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Error-Bounds for Finite Element Method*

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Dedicated to Professor L. Collatz on his 60^{th} birthday

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1. Introduction

Different variational methods are used as a tool to approximate the solution and error-estimates are studied in different norms. The purpose of this contribution is to show a generalization which gives errors in different spaces and some special applications.

As a concrete application we shall study the error of the finite element method for the Dirichlet problem for the Laplace equation on a Lipschitz domain. About other results concerning the finite element method see also [17–23].

2. Basic Theorems

Theorem 2.1. Let H_1 and H_2 be two Hilbert (complex and complete) spaces with scalar product $(\cdot,\cdot)_{H_1}$ (resp. $(\cdot,\cdot)_{H_2}$). Further let $B(u,v)$ be a bilinear form on $H_1\times H_2$, $u\in H_1$, $v\in H_2$ such that

(2.1) $$|B(u,v)|\leqq C_1\|u\|_{H_1}\|v\|_{H_2},$$

(2.2) $$\sup_{\substack{u\in H_1\\ \|u\|_{H_1}\leqq 1}}|B(u,v)|\geqq C_2\|v\|_{H_2},$$

(2.3) $$\sup_{\substack{v\in H_2\\ \|v\|_{H_2}\leqq 1}}|B(u,v)|\geqq C_3\|u\|_{H_1},$$

with $C_2>0$, $C_3>0$, $C_1<\infty$.

Further let $f\in H_2'$, i.e., f be a linear functional on H_2. Then there exists exactly one element $u_0\in H_1$ such that

(2.4) $$B(u_0,v)=\overline{f(v)}\,^{**}$$

for all $v\in H_2$ and

(2.5) $$\|u_0\|_{H_1}\leqq\frac{\|f\|_{H_2'}}{C_3}.$$

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** $\overline{f(v)}$ means complex conjugate value to $f(v)$.

Proof. The proof is adapted from [2] and [3]. We present this proof because we shall use a portion of it for the proof of the next theorem.

1. Because of (2.1), for every $u \in H_1$

$$\phi_u(v) = \overline{B(u, v)}$$

is a linear functional on H_2 with the norm

$$\|\phi_u\|_H = \sup_{\substack{v \in H_2 \\ \|v\|_{H_2} \leqq 1}} |B(u, v)| \leqq C_1 \|u\|_{H_1}.$$

So we may write

$$(z, v)_{H_2} = B(u, v), \quad z \in H_2 \tag{2.6}$$

i.e., there exists a mapping $R(u)$ of H_1 into H_2 such that

$$\big(R(u), v\big)_{H_2} = B(u, v) \tag{2.7}$$

and

$$\|R(u)\|_{H_1 \to H_2} \leqq C_1. \tag{2.8}$$

Obviously R is linear and continuous.

2. Now let us show that $R(H_1)$ is a closed set in H_2.

Because

$$\|R(u)\|_{H_2} = \sup_{\substack{v \in H_2 \\ \|v\|_{H_2} \leqq 1}} \big|\big(R(u), v\big)\big| = \sup_{\substack{v \in H_2 \\ \|v\|_{H_2} \leqq 1}} |B(u, v)|,$$

by (2.3) we have

$$\|R(u)\|_{H_2} \geqq C_3 \|u\|_{H_1}. \tag{2.9}$$

Now let $\{R(u_n)\}$ be a Cauchy-sequence in H_2, then $\{u_n\}$ is a Cauchy-sequence in H_1. In fact

$$\|R(u_n) - R(u_m)\|_{H_2} = \|R(u_n - u_m)\| \geqq C_3 \|u_n - u_m\|_{H_1}.$$

It follows immediately that $R(H_1)$ is closed.

3. Let us now show that $R(H_1) = H_2$. If this were not so, then because $R(H_1)$ is a subspace there exists $v_0 \in H_2$, $(v_0 \neq 0)$ such that

$$0 = \big(R(u), v_0\big)_{H_2} \quad \text{for all} \quad u \in H_1.$$

But by (2.7)

$$\big(R(u), v_0\big)_{H_2} = B(u, v_0)$$

and therefore by (2.2) there exists $u' \in H_1$ such that

$$|B(u', v_0)| \geqq \tfrac{1}{2} C_2 \|v_0\|_{H_2}$$

and this is a contradiction.

4. So the equation $R(x) = y$ has a solution for every $y \in H_2$ and (2.9) holds. Thus there exists a linear continuous inverse R^{-1} and

$$\|R^{-1}\|_{H_2 \to H_1} \leqq \frac{1}{C_3}$$

[see e.g. [4], p. 168].

5. Let $f \in H_2'$. So we have

$$\bar{f}(v) = (v_0, v)_{H_2}$$

with $\|v_0\|_{H^2} = \|f\|_{H_2'}$ and putting $u_0 = R^{-1} v_0$ we have our theorem, because uni queness is obvious.

