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An optimal design problem with perimeter penalization*

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Abstract. We study the optimal design problem of finding the minimal energy configuration for a mixture of two conducting materials when a perimeter penalization of the unknown domain is added. We show that in this situation an optimal domain exists and that, under suitable assumptions on the data, it is an open set.

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1. Introduction

The problem of finding the minimal energy configurations of a mixture of two conducting materials has been widely studied in the literature (see for instance Kohn and Strang [13] and Murat and Tartar [15]). Denoting by α and β the conductivities of the two materials, and by Ω the prescribed container, the problem consists in finding a domain $A \subset \Omega$ of prescribed volume minimizing the quantity

$$-\int_\Omega f(x)u_A(x)\,dx \tag{1.1}$$

where $f(x)$ denotes the source density, and u_A is the solution of the problem

$$\begin{cases} -\operatorname{div}\big((\alpha 1_A + \beta 1_{\Omega\setminus A})Du\big) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Since for every solution u of (1.2) we have

$$\int_\Omega (\alpha 1_A + \beta 1_{\Omega\setminus A})|Du|^2\,dx = \int_\Omega fu\,dx$$

the problem can be reformulated by the minimization of $E(u, A)$, where

$$E(u,A) = \int_\Omega (\alpha 1_A + \beta 1_{\Omega\setminus A})|Du|^2 - 2fu\,dx, \tag{1.3}$$

u varies in $H_0^1(\Omega)$, and A varies in the open subsets of Ω.

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It is well known that in general the optimal configuration does not exist if we only prescribe the volume of the regions occupied by the two materials. On the other hand, it is easy to show (see the proof of Theorem 2.1) that an upper bound on the perimeter of surmentioned regions gives an extra compactness property which is enough to imply the existence of a weak solution for the optimal design problem (1.1) which merely belongs to the class of sets with finite perimeter in Ω.

This paper is concerned with the minimization of functionals of the form

$$E(u,A) = \int_A \alpha|Du|^2 + g_1(x,u)\,dx + \int_{\Omega\setminus A} \beta|Du|^2 + g_2(x,u)\,dx + \sigma P(A,\Omega)$$

where $\sigma > 0$ and $P(A,\Omega)$ denotes the perimeter of A in Ω. Under suitable growth assumptions on g_1 and g_2 we are able to show that the minimization problem

$$\min\Big\{E(u,A)\ :\ u\in H_0^1(\Omega),\ A\subset\Omega\Big\} \tag{1.4}$$

has a solution. Moreover, we prove that if (u,A) is a minimizing pair then u is Hölder continuous and A is (equivalent to) an open set.

Our techniques are based on energy estimates, and we follow the methods of De Giorgi [4], De Giorgi et al. [5], Giaquinta and Giusti [8], and Ladyzhenskaya and Uraltseva [14]. In particular, we do not make any differentiability assumption on the functions g_1 and g_2 in (1.4).

2. Notation and statement of the results

In all the paper Ω will denote a bounded open subset of $\mathbf{R}^n$, and α,β two real numbers with $0<\alpha<\beta$. For every subset A of Ω we denote by $a_A(x)$ the function defined for every $x\in\Omega$ and $z\in\mathbf{R}^n$ by

$$a_A(x) = \alpha 1_A(x) + \beta 1_{\Omega\setminus A}(x)$$

where for every set E we denoted by 1_E the indicator function of E

$$1_E(x) = \begin{cases} 1 & \text{if } x\in E \\ 0 & \text{if } x\notin E. \end{cases}$$

Finally, we denote by $\mathcal{H}^{n-1}$ the Hausdorff $(n-1)$-dimensional measure in $\mathbf{R}^n$.

Given two Borel functions $g_1, g_2 : \Omega\times\mathbf{R}\to\mathbf{R}$ we may consider, for every subset A of Ω, the minimization problem

$$\min\left\{\int_\Omega \left[a_A(x)|Du|^2 + g_A(x,u)\right]dx\ :\ u\in H_0^1(\Omega)\right\} \tag{2.1}$$

where we set

$$g_A(x,s) = 1_A(x)g_1(x,s) + 1_{\Omega\setminus A}(x)g_2(x,s).$$

We assume that g_1 and g_2 satisfy the following assumption:

$$g_i(x,s) \ge \gamma(x) - k|s|^2 \qquad i=1,2 \tag{2.2}$$

where $\gamma\in L^1(\Omega)$ and $k<\alpha\lambda_1$, being λ_1 the first eigenvalue of $-\Delta$ on Ω.

It is well known (see for instance Ioffe [11] or Buttazzo [2]) that, if $g_1(x,s)$ and $g_2(x,s)$ are lower semicontinuous in s, then problem (2.1) admits at least a solution u_A. We denote by $E(A)$ the minimum value of problem (2.1).

The optimal design problem we are interested in is then

$$\min\Big\{E(A)+\sigma P(A,\Omega)\ :\ A\in\mathcal{A}(\Omega)\Big\} \tag{2.3}$$

where $\sigma>0$, $P(A,\Omega)$ denotes the perimeter of A in Ω, and $\mathcal{A}(\Omega)$ is the class of all subsets of Ω with finite perimeter. For the sake of completeness we recall that the perimeter $P(A,\Omega)$ is defined for any Borel set $A\subset\mathbf{R}^n$ by

$$P(A,\Omega)=\sup\left\{\int_A \operatorname{div}\phi\,dx\ :\ \phi\in C_c^1(\Omega;\mathbf{R}^n),\ |\phi|\le 1\right\}.$$

