Calc. Var. 1, 87–121 (1993)

Variational problems for maps of bounded variation with values in S^1 *

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Received August 4, 1992/Accepted September 6, 1992

Abstract. The main goal of this paper is to characterize the weak limits of sequences of smooth maps from a Riemannian manifold into S^1. This is achieved in terms of Cartesian currents. Applications to the existence of minimizers of area type functionals in the class of maps with values in S^1 satisfying Dirchlet and homological conditions are then discussed. The so called dipole problem is solved, too.

Mathematics Subject Classification: 49Q20

The main goal of this paper is to identify the weak limits of sequences of smooth maps from a Riemannian manifold into S^1 with equibounded total variations of the gradients. In doing that we have in mind and we aim to study variational problems for integrals of the type of the total variation

$$\int_{\Omega} |Du(x)|\, dx \tag{1}$$

among maps from some open set Ω of a Riemannian manifold $\mathcal{X}$, with or without boundary, into S^1, satisfying suitable homological restrictions and boundary conditions, if $\partial\Omega$ is not empty. Typical problems we want to consider are such as minimizing the integral (1)

(i) among maps satisfying periodicity conditions as $u(x + 2\pi) = u(x)$, i.e. among maps from a torus into S^1.
(ii) among maps from S^1 into S^1 with prescribed degree and prescribed values on a fixed subset Σ of S^1.
(iii) among maps $u : \mathcal{X} \to S^1$ with prescribed homology map

$$u_* \;:\; H_1(\mathcal{X}, \mathbb{Z}) \longrightarrow H_1(S^1, \mathbb{Z}) \simeq \mathbb{Z}$$

(iv) among maps with prescribed homological singularities, the so-called dipole problem.

* This work has been partially supported by the Ministero dell'Universitá e della Ricerca Scientifica, by C.N.R. contract n. 91.01343.CT01, and by the European Research Project GADGET. It was partially carried out while the first and the third author were visiting the Mathematisches Institut der Universität Bonn under the support respectively of the Alexander von Humboldt-Stiftung and of the SBF 256.

When dealing with the integral (1) in the *scalar case*, i.e. for maps $u : \Omega \to \mathbb{R}$, it is well known that the natural space to work in is the space $BV(\Omega, \mathbb{R})$ of L^1-functions with distributional derivatives which are Radon measures of bounded total variation. In fact, modulo passing to subsequences, sequences with equibounded L^1-norms and total variations weakly converge to some function $u \in BV(\Omega, \mathbb{R})$ and, on the contrary, every $u \in BV(\Omega, \mathbb{R})$ is the weak limit of one such a sequence.

It is also well-known that the class $BV(\Omega, \mathbb{R})$ can be described as the class of n-dimensional rectifiable currents in $\Omega \times \mathbb{R}$ which arise as boundaries of $(n+1)$-dimensional currents defined by integration of $(n+1)$-forms over subgraphs of functions u, more precisely as the class

$$\left\{ G_u := (-1)^n \partial [\![SG_u]\!] \mid G_u \text{ is } n\text{-rectifiable, } \mathbf{M}(G_u) < \infty \right\} \tag{2}$$

where

$$SG_u \; := \; \left\{ (x,y) \in \Omega \times \mathbb{R} \mid y < u(x) \right\} \tag{3}$$

compare [17] [8] [9] [10].

There is still another equivalent way of defining the class $BV(\Omega, \mathbb{R})$ in terms of the so-called Cartesian currents, introduced and studied in [13] [14], compare also [16]. One sees, compare Sect. 3, that $BV(\Omega, \mathbb{R})$ can be identified as the class

$$\begin{aligned} \text{cart}(\Omega \times \mathbb{R}) := \big\{ & T \in \mathcal{D}_n(\Omega \times \mathbb{R}) | T \text{ is } n\text{-rectifiable, } \partial T \llcorner (\Omega \times \mathbb{R}) = 0, \\ & \pi_\# T = [\![\Omega]\!], T \llcorner dx \geq 0, \; \|T\|_1 < \infty, \; \mathbf{M}(T) < \infty \big\} \end{aligned} \tag{4}$$

where π denotes the linear projection of $\Omega \times \mathbb{R}$ into Ω, $\mathbf{M}(T)$ the mass of T, and

$$\|T\|_1 \; := \; \sup \left\{ T(\phi(x,y)|y| \, dx) \mid \phi \in C^\infty_c(\Omega \times \mathbb{R}), \; |\phi| \leq 1 \right\}$$

is the equivalent of the L^1-norm of u.

When dealing with variational problems for maps into S^1, one can think of lifting the problem of functions v with real values by setting

$$u = (\cos v, \sin v) \tag{5}$$

and by considering the lifted variational problem. Unfortunately one cannot always write a function $u \in C^1(\mathcal{X}, S^1)$ in the form (5), $v \in C^1(\mathcal{X}, \mathbb{R})$, an obstruction (and in fact the only one, compare Sect. 4) being the non triviality of the homology map

$$u_* : H_1(\mathcal{X}, \mathbb{Z}) \longrightarrow H^1(S^1, \mathbb{Z})$$

as the identity map from S^1 into S^1 shows. We are then forced to work with maps with values into S^1; but, in this case, it turns out that the three ways of describing functions with bounded variation are *not* equivalent.

Essentially because of the previous obstruction we cannot define BV-maps with values into S^1 as *boundaries of relative subgraphs* as the identity map from S^1 into S^1 again shows, compare Sect. 4. Naively, we may think of defining $BV(\Omega, S^1)$ as the class of maps $BV(\Omega, \mathbb{R}^2)$ such that $u(x) \in S^1$ for almost every $x \in \Omega$. However such a definition is not intrinsic. In fact in this case, while the absolutely continuous part of the gradient of u, $(Du)^a(x)$ maps $T_x\mathcal{X}$ to $T_{u(x)}S^1$, for the jump part $(Du)^J$ we may find

$$(Du)^J = \nu_{J_u} \otimes \mathbf{t}$$

where ν_{J_u} is the normal to the *jump set* J_u of u, but $\mathbf{t}$ is *not* tangential to S^1. Alternatively we may consider the absolutely continuous part of u and try to *close its jumps in* S^1. But soon one realizes that there are several possible ways of closing jumps and different sequences may have graphs converging to substantially different objects which however have the same absolutely continuous part. Thus whatever is the chosen way of closing the jumps in S^1 of a function $u \in BV(\Omega, \mathbb{R}^2)$ with $u(x) \in S^1$ for a.e. x, this will not allow to identify the limits of smooth sequences.

As pointed out and discussed in [14] [13] [16] in connection with the Dirichlet integral in the vector valued case, as it is essentially the case if we map into a manifold, even a one-dimensional manifold, weak convergence produces in general on one hand *concentrations* which cannot be described in terms of their projections into Ω and on the other hand loss in the limit of homological properties of the approximating maps. As already showed in those papers a reasonable way of overcoming these difficulties is to work in the setting of Cartesian currents.

Our aim is to illustrate this approach and show how it works in the case of maps of bounded total variation with values into S^1. Despite the simple geometric structure of the target manifold S^1, in the present case, in contrast with the case of maps with equibounded Dirichlet energies, we will have to confront further difficulties due to the appearence of jumps and Cantor parts in the limit. However we shall see that there are strong similarities, if not identity, with the case of maps into S^2 with bounded Dirichlet energies.

Before briefly discussing our results, let us mention two simple examples which may motivate further our approach and clarify the context in which we are going to work.

Example 1 Consider the sequence of smooth maps $u_k^{(1)}(t) := (\cos kt, \sin kt)$ for $0 \leq t \leq 2\pi/k$ and by $u_k^{(1)}(t) := (1,0)$ otherwise. Clearly $\{u_k^{(1)}\}$ is a sequence of maps with equibounded total variation

$$V(u_k^{(1)}) := \int_{-\pi}^{\pi} |Du_k^{(1)}|\, dx = 2\pi$$

which converges weakly in $BV((-\pi,\pi), \mathbb{R}^2)$ to the constant map $u_0 : [-\pi,\pi] \to S^1$, $u_0(t) = (1,0)$; of course $V(u_0) = 0$. Also regarding the $u_k^{(1)}$ as maps from S^1 into S^1, each $u_k^{(1)}$ has degree 1 while the BV-limit is constant, hence of degree zero.

If we instead look at the *line integrals* over the graphs of the $u_k^{(1)}$'s, $G_{u_k^{(1)}}$, it is easily seen that

$$G_{u_k^{(1)}} \rightharpoonup G_{(1,0)} + \delta_0 \times [\![S^1]\!], \qquad \delta_0 = \text{Dirac mass at zero}$$

in the sense of currents in $[-\pi,\pi] \times S^1$, or better in $S^1 \times S^1$, the degree of the limit is again 1, and computing the total variation of $T := G_{(1,0)} + \delta_0 \times S^1$ in the Lebesgue spirit as the *relaxed functional*

$$V(T) := \inf \{ \liminf_{k\to\infty} V(v_k) \,|\, v_k \in C^1([-\pi,\pi], S^1), G_{v_k} \rightharpoonup T \},$$

then $V(T) = 2\pi$, compare Sect. 5.

Consider now the sequence $\{u_k^{(2)}\}$ defined as $u_k^{(2)}(t) := (\cos kt, \sin kt)$ for $0 \leq t \leq 4\pi/k$ and as $u_k^{(2)}(t) = (1,0)$ otherwise. Clearly $V(u_k^{(2)}) = 4\pi$, $\deg u_k^{(2)} = 2$, but again

$u_k^{(2)}$ converge weakly in BV to the constant map $u_0(t) = (1, 0)$. If we instead consider the limit of the graphs of the $u_k^{(2)}$'s we have

$$G_{u_k^{(2)}} \rightharpoonup G_{u_0} + 2\delta_0 \times [\![S^1]\!].$$

Example 1 shows in a very simple case concentration, loss of degree and energy under BV-weak convergence and how those phenomena are handled by the *stronger* convergence of the graphs G_{u_k} with respect to the BV-convergence of the components of u_k.

Example 2 Consider the map $u(x) := \frac{x}{|x|}$ from the unit disk $B(0,1)$ of $\mathbb{R}^2$ into S^1. Clearly $u \in W^{1,p}(B(0,1), S^1)$ for all $p < 2$. Its graphs has a "hole" over 0 and in fact we have in $B(0,1) \times S^1$

$$\partial G_{\frac{x}{|x|}} = -\delta_0 \times [\![S^1]\!].$$

Let $\{u_k\}$ be a sequence of smooth maps from $B(0,1)$ into S^1 such that

$$\begin{aligned} &u_k \longrightarrow u \quad \text{strongly in} \quad L^1(B(0,1), S^1) \\ &\sup_k \int_{B(0,1)} |Du_k(x)|\, dx < \infty\,. \end{aligned} \tag{6}$$

We shall see that there exists a 1-dimensional current L in $B(0,1)$ such that

$$G_{u_k} \rightharpoonup G_{\frac{x}{|x|}} + L \times S^1\,. \tag{7}$$

This time the concentration occurs over a 1-dimensional rectifiable current L with the property that $\partial L \llcorner B(0,1) = \delta_0$. Actually we shall see that for every 1-dimensional rectifiable current L in $B(0,1)$ with $\partial L \llcorner B(0,1) = \delta_0$, one can find a sequence of smooth maps $\{u_k\}$ such that (6) and (7) hold. Notice that instead we have

$$u_k \rightharpoonup \frac{x}{|x|} \quad \text{in } BV(B(0,1), \mathbb{R}^2).$$

This paper is organized as follows. After the two preliminary Sects. 2 and 3, where we prove a few simple results to be used later and that

$$BV(\Omega, \mathbb{R}) \simeq \text{cart}(\Omega \times \mathbb{R}),$$

we shall discuss the class $\text{cart}(\Omega \times S^1)$ in Sect. 4. There we shall prove a *structure theorem* for Cartesian currents $T \in \text{cart}(\Omega \times S^1)$ and an *approximation theorem*, stating that each $T \in \text{cart}(\Omega \times S^1)$ can be approximated by a sequence of smooth maps $u_k : \Omega \to S^1$ in such a way that

$$\begin{aligned} &G_{u_k} \rightharpoonup T \quad \text{in } \mathcal{D}_n(\Omega \times S^1) \\ &\mathbf{M}(G_{u_k}) \longrightarrow \mathbf{M}(T). \end{aligned} \tag{8}$$

In Sect. 5 we shall deal with the *relaxed functional* of general integrals and in particular of the area of graphs of maps with values in S^1. We shall give an integral representation of the relaxed functional in $\text{cart}(\Omega \times S^1)$; instead, we shall see that the relaxed functional in $\{\, u \in BV(\Omega, \mathbb{R}^2) \mid |u(x)| = 1 \text{ a.e. } x \,\}$ or in $L^1(\Omega, S^1)$ are not integral functionals. Applications to the existence of minimizers under Dirichlet and homological conditions will be presented in Sect. 6, while in Sect. 7 we shall discuss the so-called dipole problem in the case of maps with values in S^1.

We would like to thank Michael Struwe who read the manuscript and suggested a few corrections.

1. Preliminaries

In the sequel we shall think of S^1 as isometrically embedded in $\mathbb{R}^2$ as the unit circle $S^1 := \{(y^1, y^2) \in \mathbb{R}^2 \mid (y^1)^2 + (y^2)^2 = 1\}$. We denote by $\varepsilon_1, \varepsilon_2$ the standard basis in $\mathbb{R}^2$, by e_θ,

$$e_\theta := -y^2\varepsilon_1 + y^1\varepsilon_2,$$

the unit tangent vector which orients S^1 in the usual anticlockwise way, and by Θ the volume 1-form on S^1 chosen in such a way that

$$\int_{S^1} \Theta = 2\pi.$$

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$. We denote by $\mathcal{D}^k(\Omega \times S^1)$, $1 \leq k \leq n+1$, the space of all k-forms with smooth and compactly supported coefficients in $\Omega \times S^1$. If $e_1, \dots, e_n$, denote the standard basis in $\mathbb{R}^n$ and $(x^1, \dots, x^n)$ the coordinates in $\mathbb{R}^n$, any $\omega \in \mathcal{D}^k(\Omega \times S^1)$ can be written as

$$(1)\qquad \omega = \sum_{|\alpha|=k} \omega_\alpha(x,\theta)\, dx^\alpha + \sum_{|\alpha|=k-1} \eta_\alpha(x,\theta)\, dx^\alpha \wedge \Theta =: \omega^{(0)} + \omega^{(1)}$$

with

$$\omega_\alpha(x,\theta), \eta_\alpha(x,\theta) \in C_c^\infty(\Omega \times S^1, \mathbb{R}).$$

We use the standard notation for multiindeces: $\alpha = (\alpha_1, \dots, \alpha_p)$, $1 \leq \alpha_1 \leq \dots \leq \alpha_p \leq n$, $\alpha_i \in \mathbb{N}$, $|\alpha| := p$ and $dx^\alpha := dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p}$.