Theorem 2.2. Let the assumptions of Theorem 1 be fulfilled. Further let there be given the subspaces (closed) $M_1 \subset H_1$ and $M_2 \subset H_2$ and let for every $v \in M_2$

$$\sup_{\substack{u \in M_1 \\ \|u\|_{H_1} \leq 1}} |B(u, v)| \geqq d_2(M_1, M_2) \|v\|_{H_2} \tag{2.10}$$

with $d_2(M_1, M_2) > 0$, and for every $u \in M_1$

$$\sup_{\substack{v \in M_2 \\ \|v\|_{H_2} \leq 1}} |B(u, v)| \geqq d_3(M_1, M_2) \|u\|_{H_1} \tag{2.11}$$

with $d_3(M_1, M_2) > 0$.

Let $f \in H_2'$ be given, let u_0 denote element of H_1 such that

$$B(u_0, v) = \overline{f(v)} \tag{2.12}$$

holds for all $v \in H_2$ [such an element exists and is unique by Theorem 2.1].

Let there exists $\omega \in M_1$ such that

$$\|u_0 - \omega\|_{H_1} \leqq \delta. \tag{2.12}$$

Further let $\hat{u}_0 \in M_1$ be such that

$$B(\hat{u}_0, v) = \bar{f}(v) \tag{2.13}$$

for all $v \in M_2$. Then

$$\|u_0 - \hat{u}_0\|_{H_1} \leqq \left[1 + \frac{C_1}{d_3(M_1, M_2)}\right] \delta. \tag{2.14}$$

Proof. 1. First let us remark that there exists exactly one u_0 fulfilling (2.13) because of Theorem 2.1.

2. Now let P denote the projector (orthogonal) in H_2 which maps H_2 onto M_2 and let R be the mapping constructed in the proof of Theorem 2.1, i.e.,

$$(R(u), v)_{H_2} = B(u, v). \tag{2.15}$$

Then let us put $Q = PR$. Its contraction is obviously a continuous mapping of M_1 into M_2.

Now let S be a mapping of M_1 onto M_2 such that

$$(Su, v)_{H_2} = B(u, v) \tag{2.16}$$

for every $u \in M_1$ and $v \in M_2$. Such a mapping will be constructed in the same way as in the proof of the Theorem 2.1 and it is unique.

Let us show that $S = PR = Q$ after contraction on M_1. We have to verify

$$(PRu, v)_{H_2} = B(u, v) \tag{2.17}$$

for all $u \in M_1$ and $v \in M_2$. It is clear that

$$(PRu, v)_{H_2} = (Ru, Pv)_{H_2} = (Ru, v)_{H_2} = B(u, v),$$

because of symmetry of P, and $Pv = v$ for $v \in M_2$. So let $v_0 \in M_2$ be such that $(v_0, v)_{H_2} = \bar{f}(v)$ for every $v \in M_2$. Obviously $v_0 = PRu_0$. So

$$\hat{u}_0 = S^{-1} P R u_0. \tag{2.18}$$

Now we have $(u_0 - \omega = z)$

$$\hat{u}_0 - u_0 = S^{-1} P R u_0 - u_0 = S^{-1} P R \omega + S^{-1} P R z - \omega - z = -z + S^{-1} P R z$$

and therefore

$$\|u_0 - \hat{u}_0\|_{H_1} \leqq \delta \left[1 + \frac{C_1}{d_3(M_1, M_2)}\right]$$

which was to be proved.

3. Hilbert Scales

Let R_n be the n-dimensional Euclidean space $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\|\boldsymbol{x}\|^2 = \sum x_i^2$, $|\boldsymbol{x}| = \max\{|x_i|\}$. A domain $\Omega \subset R_n$ will be said to be an L-domain [Lipschitz domain] if there exist numbers $\alpha > 0$, $\beta > 0$ and the systems of coordinates $(x_{r,1}, \ldots, x_{r,n}) = (\boldsymbol{x}'_r, x_{r,n})$, $r = 1, 2, \ldots, M$ and Lipschitz functions a_r defined on the cubes $|x_{r,i}| < \alpha$, $i = 1, \ldots, n-1$, $r = 1, \ldots, M$, so that each point x of the boundary Ω could be expressed at least in one system in the form of $(\boldsymbol{x}'_r, a_r(x'_r))$, and for each system the points $(x'_r, a_r(x'_r))$ are inside (resp. outside) of Ω for

$$a_r(x'_r) < x_{r,n} < a_r(x'_r) + \beta \left(\text{resp. } a_r(x'_r) - \beta < x_{r,n} < a_r(x'_r)\right).$$

In the paper we shall assume that all domains will be Lipschitz domains.