Theorem 2.1. *The minimum problem (2.3) admits at least a solution.*

Proof. Let (A_h) be a minimizing sequence for problem (2.3); then $P(A_h,\Omega)$ are bounded, so that, up to extracting subsequences, we may assume (A_h) is strongly convergent in the L^1_{loc} sense to some $A\in\mathcal{A}(\Omega)$, that is $1_{A_h}\to 1_A$ in $L^1_{loc}(\Omega)$. We claim that A is a solution of problem (2.3). Let us denote by u_h a solution of problem (2.1) associated to A_h; then by (2.2) (u_h) is bounded in $H_0^1(\Omega)$ and we may assume it converges weakly to some $u\in H_0^1(\Omega)$. Then

$$E(A)\le\int_\Omega\big[a_A(x)|Du|^2+g_A(x,u)\big]\,dx.$$

Recalling the expressions of a_A and g_A, and applying the Ioffe lower semicontinuity result (see Ioffe [11]) to the integrand

$$\Phi(x,s_1,s_2,z)=\alpha s_1|z|^2+\beta(1-s_1)|z|^2+s_1g_1(x,s_2)+(1-s_1)g_2(x,s_2)$$

where $x\in\Omega$, $s_1\in[0,1]$, $s_2\in\mathbf{R}$, $z\in\mathbf{R}^n$, we obtain

$$\begin{aligned}E(A)&\le\int_\Omega\big[a_A(x)|Du|^2+g_A(x,u)\big]\,dx=\\&=\int_\Omega\Phi(x,1_A,u,Du)\,dx\le\liminf_{h\to+\infty}\int_\Omega\Phi(x,1_{A_h},u_h,Du_h)\,dx=\\&=\liminf_{h\to+\infty}\int_\Omega\big[a_{A_h}(x)|Du_h|^2+g_{A_h}(x,u_h)\big]\,dx=\liminf_{h\to+\infty}E(A_h).\end{aligned}$$

Therefore, by the lower semicontinuity of the perimeter with respect to L^1_{loc} convergence,

$$E(A)+\sigma P(A,\Omega)\le\liminf_{h\to+\infty}\big[E(A_h)+\sigma P(A_h,\Omega)\big],$$

which proves that A is a solution of (2.3). □

The main result of the paper is the following:

Theorem 2.2. *Let us assume that g_1 and g_2 satisfy (2.2) and*

$$|g_i(x,s)|\le C\big(1+|s|^q\big)\qquad i=1,2 \tag{2.4}$$

where $q \geq 2$ if $n = 2$ and $2 \leq q < 2n/(n-2)$ if $n > 2$. Then, for any set $A \subset \Omega$, any minimizer u_A in (2.1) is locally Hölder continuous. Moreover, for every solution A of problem (2.3) there exists an open set $\tilde{A}$ such that

$$\operatorname{meas}(A \triangle \tilde{A}) = 0 \qquad \textit{and} \qquad P(A, \Omega) = P(\tilde{A}, \Omega) = \mathcal{H}^{n-1}(\partial \tilde{A} \cap \Omega).$$

Remark 2.3. By a careful inspection of the proof it is possible to see that Theorem 2.2 still holds if instead of (2.4) we make the following weaker assumption:

$$|g_i(x,s)| \leq C\big(a(x) + |s|^q\big) \qquad i = 1,2 \tag{2.5}$$

where q is as before and $a \in L^p(\Omega)$ with $p > n$. However, we have preferred the stronger condition (2.4) in order to simplify the estimates we shall obtain in Sections 3 and 4.

Remark 2.4. Theorem 2.2 can be extended, with minor changes in the proof, to mixtures of more than two materials. More generally, (2.3) can be generalized as follows:

$$\min \left\{ \int_\Omega \big[v(x)|Du|^2 + g(x,u,v)\big]\,dx + \sigma|Dv|(\Omega) \ : \right.$$
$$\left. : \ (u,v) \in H_0^1(\Omega) \times BV(\Omega),\ v(x) \in K \text{ a.e.} \right\},$$

where $K \subset]0,+\infty[$ is a fixed compact set, $BV(\Omega)$ is the space of all functions in Ω with finite total variation, and $|Dv|(\Omega)$ is the total variation of v in Ω. In this case, the jump set of the optimal solution v turns out to be equivalent (with respect to $\mathcal{H}^{n-1}$) to a closed subset C of $\mathbf{R}^n$; moreover, v is equivalent (with respect to the Lebesgue measure) to a continuous function $w : \Omega \setminus C \to K$. Similar results for the class of "free discontinuity problems" have been obtained by De Giorgi et al. [5].

Example 2.5. Consider the minimum problem

$$\min_{A \in \mathcal{A}(\Omega)} \left\{ P(A,\Omega) + \int_A h(x)\,dx + \int_\Omega g(x) u_A\,dx \right\} \tag{2.6}$$

where u_A is the unique solution of the elliptic problem

$$\begin{cases} -\operatorname{div}\big(a_A(x)Du\big) + g(x) = 0 & \text{in } \Omega \\ u \in H_0^1(\Omega). \end{cases}$$

It is easy to see that problem (2.6) can be written in the form

$$\min_{A \in \mathcal{A}(\Omega)} \min_{u \in H_0^1(\Omega)} \left\{ P(A,\Omega) + \int_A h(x)\,dx + \int_\Omega a_A(x)|Du|^2\,dx + \int_\Omega 2g(x)u_A\,dx \right\}$$

hence it is of the form (2.3) with

$$g_1(x,s) = h(x) + 2g(x)s \qquad g_2(x,s) = 2g(x)s.$$

By Theorem 2.1, when $g \in L^2(\Omega)$ and $h \in L^1(\Omega)$ problem (2.6) admits a solution, and by Theorem 2.2, when $g \in L^\infty(\Omega)$ and $h \in L^\infty(\Omega)$ every solution is (equivalent to) an open set.

The main tool for the proof of Theorem 2.2 is the following density estimate.

Proposition 2.6. *Let A be a solution of (2.3). Then, for every compact set $K \subset \Omega$ there exists a constant $\xi \in]0, \mathrm{dist}(K, \partial\Omega)[$ such that for every $y \in K$*

$$\lim_{\eta \to 0} \eta^{1-n} \left\{ \int_{B_\eta(y)} a_A(x)|Du|^2\,dx + \sigma P(A, B_\eta(y)) \right\} = 0$$

whenever, for some $\rho < \xi$, it is

$$\int_{B_\rho(y)} a_A(x)|Du|^2\,dx + \sigma P(A, B_\rho(y)) < \xi \rho^{n-1}.$$

The proof of Proposition 2.6 is rather technical; we shall devote to it the last section of the paper.