The expression (1) says in particular that the product structure in $\Omega \times S^1$ yields a canonical splitting of ω as $\omega = \omega^{(0)} + \omega^{(1)}$ where $\omega^{(0)}$ is a k-form with no differential with respect to θ and $\omega^{(1)}$ is a linear combination of k-differentials which contain Θ as a factor. For convenience we also introduce

$$\mathcal{D}^{k,0}(\Omega \times S^1) := \{\omega \in \mathcal{D}^k(\Omega \times S^1) \mid \omega = \omega^{(0)}\}$$
$$\mathcal{D}^{k,1}(\Omega \times S^1) := \{\omega \in \mathcal{D}^k(\Omega \times S^1) \mid \omega = \omega^{(1)}\}$$

so that

$$\mathcal{D}^k(\Omega \times S^1) = \mathcal{D}^{k,0}(\Omega \times S^1) \oplus \mathcal{D}^{k,1}(\Omega \times S^1).$$

Similarly the exterior differential operator

$$d \,:\, \mathcal{D}^k(\Omega \times S^1) \longrightarrow \mathcal{D}^{k+1}(\Omega \times S^1) \qquad 0 \leq k \leq n$$

splits as the sum

$$d = d_x + d_\theta.$$

For a function $f \in \mathcal{D}^0(\Omega \times S^1)$

$$d_x f(x,\theta) = \sum_i \frac{\partial f}{\partial x^i}\, dx^i, \qquad d_\theta f(x,\theta) = \frac{\partial f}{\partial \theta}\, \Theta.$$

We note that for $0 \le k \le n$

$$\begin{aligned} d_x &: \mathcal{D}^{k,0}(\Omega \times S^1) \longrightarrow \mathcal{D}^{k+1,0}(\Omega \times S^1) \\ d_x &: \mathcal{D}^{k,1}(\Omega \times S^1) \longrightarrow \mathcal{D}^{k+1,1}(\Omega \times S^1) \\ d_\theta &: \mathcal{D}^{k,0}(\Omega \times S^1) \longrightarrow \mathcal{D}^{k+1,1}(\Omega \times S^1) \end{aligned}$$

and

$$d_\theta \omega = 0 \qquad \forall \omega \in \mathcal{D}^{k,1}(\Omega \times S^1).$$

The orthogonal projection $\pi : \Omega \times S^1 \to \Omega$, $\pi(x,\theta) = x$ yields a natural injection

$$\pi^{\#} : \mathcal{D}^k(\Omega) \longrightarrow \mathcal{D}^k(\Omega \times S^1)$$

by just considering the coefficients $f(x)$ of $\omega \in \mathcal{D}^k(\Omega)$ as functions of two variables (x,θ) constant in θ. Of course $f(x)$ has compact support in $\Omega \times S^1$ since S^1 is compact. For the sake of simplicity in the sequel the injection $\pi^{\#}$ will be understood and we shall regard forms in Ω as forms in $\Omega \times S^1$. Analogously, the projection $\hat{\pi} : \Omega \times S^1 \to S^1$ yields an injection from $\mathcal{D}^k(S^1)$, $k = 0, 1$, but in $\mathcal{B}^k(\Omega \times S^1)$, the space of k-forms with *bounded* and smooth coefficients. In fact, regarding coefficients as functions in (x,θ) which are constant with respect to x they will not have compact support since Ω is not compact.

In dealing with k-forms in $\Omega \times S^1$, besides considering them as restriction of forms in $\Omega \times \mathbb{R}^2$, it is convenient to introduce the *covering map*

$$i : \Omega \times \mathbb{R} \longrightarrow \Omega \times S^1$$

given by

$$i(x,t) = (x, \cos t, \sin t).$$

Clearly i defines an isomorphism between smooth functions in $\Omega \times S^1$ and smooth functions $f(x,t)$ in $\Omega \times \mathbb{R}$ which are 2π-periodic in the second variable $t \in \mathbb{R}$. Also, if $f : \Omega \times S^1 \to \mathbb{R}$ has compact support in $\Omega \times S^1$, then the lift $f \circ i$ in $\Omega \times \mathbb{R}$ has support in $\widetilde{\Omega} \times \mathbb{R}$ where $\widetilde{\Omega} \subset\subset \Omega$. Denoting by $\mathcal{B}^k_{2\pi}(\Omega \times \mathbb{R})$ the set of all k-forms with smooth coefficients $f_\alpha(x,t)$ which are bounded, 2π-periodic in t and are supported in $\widetilde{\Omega} \times \mathbb{R}$ for some $\widetilde{\Omega} \subset\subset \Omega$, then the lift

$$i^{\#} : \mathcal{D}^k(\Omega \times S^1) \longrightarrow \mathcal{B}^k_{2\pi}(\Omega \times \mathbb{R})$$

yields a bijection between the two spaces. Moreover $i^{\#}$ commutes with the exterior differential operator and with respect to the decomposition $\omega = \omega^{(0)} + \omega^{(1)}$, and

$$\sup |\omega| = \sup |i^{\#}\omega|$$

Consider now a 1-form α on S^1 and denote by $j : \mathbb{R} \to S^1$ the map

$$j(t) := (\cos t, \sin t).$$

Then we have $j^{\#}(\alpha) = f(t)\, dt$ for some smooth, 2π-periodic function $f(t)$. Set

$$\bar{f} := \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt, \qquad g(t) := \int_0^t [f(s) - \bar{f}]\, ds.$$

Then clearly $f(t) = \bar{f} + g'(t)$ and we can write $j^{\#}(\alpha) = \bar{f}\,dt + dg$. Since g is 2π-periodic we can finally state

$$\alpha = \bar{f}\,\Theta + dg. \tag{2}$$

Finally it is easily seen that if (2) holds then $\bar{f}$ is uniquely defined by α and g is uniquely defined up to a constant. The previous relation (2), which is nothing else the Hodge-Kodaira decomposition theorem in the very special case of 1-forms in S^1, gives us a way to decompose forms in $\mathcal{D}^{k,1}(\Omega \times S^1)$ by just considering the x-variable as a parameter (compare [16]).

Proposition 1 *Let $\omega \in \mathcal{D}^{k,1}(\Omega \times S^1)$. Then there exist $\bar{\omega} \in \mathcal{D}^{k-1}(\Omega)$ and $\eta \in \mathcal{D}^{k-1,0}(\Omega \times S^1)$ such that*

$$\omega = \bar{\omega} \wedge \Theta + d_\theta \eta. \tag{3}$$

Moreover $\bar{\omega}$ is unique and η is unique in $\mathcal{D}^{k-1,0}(\Omega \times S^1)$ up to a $(k-1)$-form in Ω. In particular (3) gives a unique decomposition if we add a condition on η such as $\eta(x,\theta) = 0$ for $\theta = 0$ or $\eta(x,y) = 0$ for $y = (1,0)$.

Proof For $\omega \in \mathcal{D}^{k,1}(\Omega \times S^1)$ we can write $i^{\#}(\omega)$ as

$$i^{\#}(\omega) = \sum_{|\alpha|=k-1} \omega_\alpha(x,t)\,dx^\alpha \wedge dt$$

where $\omega_\alpha(x,t)$ are smooth, bounded, 2π-periodic, and supported in $\widetilde{\Omega} \times \mathbb{R}$ for a suitable open set $\widetilde{\Omega} \subset\subset \Omega$. Define now

$$\bar{\omega}_\alpha(x) := \frac{1}{2\pi} \int_0^{2\pi} \omega_\alpha(x,t)\,dt, \qquad \eta_\alpha(x,t) = \int_0^t [\omega_\alpha(x,s) - \bar{\omega}(x)]\,ds$$

and

$$\bar{\omega}(x) := \sum_{|\alpha|=k-1} \bar{\omega}_\alpha(x)\,dx^\alpha \qquad \eta(x,t) := (-1)^{k-1} \sum_{|\alpha|=k-1} \eta_\alpha(x,t)\,dx^\alpha.$$

Then clearly $\bar{\omega}(x) \in \mathcal{D}^{k-1}(\Omega)$, $\eta(x,t) \in \mathcal{D}^{k-1,0}(\Omega \times \mathbb{R})$ and

$$i^{\#}(\omega) = \bar{\omega} \wedge dt + d_t \eta. \tag{4}$$

Since the $\eta_\alpha(x,t)$ are 2π-periodic in t we may think of η as a form in $\mathcal{D}^{k-1,0}(\Omega \times S^1)$, thus (4) is equivalent to (3). This concludes the proof as the unicity follows from the above construction. □

A simple consequence of the previous decomposition theorem is the following

Proposition 2 *Let $T \in \mathcal{D}_k(\Omega \times S^1)$ be a k-current in $\Omega \times S^1$ without boundary in $\Omega \times S^1$, i.e., $\partial T \llcorner \Omega \times S^1 = 0$. Then $T = 0$ if and only if*

$$T(\omega) = 0 \qquad \forall \omega \in \mathcal{D}^{k,0}(\Omega \times S^1) \quad \text{and} \quad T(\alpha(x) \wedge \Theta) = 0 \qquad \forall \alpha \in \mathcal{D}^{k-1}(\Omega)\,.$$

Proof Split $\omega \in \mathcal{D}^k(\Omega \times S^1)$ as $\omega = \omega^{(0)} + \omega^{(1)}$, and decompose $\omega^{(1)}$ as in Proposition 1. We get

$$\omega = \omega^{(0)} + \omega^{(1)} = \omega^{(0)} + \alpha(x) \wedge \Theta + d_\theta \eta =$$
$$= \omega^{(0)} - d_x \eta + \alpha(x) \wedge \Theta + d\eta$$

for suitable $\alpha \in \mathcal{D}^{k-1}(\Omega)$ and $\eta \in \mathcal{D}^{k-1,0}(\Omega \times S^1)$. Thus

$$T(\omega) = T(\omega^{(0)} - d_x\eta) + T(\alpha(x) \wedge \Theta) + \partial T(\eta) =$$
$$= T(\omega^{(0)} - d_x\eta) + T(\alpha(x) \wedge \Theta).$$

The result then readily follows as $d_x\eta \in \mathcal{D}^{k,0}(\Omega \times S^1)$. □

Denoting by $T_{(0)}$ the zero component of $T \in \mathcal{D}_k(\Omega \times S^1)$

$$T_{(0)}(\omega) := T(\omega^{(0)}), \qquad \omega = \omega^{(0)} + \omega^{(1)}$$

and introducing the $(k-1)$-dimensional current in Ω

$$L_T := \frac{1}{2\pi}\, \pi_\#(T \llcorner \Theta),$$

more explicitly

$$L_T(\alpha) := \frac{1}{2\pi}\, T(\alpha(x) \wedge \Theta),$$

we are then led to the following decomposition theorem for k-currents in $\Omega \times S^1$

Proposition 3 *Let $T \in \mathcal{D}_k(\Omega \times S^1)$ and $\partial T \llcorner \Omega \times S^1 = 0$. Then for any $\omega \in \mathcal{D}^k(\Omega \times S^1)$ we have*

$$T(\omega) = T_{(0)}(\omega - d_x\eta) + L \times [\![S^1]\!](\omega - d_\theta\eta)$$

η being the form in the decomposition

$$\omega = \omega^{(0)} + \tilde{\omega}(x) \wedge \Theta + d_\theta\eta.$$

Proof From the proof of Proposition 2 we get

$$T(\omega) = T(\omega^{(0)} - d_x\eta) + T(\tilde{\omega} \wedge \Theta).$$

On the other hand

$$L_T \times [\![S^1]\!](\omega - d\eta) = L_T \times [\![S^1]\!](\omega^{(1)} - d_\theta\eta) = L_T \times [\![S^1]\!](\tilde{\omega} \wedge \Theta) =$$
$$= L_T(\tilde{\omega})\, [\![S^1]\!](\Theta) = 2\pi\, L_T(\tilde{\omega}) = T(\tilde{\omega} \wedge \Theta).$$

□

2. Graphs of functions in $BV(\Omega, \mathbb{R})$

Let Ω be a domain in $\mathbb{R}^n$. To each function $u \in L^1(\Omega)$ we can associate its *subgraph* defined by

(1) $$SG_u := \{(x, y) \in \Omega \times \mathbb{R} \mid y < u(x)\}.$$

We then denote by G_u the n-dimensional current in $\mathcal{D}_n(\Omega \times \mathbb{R})$

$$G_u := (-1)^n \partial [\![SG_u]\!]. \tag{2}$$

In the case that u is a smooth function, it is easily seen that G_u is just the current given by integration of n-forms in $\Omega \times \mathbb{R}$ over the graph of u, given by the formulas

$$\begin{aligned} G_u(\phi(x,y)\,dx) &= \int_\Omega \phi(x,u(x))\,dx \\ G_u(\phi(x,y)\,\widehat{dx^i} \wedge dy) &= (-1)^{n-i} \int_\Omega \phi(x,u(x))\,D_i u\,dx \end{aligned} \tag{3}$$

In fact, if $\omega = \phi(x,y)\,dx$, then

$$d\omega = (-1)^n \phi_y(x,y)\,dx \wedge dy,$$

thus

$$G_u(\omega) = (-1)^n [\![SG_u]\!](d\omega) = \int_\Omega dx \int_{-\infty}^{u(x)} \phi_y(x,y)\,dy = \int_\Omega \phi(x,u(x))\,dx. \tag{4}$$

If $\omega = \phi(x,y)\,\widehat{dx^i} \wedge dy$,

$$d\omega = (-1)^{i-1} \phi_{x^i}(x,y)\,dx \wedge dy,$$

thus

$$G_u(\omega) = (-1)^n [\![SG_u]\!](d\omega) = (-1)^{n+i-1} \int_\Omega dx \int_{-\infty}^{u(x)} \phi_{x^i}(x,y)\,dy.$$

The function

$$x \longrightarrow \int_{-\infty}^{u(x)} \phi_{x^i}(x,y)\,dy$$

has compact support in Ω and

$$D_i \int_{-\infty}^{u(x)} \phi(x,y)\,dy = \int_{-\infty}^{u(x)} \phi_{x^i}(x,y)\,dy + \phi(x,u(x))\,D_i u(x).$$

Thus the second formula of (3) follows at once.