Let $D(\Omega)$ denote the set of all functions with compact support in Ω, and continuous derivatives of all orders.

$$W_2^\alpha(\Omega) \left(\text{resp. } W_2^\alpha(R_n)\right), \quad \alpha \geqq 0$$

are the Sobolev fractional spaces on Ω (resp. R_n). For α an integer we have

$$\|u\|^2_{W_2^\alpha(\Omega)} = \sum_{|k| \leqq \alpha} \|D^k u\|^2_{L_2(\Omega)} \tag{3.1}$$

where the sum is over all derivatives of order $k \leqq \alpha$.

The definition in R_n is analogous. For $\alpha = [\alpha] + \sigma$, $0 < \sigma < 1$, we introduce the fractional norm due to Arouszajn [5] and Slobodetskij [6].

$$\|u\|^2_{W_2^\alpha(\Omega)} = \|u\|^2_{W_2^{[\alpha]}(\Omega)} + \sum_{|k| = [\alpha]} \|D^k u\|^2_{W_2^\sigma(\Omega)} \tag{3.2}$$

where

$$\|u\|^2_{W_2^\sigma(\Omega)} = \int_\Omega \int_\Omega \frac{|u(\boldsymbol{t}) - u(\boldsymbol{\tau})|^2}{|\boldsymbol{t} - \boldsymbol{\tau}|^{n+2\alpha}} \, d\boldsymbol{t} \, d\boldsymbol{\tau}. \tag{3.3}$$

The same definition is valid for $\Omega = R_n$. In R_n we introduce another norm

$$\|u\|^2_{W_2^{\alpha,1}(R_n)} = \int_{R_n} (1 + \|\boldsymbol{\sigma}\|^2)^\alpha |F(u)(\boldsymbol{\sigma})|^2 \, d\boldsymbol{\sigma} \tag{3.4}$$

where $F(u)$ is the Fourier transform of u. The norms $\|\cdot\|_{W_2^{\alpha,1}(R_n)}$ and $\|\cdot\|_{W_2^{\alpha}(R_n)}$ are equivalent (see e.g. [6]). The space $\mathring{W}_2^{\alpha}(\Omega)$ will be the closure of $D(\Omega)$ in $W_2^{\alpha}(\Omega)$. The space $W_2^{-\alpha}$ will be the dual space to W_2^{α}. Let us now quote some results which we shall use.

Lemma 3.1. Let $f\in\mathring{W}_2^{\alpha}(\Omega)$, $\frac{1}{2}<\alpha<\frac{3}{2}$. Then the operator T of continuation by zero $\big($i.e., $(Tf)(x)=f(x)$ for $x\in\Omega$, $(Tf)(x)=0$ for $x\notin\Omega\big)$ is a linear continuous operator from $\mathring{W}_2^{\alpha}(\Omega)$ into $W_2^{\alpha}(R_n)$. See e.g. Lions Magenes [7] and Nečas [1].

Lemma 3.2. Denoting by C(resp. $\|\cdot\|_C$) the usual space of continuous functions (resp. the usual norm in C), we have

$$\|u\|_{C(R_n)}\leqq K\|u\|_{W_2^{\alpha}(R_n)} \tag{3.5}$$

if $\alpha>n/2$, see e.g. [8]. We shall use this lemma especially for $n=2$.

Lemma 3.3. We have

$$\|u\|_{W_2^{\gamma,1}(R_n)}\leqq\|u\|^{\mu}_{W_2^{\alpha,1}(R_n)}\,\|u\|^{\nu}_{W_2^{\beta,1}(R_n)} \tag{3.6}$$

$0\leqq\alpha\leqq\gamma\leqq\beta\leqq\frac{3}{2}$ or $-\frac{3}{2}\leqq\alpha\leqq\gamma\leqq\beta\leqq 0$.

$$\mu=\frac{\beta-\gamma}{\beta-\alpha},\qquad \nu=\frac{\gamma-\alpha}{\beta-\alpha}, \tag{3.7}$$

see e.g. [9].

Lemma 3.4. Let $-\frac{3}{2}\leqq\alpha_1\leqq\beta_1\leqq 0$, $0\leqq\alpha_2\leqq\beta_2\leqq\frac{3}{2}$. Let A be a linear operator defined on $W_2^{\beta_1,1}(R_n)$ which is a bounded operator from $W_2^{\beta_1,1}(R_n)$ into $W_2^{\beta_2,1}(R_n)$ and from $W_2^{\alpha_1,1}(R_n)$ into $W_2^{\alpha_2,1}(R_n)$. Then A is also a bounded operator from $W_2^{\gamma_1,1}(R_n)$ into $W_2^{\gamma_2,1}(R_n)$, $\alpha_i\leqq\gamma_i\leqq\beta_i$, $i=1,2$, where $\dfrac{\gamma_1-\alpha_1}{\beta_1-\alpha_1}=\dfrac{\gamma_2-\alpha_2}{\beta_2-\alpha_2}$ and