Proof of Theorem 2.2. The Hölder continuity of solutions of (2.1) follows by Theorem 3.1 below. In order to show that any solution of (2.1) is equivalent to an open set, let us define

$$\Omega_0 = \left\{ y \in \Omega \ : \ \lim_{\rho \to 0} \rho^{1-n} \left[\int_{B_\rho(y)} a_A(x)|Du|^2\,dx + \sigma P(A, B_\rho(y)) \right] = 0 \right\};$$

we show that Ω_0 is open. Let $x_0 \in \Omega_0$, let

$$K = \{ x \in \Omega \ : \ 2\,\mathrm{dist}(x, \partial\Omega) \geq \mathrm{dist}(x_0, \partial\Omega) \},$$

and let $\xi > 0$ be given by Proposition 2.6. We can find a sufficiently small $\rho_0 > 0$ with $\rho_0 < \xi$ such that

$$\int_{B_{\rho_0}(x_0)} a_A(x)|Du|^2\,dx + \sigma P\big(A, B_{\rho_0}(x_0)\big) < \xi \left(\frac{\rho_0}{2} \right)^{n-1}.$$

Since for any $y \in B_{\rho_0/2}(x_0)$

$$\int_{B_{\rho_0/2}(y)} a_A(x)|Du|^2\,dx + \sigma P\big(A, B_{\rho_0/2}(y)\big) < \xi \left(\frac{\rho_0}{2} \right)^{n-1},$$

by Proposition 2.6 we get $B_{\rho_0/2}(x_0) \subset \Omega_0$. It is known (see for instance Federer [6], 4.5.6) that for any set of finite perimeter, the set function $B \mapsto P(A, B)$ is representable by a measure supported by $\partial^* A$, where

$$\partial^* A = \left\{ x \in \Omega \ : \ \limsup_{\rho \to 0^+} \frac{P\big(A, B_\rho(x)\big)}{\rho^{n-1}} > 0 \right\}$$

and

$$P(A, \Omega) = \mathcal{H}^{n-1}(\partial^* A) < +\infty.$$

In particular, if $x \in \Omega_0$ and $B_\rho(x) \subset \Omega_0 \subset \Omega \setminus \partial^* A$, by the isoperimetric inequality

$$\min\left\{ \mathrm{meas}\{B_\rho(x) \cap A\}, \mathrm{meas}\{B_\rho(x) \setminus A\} \right\} \leq c(n) P\big(A, B_\rho(x)\big)^{n/(n-1)} = 0$$

we infer that either $1_A = 1$ a.e. or $1_A = 0$ a.e. in $B_\rho(x)$. Set now

$$\tilde{A} = \{x \in \Omega_0 \ : \ 1_A = 1 \text{ a.e. in a neighbourhood of } x\}.$$

We claim that $\mathcal{H}^{n-1}\big((\Omega \setminus \Omega_0)\Delta\partial^* A\big) = 0$. Indeed, $\partial^* A \subset \Omega \setminus \Omega_0$, and

$$(\Omega \setminus \Omega_0) \setminus \partial^* A \subset \bigcup_{\varepsilon>0} S_\varepsilon,$$

where

$$S_\varepsilon = \left\{ y \in \Omega \ : \ \limsup_{\rho \to 0^+} \rho^{1-n} \int_{B_\rho(y)} a_A(x)|Du|^2\,dx > \varepsilon \right\}.$$

By general results on the differentiation of measures (see for instance Federer [6], 2.10.18(1)) we infer

$$C \int_{S_\varepsilon} a_A(x)|Du|^2\,dx \geq \varepsilon \mathcal{H}^{n-1}(S_\varepsilon) \tag{2.7}$$

with C depending only on n. By (2.7) we infer that $\mathcal{H}^{n-1}(S_\varepsilon) < +\infty$ hence meas $(S_\varepsilon) = 0$ for any $\varepsilon > 0$; by (2.7) again we get $\mathcal{H}^{n-1}(S_\varepsilon) = 0$, and this shows the claim.

The set $\tilde{A}$ is clearly open, and equivalent to A because $\Omega \setminus \Omega_0$ is negligible. Moreover, since $\partial\tilde{A} \cap \Omega \subset \Omega \setminus \Omega_0$, we have

$$\mathcal{H}^{n-1}(\Omega \cap \partial\tilde{A}) \leq \mathcal{H}^{n-1}(\Omega \setminus \Omega_0) = \mathcal{H}^{n-1}(\partial^* A) = P(A, \Omega) = P(\tilde{A}, \Omega).$$

On the other hand, for any Borel set $C \subset \mathbf{R}^n$, it holds (Federer [6], 4.5.6)

$$P(C, \Omega) \leq \mathcal{H}^{n-1}(\Omega \cap \partial C),$$

so that $\mathcal{H}^{n-1}(\Omega \cap \partial\tilde{A}) = P(\tilde{A}, \Omega)$. □

3. Higher integrability and Hölder continuity

In this section we show that minimizers in (2.1) are locally Hölder continuous, and the higher integrability property for the gradient holds. The Hölder continuity is obtained by using the results of Giaquinta and Giusti [8] and Ladyzhenskaya and Uraltseva [14], which are based on De Giorgi's truncation argument [4]. The higher integrability of the gradient follows by a reverse Hölder inequality as in [8], Theorem 4.1.

In the next section it will be useful for us to have uniform estimates on u. Hence, we assume (up to a rescaling of the other constants) that $\sigma = 1$, and we denote by $\mathcal{K}$ the set of constants $\{n, \alpha, \beta, C, q\}$.