Let $u \in BV(\Omega,\mathbb{R})$. The relationships between u and G_u defined in (2) are well understood, compare [8] [17] [10, Theorem 4.5.9] [7]. For our purposes we recall the following facts which we collect as Theorem 1.

Denote by J_u the set of points in Ω for which the *approximate limit of* u does not exist and, for $x \in J_u$, set

$$u_+(x) := \operatorname{ap}\limsup_{y \to x} u(y), \qquad u_-(x) := \operatorname{ap}\liminf_{y \to x} u(y).$$

Theorem 1 *Let $u \in BV(\Omega, \mathbb{R})$. Then we have*

(i) J_u is countably $(n-1)$-rectifiable. For $\mathcal{H}^{n-1}$-a.e. $x \in J_u$ the approximate tangent plane to J_u at x splits $\mathbb{R}^n$ into two half-spaces P^+ and P^- so that

$$\operatorname{ap}\limsup_{\substack{y\to x\\ y\in P^+}} u(y) = u_+(x), \quad \operatorname{ap}\liminf_{\substack{y\to x\\ y\in P^-}} u(y) = u_-(x).$$

(ii) Denote by ν_{J_u} the unit normal vector to J_u oriented so that

$$x + \nu_{J_u} \in P^+.$$

Then the distributional gradient of u splits into the mutually singular measures

$$Du = (Du)^{(a)}\, dx + (Du)^{(j)} + (Du)^{(C)},$$

where $(Du)^{(a)}$ is the density of Du with respect to Lebesgue's measure, $(Du)^{(j)}$ corresponds to the jump part

$$(Du)^{(j)} = (u_+ - u_-)\,\nu_{J_u}\, d\mathcal{H}^{n-1},$$

and finally $(Du)^{(C)}$ is the Cantor part which is singular with respect to Lebesgue measure and $(Du)^{(C)}(B) = 0$ for any B with $\mathcal{H}^{n-1}(B) < \infty$. In particular $\mathcal{H}^{n-1}$-a.e.

$$J_u = \{x \mid \frac{d|Du|}{d\mathcal{H}^{n-1}}(x) > 0\}.$$

(iii) SG_u is a set of finite perimeter.

(iv) The current G_u is a rectifiable n-dimensional current with integer multiplicity equal 1. Moreover

$$G_u = \tau(\partial^- SG_u, 1, \mathbf{G}_u)$$

where $\partial^- SG_u$ is the so called reduced boundary of SG_u in $\Omega \times \mathbb{R}$ and $\mathbf{G}_u$ is the unit tangent n-vector to $\partial^- SG_u$ oriented in such a way that

$$\mathbf{G}_u \wedge n(\cdot, \partial^- SG_u) = e_1 \wedge \ldots \wedge e_n \wedge \varepsilon,$$

$n(\cdot, \partial^- SG_u)$ being the inward normal to $\partial^- SG_u$, and ε being the unit vector in the target $\mathbb{R}$.

(v) The inward normal to $\partial^- SG_u$ at $(x, u(x))$ is given by

$$n((x,u(x)), \partial^- SG_u) = \frac{d(Du, -\mathcal{L}^n)}{d\,\|\,(Du, -\mathcal{L}^n)\,\|}(x);$$

in particular for $x \in J_u$

$$n((x,s), \partial^- SG_u) = (\nu_{J_u}(x), 0)$$

for all s with $u_-(x) \le s \le u_+(x)$.

(vi) The total variation of G_u, or mass of G_u, can be computed by testing G_u on forms with coefficients which are constant in t

$$\mathbf{M}(G_u) = \sup\{G_u(\omega) \mid \omega = \sum_{i-1}^{n} \omega_i(x)\,\widehat{dx^i} \wedge dt + \omega_0(x)\, dx, \sum_{i=0}^{n} \omega_i^2 \le 1\}$$

or, in other words,

$$\| G_u \| = \| (Du, -\mathcal{L}^n) \|.$$

In dealing with variational problems for vector valued maps we introduced in [14] [13] [16] the so called class of *Cartesian currents* which may be regarded as a class of generalized graphs with almost everywhere tangent plane and without "interior" boundaries. In the special case of maps from Ω into $\mathbb{R}$ we have

Definition 1 *The class of Cartesian currents in $\Omega \times \mathbb{R}$ is given by*

$$\begin{aligned} \mathrm{cart}(\Omega \times \mathbb{R}) := \{ T \in \mathcal{D}_n(\Omega \times \mathbb{R}) \mid & T \text{ is } n\text{-rectifiable}, \partial T \llcorner (\Omega \times \mathbb{R}) = 0 \\ & \pi_\# T = [\![\Omega]\!], T \llcorner dx \geq 0, \mathbf{M}(T) < \infty, \| T \|_1 < \infty \} \end{aligned}$$

where

$$\| T \|_1 := \sup\{ T(\phi(x,y)|y|\,dx) \mid \phi \in C_c^\infty(\Omega \times \mathbb{R}), |\phi| \leq 1 \}.$$

In [14] [13] we proved that to any $T \in \mathrm{cart}(\Omega \times \mathbb{R})$ we can associate a function $u_T \in BV(\Omega, \mathbb{R})$ such that

$$\begin{aligned} T(\phi(x,y)\,dx) &= \int_\Omega \phi(x, u_T(x))\,dx \\ \| T \|_1 &= \int_\Omega |u_T|\,dx. \end{aligned} \tag{5}$$

Actually, in the scalar case as we are dealing in this section, we have that $\mathrm{cart}(\Omega \times \mathbb{R})$ agrees with the space of "graphs" of BV-functions in the sense made precise by the next theorem.

Theorem 2 *The map*

$$G : u \longrightarrow G_u$$

which associates to each function u the current G_u defined in (2) *maps $BV(\Omega, \mathbb{R})$ into* $\mathrm{cart}(\Omega \times \mathbb{R})$. *Moreover $G : BV(\Omega, \mathbb{R}) \to \mathrm{cart}(\Omega \times \mathbb{R})$ is onto and injective. More precisely, for every $T \in \mathrm{cart}(\Omega \times \mathbb{R})$ we have $T = G_{u_T}$, u_T being the function in $BV(\Omega, \mathbb{R})$ defined in* (5).

Proof Given $u \in BV(\Omega)$, the rectifiability of G_u follows from Theorem 1 (i) (iv). The computation in (4) yields

$$G_u(\phi(x,y)\,dx) = \int_\Omega \phi(x, u(x))\,dx,$$

thus $G_u \llcorner dx \geq 0$, $\pi_\# G_u = [\![\Omega]\!]$ and $[\![G_u]\!]_1 = \int_\Omega |u|\,dx$, i.e., $G_u \in \mathrm{cart}(\Omega \times \mathbb{R})$. Let $u, v \in BV(\Omega, \mathbb{R})$ and $G_u = G_v$. Then in particular $G_u = G_v$ on forms of the type $\phi(x,y)\,dx$, hence

$$\int_\Omega \phi(x,u)\,dx = \int_\Omega \phi(x,v)\,dx \qquad \forall \phi \in C_c^\infty(\Omega \times \mathbb{R}).$$

This obviously implies $u = v$.

Finally, let $T \in \mathrm{cart}(\Omega \times \mathbb{R})$ and let $u_T \in BV(\Omega, \mathbb{R})$ be the function in (5). For any $\omega \in \mathcal{D}^{n,0}(\Omega \times S^1)$, $\omega = \phi(x,y)\,dx$, we have

$$G_{u_T}(\omega) = \int_\Omega \phi(x, u_T)\, dx = T(\omega).$$

Next proposition states then that $G_{u_T} = T$. This shows that G maps $BV(\Omega, \mathbb{R})$ onto $\mathrm{cart}(\Omega \times \mathbb{R})$ and concludes the proof. □

Proposition 1 *Let T be a boundaryless n -dimensional current in $\Omega \times \mathbb{R}$ with finite mass. If $T(\omega) = 0\ \forall \omega \in \mathcal{D}^{n,0}(\Omega \times S^1)$, then $T = 0$.*

Proof Clearly, it suffices to show that $T(\omega) = 0$ for any ω of the type

$$\omega(x, y) = \varphi(x, y)\, \widehat{dx^i} \wedge dy \qquad \varphi \in C_c^\infty(\Omega \times \mathbb{R}), i = 1, \dots, n.$$

Set

$$\phi(x, y) := \int_{-\infty}^{y} \varphi(x, s)\, ds$$
$$\xi := \phi(x, y)\, \widehat{dx^i}.$$

As ϕ is bounded and supported in $\widetilde{\Omega}\times\mathbb{R}$ for some $\widetilde{\Omega}\subset\subset\Omega$, ξ belongs to $\mathcal{B}^{n-1,0}(\Omega\times\mathbb{R})$. We have

$$d_y \xi = (-1)^{n-1} \omega$$

thus

$$d\xi = d_x \xi + (-1)^{n-1} \omega.$$

T being of finite mass, we therefore conclude

$$T(\omega) = (-1)^{n-1} [T(d\xi) - T(d_x \xi)] = (-1)^n T(d_x \xi) = 0,$$

i.e. the claim follows, since $d_x \xi \in \mathcal{D}^{n,0}(\Omega \times \mathbb{R})$. □

Remark 1 The previous two results show that G_u is completely identified by the function $u \in BV(\Omega, \mathbb{R})$ and that every $T \in \mathrm{cart}(\Omega \times \mathbb{R})$ is completely identified by the function $u_T \in BV(\Omega, \mathbb{R})$ associated to T by (5). In the next section we shall see that this is not anymore true if we replace $\mathbb{R}$ by S^1.

We conclude this section by stating a *structure theorem* for Cartesian currents in $\mathrm{cart}(\Omega \times \mathbb{R})$, which is a simple consequence of Theorem 1.

Let $T \in \mathrm{cart}(\Omega \times \mathbb{R})$. We know that $T = G_u$ for some $u \in BV(\Omega, \mathbb{R})$ and that $T = \tau(\mathcal{M}, 1, \mathbf{T})$ for some n-rectifiable set $\mathcal{M}$. Denote by $\mathcal{M}_+$ the set of points $z \in \mathcal{M}$ at which the tangent plane $\mathrm{Tan}_z \mathcal{M}$ is not vertical or equivalently the projection map π restricted to $\mathrm{Tan}_z \mathcal{M}$ has maximal rank. Then by Theorem 1 (ii)

$$J_u = \{x \mid \frac{d\pi_\# \| T \|}{d\mathcal{H}^{n-1}}(x) > 0\}.$$

We then set

$$T^{(a)} := T \llcorner \mathcal{M}_+$$
$$T^{(j)} := T \llcorner (J_u \times \mathbb{R})$$
$$T^{(C)} := T \llcorner (\mathcal{M} \setminus (\mathcal{M}_+ \cup J_u \times \mathbb{R})).$$

Obviously T decomposes as

$$T = T^{(a)} + T^{(j)} + T^{(C)}$$

and the three measures $\| T^{(a)} \|$, $\| T^{(j)} \|$ and $\| T^{(C)} \|$ are mutually singular.

On n-forms of the type $\phi(x,t)\,dx$ we have

$$T^{(a)}(\phi(x,t)\,dx) = \int_\Omega \phi(x,u(x))\,dx$$
$$T^{(j)}(\phi(x,t)\,dx) = T^{(C)}(\phi(x,t)\,dx) = 0,$$

while we have

Proposition 2 *For any n-form $\omega = \phi(x,t)\widehat{dx^i} \wedge dt$, $i = 1, \ldots, n$, we have*

$$T^{(a)}(\omega) = (-1)^{n-i} \int_\Omega \phi(x,u(x))\,(D_i u)^a(x)\,dx$$

$$T^{(j)}(\omega) = (-1)^{n-i} \int \Big(\int_{u_-(x)}^{u_+(x)} \phi(x,s)\,ds \Big)\, \nu_{J_u i}(x)\, d\mathcal{H}^{n-1} \llcorner J_u$$

$$T^{(C)}(\omega) = (-1)^{n-i} \int \phi(x, u_+(x))\,(D_i u)^{(C)}.$$

3. The class cart($\Omega \times S^1$)

The class of *Cartesian currents* in $\Omega \times S^1$ has been introduced in [14], compare also [13] [16] as follows. We consider the class

$$\text{cart}(\Omega \times \mathbb{R}^2) := \{T \in \mathcal{D}_n(\Omega \times \mathbb{R}^2) \mid T \text{ is } n\text{-rectifiable}, \partial T \llcorner (\Omega \times \mathbb{R}^2) = 0,$$
$$\pi_\# T = [\![\Omega]\!], T \llcorner dx \geq 0, \| T \|_1 < \infty, \mathbf{M}(T) < \infty\}.$$

Then we set

Definition 1 cart$(\Omega \times S^1) := \{T \in \text{cart}(\Omega \times \mathbb{R}^2) \mid \text{spt}\, T \subset \bar{\Omega} \times S^1\}$

From [14], compare also [13] we know

Theorem 1 *Let $T \in$ cart$(\Omega \times S^1)$, $T = \tau(\mathcal{M}, \theta, \mathbf{T})$, and let $\mathcal{M}_+$ denote the set of points z in $\mathcal{M}$ at which the tangent plane* Tan$_z\mathcal{M}$ *does not contain vertical vectors, or, in other words, the Jacobian of the projection map π restricted to* Tan$_z\mathcal{M}$ *has maximal rank. Then we have*

(i) There exists a map $u_T \in BV(\Omega, \mathbb{R}^2)$ with $|u_T| = 1$ a.e. in Ω such that

$$(1) \qquad T(\phi(x,y)\,dx) = \int_\Omega \phi(x, u_T(x))\,dx \qquad \forall \phi \in C_c^\infty(\Omega \times S^1)$$

(ii) $\mathcal{H}^n(\Omega \setminus \pi(\mathcal{M}_+)) = 0$, $\theta(z) = 1$ $\mathcal{H}^n \llcorner \mathcal{M}_+$ a.e., and $\mathcal{H}^n \llcorner \mathcal{M}_+$-a.e.

$$\mathcal{M}_+ = \{(x,y) \mid x \in \pi(\mathcal{M}_+), y = u_T(x)\}.$$

(iii) $T \llcorner \mathcal{M}_+$ is the current with component $T_{(0)}$ given by (1) and first component given by

$$T^{\bar{i}j} \llcorner \mathcal{M}_+(\phi(x,y)) = (-1)^{i-1} \int_\Omega \phi(x, u_T(x))\,(D_i u_T^j)^{(a)}\,dx,$$

$(D_i u_T^j)^{(a)}$ denoting the absolutely continuous part of $D_i u_T^j$ with respect to Lebesgue measure.

