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Bounds on the Error of Gauss-Type Quadratures

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1. Introduction

This paper gives some bounds for

$$E_n(f) = \int_{-1}^{1} w(x)\, f(x)\, dx - \sum_{k=1}^{n} w_{n,k}\, f(x_{n,k}) \tag{1}$$

in the case when $w(x)$ is an even function of x, positive and Lebesque integrable over $(-1, 1)$. The $x_{n,k}$ $(n=1, 2, 3, \ldots;\ k=1, 2, \ldots, n)$ are the n zeros of the polynomial $p_n(x)$ of degree n which is a member of the sequence $\{p_n(x)\}$ orthogonal over the interval $(-1, 1)$ with respect to $w(x)$, and the $w_{n,k}$ are the corresponding weights. The function $f(z)$ is assumed to be an analytic function of z, regular in $|z| < 1+2\varepsilon$ where $\varepsilon > 0$; that is,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad |z| < 1+2\varepsilon. \tag{2}$$

The above assumptions on $f(z)$ make it possible to obtain bounds on $E_n(f)$ which depend only on the modulus of f on some set in the region of regularity of f, such as a contour enclosing the strip $(-1, 1)$; bounding $E_n(f)$ in this manner is often easier than bounding a $2n$'th order derivative of f in $(-1, 1)$.

The convergence of the quadrature scheme (1) with f as in (2) is studied by KRYLOV [1]. DAVIS (see e.g. [2]; which gives references to DAVIS's papers) appears first to have effectively used the fact that the error $E_n(f)$ is a linear functional in f, and thereby obtained some very sharp bounds of the form $E_n(f) \leqq \sigma_n \|f\|$ in the case $w(x)=1$.

The work of DAVIS was extended by HAMMERLIN [3]; HAMMERLIN, in considering quadrature formulae requiring equi-spaced abscissae $x_{n,k}$, concentrated on obtaining general expressions for σ_n which are easy to evaluate.

WILF [4] also considered the case of $w(x)=1$. For the particular norm $\|f\| = \left(\sum_{k=0}^{\infty} |a_k|^2\right)^{\frac{1}{2}}$ WILF chose his $x_{n,k}$ and $w_{n,k}$ such that σ_n was a minimum for each fixed n. He thus obtained a new class of integration formulae.

MCNAMEE [5] gave a somewhat different procedure of bounding $E_n(f)$ for the case $w(x)=1$. His method depends on expressing the error in the form of a contour integral

$$E_n(f) = \frac{1}{2\pi i} \int_C f(z)\, q_n(z)\, dz;$$

expanding $q_n(z)$ in powers of $1/z$, approximating $q_n(z)$ by the dominating term of this expansion on a contour C far from the origin, then minimizing the modulus of the resulting integrand with respect to a particular family of contours C (e.g. a family of concentric circles with center at the origin).

The bounds we shall obtain are as easy to evaluate as those of the above authors. In addition, they all make use of the fact that $E_n(f)$ in (1) neither depends on a_{2k+1} ($k=0, 1, 2, \ldots$) nor on $a_0, a_2, \ldots, a_{2n-2}$. It follows from this that the bounds obtained by DAVIS[1], HAMMERLIN[2] and MCNAMEE[3] can (at least in theory) be made sharper.

We shall also determine the sign of $E_n(f)$ and show that $E_n(f)$ tends to zero monotonically for the class $\{f\}$ for which a_{2k} is of the same sign for all k sufficiently large.

2. General Developments

$p_n(x)$ is assumed to have the properties described in the introduction above; we shall also require in this section that k_n, the coefficient of x^n in $p_n(x)$, be positive.

For $n \geqq 1$ consider the contour integral

$$J = \frac{1}{2\pi i} \int_C \frac{w(x)\, f(z)\, p_n(x)}{(z-x)\, p_n(z)}\, dz \tag{3}$$

where the contour C is the circle of radius $1+\varepsilon$ about $z=0$ and x is any point in $-1<x<1$. Employing Cauchy's theorem of residues we find that

$$J = w(x)\, f(x) - w(x)\, p_n(x) \sum_{k=1}^{n} \frac{f(x_{n,k})}{(x-x_{n,k})\, p_n'(x_{n,k})} \tag{4}$$

since under our assumptions on $w(x)$, the zeros $x_{n,k}$ of $p_n(x)$ are distinct and located in the open interval $(-1, 1)$.