Theorem 3.1. *Let u be a solution of (2.1), and let us assume that g_1, g_2 satisfy (2.4). Then the following facts hold.*

(i) For any open set $\Omega_0 \subset\subset \Omega$ the quantity $\|u\|_{L^\infty(\Omega_0)}$ is bounded by a constant depending only on $\mathcal{K}$ and $\|u\|_{L^2(\Omega)}$.

(ii) u is locally Hölder continuous in Ω.

(iii) Let $\Omega_0 \subset\subset \Omega$, let $\tau = \operatorname{dist}(\Omega_0, \partial\Omega)$, and let

$$K = \{x \in \Omega : \text{dist}(x, \Omega_0) \leq \tau/2\}.$$

Then there are constants $\gamma > 0$ and $r > 2$ depending only on $\mathcal{K}$ and $\|u\|_{L^\infty(K)}$ such that

$$\int_{Q_{R/2}(y)} |Du|^r \, dx \leq \gamma \left[R^{n(1-r/2)} \left(\int_{Q_R(y)} |Du|^2 \, dx \right)^{r/2} + R^n \right] \tag{3.1}$$

whenever $y \in \Omega_0$ and $Q_R(y) \subset K$, $Q_R(y)$ being the standard n-cube centered at y with sides of length R.

Proof. Let us prove (i). Let $y \in \Omega$, and let $R \in]0, \text{dist}(y, \partial\Omega)[$. In Theorem 2.1 of [8], it is shown that there are constants $\gamma > 0$ and $0 < \theta < R$ depending only on $\mathcal{K}$ and $\|u\|_{L^2(\Omega)}$ such that

$$\int_{A_{k,\rho}} |Du|^2 \, dx \leq \gamma \left\{ (R - \rho)^{-2} \int_{A_{k,R}} |u - k|^2 \, dx + k^2 \left(\text{meas}\, A_{k,R} \right)^{1-2/n+\varepsilon} \right\}$$

and

$$\int_{B_{k,\rho}} |Du|^2 \, dx \leq \gamma \left\{ (R - \rho)^{-2} \int_{B_{k,R}} |u + k|^2 \, dx + k^2 \left(\text{meas}\, B_{k,R} \right)^{1-2/n+\varepsilon} \right\},$$

where $k \geq 1/\theta$, $\theta/2 \leq \rho \leq \theta$, $\varepsilon = (1 - q/2^*)$,

$$A_{k,\rho} = \{y \in B_\rho(x) : u(x) > k\}, \qquad B_{k,\rho} = \{y \in B_\rho(x) : u(x) < -k\}.$$

Then, the statement follows by Lemma 5.4 of [14].

Let us prove (ii). We briefly sketch the proof given in Theorem 3.1 of [8]. Let Ω_0, τ, K as in (ii); there is a constant γ depending only on $\mathcal{K}$ and $\|u\|_{L^\infty(K)}$ such that

$$\int_{A_{k,\rho}} |Du|^2 \, dx \leq \gamma \left\{ (R - \rho)^{-2} \int_{A_{k,R}} |u - k|^2 \, dx + \text{meas}(A_{k,R}) \right\}$$

and

$$\int_{B_{k,\rho}} |Du|^2 \, dx \leq \gamma \left\{ (R - \rho)^{-2} \int_{B_{k,R}} |u + k|^2 \, dx + \text{meas}(A_{k,R}) \right\},$$

whenever $k > -\|u\|_{L^\infty(K)}$, $y \in \Omega_0$ and $0 < \rho < R < \tau/2$. By using the inequality

$$\int_{A_{k,R}} |u - k|^2 \, dx \leq \max_{A_{k,R}} |u - k|^2 \, \text{meas}(A_{k,R}),$$

we get

$$\int_{A_{k,\rho}} |Du|^2 \, dx \leq \gamma \left[(R - \rho)^{-2} \max_{A_{k,R}} |u - k|^2 + 1 \right] \text{meas}(A_{k,R}).$$

Similarly, we get

$$\int_{B_{k,\rho}} |Du|^2\,dx \le \gamma\left[(R-\rho)^{-2}\max_{B_{k,R}}|u+k|^2+1\right]\text{meas}(B_{k,R}).$$

Hence, u belongs to a class $\mathcal{B}_2$ which has been introduced in [4] and [14] to prove the Hölder continuity of solutions to quasi-linear elliptic equations. The Hölder continuity follows by Theorem 6.1 of [14].

Let us prove (iii). As for (ii), we briefly sketch the proof of Theorem 4.1 of [8]. In fact, arguing exactly as in [8], we get the Caccioppoli inequality

$$\int_{Q_{R/2}(y)} |Du|^2\,dx \le \gamma\left[R^{-2}\int_{Q_R(y)}|u-u_{R,y}|^2\,dx+R^n\right],$$

where $y \in \Omega_0$, $Q_R(y) \subset K$, $u_{R,y}$ is the average of u in $Q_R(y)$ and γ depends only on $\mathcal{K}$ and $\|u\|_{L^\infty(K)}$. By the Sobolev-Poincaré inequality

$$R^{-2}\int_{Q_R(y)}|u-u_{R,y}|^2\,dy \le C\left(\int_{Q_R(y)}|Du|^{2n/(n+2)}\,dx\right)^{1+2/n}$$

we get

$$\int_{Q_{R/2}(y)} |Du|^2\,dx \le \gamma\left[C\left(\int_{Q_R(y)}|Du|^{2n/(n+2)}\,dx\right)^{1+2/n}+R^n\right].$$

Hence, $|Du|^{2n/(n+2)}$ satisfies a reverse Hölder inequality. By applying Proposition 1.1, page 122 of Giaquinta [7] (see also Giaquinta and Modica [9]) we get (3.1). □