(iv) $T \llcorner \mathcal{M}_+ = T$ if and only if

$$(\pi_\# \| T \|)^s = 0.$$

where μ^s denotes the singular part of the measure μ.

The examples in the introduction show that in general $T \llcorner (\mathcal{M} \setminus \mathcal{M}_+)$ is non zero and cannot be recovered from $T \llcorner \mathcal{M}_+$, i.e., from u_T; in other words, it is impossible to describe concentrations of limits of sequences of smooth functions with values in S^1 in terms of the BV-limits in $\mathbb{R}^2$.

In order to understand the structure of the elements of cart$(\Omega \times S^1)$ we shall use the covering map

$$i \;:\; \Omega \times \mathbb{R} \longrightarrow \Omega \times S^1$$

and the lift operator $i_\#$.

Proposition 1 *We have*

(i) The lift $i_\#$ maps cart$(\Omega \times \mathbb{R})$ *into* cart$(\Omega \times S^1)$.

(ii) If $T \in$ cart$(\Omega \times S^1)$ is such that $T = i_\# G_u$ for some $u \in BV(\Omega, \mathbb{R})$ then

$$u_T = (\cos u, \sin u), \tag{2}$$

and

$$\mathbf{M}(T) = \mathbf{M}(G_u). \tag{3}$$

(iii) If $T, T' \in$ cart$(\Omega \times \mathbb{R})$ and $i_\# T = i_\# T'$, then

$$T' = \tau_{k\#} T$$

for some $k \in \mathbb{Z}$, where τ_k denotes the translation map $(x,t) \to (x, t + 2k\pi)$.

Proof Let $T \in$ cart$(\Omega \times \mathbb{R})$. Since T has finite mass, it acts on all forms ω with bounded and continuous coefficients in $\Omega \times \mathbb{R}$, in particular on forms in $\mathcal{B}^n_{2\pi}(\Omega \times \mathbb{R})$. Thus $i_\# T \in \mathcal{D}_n(\Omega \times S^1)$. Since $\pi \circ i = \pi$, we deduce $\pi_\# i_\# T = \pi_\# T = [\![\Omega]\!]$ and $i_\# T \llcorner dx \geq 0$. Finally from $\| \Lambda_n(i) \| = 1$ we deduce

$$\mathbf{M}(i_\# T) \leq \mathbf{M}(T);$$

on the other hand, taking into account Theorem 1 (iii) of Sect. 2, we have

$$\begin{aligned}\mathbf{M}(T) &= \sup\{T(\omega) \mid \omega = \omega_0(x)dx + \sum_{i=1}^n \omega_i(x)\widehat{dx^i} \wedge dt, \sum_{i=0}^n \omega_i^2 \leq 1\} \leq \\ &\leq \sup\{T(\omega) \mid \omega = \omega_0(x,t)dx + \sum_{i=1}^n \omega_i(x,t)\widehat{dx^i} \wedge dt, \sum_{i=0}^n \omega_i^2 \leq 1, \\ &\qquad \omega \in \mathcal{B}^n_{2\pi}(\Omega \times \mathbb{R})\} = \mathbf{M}(i_\# T),\end{aligned}$$

hence $\mathbf{M}(T) = \mathbf{M}(i_\# T)$. This proves (i) and (ii), as (2) is trivial.

Let us finally prove (iii). Denote by u_T and $u_{T'}$ the BV-functions associated to T and T' in such a way that $T = G_{u_T}$ and $T' = G_{u_{T'}}$. Since $i_\# T = i_\# T'$, then T and T' agrees on all forms with coefficients which are independent of t, thus $Du_T = Du_{T'}$, i.e., $u_T - u_{T'} = c \in \mathbb{R}$. On the other hand $(\cos u_T, \sin u_T) = (\cos u_{T'}, \sin u_{T'})$ for a.e. x, hence $c = 2k\pi$ for some $k \in \mathbb{Z}$. □

Remark 1 We notice that in fact the proof of Proposition 1 (iii) yields also
(iii)′ *If* $T, T' \in \text{cart}(\Omega \times \mathbb{R})$ *and* $u_{i_\# T} = u_{i_\# T'}$, *then*

$$T' = \tau_{k\#} T$$

for some $k \in \mathbb{Z}$.

We now ask whether the lift $i_\#$ is onto, equivalently whether every $T \in \text{cart}(\Omega \times S^1)$ can be written as $i_\# G_u$ for some u in $BV(\Omega, \mathbb{R})$, and whether each $T \in \text{cart}(\Omega \times S^1)$ is the "boundary" of an $(n+1)$-dimensional current in $\Omega \times S^1$. The two questions are closely related and, as we shall see, actually equivalent.

Let us start from the second question which needs to be made more precise, in fact even for smooth maps u from Ω into S^1, G_u is never the boundary of an $(n+1)$-dimensional current. In this context it is convenient to replace "subgraphs" by "relative subgraphs". Thus we fix for instance the constant map u_0 which maps every point in Ω to the point $(1, 0) \in S^1 \subset \mathbb{R}^2$ or $\theta = 0$, θ being the angular variable in S^1, and we ask whether for every $T \in \text{cart}(\Omega \times S^1)$ there exists an $(n+1)$-dimensional current Σ in $\Omega \times S^1$ such that

$$T - G_{u_0} = (-1)^n \, \partial \Sigma. \tag{4}$$

Again the answer to this question is negative as the following simple example shows

Example 1 Consider the smooth map

$$u \,:\, S^1 \longrightarrow S^1, \qquad u(\theta) := \theta$$

or equivalently

$$u \,:\, S^1 \subset \mathbb{R}^2 \longrightarrow S^1 \subset \mathbb{R}^2, \qquad u(x) := \frac{x}{|x|}.$$

Similarly we could consider $u(x) := \frac{x}{|x|}$ from the anulus $B_R \setminus B_r \subset \mathbb{R}^2$, $0 < r \leq R$, into S^1. Clearly the graphs of u and u_0 do not bound any region in $S^1 \times S^1$. Notice that G_u is a Cartesian current in $(B_1 \setminus \{0\}) \times S^1$ but not in $B_1 \times S^1$. In $B_1 \times S^1$, G_u has non-zero boundary.

There is a *homological* obstruction to (4). Of course for any exact form $\alpha = d\beta$, $\beta \in \mathcal{D}^{n-2}(\Omega)$, we have

$$T(\alpha \wedge \Theta) = \partial T(\beta \wedge \Theta) = 0 \quad \text{as}$$

$$\alpha \wedge \Theta = d\beta \wedge \Theta = d(\beta \wedge \Theta).$$

However, if (4) holds, we must also have

$$T(\alpha(x) \wedge \Theta) = (T - G_{u_0})(\alpha(x) \wedge \Theta) = (-1)^n \, \partial \Sigma(\alpha(x) \wedge \Theta) = 0,$$

i.e.,

(5) $$T(\alpha(x) \wedge \Theta) = 0$$

for all closed $(n-1)$-forms α in Ω, $d\alpha = 0$. Clearly (5) does not hold for the current G_u in Example 1 above, but it is the only obstruction to (4). In fact we have

Theorem 2 *Let* $T \in \text{cart}(\Omega \times S^1)$. *Then*

(6) $$T - G_{u_0} = (-1)^n \, \partial\Sigma$$

for some $\Sigma \in \mathcal{D}_{n+1}(\Omega \times S^1)$ *if and only if*

(7) $$T(\alpha(x) \wedge \Theta) = 0 \qquad \forall \alpha \in \mathcal{D}^{n-1}(\Omega) \textit{ with } d\alpha = 0.$$

Proof It suffices to show that, assuming (7), we can construct Σ so that (6) holds.

Let $\omega \in \mathcal{D}^{n+1}(\Omega \times S^1)$. Using Proposition 1 of Sect. 1 we decompose ω uniquely as

$$\omega = \bar{\omega} \wedge \Theta + d_\theta \eta$$

where $\bar{\omega} \in \mathcal{D}^n(\Omega)$, $\eta \in \mathcal{D}^{n,0}(\Omega \times S^1)$ and $\eta(x, 0) = 0$. As $\bar{\omega} = f(x)\, dx$, $f(x) \in C_c^\infty(\Omega)$, we can find, compare e.g. Theorem 2 of [6] a smooth vector field $b = (\beta_1, \ldots, \beta_n)$ such that

(8) $$\begin{cases} b \in C^1(\Omega) \cap C^0(\bar{\Omega}), & b = 0 \text{ on } \partial\Omega \\ \operatorname{div} b = f - f_\Omega & \text{in } \Omega \end{cases}$$

where f_Ω denotes the mean value of f in Ω. Setting

$$\beta := \sum_{i=1}^{n} (-1)^{i-1} \beta_i(x)\, \widehat{dx^i}$$

we then find

$$\bar{\omega} = f_\Omega \, dx + d\beta \qquad \text{in } \Omega,$$

and we define $\Sigma \in \mathcal{D}^{n+1}(\Omega \times S^1)$ by

(9) $$\Sigma(\omega) := T(\eta + (-1)^n \beta \wedge \Theta + f_\Omega \, x^1 \widehat{dx^1} \wedge \Theta).$$

We claim that Σ is well defined, i.e., it does not depend on the solution b of (8). Suppose in fact that $\bar{\omega} = f_\Omega dx + d\tilde{\beta}$, $\tilde{\beta} \in C^1(\Omega) \cap C^0(\bar{\Omega})$, $\tilde{\beta} = 0$ on $\partial\Omega$ be another decomposition of $\bar{\omega}$. Then $d(\beta - \tilde{\beta}) = 0$, $\beta - \tilde{\beta} = 0$ on $\partial\Omega$ and we can find a sequence $\{\beta_k\}$ of forms in $\mathcal{D}^{n-1}(\Omega)$ such that $d\beta_k = 0$ and β_k converge unifomly to $\beta - \tilde{\beta}$ in $\bar{\Omega}$. From

$$0 = T(\beta_k \wedge \Theta) \longrightarrow T((\beta - \tilde{\beta}) \wedge \Theta)$$

we then deduce $T(\beta \wedge \Theta) = T(\tilde{\beta} \wedge \Theta)$.

In order to prove (6) it suffices now, by Proposition 2 of Sect. 1, to show that $T - G_{u_0}$ and $(-1)^n \partial\Sigma$ agree on $\mathcal{D}^{n,0}(\Omega \times S^1)$ and on forms of the type $\alpha \wedge \Theta$, $\alpha \in \mathcal{D}^{n-1}(\Omega)$.

Let $\omega \in \mathcal{D}^{n,0}(\Omega \times S^1)$. We have $\omega = \phi(x, \theta)\, dx$, $d_x\omega = 0$ and $d\omega = d_\theta\omega = d_\theta\eta$ with $\eta(x, \theta) := \omega - \omega(x, 0)$. Since $\eta(x, 0) = 0$

$$\Sigma(d\omega) = T(\eta) = T(\omega - \omega(x,0)) = T(\omega) - \int_\Omega \phi(x,0)\,dx =$$
$$= (T - G_{u_0})(\omega).$$

Let $\alpha \in \mathcal{D}^{n-1}(\Omega)$. We have $d(\alpha \wedge \Theta) = d\alpha \wedge \Theta$. Therefore the definition of Σ yields

$$\Sigma(d(\alpha \wedge \Theta)) = T((-1)^n \alpha \wedge \Theta) = (-1)^n T(\alpha \wedge \Theta) = (-1)^n (T - G_{u_0})(\alpha \wedge \Theta)$$

as $G_{u_0}(\alpha \wedge \Theta) = 0$ since u_0 is a constant map. Therefore

$$(-1)^n\, \partial\Sigma = T - G_{u_0}.$$

Theorem 3 *Let $T \in \operatorname{cart}(\Omega \times S^1)$ be such that $T(\alpha \wedge \Theta) = 0$ for all $\alpha \in \mathcal{D}^{n-1}(\Omega)$ with $d\alpha = 0$. Then there exists $u \in BV(\Omega, \mathbb{R})$ such that*

$$i_\# G_u = T,$$

and in particular

$$\mathbf{M}(G_u) = \mathbf{M}(T).$$

Proof Consider the current Σ defined in (4.7) for which

$$T - G_{u_0} = (-1)^n\, \partial\Sigma.$$

From [19, 26.28] there exists a function $\tilde{g} : \Omega \times S^1 \to \mathbb{R}$, $\tilde{g} \in BV_{\text{loc}}(\Omega \times S^1)$ such that for any smooth function $\tilde{f} : \Omega \times S^1 \to \mathbb{R}$ with compact support the following holds

$$\Sigma(\tilde{f}(x,\theta)\,dx \wedge \Theta) = \int_{\Omega \times S^1} \tilde{f}(x,\theta)\,\tilde{g}(x,\theta)\,d\mathcal{H}^{n+1} = \tag{10}$$
$$= \int_\Omega dx \int_0^{2\pi} f(x,t)\,\bar{g}(x,t)\,dt$$

where in the last term we have set

$$f(x,t) := \tilde{f}(x, (\cos t, \sin t)), \qquad \bar{g}(x,t) := \tilde{g}(x, (\cos t, \sin t)).$$

Moreover again from [19, 26.28]

$$|D\bar{g}| = \|\,\partial\Sigma\,\|.$$

Thus $\partial\Sigma$ being rectifiable, $\partial\Sigma = \tau(\mathcal{S}, \theta, \boldsymbol{S})$, we infer that

$$|D\bar{g}| = \theta\,\mathcal{H}^n \llcorner \mathcal{S}$$

where θ is an integer valued function. From this we deduce that

$$\bar{g}(x,t) = r_0 + g(x,t)$$

where r_0 is a real number and g is an integer valued BV function. Clearly f and g are 2π-periodic in t, and $g \in BV_{\text{loc}}(\Omega \times (0, 2\pi))$.