Integrating (4) with respect to x over $(-1, 1)$ we have

$$E_n(f) = \int_{-1}^{1} w(x)\, f(x)\, dx - \sum_{k=1}^{n} w_{n,k}\, f(x_{n,k}) \tag{5}$$

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[1] DAVIS puts $f(z) = \sum_{k=0}^{\infty} \alpha_k U_k(z)$, where $U_k(x)$ belongs to the sequence of Chebyshev polynomials orthonormal over $(-1, 1)$ with respect to $\sqrt{1-x^2}$. Thus his estimate $|E_n(f)| \leqq \sigma_n \left(\sum_{k=0}^{\infty} |\alpha_k|^2\right)^{\frac{1}{2}}$ can be sharpened to $|E_n(f)| \leqq \sigma_n \left(\sum_{k=0}^{\infty} |\alpha_{2n+2k}|^2\right)^{\frac{1}{2}}$.

[2] The author is grateful to the referee for pointing out that in his latest paper [Num. Math. 7, 232–237 (1965)] HAMMERLIN has in fact replaced $\|f\|$ by $\inf_a \|f-a\|$ where a is a suitable polynomial.

[3] MCNAMEE's expression for $E_n(f)$ can be altered to

$$E_n(f) = \frac{1}{2\pi i} \int_C \frac{1}{2} \{f(z) + f(-z) - p_{n-1}(z^2)\}\, q_n(z)\, dz.$$

The polynomial $p_{n-1}(z^2)$ of degree $n-1$ in z^2 can for example be chosen so that $|\frac{1}{2}[f(z)+f(-z)] - p_{n-1}(z^2)|$ takes on a minimum value on a particular contour C.

where $w_{n,k}$ are the positive weights given by

$$w_{n,k} = \int_{-1}^{1} \frac{w(x)\, p_n(x)\, dx}{(x - x_{n,k})\, p_n(x_{n,k})}. \tag{6}$$

On interchanging the order of integration in the resulting repeated integral, we find that $E_n(f)$ is also given by

$$E_n(f) = \frac{1}{2\pi i} \int_C \int_{-1}^{1} \frac{w(x)\, f(z)\, p_n(x)}{(z-x)\, p_n(z)}\, dx\, dz; \tag{7}$$

in what follows we shall examine this latter double integral more closely.

Lemma 1. *In the expansion*

$$\frac{1}{p_n(z)} = \sum_{j=0}^{\infty} b_{n,j} z^{-n-j}, \qquad |z| \geqq 1 \tag{8}$$

and $n > 1$, *we have* $b_{n,2j+1} = 0$, $b_{n,2j} > 0$ $(j = 0, 1, 2, \ldots)$. *For* $n = 1$ *we trivially have* $\frac{1}{p_1(z)} = b_{1,0} z^{-1}$ *where* $b_{1,0} = \frac{1}{k_1} > 0$.

Proof. We shall assume that $n \geqq 2$. Since $w(x)$ is an even function of x in the interval $(-1, 1)$, each $p_n(x)$ is either even or odd with n. It is thus clear that $b_{n,2j+1} = 0$. To prove that $b_{n,2j} > 0$ we observe that the zeros of $p_n(x)$ are symmetric about $z = 0$ and so we have

$$\frac{1}{p_n(z)} = k_n^{-1} \left\{ z^n \sum_{j=1}^{[n/2]} \left[1 - \left(\frac{x_{n,j}}{z} \right)^2 \right] \right\}^{-1}. \tag{9}$$

Expanding the right of (9) we have

$$\frac{1}{p_n(z)} = \frac{k_n^{-1}}{z^n} \prod_{j=0}^{[n/2]} \left\{ \sum_{k=0}^{\infty} \left(\frac{x_{n,j}}{z} \right)^{2k} \right\}; \tag{10}$$

the right hand side of (10) is a product of $[n/2]$ power series each of which only has positive coefficients. The statement of Lemma 1 follows.