$$\|A\|_{W_2^{\gamma_1,1}(R_n)\to W_2^{\gamma_2,1}(R_n)}\leqq C\big[\|A\|_{W_2^{\alpha_1,1}(R_n)\to W_2^{\alpha_2,1}(R_n)}\big]^{\mu}\cdot\big[\|A\|_{W_2^{\beta_1,1}(R_n)\to W_2^{\beta_2,1}(R_n)}\big]^{\nu} \tag{3.8}$$

where

$$\mu=\frac{\beta_1-\gamma}{\beta_1-d_1},\qquad \nu=\frac{\gamma-\alpha_1}{\beta_1-\alpha_1},$$

and C does not depend on γ_1 and γ (see e.g. [9]).

4. Dirichlet Problem

Let us solve the Dirichlet problem for the Laplace's equation.

$$\begin{aligned} -\Delta u&=f\\ u&=0 \quad\text{on } \Omega^{\bullet}. \end{aligned} \tag{4.1}$$

We will be looking for a weak solution, i.e., $u\in\mathring{W}_2^{\alpha}(\Omega)$ will be the solution if

$$\int_{\Omega}\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_i}\,dx=\int_{\Omega}f\,v\,dx$$

for every $v\in D(\Omega)$.

Now the following theorem holds.

Theorem 4.1. Let $-\frac{1}{2}<\alpha<\frac{1}{2}$, then there exists a unique solution in $\mathring{W}_2^{1+\alpha}(\Omega)$ for every $f\in W_2^{-1+\alpha}(\Omega)$ and we have

$$\|u\|_{W_2^{1+\alpha}(\Omega)}\leqq K\,\|f\|_{W_2^{-1+\alpha}(\Omega)}.$$

See Nečas [1].

Let $Q\subset R_n$ be a set, then a—neighborhood of Q will be the set $E[\boldsymbol{x},\ \|\boldsymbol{x}-\boldsymbol{y}\|<a,\ \boldsymbol{y}\in Q]$.

Definition 4.1. We shall say that $\omega(\boldsymbol{x})$ is a proper function if

a) $\omega(\boldsymbol{x})\in W_2^2(R_n)$.

b) $\omega(\boldsymbol{x})$ has compact support.

c) There exist constants C and L such that for every $f\in W_2^{\alpha}(R_n)$ and $h>0$, there exist $C_h(\boldsymbol{k})$, $\boldsymbol{k}=(k_1,\ldots,k_n)$, k_i integers such that

$$\Big\|f-\sum_{\boldsymbol{k}}C_h(\boldsymbol{k})\,\omega\Big(\frac{\boldsymbol{x}-h\,\boldsymbol{k}}{h}\Big)\Big\|_{W_2^{\alpha}(R_n)}\leqq C\,\|f\|_{W_2^{\beta}(R_n)}\,h^{\beta-\alpha}$$

for every $0\leqq\alpha\leqq\beta\leqq 2$, and if f has compact support S, then the function $\sum_{\boldsymbol{k}}C_h(\boldsymbol{k})\,\omega\Big(\frac{\boldsymbol{x}-h\,\boldsymbol{k}}{h}\Big)$ has the support in the Lh neighborhood of S. Such functions exist. For more information, see [10].

Now let us study the finite elements method for solving (4.1) provided that $f\in W_2^{\beta}(\Omega)$, $-\frac{3}{2}<\beta$. The method is the following:

For a given $h>0$ let us take the linear space M_h of all functions of the type $g_h=\sum C(\boldsymbol{k})\,\omega\Big(\frac{\boldsymbol{x}-h\,\boldsymbol{k}}{h}\Big)$ such that g_h has its support inside $\bar{\Omega}$ and ω is a proper function (Definition 4.1). Now the approximate solution u_h will be such that

a) $u_h\in M_h$;

b) $\displaystyle\int_{\Omega}\Big(\sum\frac{\partial u_h}{\partial x_i}\,\frac{\partial g_h}{\partial x_i}\Big)\,d\boldsymbol{x}=\int f\,g_h\,d\boldsymbol{x}$

for every $g_h\in M_h$. The integral on the right-hand side is to be taken in the sense of generalized functions (distributions). Because $g_h\in\mathring{W}_2^{2}(\Omega)$ and $f\in W_2^{\beta}(\Omega)$, $-\frac{3}{2}<\beta$, the right-hand side makes sense. Also because the form on the left-hand side is positive definite there exists exactly one solution u_h. Denoting by u_0 the weak solution of 4.1, we shall be interested in the estimation of the type

$$\|u_0-u_h\|_{W_2^{\beta}(\Omega)}\leqq K\cdot h^{\mu}\,\|f\|_{W_2^{\alpha}(\Omega)}.$$