Consider the function $u(x) \in BV_{\text{loc}}(\Omega, \mathbb{R})$ defined by

$$u(x) := \int_0^{2\pi} g(x,s)\,ds$$

and the $(n+1)$-dimensional current S_u given by

$$\varphi(x,t) \longrightarrow S_u(\varphi(x,t)\,dx \wedge dt) := \int_\Omega dx \int_0^{u(x)} \varphi(x,t)\,dt.$$

We have, compare the computations in the beginning of Sect. 2

$$G_u - G_0 = (-1)^n\,\partial S_u, \tag{11}$$

and we claim that

$$i_\# S_u + r_0 [\![\Omega \times S^1]\!] = \Sigma. \tag{12}$$

From (11) and (12), as $i_\# G_0 = G_{u_0}$, we conclude

$$i_\# G_u = i_\# G_0 + (-1)^n i_\# \partial S_u = G_{u_0} + (-1)^n \partial\Sigma = T;$$

in particular $\mathbf{M}(G_u) = \mathbf{M}(T)$. Therefore the total variation of u in Ω is finite and, consequently, by Poincaré type inequality (compare e.g. [19, 6.4]) $u \in BV(\Omega, \mathbb{R})$. To prove (12) we observe that for any smooth function $\tilde{f}(x,\theta)$ with compact support by Theorem 1 we have

$$\begin{aligned} &\int_\Omega [\tilde{f}(x,u_T(x)) - \tilde{f}(x,0)]\,dx = (T - G_{u_0})(\tilde{f}(x,\theta)\,dx) = \\ &(-1)^n\,\partial\Sigma(\tilde{f}(x,\theta)\,dx) = \Sigma(\tilde{f}_\theta\,dx \wedge d\theta) = \int_{\Omega\times S^1} \tilde{f}_\theta(x,\theta)\,\tilde{g}(x,\theta)\,dx\,d\theta. \end{aligned} \tag{13}$$

We then denote by $\ell(x)$ the point in $[0,2\pi)$ such that $(\cos\ell(x), \sin\ell(x)) = u_T(x)$ and rewrite (13) as

$$\int_\Omega [f(x,\ell(x)) - f(x,0)]\,dx = \int_\Omega dx \int_0^{2\pi} f_t(x,t)\,g(x,t)\,dt$$

or

$$\int_\Omega dx \int_0^{2\pi} [g(x,t) - \chi_{[0,\ell(x)]}(t)]\,f_t(x,t)\,dt = 0.$$

From the last equality we deduce that

$$\int_\Omega dx \int_0^{2\pi} [g(x,t) - \chi_{[0,\ell(x)]}(t)]\,\varphi(x,t)\,dt = 0$$

for all φ of class C^∞ with compact support in x, 2π-periodic in t and such that

$$\int_0^{2\pi} \varphi(x,t)\,dt = 0 \qquad \forall x,$$

and, consequently that for almost every x in Ω

$$g(x,t) = c(x) + \chi_{[0,\ell(x))]}(t),$$

in particular $c(x)$ is integer valued.

Integrating the last equality with respect to t in $[0, 2\pi]$ we then get

$$u(x) := \int_0^{2\pi} g(x,t)\, dt = 2\pi\, c(x) + \ell(x)$$

and finally

$$\int_\Omega dx \int_0^{2\pi} f(x,t)\, g(x,t)\, dt = \int_\Omega dx \int_0^{2\pi} f(x,t)\, (c(x) + \chi_{[0,\ell(x))]}(t))\, dt =$$
$$= \int_\Omega \left\{ c(x) \int_0^{2\pi} f(x,t)\, dt + \int_0^{\ell(x)} f(x,t)\, dt \right\} dx =$$
$$= \int_\Omega dx \int_0^{u(x)} f(x,t)\, dt = S_u(i_\#(\tilde{f}\, dx \wedge \Theta))$$

taking into account the periodicity of f in t. □

Summarizing Theorems 2 and 3 we have

Theorem 4 *Let $T \in \mathrm{cart}(\Omega \times S^1)$. Then the following three claims are equivalent*
(i) $T(\alpha \wedge \Theta) = 0\ \forall \alpha \in \mathcal{D}^{n-1}(\Omega)$ *with* $d\alpha = 0$
(ii) $T - G_{u_0} = (-1)^n \partial \Sigma$ *for some* $\Sigma \in \mathcal{D}_{n+1}(\Omega \times S^1)$
(iii) $T = i_\# G_u$ *for some* $u \in BV(\Omega, \mathbb{R})$.

Suppose now that Ω is simply connected. Then we know that the first De Rham cohomology group is zero, $H^1_{DR}(\Omega; \mathbb{R}) = 0$. By duality then the $(n-1)$-cohomology group with compact support is zero, $H^{n-1}_c(\Omega; \mathbb{R}) = 0$; therefore any closed $(n-1)$-form α is a differential. Thus for α with $d\alpha = 0$, there is $\beta \in \mathcal{D}^{n-2}(\Omega)$ such that $\alpha = d\beta$, and as we have seen

$$T(\alpha \wedge \Theta) = T(d(\beta \wedge \Theta)) = \partial T(\beta \wedge \Theta) = 0.$$

Hence we conclude at once

Corollary 1 *Let Ω be simply connected. Then for any $T \in \mathrm{cart}(\Omega \times S^1)$ there exist an $(n+1)$-current $\Sigma \in \mathcal{D}_{n+1}(\Omega \times S^1)$ and a function $u \in BV(\Omega, \mathbb{R})$ such that*

$$T - G_{u_0} = (-1)^n\, \partial \Sigma \qquad \text{and} \qquad T = i_\#\, G_u.$$

We are now ready to discuss the structure of the currents T in $\mathrm{cart}(\Omega \times S^1)$. As our discussion is of local nature we may assume that

$$T = i_\#\, G_u \tag{14}$$

for some $u \in BV(\Omega, \mathbb{R})$. We also know that

$$T = \tau(\mathcal{M}, \vartheta, \mathbf{T}),$$

the zero component of T agrees with the zero component of $T \llcorner \mathcal{M}_+$, i.e., T acts on forms of the type $\phi(x,y)\, dx$, $\phi(x,y) \in C^\infty_c(\Omega \times S^1)$, as

$$T(\phi(x,y)\,dx) = \int_\Omega \phi(x, u_T(x))\,dx$$

where $u_T \in BV(\Omega, \mathbb{R}^2)$, $|u_T| = 1$, is the function associated to T and defining $\mathcal{M}_+$, and finally that

$$u_T = (\cos u, \sin u).$$

Thus we only need to find out how T acts on forms of the type $\phi(x,y)\widehat{dx^i} \wedge dy^j$, $\phi(x,y) \in C_c^\infty(\Omega \times \mathbb{R}^2)$, $i = 1, \dots, n$, $j = 1, 2$. In order to do that we introduce the set of *points of jumps and concentrations of* T defined as

$$Jc(T) := \{x \in \Omega \mid \frac{d\pi_\# \| T \|}{d\mathcal{H}^{n-1}}(x) > 0\}.$$

We also define

$$\mathcal{M}^{(Jc)} := (\mathcal{M} \setminus \mathcal{M}_+) \cap (Jc(T) \times S^1)$$
$$\mathcal{M}^{(C)} := \mathcal{M} \setminus (\mathcal{M}_+ \cup \mathcal{M}^{(Jc)})$$

and

$$T^{(a)} := T \llcorner \mathcal{M}_+$$
$$T^{(Jc)} := T \llcorner \mathcal{M}^{(Jc)}$$
$$T^{(C)} := T \llcorner \mathcal{M}^{(C)}$$

Trivially $\mathcal{M}$ is the disjoint union of $\mathcal{M}_+, \mathcal{M}^{(Jc)}$ and $\mathcal{M}^{(C)}$

$$\mathcal{M} = \mathcal{M}_+ \cup \mathcal{M}^{(Jc)} \cup \mathcal{M}^{(C)}$$

and

$$T = T^{(a)} + T^{(Jc)} + T^{(C)}.$$

From e.g. [13] [14], compare Theorem 1, we know how $T^{(a)}$ acts on all forms, and that $T^{(Jc)}$ and $T^{(C)}$ are completely vertical, i.e.,

$$T^{(Jc)}(\omega) = T^{(C)}(\omega) = 0$$

on horizontal forms, that is on forms of the type $\omega = \phi(x,y)\,dx$.

Theorem 5 (Structure theorem, part I) *Let* $T \in \operatorname{cart}(\Omega \times S^1)$. *Then locally* (12) *holds and we have*

$$Jc(T) = J_u. \tag{15}$$

In particular $Jc(T)$ *is countably* $(n-1)$*-rectifiable in* Ω *and, still locally,*

$$\begin{aligned} T^{(a)} &= i_\# G_u^{(a)} \\ T^{(C)} &= i_\# G_u^{(C)}, \end{aligned} \tag{16}$$

also the three measures $\| T^{(a)} \|, \| T^{(Jc)} \|$ *and* $\| T^{(C)} \|$ *are mutually singular.*

Moreover for any form of the type $\phi(x,y)\,\widehat{dx^i} \wedge dy^j$, $\phi(x,y) \in C_c^\infty(\Omega \times \mathbb{R}^2)$ *we have*

$$\begin{aligned} T^{(a)}(\phi(x,y)\widehat{dx^i} \wedge dy^j) &= (-1)^{n-i} \int_\Omega \phi(x, u_T(x))(D_i u_T^j)^{(a)}\,dx \\ T^{(C)}(\phi(x,y)\widehat{dx^i} \wedge dy^j) &= (-1)^{n-i} \int_\Omega \phi(x, u_T(x))\,d(D_i u_T^j)^{(C)} \end{aligned} \tag{17}$$

In particular the density ϑ is equal to 1 on $\mathcal{M}_+$ and $\mathcal{M}^{(C)}$ respectively $\mathcal{H}^n$-a.e. and $|(Du_T)^{(C)}|$-a.e.

Proof First we observe that $T^{(a)} = i_\# G_u^{(a)}$ as $T = i_\# G_u$. From (3) of Proposition 1 (ii) we then deduce

$$\frac{d\pi_\# \| T \|}{d\mathcal{H}^{n-1}}(x) = \frac{d\pi_\# \| G_u \|}{d\mathcal{H}^{n-1}}(x)$$

from which (15) and therefore the first part of the claim follows at once.

The second part follows easily applying the chain rule for the derivatives of the composite function $u_T = (\cos u, \sin u)$, compare [20] [21], see also [2], which states that in the sense of measures

$$Du_T = (-\sin u, \cos u)\, Du \qquad \text{on } \Omega \setminus J_u.$$

In fact, for instance for $\omega := \phi(x,y)\widehat{dx^i} \wedge dy^2$, we have

$$T^{(C)}(\omega) = G_u^{(C)}(i^\# \omega) = (-1)^{n-i} \int_\Omega \phi(x, \cos u(x), \sin u(x)) \cos u(x)\, (D_i u)^{(C)} =$$
$$= (-1)^{n-i} \int_\Omega \phi(x, u_T(x))\, (D_i u_T^2)^{(C)}.$$

□

The previous theorem says that both the absolutely continuous part and the Cantor part of the Cartesian current T are still identified, as in the scalar case, by the function u_T. But in general this does not hold for the jump-concentration part $T^{(Jc)}$.

