the resulting equation for ρ_μ, we get the parametric representation of the bifurcation curve l_1 in the form

$$l_1 : \rho_\mu = c_{02} e^{-\nu_2 \psi/\omega_2}(1 + R_2(\psi)), \quad \psi \in (2\pi N, \infty), \tag{4.1}$$

where, by virtue of (2.14),

$$|R(\psi)| + |dR_2/d\psi| + |d^2R_2/d\psi^2| \le Me^{-(\varkappa_2 - \varepsilon)\psi/\omega_2}. \tag{4.2}$$

Similarly, the equation of the bifurcation curve l_2 is defined by the relation

$$l_2 : \rho_\mu = c_{01} e^{-\nu_1(\pi+\varphi}(1 + R_1(\varphi)) \cdot \sqrt{\lambda^2 \cos^2\varphi + 1/\lambda^2 \sin^2\varphi}, \tag{4.3}$$

where

$$\varphi \in (2\pi N, \infty),\ |R_1(\varphi)| + |dR_1/d\varphi| + |d^2R_1/d\varphi^2| \le Me^{-(\varkappa-\varepsilon)\varphi}. \tag{4.4}$$

We have proved the lemma.

Theorem 4.1. *For every two-parameter family $X_\mu(x)$, $X_0(x) \in H^0$, there exists, in the plane of parameters μ, a countable set $\{\mu^k\}$ of points μ^k, for each of which the dynamical system in $\mathcal{U}(\mathcal{L})$ simultaneously has a homoclinic orbit of the saddle-focus O_1 and a homoclinic orbit of the saddle-focus O_2. In this case, when $X_0(x) \in H_1^0$, at each point μ^k the curves l_1 and l_2 are in the general position, whereas when $X_0(x) \in H_2^0$ there exists, for any family $X_\mu(x)$, an arbitrarily close $X'_\mu(x)$ for which there exists a point $\tilde{\mu} \in l_1 \cap l_2$ corresponding to the tangency of the curves l_1 and l_2.*

Proof. The simultaneous existence of homoclinic orbits of the saddle-foci O_1 and O_2 means that the curves l_1 and l_2 have the points μ^k in common. Obviously, for each of these points the right-hand sides of expressions (4.1) and (4.3) must assume the same values. Since we obtain (4.3) using transformation (2.17), each of the values $\varphi \in [2\pi n, 2\pi(n+1))$ and $\psi \in [1\pi m, 2\pi(m+1))$ is connected by the relation $\lambda^{-2}\cot\varphi = \cot\psi$ or, if we set $\psi = 2\pi m + \theta$, by the relation

$$\theta = \begin{cases} \operatorname{arccot}(\lambda^{-2}\cot\varphi), & \text{if} \quad \varphi \in [2\pi n, 2\pi(n+1/2)), \\ \operatorname{arccot}(\lambda^{-2}\cot\varphi) + \pi, & \text{if} \quad \varphi \in [2\pi(n+1/2), 2\pi(n+1). \end{cases} \tag{4.5}$$

Equating the right-hand sides of relations (4.1) and (4.3) and considering φ modulo $2\pi n$, we get an equation for θ:

$$\begin{gathered} c_{02} e^{-\nu_2(\pi m + \operatorname{arccot}(\lambda^2 \cot\theta))/\omega_2}(1 + \tilde{R}_2(m,\theta)), \\ c'_{01} e^{-\nu_1(\pi n+\theta))/\omega_1}(1 + \tilde{R}_1(n,\theta)) \cdot \sqrt{\lambda^2\cos^2\theta + 1/\lambda^2 \sin^2\theta}, \end{gathered} \tag{4.6}$$

where $c'_{01} = c_{01} e^{-\nu_1\pi/\omega_1}$, $\tilde{R}_1(n,\theta) = R_1(\pi n + \theta)$, $\tilde{R}_2(m,\theta) = R_2(\pi m + \operatorname{arccot}(\lambda^{-2}\cot\theta))$. After taking logarithms of both sides of relation (4.6), we get an equation that has the form (3.3), where the expression for $f(\theta)$ differs from the principal term in (3.4) by quantities of order $e^{-(\varkappa_1-\varepsilon)\pi n/\omega_1}$, $e^{-(\varkappa_2-\varepsilon)\pi n/\omega_2}$. Repeating verbatim all arguments that we used when proving Lemma 1, we arrive at the statement of the theorem.

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