We shall prove the above inequality

(4.2) for $-\frac{1}{2}>\alpha\geqq-1$, $\frac{3}{2}>\beta\geqq 1$, $\alpha-2\beta+3\geqq 0$, $\mu=(\alpha+1)-2(\beta-1)$,

(4.3) for $-\frac{1}{2}>\alpha\geqq-1$, $0\leqq\beta\leqq 1$, $\mu=\alpha+1+(1-\beta)\frac{1}{2}-\varepsilon$, for $\varepsilon>0$ arbitrary,

(4.4) for $\alpha\leqq-1$, $0\leqq\beta\leqq\alpha+2$, $-\frac{3}{2}<\alpha$, $\frac{5}{2}+2\alpha>\dfrac{(\frac{3}{2}+\alpha)\,\beta}{2+\alpha}$,
$\mu=\frac{5}{2}+2\alpha-\dfrac{(\frac{3}{2}+\alpha)\,\beta}{2+\alpha}-\varepsilon$, for $\varepsilon>0$, arbitrary.

By using Lemma 3.2 we have immediately for $n=2$ (i.e. in the plane) the following result

$$\|u_0 - u_h\|_{C(\Omega)} \leqq K h^{\mu'} \|f\|_{W_2^{\alpha'}(\Omega)}$$

$-\frac{1}{2} > \alpha' > -1$, $\mu' = \alpha' + 1 - \varepsilon$, $\varepsilon > 0$, arbitrary. So e.g. using the imbedding theorem we have

$$\|u_0 - u_h\|_{C(\Omega)} \leqq C h^{\frac{1}{2}-\varepsilon} \|f\|_{W_2^{-\frac{1}{2}}(\Omega)} \tag{4.5}$$

for $\varepsilon > 0$, arbitrary. It is interesting to compare this result with Laasonen's [11, 12] upper bound and Volkov's [16] lower bound of the rate of convergence of the finite differences method—five point formula. It is easy to see that we may select the function $\omega(x) \in W_2^{\frac{3}{2}-\eta}(R_n)$, $0 < \eta < \frac{1}{2}$, so that we get classical five point formula (see e.g. [17], p. 305) and the theorem holds in the same sense. So our results are almost the best possible results with respect to the convergence in C. See also [13].

Another interesting case for $n=2$ is the case $f = \theta$, θ is a Dirac function. Then the solution is the Green function. We see that the convergence is of order $h^{\frac{1}{2}-\varepsilon}$ in L_2. For the one dimensional case and special trial function see [14]. We have shown a very special case. With obvious changes in the approach which we used it is possible to study nonselfadjoint equations, equations with nonconstant coefficients, higher order equation and systems of equations.

5. Some Lemmas

Lemma 5.1. Let $f \in W_2^{\beta}(R_n)$, $0 \leqq \alpha \leqq \beta$. Denoting $f_{h\boldsymbol{a}}(\boldsymbol{x}) = f(\boldsymbol{x} + h\boldsymbol{a})$, $\|\boldsymbol{a}\| = 1$, we have

$$\|f - f_{h\boldsymbol{a}}\|_{W_2^{\alpha}(R_n)} \leqq K \cdot h^{\mu} \|f\|_{W_2^{\beta}(R_n)} \tag{5.1}$$

where

$$\mu \leqq \min[1, \beta - \alpha].$$

Proof. We shall prove it for the norm $W_2^{\alpha,1}(R_u)$. We have

$$\begin{aligned}
\|f_{h\boldsymbol{a}} - f\|^2_{W_2^{\alpha,1}(R_n)} &= \int_{R_n} (1 + \|\boldsymbol{x}\|^2)^{\alpha} |(1 - e^{i\boldsymbol{a}h\boldsymbol{x}})^2 F(f)(\boldsymbol{x})|^2 \, d\boldsymbol{x} \\
&= \int_{\|\boldsymbol{x}h\| \geqq 1} \cdots + \int_{\|\boldsymbol{x}h\| \leqq 1} \cdots \leqq C \int_{\|\boldsymbol{x}h\| < 1} h^{2\mu} \|\boldsymbol{x}\|^{2\mu} |(1 + \|\boldsymbol{x}\|^2)|^{\alpha} |F(f)|^2 \, d\boldsymbol{x} \\
&\quad + C' \int_{\|\boldsymbol{x}h\| \geqq 1} (1 + \|\boldsymbol{x}\|^2)^{\alpha} \|\boldsymbol{x}\|^{2\mu} \cdot h^{2\mu} |F(f)|^2 \, d\boldsymbol{x}
\end{aligned}$$