Let us compute $T^{(Jc)}$ on the n-form $\omega := \phi(x, y^1, y^2)\widehat{dx^i} \wedge dy^2$. Using Proposition 2 of Sect. 2 we find

$$T^{(Jc)}(\omega) = i_\# G_u \llcorner (J_u \times \mathbb{R})(\omega) =$$
$$= (-1)^{n-i} \int \Big(\int_{u_-(x)}^{u_+(x)} \phi(x, \cos s, \sin s) \cos s\, ds \Big)\, \nu_{J_{u}i}(x)\, d\mathcal{H}^{n-1} \llcorner J_u.$$

For every x denote by $p_+(x)$ and $k(x)$ respectively the real number and the non negative integer such that

$$u_+(x) = p_+(x) + 2k(x)\pi, \qquad 0 \le p_+(x) - u_-(x) < 2\pi;$$

also denote by $\gamma_{u_-(x), u_+(x)}$ the oriented arc of S^1 which connects the points $i(u_-(x))$ and $i(p_+(x))$. Then we can write, taking into account the periodicity of ϕ,

$$\int_{u_-(x)}^{u_+(x)} \phi(x, \cos s, \sin s) \cos s\, ds =$$
$$= \int_{u_-(x)}^{p_+(x)} \phi(x, \cos s, \sin s) \cos s\, ds + \int_{p_+(x)}^{u_+(x)} \phi(x, \cos s, \sin s) \cos s\, ds =$$
$$= \int_{\gamma_{u_-(x), u_+(x)}} \phi(x, y^1, y^2)\, dy^2 + k(x) \int_{S^1} \phi(x, y^1, y^2) dy^2.$$

Therefore we can conclude

$$
(18)\qquad \begin{aligned} T^{(Jc)}(\omega) &= (-1)^{n-i}\int\Big(\int_{\gamma_{u_-(x),u_+(x)}} \phi(x,y)\,dy^2\Big)\,\nu_{J_u i}(x)\,d\mathcal{H}^{n-1} \llcorner J_u + \\ &+ (-1)^{n-i}\int k(x)\Big(\int_{S^1} \phi(x,y)\,dy^2\Big)\,\nu_{J_u i}(x)\,d\mathcal{H}^{n-1} \llcorner J_u. \end{aligned}
$$

Consider now the $(n-1)$-dimensional current in Ω

$$
L_T^{(Jc)} := \frac{1}{2\pi}\,\pi_\#(T^{(Jc)} \llcorner \Theta).
$$

For $\omega := \phi(x)\widehat{dx^i}$ we find

$$
(19)\qquad \begin{aligned} L_T^{(Jc)}(\omega) &= \frac{1}{2\pi}\,T^{(Jc)}(\omega \wedge \Theta) = \frac{1}{2\pi}\,i_\# G_u^{(j)}(\omega \wedge \Theta) = \frac{1}{2\pi}\,G_u^{(j)}(\phi(x)\widehat{dx^i} \wedge dt) = \\ &= \frac{1}{2\pi}\int \phi(x)(u_+(x) - u_-(x))\mathbf{J}_{u\bar{i}}\,d\mathcal{H}^{n-1} \llcorner J_u \end{aligned}
$$

where $\mathbf{J}_u$ is the tangent $(n-1)$-vector to J_u oriented in such a way that (19) holds. As the component $\mathbf{J}_{u\bar{i}}$, $\bar{i}$ being the multiindex which complements i, is given in terms of the normal ν_{J_u} by

$$
\mathbf{J}_{u\bar{i}} = (-1)^{n-i}\,\nu_{J_u i}
$$

we finally get

$$
(20)\qquad \begin{aligned} L_T^{(Jc)}(\phi(x)\,\widehat{dx^i}) &= \frac{(-1)^{n-i}}{2\pi}\int \phi(x)(u_+(x) - u_-(x))\,\nu_{J_u i}\,d\mathcal{H}^{n-1} \llcorner J_u = \\ &= (-1)^{n-i}\int \phi(x)\,k(x)\,\nu_{J_u i}\,d\mathcal{H}^{n-1} \llcorner J_u + \\ &+ \frac{(-1)^{n-i}}{2\pi}\int \phi(x)(p_+(x) - u_-(x))\,\nu_{J_u i}\,d\mathcal{H}^{n-1} \llcorner J_u. \end{aligned}
$$

Let us denote by $\mathcal{C}$ the subset of the countably rectifiable set J_u on which $k(x) \geq 1$ with orientation $\boldsymbol{\mathcal{C}}$ given by the orientation of J_u chosen in such a way that (17) holds, and let us define

$$
(21)\qquad \begin{aligned} L_T^{(C)} &:= \tau(\mathcal{C}, k(x), \boldsymbol{\mathcal{C}}) \\ L_T^{(j)} &:= \tau(J_u, p_+(x) - u_-(x), \mathbf{J}_u). \end{aligned}
$$

Taking into account that $L_T^{(Jc)}$ has finite mass, we can collected our information on $L_T^{(Jc)}$ as

Proposition 2 *The $(n-1)$-dimensional current $L_T^{(Jc)}$ in Ω is locally the current integration over the rectifiable set J_u with real density $u_+(x) - u_-(x)$. It can be splited as the sum of the integer rectifiable current $L_T^{(C)}$ and the current $L_T^{(j)}$ integration over J_u but with real density $p_+ - u(x)$.*

From (18) and Proposition 2 we now readily deduce

Theorem 6 (Structure theorem, part II) *The jump-concentration part of a current $T \in \text{cart}(\Omega \times S^1)$ splits into the sum*

$$T^{(Jc)} = T^{(C)} + T^{(j)} \tag{22}$$

where $T^{(C)}$ takes into account the concentration part and is given by

$$T^{(C)} = L_T^{(C)} \times [\![S^1]\!]$$

where $L_T^{(C)}$ is the rectifiable $(n-1)$-dimensional current defined in $(21)_1$ and $T^{(j)}$ takes into account the jump part and is defined by

$$T^{(j)}(\phi(x,y)\,\widehat{dx^i} \wedge dy^j) = (-1)^{n-i} \int \left\{ \int_{\gamma_{u_-(x),u_+(x)}} \phi(x,y)\,dy^j \right\} \nu_{J_u i}(x)\,d\mathcal{H}^{n-1} \llcorner J_u$$

Remark 2 In conclusion we see that every $T \in \text{cart}(\Omega \times S^1)$ has the same structure of the elements in $\text{cart}(\Omega \times \mathbb{R})$, i.e. of functions in $BV(\Omega, \mathbb{R})$, apart from the concentration term $L_T^{(C)} \times S^1$. However there is an important difference. The jump-concentration term $T^{(Jc)}$ cannot be written in terms of the BV-function u_T associated to T and the decomposition (22) is well defined only in terms of u; moreover one cannot separate the sets of integrations of $T^{(j)}$ and $T^{(C)}$.

Remark 3 We point out the similarity, and actually the formal equivalence, between the structure theorem in $\text{cart}(\Omega \times S^1)$ and in $\text{cart}^{2,1}(\Omega \times S^2)$ which is the space of the limits of sequences of graphs of smooth maps with equibounded Dirichlet's energies, compare [15] [14] [16].

Remark 4 We observe that, in the case that the function u_T associated to $T \in \text{cart}(\Omega \times S^1)$ is in $W^{1,1}(\Omega, \mathbb{R}^2)$, it is not difficult to deduce that T must have the form $T = G_{u_T} + L \times S^1$, in fact in this case no jump can occur, but only boundaries of G_{u_T} to be compensated by $\partial L \times [\![S^1]\!]$. Consequently, every $T \in \text{cart}(\Omega \times S^1)$ with $u_T = \frac{x}{|x|}$ must have the form

$$T = G_{\frac{x}{|x|}} + L \times [\![S^1]\!]$$

where L is a 1-dimensional rectifiable current in Ω with $\partial L \llcorner B = \delta_0 \times S^1$.

We conclude this section by proving that, not only limits of smooth functions from Ω into S^1 with bounded variations give rise to Cartesian currents in $\text{cart}(\Omega \times S^1)$, but also that every $T \in \text{cart}(\Omega \times S^1)$ is the limit of such a sequence. This way we fully answer the initial question of identifying limits of smooth maps with equibounded total variations.

Theorem 7 (Approximation theorem) *Let $T \in \text{cart}(\Omega \times S^1)$. Then there exists a sequence of smooth maps $v_h \in C^\infty(\Omega, S^1)$ such that*

$$G_{v_h} \rightharpoonup T$$

weakly in the sense of currents in $\Omega \times S^1$ and

$$\mathbf{M}(G_{v_h}) \longrightarrow \mathbf{M}(T)$$

Proof Using Whitney's covering argument, we can write Ω as the union of dyadic cubes $Q(x_j, r_j)$ in such a way that the doubles $Q(x_j, 2r_j)$ are still inside Ω, do not overlap more than $c_1(n)$ times, and for which the radii r_j are approximately equal to the distance of $Q(x_j, r_j)$ from $\partial\Omega$, i.e.,

$$c_2(n) \leq \frac{\operatorname{dist}(Q(x_j, r_j), \partial\Omega)}{r_j} \leq c_3(n).$$

For each j we choose a simply connected domain Ω_j with smooth boundary in such a way that $Q(x_j, \frac{3}{2}r_j) \subset\subset \Omega_j \subset Q(x_j, 2r_j)$ (for instance we can take as Ω_j the cube $Q(x_j, 2r_j)$ with rounded edges), and we note that there is $\gamma = \gamma(n)$ such that

$$\operatorname{diam}(Q(x, \frac{3}{4}r_j) \geq \gamma \operatorname{dist}(x, \partial\Omega) \qquad \forall x \in Q(x_j, \frac{3}{2}r_j).$$

Applying Corollary 1 and Theorem 1 we then find for each j a function $u_j \in BV(\Omega_j, \mathbb{R})$ such that

$$\begin{aligned} & i_\# G_{u_j} = T \llcorner (\Omega_j \times S^1) \\ & \mathbf{M}(G_{u_j}) = \mathbf{M}(T \llcorner (\Omega_j \times S^1)). \end{aligned} \tag{23}$$

For $l = 1, 2, \ldots$ and $x \in Q(x_j, \frac{3}{2}r_j)$ set

$$u_{j,\ell}(x) := u_j * \varphi_{\varepsilon(x)} \qquad \varepsilon(x) := \frac{\gamma}{\ell} \operatorname{dist}(x, \partial\Omega) \tag{24}$$

where φ is a standard mollifier. From [3], [22], we then deduce for $\ell \to \infty$

$$G_{u_{j,\ell}} \rightharpoonup G_{u_j} \qquad \text{in } Q(x_j, \frac{3}{2}r_j) \times \mathbb{R}, \tag{25}$$

and also we can find $\rho_j \in (\frac{5}{4}r_j, \frac{3}{2}r_j)$ so that

$$\mathbf{M}(G_{u_{j,\ell}} \llcorner Q_j \times \mathbb{R}) \longrightarrow \mathbf{M}(G_{u_j} \llcorner Q_j \times \mathbb{R}) \tag{26}$$

where we have set $Q_j := Q(x_j, \rho_j)$. Moreover, if $Q_j \cap Q_k$ is non empty we have

$$(i_\# G_{u_j} - i_\# G_{u_k}) \llcorner ((Q_j \cap Q_k) \times S^1) = 0.$$

Thus, by Proposition 1 (iii), we deduce that $u_j - u_k$ is an integer multiple of 2π, and consequently also $u_{j,\ell} - u_{k,\ell}$ is an integer multiple of 2π.

From the above we conclude that the maps from Ω into S^1 given by

$$v_\ell := (\cos u_{j,\ell}(x), \sin u_{j,\ell}(x)) \qquad \text{for } x \in Q_j$$

are well defined and smooth for all ℓ. Also from (24) and (25)

$$G_{v_\ell} \llcorner Q_j \times S^1 = i_\# (G_{u_{j,\ell}} \llcorner Q_j \times \mathbb{R}) \rightharpoonup i_\# (G_{u_j} \llcorner Q_j \times \mathbb{R}) = T \llcorner Q_j \times S^1$$

and from (26) and (23)

$$\mathbf{M}(G_{v_\ell} \llcorner Q_j \times S^1) = \mathbf{M}(G_{u_{j,\ell}} \llcorner Q_j \times \mathbb{R}) \rightharpoonup \mathbf{M}(G_{u_j} \llcorner Q_j \times \mathbb{R}) = \mathbf{M}(T \llcorner Q_j \times S^1).$$

The proof of the theorem is then easily completed by observing that the covering $\{Q_j\}$, $Q_j := Q(x_j, \rho_j)$, is locally finite. □

4. Relaxed energies

Let us begin by considering the area functional for maps u from a bounded domain Ω of $\mathbb{R}^n$ into S^1

$$\mathcal{A}(u, \Omega) := \int_\Omega \sqrt{1 + |Du|^2}\, dx.$$

In the same spirit of Lebesgue's area for continuous functions, the *relaxed area* of graphs in $\Omega \times S^1$ is given for $T \in \mathcal{D}_n(\Omega \times S^1)$ by

(1) $$\mathcal{A}(T, \Omega) := \inf\{\liminf_{k\to\infty} \int_\Omega \sqrt{1 + |Du_k|^2}\, dx \mid u_k \in C^1(\Omega, S^1), G_{u_k} \rightharpoonup T\}.$$

An immediate consequence of the approximation theorem in Sect. 3 is that

(2) $$\mathcal{A}(T, \Omega) = \begin{cases} \mathbf{M}(T) & \text{if } T \in \text{cart}(\Omega \times S^1) \\ +\infty & \text{otherwise} \end{cases}$$

Taking into account the structure theorem for Cartesian currents in $\Omega \times S^1$, we can also write for $T \in \text{cart}(\Omega \times S^1)$

(3) $$\mathcal{A}(T, \Omega) = \int_\Omega \sqrt{1 + |(Du)^{(a)}|^2}\, dx + \int_\Omega |(Du)^{(C)}| + \int_\Omega d\| T^{(Jc)} \|.$$

In particular we see that $\mathcal{A}(T, \Omega)$ is a *local* functional.

Similarly we may consider the *relaxed area* of the "graphs" of L^1-functions $u : \Omega \to S^1$, defined by

(4) $$\bar{\mathcal{A}}(u, \Omega) := \inf\left\{ \liminf_{k\to\infty} \int_\Omega \sqrt{1 + |Du_k|^2}\, dx \mid u_k \in C^1(\Omega, S^1), u_k \to u \text{ in } L^1 \right\}.$$

However, it turns out that in this case $\bar{\mathcal{A}}$ *is not local*, i.e. $\bar{\mathcal{A}}(u, \cdot)$ is not a measure in Ω.

Proposition 1 *The following facts are equivalent*
(i) $u \in BV(\Omega, \mathbb{R}^2)$, $|u(x)| = 1$ *a.e. in* Ω
(ii) $\bar{\mathcal{A}}(u, \Omega) < \infty$
(iii) There exists $T \in \text{cart}(\Omega \times S^1)$ *such that* $u_T = u$ *in* Ω.