Lemma 2.

$$x^n = \begin{cases} \sum_{j=0}^{k} \alpha_{n,2j}\, p_{2j}(x); & n = 2k \\ \sum_{j=0}^{k} \alpha_{n,2j+1}\, p_{2j+1}(x); & n = 2k+1 \end{cases} \tag{11}$$

where

$$\alpha_{n,2j} > 0, \qquad \alpha_{n,2j+1} > 0, \qquad n \geqq 0; \qquad j = 0, 1, \ldots, k.$$

Proof. The proof is by induction. Clearly we can choose $p_0(x) = k_0$, $p_1(x) = k_1 x$ where k_0 and k_1 are positive. In addition $p_2(x) = k_2 x^2 - a$ where k_2 and a are both positive. Now suppose that for $n = 2k \geqq 0$ we have the first of (11) with all the coefficients $\alpha_{n,2j}>0$. Multiplying this equation through by x we get

$$x^{2k+1}=\sum_{j=0}^{k}\alpha_{n,2j}\,x\,p_{2j}(x). \tag{12}$$

We now use the three term recurrence relation

$$x\,p_n(x)=A_n\,p_{n+1}(x)+B_n\,p_{n-1}(x), \qquad n\geqq 1 \tag{13}$$

in (12) $\left(A_n=k_n/k_{n+1},\ B_n=k_{n-1}/k_n,\ k_n\right.$ being the positive coefficient of x^n in $\left.p_n(x)\right)$ to obtain an expansion having the form of the second of (11) with all the coefficients positive. The proof for $n=2k+1$ is similar and is omitted.

Lemma 3.

$$\int_{-1}^{1}\frac{w(x)\,p_n(x)}{z-x}\,dx=\sum_{j=0}^{\infty}c_{n,2j}\,z^{-n-2j-1}, \qquad |z|>1, \tag{14}$$

where $c_{n,2j}>0$, $n=0,1,2,\ldots;$ $j=0,1,2,\ldots$.

Proof. Expanding the denominator of the integrand in (14) we obtain

$$\int_{-1}^{1}\frac{w(x)\,p_n(x)}{z-x}\,dx=\int_{-1}^{1}\frac{w(x)\,p_n(x)}{z}\sum_{j=0}^{\infty}\left(\frac{x}{z}\right)^j dx. \tag{15}$$

If we now substitute equation (11) into the sum on the right of (15), use the orthogonality property of the polynomials $p_n(x)$ together with the results of Lemma 2, we obtain the expansion on the right of (14) with $c_{n,2j}>0$, $j=0,1,2,\ldots$.

Theorem 1. *Let* $E_n(f)$ *be given by equation* (7), *let* a_k *be defined by equation* (2), *and let* $e_{n,k}$ *be defined by* (18) *below. Then, for* $n\geqq 1$,

$$E_n(f)=\sum_{k=0}^{\infty}a_{2n+2k}\,e_{n,k} \tag{16}$$

where $e_{n,k}>0$, $k=0,1,2,\ldots$.

Proof. Combining the results of Lemmas 1 and 3 we have

$$\int_{-1}^{1}\frac{w(x)\,p_n(x)}{(z-x)\,p_n(z)}\,dx=\sum_{k=0}^{\infty}e_{n,k}\,z^{-2n-2k-1}, \qquad |z|>1 \tag{17}$$

where

$$e_{n,k}=\sum_{j=0}^{k}b_{n,2j}\,c_{n,2k-2j}. \tag{18}$$

By Lemmas 1 and 2 it is clear that $e_{n,k}>0$. Substituting (17) into (7) we obtain

$$E_n(f)=\frac{1}{2\pi i}\int_C\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_j\,e_{n,k}\,z^{j-2n-2k-1}\,dz. \tag{19}$$

Both series, (2) and that on the right of (17), converge uniformly and absolutely in the annulus $1+\frac{1}{2}\varepsilon<|z|<1+\frac{3}{2}\varepsilon$ and hence the resulting double sum in (19) also converges uniformly and absolutely in this annulus. With C the circle $|z|=1+\varepsilon$, we may thus interchange integration and summation in (19). All terms integrate to zero except those for which $j=2n+2k$; summing the residues for $j-2n-2k-1=-1$ we obtain (16). This completes the proof of the theorem.

With μ_k defined by

$$\mu_k=\int_{-1}^{1} w(x)\, x^k\, dx, \qquad k=0,1,2,\ldots, \tag{20}$$

we have the following corollary, which follows from the above theorem and equation (5).