and

$$2\mu + 2\alpha \leqq 2\beta.$$

Lemma 5.2. Let $f \in \mathring{W}_2^{\beta}(\Omega)$, $\frac{1}{2} < \beta < \frac{3}{2}$. Then for every $h > 0$ there exists $f_h \in \mathring{W}_2^{\alpha}(\Omega)$, such that $f_h = 0$ in a h-neighborhood of $\Omega^{\boldsymbol{\cdot}}$ and

$$\|f_h - f\|_{W_2^{\alpha}(\Omega)} \leqq K h^{\mu} \|f\|_{W_2^{\beta}(\Omega)} \tag{5.2}$$

where

$$\mu \leqq \min[1, \beta - \alpha], \quad 0 \leqq \alpha \leqq \beta.$$

Proof. Because $\frac{1}{2}<\beta<\frac{3}{2}$ we may use Lemma 3.1 and take f as a function in $W_2^\beta(R_n)$ with support in $\bar{\Omega}$. Because Ω is a Lipschitz domain using the common approach of partitions of unity, we may write

$$f=\sum f_i, \qquad \|f_i\|_{W_2^\beta} \leqq C\,\|f\|_{W_2^\beta}$$

with $f_i \in \mathring{W}_2^\beta(\Omega)$ and either the support inside Ω or $(f_i)_{ah} \in \mathring{W}_2^\beta(\Omega)$ and $(f_i)_{ah}(\boldsymbol{x})=0$ for $\boldsymbol{x}$ in a h-neighborhood of $\Omega^\cdot$. Using the previous lemma, we prove the statement.

Lemma 5.3. Let $\omega(\boldsymbol{x}) \in W_2^2(R_n)$ with compact support. Further let

$$g_h(\boldsymbol{x})=\sum C_{\boldsymbol{k}}\,\omega\left(\frac{\boldsymbol{x}-\boldsymbol{k}\,h}{h}\right) \tag{5.3}$$

where $\boldsymbol{k}=(k_1,\ldots,k_n)$ and the k_i are integers and g_h has compact support (i.e. 5.3 is finite sum). Then there exists K independent of h such that

$$\|g_h(\boldsymbol{x})\|_{W_2^2(R_n)} \leqq \frac{K}{h}\,\|g_h(\boldsymbol{x})\|_{W_2^1(R_n)}. \tag{5.4}$$

Proof. 1. Let us take

$$K_1=\sup_{C_{\boldsymbol{k}}} \frac{\|\sum C_{\boldsymbol{k}}\,\omega(\boldsymbol{x}-\boldsymbol{k})\|_{W_2^2(\Omega_1^0)}}{\|\sum C_{\boldsymbol{k}}\,\omega(\boldsymbol{x}-\boldsymbol{k})\|_{W_2^1(\Omega_1^0)}}, \tag{5.5}$$

where

$$\Omega_1^0=E[x_i,\,|x_i|<\tfrac{1}{2}].$$

Let us show that $K_1<\infty$. Let us take M, the set of those k, such that the $\omega(\boldsymbol{x}-\boldsymbol{k})$, for $k\in M$ are linear independent on Ω_1^0. Then every $g=\sum C_{\boldsymbol{k}}\,\omega(\boldsymbol{x}-\boldsymbol{k})$ can be written on Ω_1^0 in the form

$$g=\sum_{\boldsymbol{k}\in M} C'_{\boldsymbol{k}}\,\omega(\boldsymbol{x}-\boldsymbol{k}).$$

Because of the compact support of ω, the set M is finite. To prove $K_1<\infty$, we may restrict ourself to the case when

$$\|\sum C_{\boldsymbol{k}}\,\omega(\boldsymbol{x}-\boldsymbol{k})\|_{W_2^2(\Omega_1^0)}=1 \tag{5.6}$$

and $\boldsymbol{k}\in M$. If K_1 were ∞, then because of compactness, there must exist $\{C_k^*\}$ such that

$$\|\sum C_{\boldsymbol{k}}^*\,\omega(\boldsymbol{x}-\boldsymbol{k})_{W_2^1(\Omega_1^0)}=0$$

and

$$\|\sum C_{\boldsymbol{k}}^*\,\omega(\boldsymbol{x}-\boldsymbol{k})\|_{W_2^2(\Omega_1^0)}=1.$$

This would obviously be a contradiction.