Proof Suppose that (iii) holds. From the approximation theorem there is a sequence $\{u_k\} \subset C^1(\Omega, S^1)$ such that

$$G_{u_k} \rightharpoonup T, \qquad \mathbf{M}(G_{u_k}) \to \mathbf{M}(T).$$

Therefore $u_k \to u$ strongly in L^1, and the semicontinuity of the mass yields

$$\mathcal{A}(T,\Omega) \le \liminf_{k\to\infty} \mathbf{M}(G_{u_k}) = \mathbf{M}(T).$$

On the contrary, if $\{u_k\} \subset C^1(\Omega, S^1)$, $u_k \to u$ in L^1, is such that $\bar{\mathcal{A}}(u,\Omega) = \liminf_{k\to\infty} \mathbf{M}(G_{u_k})$, then G_{u_k} converge weakly to some $T \in \text{cart}(\Omega \times S^1)$ and

$$\mathbf{M}(T) \le \liminf_{k\to\infty} \mathbf{M}(G_{u_k}) = \bar{\mathcal{A}}(u).$$

This proves that (ii) is equivalent to (iii).
From Theorem 1 of Sect. 3 it follows that (iii) implies (i). Therefore it remains to show that (i) implies (iii).
Let $u \in BV(\Omega, \mathbb{R}^2)$, $|u| = 1$. For $\beta \in (0, \frac{\pi}{2})$ we set

$$\begin{aligned}
E_1 &:= \{x \mid u^2(x) \ge \sin\beta\}\\
E_2 &:= \{x \mid u^1(x) < -\cos\beta\}\\
E_3 &:= \{x \mid u^2(x) \le -\sin\beta\}\\
E_4 &:= \{x \mid u^1(x) > \cos\beta\}.
\end{aligned}$$

For almost every $\beta \in (0, \frac{\pi}{2})$ we have

$$\int_{\partial E_i} |Du| = 0 \qquad i = 1,2,3,4.$$

Therefore, defining for one such a β $\hat{u} : \Omega \to \mathbb{R}$ as

$$\begin{aligned}
&\text{for } x \in E_1 \cos\hat{u}(x) := u^1(x), \beta \le \hat{u}(x) \le \pi - \beta\\
&\text{for } x \in E_2 \sin\hat{u}(x) := u^2(x), \pi - \beta \le \hat{u}(x) \le \pi + \beta\\
&\text{for } x \in E_3 \cos\hat{u}(x) := u^1(x), \pi + \beta \le \hat{u}(x) \le 2\pi - \beta\\
&\text{for } x \in E_4 \sin\hat{u}(x) := u^2(x), 2\pi - \beta \le \hat{u}(x) \le 2\pi + \beta
\end{aligned}$$

we deduce that $\hat{u}$ belongs to $BV(\Omega, \mathbb{R})$, $\beta \le \hat{u}(x) < \beta + 2\pi$. To conclude the proof it suffices now to take as T the Cartesian current $i_\# G_{\hat{u}}$.

From Proposition 1, (2) and (3) we deduce

Proposition 2 *$\bar{\mathcal{A}}(u)$ is finite if and only if $u \in BV(\Omega, \mathbb{R}^2)$, $|u| = 1$. Moreover, for $u \in BV(\Omega, \mathbb{R}^2)$, $|u| = 1$, we have*

$$\begin{aligned}
\bar{\mathcal{A}}(u,\Omega) &= \inf\{\mathbf{M}(T) \mid T \in \text{cart}(\Omega \times S^1), u_T = u\} =\\
&= \int_\Omega \sqrt{1 + |(Du)^{(a)}|^2}\, dx + \int_\Omega |(Du)^{(C)}| +\\
&\quad + \inf\{\int d\|T^{(Jc)}\| \mid T \in \text{cart}(\Omega \times S^1) \textit{such that } u_T = u\}
\end{aligned}$$

From Remark 1 Sect. 3 and the structure theorem we also deduce

Proposition 3 *Let $T, T' \in \text{cart}(\Omega \times S^1)$. We have $u_T = u_{T'}$ if and only if there exists an integer rectifiable $(n-1)$-dimensional current L in Ω such that*

$$T - T' = L \times [\![S^1]\!].$$

Combining Propositions 1 and 2 we finally find the following representation formula for the relaxed area with respect to the L^1-convergence.

Theorem 1 *Let $u \in BV(\Omega, \mathbb{R}^2), |u| = 1$. Then*

$$\begin{aligned}\bar{\mathcal{A}}(u, \Omega) = \int_\Omega \sqrt{1 + |(Du)^{(a)}|^2}\, dx + \int_\Omega |(Du)^{(C)}| + \\ + 2\pi \inf\{\mathbf{M}(L_T^{(Jc)} + \widetilde{L}) \mid \widetilde{L} \text{ integer rectifiable with } \partial\widetilde{L} \llcorner \Omega = 0\}\end{aligned} \tag{5}$$

where T is any current in $\operatorname{cart}(\Omega \times S^1)$ *such that $u_T = u$.*

Note that $L_T^{(Jc)}$ is the current integration on J_u but with in general a *real* density. Of course, if it happens to exist a $T \in \operatorname{cart}(\Omega \times S^1), u_T = u$, such that $L_T^{(Jc)}$ is an integer rectifiable current, then we have (compare [15] for the case of maps into S^2)

$$\begin{aligned}\bar{\mathcal{A}}(u, \Omega) = \mathbf{M}(T^{(a)}) + \mathbf{M}(T^{(C)}) + \\ + 2\pi \inf\{\mathbf{M}(L) \mid \partial L \times [\![S^1]\!] = -\partial(T^{(a)} + T^{(C)})\}.\end{aligned} \tag{6}$$

In the special case of the function $u : \Omega \subset \mathbb{R}^2 \to S^1, u(x) = \frac{x}{|x|}$, we deduce from Remark 4 Sect. 3 that

$$\bar{\mathcal{A}}(\frac{x}{|x|}, B_R) = \int_{B_R} \sqrt{1 + |D\frac{x}{|x|}|^2}\, dx + 2\pi R$$

$$\bar{\mathcal{A}}(\frac{x}{|x|}, B_R \setminus B_r) = \int_{B_R \setminus B_r} \sqrt{1 + |D\frac{x}{|x|}|^2}\, dx \quad r > 0,$$

in particular

$$\bar{\mathcal{A}}(\frac{x}{|x|}, B_1) > \bar{\mathcal{A}}(\frac{x}{|x|}, B_r) + \bar{\mathcal{A}}(\frac{x}{|x|}, B_1 \setminus B_r) \quad 0 < r < 1.$$

This shows that $\bar{\mathcal{A}}(\frac{x}{|x|}, \cdot)$ is not subadditive, i.e., $\bar{\mathcal{A}}(\frac{x}{|x|}, \cdot)$ is not a measure in Ω.

In the same way as previously we can deal with general smooth integrands of the type $f(x, u, p)$ such that

(i) $f(x, u, p)$ is convex with respect to p
(ii) $|p| \le f(x, u, p) \le c_0(1 + |p|)$
(iii) the so-called *recession function*

$$f^\infty(x, u, p) := \lim_{p_0 \to 0^+} p_0\, f(x, u, \frac{p}{p_0})$$

is well defined.
Setting

$$F(x, u, p_0, p) = \begin{cases} p_0\, f(x, u, 1, p/p_0) & \text{if } p_0 > 0 \\ f^\infty(x, u, p) & \text{if } p_0 = 0 \end{cases}$$

we may define for each $T \in \text{cart}(\Omega \times S^1)$ the parametric integral

$$\mathcal{F}(T, \Omega) = \int F(x, y, \mathbf{T})\, d\| T \|. \tag{7}$$

As it is standard, see e.g. [7] [12], using a well-known theorem by Reshetnyak [18], and the approximation theorem of Sect. 3, we infer

Theorem 2 *$\mathcal{F}(T, \Omega)$ in (7) is the relaxed functional of $\int_\Omega f(x, u, Du)\, dx$.*

Of course

$$\begin{aligned}\mathcal{F}(T, \Omega) = \int_\Omega f(x, u_T(x), (Du)^{(a)})\, dx + \int_\Omega f^\infty(x, u_T(x), \frac{(Du)^{(C)}}{|(Du)^{(C)}|})\, |(Du)^{(C)}| + \\ + \int F(x, y, \mathbf{T}^{(\mathbf{Jc})})\, d\| T^{(Jc)} \|. \end{aligned} \tag{8}$$

We leave to the reader the formulation of analogous results to the ones of this section in the general case of integrable $\mathcal{F}(T, \Omega)$. We only remark that the last term in (8) can be written in terms of the functions $u \in BV(\Omega, \mathbb{R})$ for which locally $T = i_\# G_u$ as

$$\begin{aligned}\int F(x, y, \mathbf{T}^{(\mathbf{Jc})})\, d\| T^{(Jc)} \| = \int k(x)\, d\mathcal{H}^{n-1} \llcorner \mathcal{L} \int_{S^1} f^\infty(x, y, \boldsymbol{\mathcal{L}} \wedge \varepsilon_{S^1})\, dy + \\ + \int d\mathcal{H}^{n-1} \llcorner J_u \int_{\gamma_{u_-(x), u_+(x)}} f^\infty(x, y, \mathbf{J}_u \wedge \varepsilon_{S^1})\, dy \end{aligned} \tag{9}$$

where $\mathcal{L}$, $\boldsymbol{\mathcal{L}}$, $k(x)$, J_u and $\gamma_{u_-(x), u_+(x)}$ have the same meaning as in Sect. 3 and ε_{S^1} is the 1-vector orienting S^1.

5. Variational problems

The results of Sect. 4 allow us to readily solve variational problems in a weak sense, i.e., in suitable subclasses of $\text{cart}(\Omega \times S^1)$,for integrals of the type $\mathcal{F}(T, \Omega)$ considered in the end of Sect. 4. In fact those integrals, and in particular the area, are lower semicontinuous with respect to the weak convergence of currents with equibounded masses and coercive in $\text{cart}(\Omega \times S^1)$.

For instance let us consider the Dirichlet problem in Ω, which in analogy with the BV-case consists in the following. Given a bounded domain $\widetilde{\Omega}$, $\widetilde{\Omega} \supset\supset \Omega$, for example a normal ε-neighbourhood of Ω, and a smooth function $u_0 : \widetilde{\Omega} \to S^1$, equivalently $T_0 = G_{u_0} \in \text{cart}(\widetilde{\Omega} \times S^1)$, find a minimizer of $\mathcal{F}(T, \widetilde{\Omega})$ in the class

$$\text{cart}_{T_0}(\widetilde{\Omega} \times S^1) := \{T \in \text{cart}(\widetilde{\Omega} \times S^1) \mid (T - T_0) \llcorner (\widetilde{\Omega} \setminus \bar{\Omega}) \times S^1 = 0\}.$$

Then we get

Theorem 1 *There exists a minimizer of $\mathcal{F}(T, \Omega)$ in* $\text{cart}_{T_0}(\widetilde{\Omega} \times S^1)$.

Note that, if we assume $u_{0|\partial\Omega}$ to be extended to $\widetilde{\Omega} = \Omega_\varepsilon$ constantly along the normal to $\partial\Omega$, by retracting Ω_ε to Ω and applying the approximation theorem of Sect. 3, we can always find a sequence of smooth maps $u_k : \bar{\Omega} \to S^1$, with $u_k = u_0$ on $\partial\Omega$ such that

$$G_{u_k|\Omega} + G_{u_0|\widetilde{\Omega}\setminus\bar{\Omega}} \rightharpoonup T \in \operatorname{cart}_{G_{u_0}}(\widetilde{\Omega} \times S^1)$$

and

$$\mathcal{F}(G_{u_k|\Omega} + G_{u_0|\widetilde{\Omega}\setminus\bar{\Omega}}) \longrightarrow \mathcal{F}(T, \widetilde{\Omega}).$$

Maybe more interesting is the fact that we can minimize integrals of the type $\mathcal{F}(T, \Omega)$, as for instance the mass, in classes of mappings with prescribed homology or degree maps.

Suppose that $\mathcal{X}$ is a compact oriented Riemannian manifold without boundary and let $u : \mathcal{X} \to S^1$ be a smooth map. It is well known that the map $u_\#$ which maps any 1-dimensional cycle $[\![C]\!]$ in $\mathcal{X}$ into the 1-dimensional cycle $u_\#[\![C]\!]$ in S^1 defines a map u_* between the real (integer) homology groups of degree 1 of $\mathcal{X}$ and S^1

$$u_* : H_1(\mathcal{X}, \mathbb{R}) \longrightarrow H_1(S^1, \mathbb{R}) \simeq \mathbb{R}, \qquad u_*([C]) = [u_\# C]$$

called the *homology* or *degree map*. Assuming C regular, by means of Poincaré duality which associates to C a $(n-1)$-form so that

$$\int_C \eta = \int_{\mathcal{X}} \omega_C \wedge \eta \qquad \forall \eta \in \mathcal{D}^1(\mathcal{X})$$

we see that

$$u_\#[\![C]\!](\eta) = [\![C]\!](u^\# \eta) = \int_{\mathcal{X}} \omega_C \wedge u^\# \eta = G_u(\omega_C \wedge \eta) \quad \forall \eta \in \mathcal{D}^1(\mathcal{X}).$$

Similarly, for every Cartesian current $T \in \operatorname{cart}(\mathcal{X}, S^1)$ the *matrix of periods*

$$\frac{1}{2\pi} T(\omega \wedge \Theta) \qquad \omega \in \mathcal{D}^{n-1}(\mathcal{X}) \qquad d\omega = 0$$

defines a homology map

$$T_* : H_1(\mathcal{X}, \mathbb{R}) \longrightarrow H_1(S^1, \mathbb{R})$$

as follows. Consider a 1-dimensional normal cycle S and its regularization S_ε, $0 < \varepsilon < 1$. The normal current S_ε is homologous to S [10, 4.1.18] and can be written as

$$S_\varepsilon(\omega) = \int_{\mathcal{X}} \omega_{S_\varepsilon} \wedge \omega$$

where ω_{S_ε} is a smooth closed $(n-1)$-form in $\mathcal{X}$ [10, 4.1.12]. Thus T_* is given by

$$T_*([S])(\eta) := \frac{1}{2\pi} T(\omega_{S_\varepsilon} \wedge \eta) \qquad \forall \eta \in \mathcal{D}^1(S^1).$$

Actually T_* defines a map between the singular homology groups

$$T_* : H_1(\mathcal{X}, \mathbb{Z}) \longrightarrow H_1(S^1, \mathbb{Z}) \simeq \mathbb{Z}.$$

To see this, it suffices to consider an approximating sequence of smooth maps

$$G_{u_k} \rightharpoonup T$$

and observe that for any cycle S

$$\frac{1}{2\pi} u_{k*}(S)(\Theta) = \frac{1}{2\pi} G_{u_k}(\omega_S \wedge \Theta).$$

We note in particular that if $\{T_k\} \subset \text{cart}(\mathcal{X} \times S^1)$ is a sequence which converges to $T \in \text{cart}(\mathcal{X} \times S^1)$ and all the T_k's have the same homology map $\hat{T}_*$, then also $T_* = \hat{T}_*$. Therefore we get at once

Theorem 2 *There is a minimizer of $\mathcal{F}(T, X)$ in the class*

$$\{T \in \text{cart}(\mathcal{X} \times S^1) \mid T_* = \hat{T}_*\}$$

$\hat{T}_*$ being a prescribed homology map, and even in the class

$$\{T \in \text{cart}(\mathcal{X} \times S^1) \mid T_* = \hat{T}_*, (T - T_0) \llcorner (\Sigma \times S^1) = 0\}$$

where T_0 is a given current in cart$(\mathcal{X} \times S^1)$ *with $T_{0*} = \hat{T}_*$ and Σ is a smooth open domain of $\mathcal{X}$.*

6. Minimizers with prescribed singularities

In this last section we deal with the problem of minimizing the total variation of maps which are constants near infinity, with values in S^1, and have prescribed homological singularities, i.e., we deal with the so-called dipole problem, compare [4] [5] [1] [15] [14] [16] [11], for the case of Dirichlet integral.