Corollary 1.

$$e_{n,k}=\mu_{2n+2k}-\sum_{j=1}^{n} w_{n,j}\, x_{n,j}^{2n+2k}. \tag{21}$$

Equation (21) provides a useful method for computing the coefficients $e_{n,k}$. It further enables us to establish

Corollary 2.

$$e_{n,k}=\mu_{2n+2k}[1+o(1)], \quad n \text{ fixed}, \quad k\to\infty. \tag{22}$$

Proof. Let $x_0=\max_{k=1,2,\ldots,n}|x_{n,k}|$ where $p_n(x_{n,k})=0$. We define a constant K_n by $K_n x_0^{2n}=\sum_{j=1}^{n} w_{n,j}\, x_{n,j}^{2n}$. Since each $w_{n,j}$ is positive, K_n is also positive. We first show that $\sum_{j=1}^{n} w_{n,j}\, x_{n,j}^{2n+2k}\leqq K_n x_0^{2n+2k}$, $k=0,1,2,\ldots$. For suppose this is true for some integer $k\geqq 0$. Then, since $x_0<1$ and $w_{n,j}>0$, $\sum_{j=1}^{n} w_{n,j}\, x_{n,j}^{2n+2k+2}\leqq \sum_{j=1}^{n} w_{n,j}\, x_{n,j}^{2n+2k}$ $x_0^2\leqq K_n x_0^{2n+2k+2}$, establishing the above inequality for all $k\geqq 0$. Again, since $x_0<1$, we have

$$\mu_{2n+2k}>2\int_{\frac{1}{2}(1+x_0)}^{1} w(x)\, x^{2n+2k} dx\geqq A\left[\tfrac{1}{2}(1+x_0)\right]^{2n+2k}$$

where

$$A=\int_{\frac{1}{2}(1+x_0)}^{1} w(x)\, dx>0.$$

Hence

$$\sum_{j=1}^{n} w_{n,j}\, x_{n,j}^{2n+2k}\leqq \mu_{2n+2k}\left(\frac{K_n}{A}\right)\left[\frac{2x_0}{1+x_0}\right]^{2n+2k}=\mu_{2n+2k}\big(o(1)\big)$$

as $k\to\infty$.

In order to prove that $E_n(f)$ decreases monotonically to zero in the case when all the coefficients a_{2k} in the expansion (2) are non-negative we shall require the following lemma.

Lemma 4. *For all integers $n\geqq 1$, $k\geqq 0$, we have*

$$e_{n,k+1}-e_{n+1,k}>0. \tag{23}$$

Proof. By Theorem 1 and equation (7)

$$e_{n,k}=\frac{1}{2\pi i}\int_C\int_{-1}^{1}\frac{w(x)\, z^{2n+2k}\, p_n(x)}{(z-x)\, p_n(z)}\, dx\, dz \tag{24}$$

where the contour C is the circle of radius $1+\varepsilon>1$. It follows from (24) that

$$(25)\qquad e_{n,k+1}-e_{n+1,k}=\frac{1}{2\pi i}\int_C\int_{-1}^{1}\frac{w(x)\,z^{2n+2k+2}}{p_n(z)\,p_{n+1}(z)}\,\frac{p_{n+1}(z)\,p_n(x)-p_n(z)\,p_{n+1}(x)}{z-x}\,dx\,dz.$$

By the Christoffel-Darboux formula (see e.g. [6]) we may write this as

$$(26)\qquad \begin{aligned} e_{n,k+1}-e_{n+1,k}&=\frac{k_{n+1}}{2\pi k_n i}\int_C\int_{-1}^{1}\frac{w(x)\,z^{2n+2k+2}}{p_n(z)\,p_{n+1}(z)}\sum_{k=0}^{n}p_k(x)\,p_k(z)\,dx\,dz\\ &=\frac{k_0^2 k_{n+1}\mu_0}{2\pi k_n i}\int_C\frac{z^{2n+2k+2}}{p_n(z)\,p_{n+1}(z)}\,dz \end{aligned}$$

using the orthogonality property of the polynomials. Using the expansion for $1/p_n(z)$ as given by Lemma 1, it follows that if $n\geqq 1$ the coefficient of z^{-1} in the expansion of $z^{2n+2k+2}/[p_n(z)\,p_{n+1}(z)]$ in powers of $1/z$ is positive. Hence $e_{n,k+1}-e_{n+1,k}>0$ for all $n\geqq 1$, $k\geqq 0$.