2. Let us now take

$$K_h=\sup_{C_{\boldsymbol{k}}} \frac{\left\|\sum C_{\boldsymbol{k}}\,\omega\left(\frac{\boldsymbol{x}-\boldsymbol{k}\,h}{h}\right)\right\|_{W_2^2(\Omega_h^0)}}{\left\|\sum C_{\boldsymbol{k}}\,\omega\left(\frac{\boldsymbol{x}-\boldsymbol{k}\,h}{h}\right)\right\|_{W_2^1(\Omega_h^0)}} \tag{5.7}$$

where

$$\Omega_h^0=E[x_i,\,|x_i|<\tfrac{1}{2}h].$$

It is easy to see that

(5.8) $$K_h \leqq \frac{K_1}{h}.$$

3. Because of the compact support of $g_h(\boldsymbol{x})$ we have

(5.9) $$\|g_h(\boldsymbol{x})\|^2_{W_2^2(R_n)} = \sum_{\boldsymbol{k}} \|g_h(\boldsymbol{x})\|^2_{W_2^2(\Omega_h^0(\boldsymbol{k}))}$$

(5.10) $$\|g_h(\boldsymbol{x})\|^2_{W_2^1(R_n)} = \sum_{\boldsymbol{k}} \|g_h(\boldsymbol{x})\|^2_{W_2^1(\Omega_h^0(\boldsymbol{k}))}$$

where

$$\Omega_h^0(\boldsymbol{k}) = E\left[x_i, \ \left|\frac{\boldsymbol{x}-\boldsymbol{k}\,h}{h}\right| < \frac{1}{2}\right]$$

with the sum finite. Obviously we have

$$\|g_h(\boldsymbol{x})\|^2_{W_2^2(R_n)} \leqq K_h^2 \|g_h\|^2_{W_2^1(R_n)}$$

and because of (5.8), our lemma is proved.

Lemma 5.4. Under the same assumptions as in Lemma 5.3 we have for $0 \leqq \beta \leqq 1$

(5.11) $$\|g_h(\boldsymbol{x})\|_{W_2^{1+\beta}(R_n)} \leqq \left(\frac{K'}{h}\right)^{\beta} \|g_h(\boldsymbol{x})\|_{W_2^1(R_n)}.$$

Proof. By Lemma 3.3 we have

$$\|g_h\|_{W_2^{1+\beta,1}(R_n)} \leqq \|g_h\|^{\mu}_{W_2^{1,1}(R_n)} \|g_h\|^{\nu}_{W_2^{2,1}(R_n)}$$

and

$$\mu = 1-\beta, \qquad \nu = \beta$$

and so

$$\|g_h\|_{W_2^{1+\beta}(R_n)} \leqq \frac{K''}{h^{\beta}} \|g_h\|_{W_2^1(R_n)}.$$

6. Rate of Convergence of the Finite Element Method

In this part we shall study the method described in Part 4 using the results of Part 2. By using Lemma 5.4 we have in (2.10) and (2.11) $d_2 \geqq C h^{\beta}$, $d_3 \geqq C h^{\beta}$, for

$$B(u, v) = \int_{\Omega} \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, d\boldsymbol{x}$$

for $H_1 = \mathring{W}_2^{1+\beta}(\Omega)$, $H_2 = \mathring{W}_2^{1-\beta}(\Omega)$, $|\beta| < \frac{1}{2}$. By [1] (2.1), (2.2) and (2.3) holds. Now let $\beta \geqq 0$, $\alpha \geqq 2\beta$, $\alpha < \frac{1}{2}$ and $f \in W_2^{-1+\alpha}$ in (4.1). Then by Theorem 4.1 we have

$$\|u_0\|_{W_2^{1+\alpha}(\Omega)} \leqq K \|f\|_{W_2^{-1+\alpha}(\Omega)}$$

and

$$u_0 \in \mathring{W}_2^{1+\alpha}(\Omega).$$

Using Lemma 3.1 we may continue u_0 by zero in R_n. Then $u_0 \in W_2^{1+\alpha}(\Omega)$ with compact support in $\bar{\Omega}$. Using Lemma 5.2 for $\alpha = \beta$ and the basic property of proper function ω and Lemma 5.2 once more we may construct

$$g_h = \sum_{\boldsymbol{k}} C_h(\boldsymbol{k})\, \omega\left(\frac{\boldsymbol{x}-\boldsymbol{k}\,h}{h}\right) \in \mathring{W}_2^2(\Omega)$$

such that

$$\|u_0 - g_h\|_{W_2^{1+\beta,1}(R_n)} \leqq C\, h^{\alpha-\beta} \|f\|_{W_2^{-1+\alpha}(\Omega)}$$

for $0 \leqq \alpha < \frac{1}{2}$, $\beta \leqq \alpha$, $\alpha - \beta < 1$. Using Theorem 2.2 we have

$$\|u_0 - u_h\|_{W_2^{1+\beta}(\Omega)} \leqq C\, h^{\alpha-2\beta} \|f\|_{W_2^{-1+\alpha}(\Omega)} \tag{6.1}$$

which is (4.2). To obtain (4.3) (i.e. $0 \leqq \beta \leqq 1$) we use another approach to obtain sharper results as in the analogous approach for (4.2). By definition we have

$$B(u_h - u_0, v) = 0 \quad \text{for every} \quad v \in M_h. \tag{6.2}$$

From 6.1 we have

$$\|u_h - u_0\|_{W_2^1(\Omega)} \leqq C\, h^{\alpha} \|f\|_{W_2^{-1+\alpha}(\Omega)}. \tag{6.3}$$