4. Blow-up and energy decay

The proof of Proposition 2.6 is based upon two main ideas. Let A be a solution of (2.3), and let $u = u_A$ be the corresponding solution of (2.1). Because of the growth condition (2.4), we are led to believe that (u, A) is almost minimizing the energy

$$\int_{B_\rho(y)} a_A(x)|Du|^2\,dx + P\big(A, B_\rho(y)\big)$$

with respect to perturbations with compact support, provided $B_\rho(y) \subset \Omega$ is sufficiently small. The minimality condition can be formulated in terms of the rescaled functions u_h and the rescaled sets A_h

$$u_{\rho,y}(x) = \frac{u(y+\rho x)}{\sqrt{\rho}}, \qquad A_{\rho,y} = \{x \in B \;:\; y+\rho x \in A\}$$

in the unit ball B. This leads to Definition 4.1 below. The second heuristic idea is concerned with the behaviour of u in balls $B_\rho(y)$ where $P\big(A, B_\rho(y)\big)/\rho^{n-1}$ is small. By the isoperimetric inequality, either $B_\rho(y) \cap A$ or $B_\rho(y) \setminus A$ are close to the empty set. Hence, the diffusion coefficient a_A is close to a constant in $B_\rho(y)$. This suggests that u should be very close to an harmonic function, and this gives informations about the decay (as $\rho \to 0^+$) of

$$\int_{B_\rho(y)} a_A(x)|Du|^2\,dx.$$

Definition 4.1. Let $u_h \in H^1(B)$, $A_h \subset B$, $\lambda_h > 0$. We say that (u_h, A_h) are λ_h-asimptotically minimizing if the following condition is fulfilled for any compact set $K \subset B$: for any bounded sequence $(v_h) \subset H^1(B)$ with $\text{supp}(v_h - u_h) \subset K$ and any sequence of sets $C_h \subset B$ with $A_h \Delta C_h \subset K$, we have

$$\int_B a_{A_h}(x)|Du_h|^2 + \lambda_h P(A_h, B) \le \int_B a_{C_h}(x)|Dv_h|^2 + \lambda_h P(C_h, B) + \eta_h$$

for a suitable infinitesimal sequence η_h.

The following theorem is concerned with the behaviour of asimptotically minimizing sequences.

Theorem 4.1. *Let $\lambda_h > 0$, $u_h \in H^1(B)$, $A_h \subset B$. Assume (u_h, A_h) is λ_h-asimptotically minimizing and*

(*i*) $\quad \displaystyle\int_B a_{A_h}(x)|Du_h|^2\,dx + \lambda_h P(A_h, B)$ *is bounded*;

(*ii*) $\quad u_h \to u$ *weakly in* $H^1(B)$;

(*iii*) $\quad 1_{A_h} \to 1_A$ *in* $L^1(B)$ *and* $\lambda_h \to +\infty$;

(*iv*) $\quad |Du_h|^2$ *is locally equi-integrable in* B.

Then we have:
(a) $u_h \to u$ *in* $H^1_{loc}(B)$;
(b) $\lambda_h P(A_h, B_\rho) \to 0$ *for any* $\rho < 1$, *either* $A = \emptyset$ *or* $A = B$, *and* u *is harmonic on* B.

Proof. Let us prove (a). By the local equi-integrability we infer

$$\lim_{h\to+\infty} \int_K \left|a_{A_h}(x)|Du_h|^2 - a_A(x)|Du_h|^2\right| dx = 0 \tag{4.1}$$

for any compact set $K \subset B$. Now, we choose a function $\psi \in C^1_c(B)$ such that $0 \le \psi \le 1$ and we compare (u_h, A_h) with $(\tilde{u}_h, A_h)$, where $\tilde{u}_h = (1-\psi)u_h + \psi u$. We get

$$\int_B a_{A_h}(x)|Du_h|^2\,dx \le \int_B a_{A_h}(x)|D\tilde{u}_h|^2\,dx + \eta_h. \tag{4.2}$$

By convexity

$$\begin{aligned}\int_B a_{A_h}(x)|D\tilde{u}_h|^2\,dx &\le \int_B a_{A_h}(x)|(1-\psi)Du_h + \psi Du|^2\,dx + \sigma_h \le \\ &\le \int_B (1-\psi)a_{A_h}(x)|Du_h|^2\,dx + \int_B \psi a_{A_h}(x)|Du|^2\,dx + \sigma_h\end{aligned} \tag{4.3}$$

with σ_h infinitesimal. Therefore, (4.1), (4.2) and (4.3) yield

$$\limsup_{h\to+\infty} \int_B \psi a_A(x)|Du_h|^2\,dx \le \int_B \psi a_A(x)|Du|^2\,dx.$$

On the other hand, the lower semicontinuity with respect to weak topology of $H^1(B)$ implies

$$\liminf_{h\to+\infty}\int_B \psi a_A(x)|Du_h|^2\,dx \geq \int_B \psi a_A(x)|Du|^2\,dx.$$

Hence,

$$\lim_{h\to+\infty}\int_B \psi a_A(x)|Du_h|^2\,dx = \int_B \psi a_A(x)|Du|^2\,dx,$$

and since $\psi \in C^1_c(B)$ is arbitrary, this implies the strong convergence of u_h to u in $H^1_{loc}(B)$.

Let us prove (b). Since $\lambda_h \to +\infty$ and the energies are bounded by some constant c, the inequality

$$P(A_h, B) \leq \frac{c}{\lambda_h}$$

implies the convergence of the perimeters to 0. By semicontinuity, $P(A,B) = 0$. Hence, either $A = \emptyset$ or $A = B$. We assume that $A = \emptyset$ (the other case is analogous). Then, the isoperimetric inequality in balls (see for instance Giusti [10]) implies

$$\text{meas}(A_h) \leq c(n)\left(\frac{c}{\lambda_h}\right)^{n/(n-1)}$$

for h large enough. Hence, denoting by $\chi_h(\rho)$ the outer trace of A_h on ∂B_ρ, we have

$$\int_0^1 \left(\int_{\partial B_\rho} \chi_h(\rho)\,d\mathcal{H}^{n-1}\right) d\rho \leq c(n)\left(\frac{c}{\lambda_h}\right)^{n/(n-1)}.$$