Let $\Gamma_1, \dots, \Gamma_k$ be a finite family of $(n-2)$-dimensional curves in $\mathbb{R}^n$ which are simple, oriented, closed, smooth, and do not intersect each other. Let $u : \mathbb{R}^n \setminus \bigcup_{i=1}^{k} \Gamma_i \to S^1$ be a smooth map which is constant near infinity. For $x \in \Gamma_i$, $i = 1, \dots, k$, we consider the 1-dimensional sphere $S^1_{x,\varepsilon}$ of radius ε around x in the oriented 2-dimensional orthogonal plane to Γ_i at x. Of course for ε sufficiently small, depending on the family of the Γ_i, $S^1_{x,\varepsilon}$ does not intersect any Γ_j for $j \neq i$, for $x \in \Gamma_i$ and for all i. As u is regular in $\mathbb{R}^n \setminus \bigcup_{s=1}^{k} \Gamma_s$, the *degree* of the map

$$u_{|S^1_{x,\varepsilon}} \; : \; S^1_{x,\varepsilon} \longrightarrow S^1$$

is a well defined integer and a trivial homotopy argument shows that it does not depend on the radius ε, provided ε is small, nor on the point $x \in \Gamma_i$ for each fixed Γ_i. We call such an integer the *degree of u at $x \in \Gamma_i$ with respect to the orientation* $\boldsymbol{\Gamma}_i$ of Γ_i.

Given now k integers $d_1, \dots, d_k$, our problem is to minimize for instance the integral total variation

$$\text{(1)} \qquad \mathcal{F}(u) := \int_\Omega |Du| \, dx, \qquad \Omega := \mathbb{R}^n \setminus \bigcup_{i=1}^{k} \Gamma_i$$

in the class E of smooth maps $u : \Omega \to S^1$ which are constant, say u_0, in a neighbourhood of infinity and such that

$$\deg(u, \Gamma_i) = d_i \qquad \text{for } i = 1, \dots, k.$$

In order to tackle this problem we first observe, compare [14] [16], that the singularities of a map $u \in E$ are in fact described by the integer rectifiable $(n-2)$-dimensional current

$$\mathbf{P} := \sum_{i=1}^{k} d_i [\![\Gamma_i]\!].$$

In fact we have

Proposition 1 *Let* $u : \mathbb{R}^n \setminus \bigcup_{i=1}^{k} \Gamma_i \to S^1$ *be a smooth map with* $\mathcal{F}(u) < \infty$. *Then* $\deg(u, \Gamma_i) = d_i$, $i = 1, \dots, k$, *if and only if*

$$\frac{1}{2\pi} \partial \pi_\#(G_u \llcorner \Theta) = \mathbf{P}. \tag{2}$$

Proof Set

$$\widetilde{\mathbf{P}}(u) := \frac{1}{2\pi} \partial \pi_\#(G_u \llcorner \Theta)$$

and denote by C_ε an ε-neighbourhood of $\Gamma = \bigcup_{i=1}^{k} \Gamma_i$, ∂C_ε smooth, ε small. Since u is regular on ∂C_ε we have

$$\partial(G_u \llcorner (C_\varepsilon \times S^1)) = \partial G_u \llcorner (C_\varepsilon \times S^1) + G_{u|\partial C_\varepsilon} \qquad \text{in } \mathbb{R}^n \times S^1.$$

Thus

$$\widetilde{\mathbf{P}}(u) = \frac{1}{2\pi} \partial \pi_\# [(G_u \llcorner (C_\varepsilon \times S^1)) \llcorner \Theta] - \frac{1}{2\pi} \pi_\# (G_{u|\partial C_\varepsilon} \llcorner \Theta). \tag{3}$$

Since $\pi_\#(G_{u|\partial C_\varepsilon} \llcorner \Theta)$ is an $(n-2)$-dimensional normal current and the mass of $\pi_\#[G_u \llcorner (C_\varepsilon \times S^1)) \llcorner \Theta]$ tends to zero as $\varepsilon \to 0$, we infer from (3) that $\widetilde{\mathbf{P}}(u)$ is a locally flat chain, see [10] for the definition. The constancy theorem, see [10, 4.1.31], then yields

$$\widetilde{\mathbf{P}}(u) = \sum_{i=1}^{k} r_i [\![\Gamma_i]\!]$$

where r_i are in principle real numbers. We shall now show that in fact $r_i = d_i$, and this will complete the proof of our claim.

Consider an $(n-2)$-form ω on Γ_i and extend it constantly in the normal direction to Γ_i in a small cylinder $C_{\varepsilon_0} := \Gamma_i \times B_{\varepsilon_0}$ and with compact support in $C_{2\varepsilon_0} := \Gamma_i \times B_{2\varepsilon_0}$ in such a way that $\Gamma_i \times B_{2\varepsilon_0}$ does not intersect Γ_j for $j \neq i$. For each $x \in \Gamma_i$ we obviously have

$$\frac{1}{2\pi} u_\# [\![S_{x,\varepsilon}]\!](\Theta) = d_i$$

hence for $\varepsilon < \varepsilon_0$

$$r_i \llbracket \Gamma_i \rrbracket(\omega) = \widetilde{\mathbf{P}}(u)(\omega) = -\frac{1}{2\pi} G_{u|\partial C_\varepsilon}(\omega \wedge \Theta) + \frac{1}{2\pi} (G_u \llcorner C_\varepsilon \times S^1)(d\omega \wedge \Theta) =$$
$$= \frac{1}{2\pi} \int_\Gamma \omega(x) \int_{S_{x,\varepsilon}} u^\#(\Theta) + \frac{1}{2\pi} (G_u \llcorner C_\varepsilon \times S^1)(d\omega \wedge \Theta) =$$
$$= d_i \llbracket \Gamma_i \rrbracket(\omega) + \frac{1}{2\pi} (G_u \llcorner C_\varepsilon \times S^1)(d\omega \wedge \Theta).$$

As the second term in the last expression tends to zero as $\varepsilon \longrightarrow 0$, we deduce that $r_i = d_i$, and also that $(G_u \llcorner C_\varepsilon \times S^1)(d\omega \wedge \Theta) = 0$. □

On account of Proposition 1 our problem can be now formulated as the problem of minimizing $\mathcal{F}(u)$ in the class

$$E_{\mathbf{P}} := \{u \in C^1(\mathbb{R}^n \setminus \bigcup_{i=1}^k \Gamma_i S^1) \mid G_u = G_{u_0} \text{ in a neighbourhood of infinity},$$
$$\frac{1}{2\pi} \partial \pi_\#(G_u \llcorner \Theta) = \mathbf{P}$$

Theorem 1 *We have*

$$\inf_{u \in E} \int_\Omega |Du|\, dx = 2\pi \min\{\mathbf{M}(L) \mid L \textit{ integer rectifiable } (n-1)\textit{-current in } \mathbb{R}^n$$
$$\textit{with } \partial L = \mathbf{P}\} = 2\pi \mathbf{M}(L_0),$$

where L_0 is the integer rectifiable $(n-1)$-current in $\mathbb{R}^n$ of least area spanning $\mathbf{P}$.

Proof First we note that L_0 exists, compare [10].

Let us prove that for all $u \in E$, equivalently $u \in E_{\mathbf{P}}$,

$$\int_\Omega |Du|\, dx \geq 2\pi \mathbf{M}(L_0).$$

As in [1] this follows by a simple use of the coarea formula. In fact we have

$$\int_\Omega |Du|\, dx = \int_{\mathbb{R}^n} J_1(u)\, dx = \int_{S^1} \mathbf{M}(u^{-1}(y)) =$$
$$= \int_{S^1} \mathbf{M}(< \mathbb{R}^n, u, y >) \geq \int_{S^1} \mathbf{M}(L_0) = 2\pi \mathbf{M}(L_0),$$

since for the slice of the current $\mathbb{R}^n$ by the map u, or equivalently for the current $\tau(u^{-1}, 1, \xi)$, ξ being the natural induced orientation on $u^{-1}(y)$, we have

$$\partial < \mathbb{R}^n, u, y >= \mathbf{P}.$$

To prove the opposite inequality we consider the current $T \in \text{cart}_{\text{loc}}(\Omega \times S^1)$ defined as

$$T := G_{u_0} + L_0 \times S^1. \qquad (4)$$

Denoting still by $\mathcal{F}$ the relaxed of the integral (1) to Cartesian currents, compare Sect. 4, we have

$$\mathcal{F}(T) = 2\pi\, \mathbf{M}(L_0).$$

From the approximation theorem of Sect. 3, compare also the observation following Theorem 1 of Sect. 5, we now deduce the existence of a sequence of smooth maps $u_k \in C^1(\Omega, S^1)$, $u_k = u_0$ near infinity such that

$$\begin{aligned} &G_{u_k} \rightharpoonup T \quad \text{in } \mathcal{D}_n(\Omega \times S^1) \\ &\mathbf{M}_{U\times S^1}(G_{u_k}) \to \mathbf{M}_{U\times S^1}(T) \quad \forall U \subset\subset \mathbb{R}^n \\ &\mathcal{F}(G_{u_k}) \longrightarrow \mathcal{F}(T) \end{aligned} \tag{5}$$

To conclude the proof it suffices to show that

$$G_{u_k} \rightharpoonup T \qquad \text{in } \mathcal{D}_n(\mathbb{R}^n \times S^1) \tag{6}$$

In this case in fact we also have

$$\partial G_{u_k} \rightharpoonup \partial T,$$

i.e.,

$$\partial \pi_\#(G_{u_k} \llcorner \frac{\Theta}{2\pi}) \rightharpoonup \partial L_0 = \sum_{i=1}^{k} d_i [\![\Gamma_i]\!],$$

and, since

$$\partial \pi_\#(G_{u_k} \llcorner \frac{\Theta}{2\pi}) = \sum_{i=1}^{k} r_i [\![\Gamma_i]\!], \quad r_i \in \mathbb{Z},$$

we conclude that for k large

$$\frac{1}{2\pi}\, \partial \pi_\#(G_{u_k} \llcorner \Theta) = \mathbf{P}.$$

Let us prove now that the first two claims in (5) imply (6). This is an immediate consequence of the following result concerning general vector valued measures.

Suppose that

$$\mu_k \rightharpoonup \mu$$

in some open set $\Omega \subset \mathbb{R}^n$, for simplicity with smooth boundary $\partial\Omega$, and that for the total variation we have

$$|\mu_k|(\Omega) \longrightarrow |\mu|(\Omega).$$

Thus regarding the measures μ_k and μ as measures in $\mathbb{R}^n$ we have

$$\mu_k \rightharpoonup \mu \qquad \text{in } \mathbb{R}^n. \tag{7}$$

For the reader's convenience we give a brief stretch of the proof of (7). Consider the set Ω_ε of points x in Ω with $\operatorname{dist}(x, \partial\Omega) < \varepsilon$. We may assume that $|\mu|(\partial\Omega_\varepsilon) = 0$ for a.e. ε. Then it easily follows that

$$|\mu_k|\,(\Omega \setminus \bar{\Omega}_\varepsilon) \longrightarrow |\mu|\,(\Omega \setminus \bar{\Omega}_\varepsilon).$$

Choose then functions χ_ε which are identically 1 in Ω_ε, with compact support in Ω, and converging to the characteristic function χ_Ω of Ω. As $\mathrm{spt}(\chi_\Omega - \chi_{\Omega_\varepsilon}) \subset \Omega \setminus \bar{\Omega}_\varepsilon$, we then see that

$$(\mu_k - \mu)((\chi_\Omega - \chi_\varepsilon)f)$$

is uniformly small with respect to k for ε sufficiently small and $|f| \leq 1$. This yields easily (7). □

Note that we have also proved

Corollary 1 *The current* $T := G_{u_0} + L_0 \times S^1 \in \mathrm{cart}_{\mathrm{loc}}(\Omega \times S^1)$ *is a minimizer of* $\mathcal{F}(T)$ *in the class of Cartesian currents in* $\mathrm{cart}_{\mathrm{loc}}(\Omega \times S^1)$ *such that* $T = G_{u_0}$ *in* $U \times S^1$, U *being a neighbourhood of infinity in* $\mathbb{R}^n$, *and*

$$\frac{1}{2\pi}\,\partial\pi_\#(T \llcorner \Theta) = \mathbf{P}.$$

We also remark that Corollary 1 implies also the known fact that the set E is not empty, i.e., given any finite family of $(n-2)$-dimensional curves Γ_i in $\mathbb{R}^n$, $i = 1, \ldots, k$, as previously, and any set of integers $d_i, i = 1, \ldots, k$, then there exists a function $u \in C^1(\mathbb{R}^n \setminus \bigcup_{i=1}^{k} \Gamma_i, S^1)$, u constant near infinity such that $\deg(u, \Gamma_i) = d_i$, for any $i = 1, \ldots, k$.

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