Because $W_2^1(\Omega) \subset W_2^{-1+\gamma}(\Omega)$, $0 \leqq \gamma < \frac{1}{2}$, there exists $V \in \mathring{W}_2^{1+\gamma}(\Omega)$ such that

$$B(V, s) = \int_{\Omega} (u_h - u_0)\, s\, dx \tag{6.4}$$

for every $s \in \mathring{W}_2^{1-\gamma}(\Omega)$ and using Theorem 4.1

$$\|V\|_{W_2^{1+\gamma}(\Omega)} \leqq C\, \|u_h - u_0\|_{L_2(\Omega)}. \tag{6.5}$$

So by the basic property of M_h and Lemma 5.2 we have

$$V = V_h + Z, \qquad V_h \in M_h$$

and

$$\|Z\|_{W_2^1(\Omega)} \leqq C\, \|V\|_{W_2^{1+\gamma}(\Omega)} \cdot h^{\gamma}. \tag{6.6}$$

By (6.2) we have [in our case $B(u, v) = B(v, u)$]

$$B(u_h - u_0, V) = B(u_h - u_0, Z). \tag{6.7}$$

But using (6.4) we have

$$B(u_h - u_0, V) = \|u_h - u_0\|^2_{L_2(\Omega)}. \tag{6.8}$$

Therefore

$$\begin{aligned}\|u_h - u_0\|^2_{L_2(\Omega)} = B(u_h - u_0, Z) &\leqq C\, \|u_h - u_0\|_{W_2^1} \|Z\|_{W_2^1} \\ &\leqq C\, \|u_h - u_0\|_{L_2(\Omega)}\, h^{\gamma+\alpha} \|f\|_{W_2^{-1+\alpha}(\Omega)}\end{aligned}$$

Using (6.6), (6.5), (6.3), we have

$$\|u_h - u_0\|_{L_2(\Omega)} \leqq C\, h^{\gamma+\alpha} \|f\|_{W_2^{-1+\alpha}(\Omega)}. \tag{6.9}$$

Using Lemma 3.4 and (6.3) and (6.9), we have

$$\|u_h - u_0\|_{W_2^{\beta}(\Omega)} \leqq C (h^{\alpha})^{\beta} \cdot (h^{\gamma+\alpha})^{1-\beta} \|f\|_{W_2^{-1+\alpha}(\Omega)} = C\, h^{\alpha+\gamma(1-\beta)} \|f\|_{W_2^{-1+\alpha}(\Omega)} \tag{6.10}$$

and this is nothing else than (4.3).

Now let $f \in W_2^{1-\alpha}$, $\frac{1}{2} > \alpha > 0$ in 4.1, then by analogous arguments in the previous we obtain

$$\|u_h - u\|_{W_2^{1-\alpha}(\Omega)} \leqq C\, h^{-\alpha} \|f\|_{W_2^{-1-\alpha}(\Omega)} \tag{6.11}$$

and

$$\|u_h - u\|^2_{W_2^0(\Omega)} \leqq C\,\|u_h - u\|_{W_2^{1-\alpha}(\Omega)}\,\|Z\|_{W_2^{1+\alpha}(\Omega)} \tag{6.12}$$

with

$$\|Z\|_{W_2^{1+\alpha}(\Omega)} \leqq \|u_h - u\|_{L_2(\Omega)} \cdot h^{\gamma-\alpha}, \qquad 0<\gamma<\tfrac{1}{2}$$

and so we have

$$\|u_h - u\|_{W_2^0(\Omega)} \leqq C\,h^{-\alpha}\,\|f\|_{W_2^{-1-\alpha}(\Omega)}\,h^{\gamma-\alpha} = C\,h^{\gamma-2\alpha}\,\|f\|_{W_2^{-1-\alpha}(\Omega)}. \tag{6.13}$$

Using Lemma 3.4 for (6.11) and (6.13) we obtain

$$\|u_h - u\|_{W_2^{\beta}(\Omega)} \leqq C\,\|f\|_{W_2^{-1-\alpha}(\Omega)}\,(h^{-\alpha})^{\frac{\beta}{1-\alpha}}\,(h^{\gamma-2\alpha})^{\frac{1-\alpha-\beta}{1-\alpha}} = C\,\|f\|_{W_2^{-1-\alpha}(\Omega)}\,h^{\gamma-2\alpha+\frac{\beta(\alpha-\gamma)}{1-\alpha}}$$

and this is exactly (4.4).

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