Let us fix $\rho \in]0,1[$. Possibly passing to subsequences we can find a sequence ρ_h such that $\rho < \rho_h < (1+\rho)/2$ and

$$\lambda_h \int_{\partial B_{\rho_h}} \chi_h(\rho_h)\,d\mathcal{H}^{n-1} \to 0.$$

Comparing A_h with $A_h \setminus B_{\rho_h}$, using the inequality

$$P(A_h \setminus B_{\rho_h}, B) \leq P(A_h, B \setminus \overline{B_{\rho_h}}) + \int_{\partial B_{\rho_h}} \chi_h(\rho_h)\,d\mathcal{H}^{n-1}$$

and using the fact that (u_h, A_h) is asimptotically λ_h-minimizing, we easily find

$$\lim_{h\to+\infty} \lambda_h P(A_h, B_{\rho_h}) = 0,$$

hence $\lambda_h P(A_h, B_\rho) \to 0$. Finally, given any function $\varphi \in C^1(B)$, with $\text{supp}(\varphi) \subset B_\rho$, comparing u_h with $\tilde{u}_h = u_h + \varphi$ we find

$$\int_{B_\rho} a_{A_h}(x)|Du_h|^2\,dx \leq \int_{B_\rho} a_{A_h}(x)|D\tilde{u}_h|^2\,dx + \eta_h.$$

Passing to the limit as $h \to +\infty$ and dividing both sides by β we get

$$\int_B |Du|^2\,dx \leq \int_B |D(u+\varphi)|^2\,dx.$$

Since $\varphi \in C_c^1(B)$ is arbitrary, this implies that u is harmonic in B. □

The following decay theorem is crucial in the proof of the density estimate. The proof is achieved by contradiction, by making use of Theorem 4.1.

Theorem 4.2. *Let $K \subset \Omega$ be a compact set, and let $\delta = \operatorname{dist}(K, \partial\Omega)$. Then, there are constants $\gamma > 0$, $\theta > 0$ satisfying the following condition: for any solution A of (2.3), any ball $B_\rho(y)$ with $y \in K$ and $\rho \in]0, \delta/2[$, denoting for simplicity by u the solution u_A, the inequalities*

$$(i) \qquad \int_{B_\rho(y)} a_A(x)|Du|^2\,dx + P(A, B_\rho(y)) < \gamma\rho^{n-1}$$

$$(ii) \qquad \rho^n \leq \theta \left[\int_{B_\rho(y)} a_A(x)|Du|^2\,dx + P(A, B_\rho(y))\right]$$

imply

$$\int_{B_{\rho/2}(y)} a_A(x)|Du|^2\,dx + P(A, B_{\rho/2}(y)) \leq$$
$$\leq \left(\frac{1}{2}\right)^{n-1/2} \left[\int_{B_\rho(y)} a_A(x)|Du|^2\,dx + P(A, B_\rho(y))\right].$$

Proof. We argue by contradiction. If the statement were not true, it would be possible to find a sequence of solutions D_h of (2.3), sequences γ_h, θ_h converging to 0, balls $B_{\rho_h}(x_h)$ with $x_h \in K$ and $\rho_h \in]0, \delta/2[$, such that, denoting by w_h the solutions of (2.1) corresponding to D_h, the following inequalities hold

$$\rho_h^n \leq \theta_h \left[\int_{B_{\rho_h}(x_h)} a_{D_h}(x)|Dw_h|^2\,dx + P(D_h, B_{\rho_h}(x_h))\right],$$
$$\int_{B_{\rho_h}(x_h)} a_{D_h}(x)|Dw_h|^2\,dx + P(D_h, B_{\rho_h}(x_h)) = \gamma_h \rho_h^{n-1},$$
$$\int_{B_{\rho_h/2}(x_h)} a_{D_h}(x)|Dw_h|^2\,dx + P(D_h, B_{\rho_h/2}(x_h)) >$$
$$> \left(\frac{1}{2}\right)^{n-1/2} \left[\int_{B_{\rho_h}(x_h)} a_{D_h}(x)|Dw_h|^2\,dx + P(D_h, B_{\rho_h}(x_h))\right].$$

We claim that w_h is bounded in $H_0^1(\Omega)$. Indeed, by (2.2) and the Poincaré inequality we infer

$$E(D_h) + P(D_h, \Omega) \geq \alpha \int_\Omega |Dw_h|^2\,dx + \int_\Omega \gamma\,dx - k\int_\Omega |w_h|^2\,dx$$
$$\geq (\alpha - k/\lambda_1)\int_\Omega |Dw_h|^2\,dx + \int_\Omega \gamma\,dx$$

Since $k < \alpha\lambda_1$ and $E(D_h) + P(D_h, \Omega) \leq E(\Omega)$, the claim is proved. By Theorem 3.1 we infer that w_h is locally equibounded in Ω.

We now rescale (w_h, D_h) in the unit ball B by setting

$$v_h(y) = w_h(x_h + \rho_h y), \qquad A_h = \{y \in B \ : \ x_h + \rho_h y \in D_h\}, \qquad \lambda_h = \frac{1}{\gamma_h}.$$

Moreover, we define

$$u_h = \frac{v_h - \bar{v}_h}{\sqrt{\gamma_h}},$$

were $\bar{v}_h$ is the average of v_h in B. Now, we claim that u_h, A_h, λ_h satisfy the hypotheses of Theorem 4.2. Indeed, we have

$$\begin{aligned} &\rho_h \le \theta_h \gamma_h, \\ (4.4) \qquad &\int_B a_{A_h}(x)|Du_h|^2\,dx + \lambda_h P(A_h, B) = 1, \\ &\int_{B_{1/2}} a_{A_h}(x)|Du_h|^2\,dx + \lambda_h P(A_h, B_{1/2}) > \left(\frac{1}{2}\right)^{n-1/2}. \end{aligned}$$

In particular, we can assume with no loss of generality that u_h weakly converges in $H^1(B)$ to u and 1_{A_h} converges in $L^1(B)$ to 1_A. We have to check that (u_h, A_h) are λ_h-asimptotically minimizing, and $|Du_h|^2$ is locally equi-integrable in B. Let $K \subset B$, (u'_h, A'_h) as in Definition 4.1. We define

$$v'_h = \sqrt{\gamma_h}u'_h + \bar{v}_h, \qquad w'_h(x) = v'_h(\frac{x - x_h}{\rho_h}), \qquad D'_h = \{x : \frac{x - x_h}{\rho_h} \in A'_h\}.$$

Then, the minimality of (w_h, D_h) and (2.4) yield

$$\begin{aligned} \int_B a_{A'_h}(x)|Du'_h|^2\,dx + \lambda_h P(A'_h, B) &= \frac{1}{\gamma_h}\left[\int_B a_{A'_h}(x)|Dv'_h|^2\,dx + P(A'_h, B)\right] = \\ &= \frac{1}{\gamma_h \rho_h^{n-1}}\left[\int_{B_{\rho_h}(x_h)} a_{D'_h}(x)|Dw'_h|^2\,dx + P(D'_h, B_{\rho_h}(x_h))\right] \ge \\ &\ge \frac{1}{\gamma_h \rho_h^{n-1}}\left[\int_{B_{\rho_h}(x_h)} a_{D_h}(x)|Dw_h|^2\,dx + P(D_h, B_{\rho_h}(x_h)) - \right. \\ &\qquad \left. - C\int_{B_{\rho_h}(x_h)} (2 + |w_h|^q + |w'_h|^q)\,dx\right] = \\ &= \int_B a_{A_h}(x)|Du_h|^2\,dx + \lambda_h P(A_h, B) \\ &\qquad - 2\frac{C\omega_n\rho_h}{\gamma_h} - \frac{C}{\gamma_h\rho_h^{n-1}}\int_{B_{\rho_h}(x_h)} (|w_h|^q + |w'_h|^q)\,dx, \end{aligned}$$

where $\omega_n = \text{meas}(B)$. Since $\rho_h/\gamma_h \le \theta_h \to 0$ and w_h is locally equibounded in Ω, we need only to verify that

$$I_h = \frac{1}{\gamma_h \rho_h^{n-1}}\int_{B_{\rho_h}(x_h)} |w'_h|^q\,dx = \frac{\rho_h}{\gamma_h}\int_B |v'_h|^q\,dx$$

is infinitesimal as $h \to +\infty$. Indeed, since $v'_h = \sqrt{\gamma_h}u'_h + \bar{v}_h$, we have

$$I_h \leq 2^{q-1}\rho_h\gamma_h^{q/2-1}\int_B |u_h|^q\,dx + 2^{q-1}\frac{\omega_n\rho_h}{\gamma_h}|\bar{v}_h|^q.$$

Since u_h is bounded in $H^1(B)$, by the Poincaré-Wirtinger inequality the first term is infinitesimal. Since w_h is locally equibounded, $\bar{v}_h$ is a bounded sequence and also the second term is infinitesimal.

Now, we show that for any $t \in]0,1[$ there is $r > 2$ such that

$$\sup\left\{\int_{B_t}|Du_h|^r\,dx : h \in \mathbf{N}\right\} < +\infty \tag{4.5}$$

In particular, $|Du_h|^2$ will be locally equi-integrable in B. By (3.1) of Theorem 3.1 we get a constant $\gamma > 0$ such that

$$\int_{Q_\rho(y)}|Dw_h|^r\,dx \leq \gamma\left\{\rho^{n(1-r/2)}\left(\int_{Q_{2\rho}(y)}|Dw_h|^2\,dx\right)^{r/2} + \rho^n\right\},$$

for any $h \in \mathbf{N}$, and any n-cube $Q_\rho(x)$ with

$$\operatorname{dist}(x,K) \leq \delta/2, \qquad 0 < \rho < \delta/2.$$

By an elementary covering argument, we get

$$\int_{B_{t\rho_h}(x_h)}|Dw_h|^r\,dx \leq c(n,t)\gamma\left\{\rho_h^{n(1-r/2)}\left(\int_{B_{\rho_h}(x_h)}|Dw_h|^2\,dx\right)^{r/2} + \rho_h^n\right\},$$

so that

$$\int_{B_t}|Dv_h|^r\,dx \leq c(n,t)\gamma\left\{\left(\int_B|Dv_h|^2\,dx\right)^{r/2} + \rho_h^r\right\},$$

and

$$\int_{B_t}|Du_h|^r\,dx \leq c(n,t)\gamma\left\{1 + \left(\frac{\rho_h}{\sqrt{\gamma_h}}\right)^r\right\}.$$

Therefore (4.5) follows by the first inequality of (4.4). By Theorem 4.1 we infer that u is harmonic and either $A = \emptyset$ or $A = B$. Let us assume that $A = \emptyset$ (the other case is analogous). Since $|Du|^2$ is a subharmonic function (see for instance Giaquinta [7], page 80) by the second inequality of (4.4) we get

$$\begin{aligned}\beta\int_{B_{1/2}}|Du|^2\,dx &\leq \beta\left(\frac{1}{2}\right)^n\int_B|Du|^2\,dx = \left(\frac{1}{2}\right)^n\int_B a_A(x)|Du|^2\,dx \leq \\ &\leq \left(\frac{1}{2}\right)^n \liminf_{h\to+\infty}\int_B a_{A_h}(x)|Du_h|^2\,dx \leq \left(\frac{1}{2}\right)^n.\end{aligned}$$

On the other hand, by Theorem 4.1 we know that

$$\int_{B_{1/2}}|Du_h - Du|^2\,dx + \lambda_h P(A_h, B_{1/2}) \to 0$$

as $h \to +\infty$, hence the third inequality of (4.4) yields

$$\beta \int_{B_{1/2}} |Du|^2\,dx = \int_{B_{1/2}} a_A(x)|Du|^2\,dx$$
$$= \lim_{h\to+\infty} \int_{B_{1/2}} a_{A_h}(x)|Du_h|^2\,dx \ge \left(\frac{1}{2}\right)^{n-1/2},$$

and this is a contradiction. □

By using Theorem 4.2 iteratively, we are able to show that if

$$\rho^{1-n}\left[\int_{B_\rho(y)} a_A|Du|^2\,dx + P\big(A, B_\rho(y)\big)\right] < \xi \qquad \text{and} \qquad \rho < \xi$$

for $\xi > 0$ sufficiently small, then

$$\rho^{1-n}\left[\int_{B_\rho(y)} a_A|Du|^2\,dx + P\big(A, B_\rho(y)\big)\right]$$

converges to 0 as $\rho \to 0^+$.

Proposition 4.3. *Let K, γ, θ be given by Theorem 4.2, and let A be a solution of (2.3). Let $y \in K$, let us denote by u_A the solution of (2.1) and*

$$F(\rho) = \int_{B_\rho(y)} a_A(x)|Du|^2\,dx + P(A, B_\rho(y)).$$

Let

$$\xi = \operatorname{dist}(y, \partial\Omega) \wedge \gamma \wedge 2^{1/2-n}\gamma\theta.$$

Then, if $F(\rho) < \xi\rho^{n-1}$ for some $\rho \in]0, \xi[$ we get

$$F(\eta) < 2^{n-1/2}\gamma\rho^{n-1}\left(\frac{\eta}{\rho}\right)^{n-1/2} \qquad \forall \eta \in]0, \rho]. \tag{4.6}$$

In particular,

$$\lim_{\eta\to 0} \eta^{1-n} F(\eta) = 0.$$

Proof. Let us assume that $F(\rho) < \xi\rho^{n-1}$ for some $\rho \in]0, \xi[$. Since F is nondecreasing, in order to prove (4.6) we need only to show by induction on $j \in \mathbf{N}$ the following inequality

$$F(\eta_j) < \gamma\rho^{n-1}\left(\frac{\eta_j}{\rho}\right)^{n-1/2}, \qquad \eta_j = 2^{-j}\rho.$$

The inequality is trivially true if $j = 0$. Let us assume that it is valid for j; in particular, we have

$$\frac{F(\eta_j)}{\eta_j^{n-1}} < \gamma\left(\frac{\eta_j}{\rho}\right)^{1/2} \le \gamma.$$

A little computation shows that

$$\frac{\eta_j^n}{\theta} < \gamma\rho^{n-1}\left(\frac{\eta_{j+1}}{\rho}\right)^{n-1/2}$$

if $\rho < 2^{1/2-n}\theta\gamma$. Hence, by our choice of ξ, if $\theta F(\eta_j) \leq \eta_j^n$ then

$$F(\eta_{j+1}) \leq F(\eta_j) < \frac{\eta_j^n}{\theta} < \gamma\rho^{n-1}\left(\frac{\eta_{j+1}}{\rho}\right)^{n-1/2}.$$

On the other hand, if $\theta F(\eta_j) > \eta_j^n$, by Theorem 4.2 we infer

$$F(\eta_{j+1}) \leq \left(\frac{1}{2}\right)^{n-1/2} F(\eta_j) < \gamma\rho^{n-1}\left(\frac{\eta_{j+1}}{\rho}\right)^{n-1/2},$$

and this achieves the proof. □

Proof of Proposition 2.6. The choice of ξ in the statement of Proposition 4.3 can be uniformly made with respect to $y \in K$ for any compact set $K \subset \Omega$. Hence, the conclusion is proved by taking

$$\xi = \operatorname{dist}(K, \partial\Omega) \wedge \gamma \wedge 2^{1/2-n}\gamma\theta,$$

where θ and γ given by Theorem 4.2. □

References

[1] Alt, H.W., Caffarelli, L.A.: Existence and regularity results for a minimum problem with free boundary. J. Reine Angew. Math. **325**, 107–144 (1981)

[2] Buttazzo, G.: Semicontinuity, relaxation and integral representation in the calculus of variations. Pitman Res. Notes Math. Ser. **207**, Harlow: Longman 1989

[3] Campanato, S.: Sistemi Ellittici in Forma Divergenza. Regolarità all'Interno. Quaderno della Scuola Normale Superiore di Pisa, Pisa 1980

[4] De Giorgi, E.: Sulla differenziabilità e l'analiticità degli integrali multipli regolari. Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **3**, 25–43 (1957)

[5] De Giorgi, E, Carrierio, M., Leaci, A.: Existence theorem for a minimum problem with free discontinuity set. Arch. Rational Mech. Anal. **108**, 195–218 (1989)

[6] Federer, H.: Geometric measure theory. Berlin Heidelberg New York: Springer 1969

[7] Giaquinta, M.: Multiple integrals in the calculus of variations. Princeton: Princeton University Press 1983

[8] Giaquinta, M, Giusti, E.: On the regularity of minima of variational integrals. Acta Math. **148**, 31–46 (1982)

[9] Giaquinta, M., Modica, G.: Regularity results for some classes of higher order nonlinear elliptic systems. J. Reine Angew. Math. **311/312**, 145–169 (1979)

[10] Giusti, E.: Minimal surfaces and functions of bounded variation. Boston: Birkhäuser 1984

[11] Ioffe, A.D.: On lower semicontinuity of integral functionals I. SIAM J. Control Optim. **15**, 521–538 (1977)

[12] Kinderlehrer, D., Stampacchia, G.: Variational inequalities and applications. New York: Academic Press 1980

[13] Kohn, R.V., Strang, G.: Optimal design and relaxation of variational problems, I,II,III. Commun. Pure Appl. Math. **39**, 113–137, 139–182, 353–377 (1986)

[14] Ladyzhenskaya, O.A., Uraltseva, N.N.: Linear and quasilinear elliptic equations. New York: Academic Press 1968

[15] Murat, F., Tartar, L.: Optimality conditions and homogenization. Proceedings of "Nonlinear Variational Problems", Isola d'Elba 1983 (Res. Notes in Math. **127**, pp. 1–8) London: Pitman 1985