We now observe by (16) that

$$(27)\qquad E_n(f)-E_{n+1}(f)=e_{n,0}\,a_{2n}+\sum_{k=0}^{\infty}(e_{n,k+1}-e_{n+1,k})\,a_{2n+2k+2}.$$

On inspecting this equation in view of Theorem 1 and Lemma 4, we have

Theorem 2. *If in the expansion* (2) $a_{2k}\geqq 0$ *for all integers* $k\geqq N\geqq 1$, *then*

$$(28)\qquad E_n(f)\geqq E_{n+1}(f)\geqq 0$$

for all $n\geqq N$. *If for some* $n=n_0\geqq N$ *and some positive integer* s *we have* $E_{n_0}(f)=E_{n_0+s}(f)$, *then* $E_n(f)=0$ *for all* $n\geqq n_0$, *and* $a_{2k}=0$ *for all* $k\geqq n_0$.

It is noteworthy that if $a_{2k}\geqq 0$ for all $k\geqq N$ and two Gaussian approximate integrations for two different $n\geqq N$ (carried out with infinite precision!) are the same then by Theorem 2 the integrations are exact. We add also on passing that both $E_n(f)$ and $E_n(f)-E_{n+1}(f)$ are continuous functions of the coefficients a_{2n+2k}, $k\geqq 0$, and that for $n\geqq N$ if $E_n(f)-E_{n+1}(f)$ is small then $E_n(f)$ is also small and conversely.

It cannot be expected that the $e_{n,k}$ would in general decrease monotonically as n remains fixed and k increases. Indeed, if this were the case we could make a statement similar to Theorem 2 for the case when for each non-negative k, a_{2n+2k} has a sign opposite to that of $a_{2n+2k+2}$. That is, in this case we could say that either $E_n(f)<0$, $E_{n+1}(f)>0,\ldots$, or this same sequence with the inequalities reversed, together with $|E_{n+1}(f)|<|E_n(f)|$. A typical graph illustrating the relationship between $e_{n,k}$ and μ_{2n+2k} is given in Fig. 1.

3. Error Bounds

By equation (16) it is evident that when the coefficients a_{2k} (see equation (2)) alternate in sign the magnitude of the error of numerical integration will be smaller than when these coefficients are of the same sign.

For example, assume that $f(z)$ has a singularity at $z=b$, i.e., that

$$f(z)=g(z)/(z-b)^a \tag{29}$$

where $|b|>1$, a is real, $g(z)$ is regular for $|z|<|b|+\varepsilon$ with $\varepsilon>0$ and $g(b)\neq 0$. Then (see e.g. [7]) the coefficient a_{2k} in the expansion (2) satisfies

$$a_{2k}=\frac{g(b)}{\Gamma(a)}\,b^{-a-2k}(2k)^{a-1}\left[1+O\left(\frac{1}{k}\right)\right] \tag{30}$$

as $k\to\infty$. Hence if b is real and positive, $E_n(f)$ will have the sign of $g(b)/\Gamma(a)$ and $|E_n(f)|$ will decrease monotonically as n increases for all n sufficiently large.

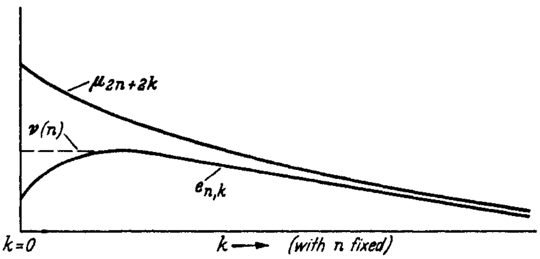

Fig. 1

Assuming the sequence $\{e_{n,k}\}_{k=0}^{\infty}$ to be in l^p $(1\leqq p\leqq\infty)$ we apply Hölder's inequality to (16) to obtain

$$|E_n(f)|\leqq\left(\sum_{k=0}^{\infty}e_{n,k}^p\right)^{1/p}\left(\sum_{k=0}^{\infty}|a_{2n+2k}|^q\right)^{1/q} \tag{31}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

The quantity $\sigma_{n,p}=\left(\sum_{k=0}^{\infty}e_{n,k}^p\right)^{1/p}$ is independent of f and can be computed once and for all. In particular

$$\nu(n)=\sigma_{n,\infty}=\sup_{k=0,1,2,\dots}e_{n,k}\qquad(<\mu_{2n}) \tag{32}$$

always exists and can be computed using equation (22). Whenever $W_n^{\frac12}=\sigma_{n,2}$ exists[4], then by proceeding similarly as in [4] the constant W_n can be shown to be given by

$$\begin{aligned}W_n=&\int_{-1}^{1}\int_{-1}^{1}\frac{w(x)\,w(y)\,(xy)^{2n}}{1-xy}\,dx\,dy-\\&-2\int_{-1}^{1}\sum_{k=1}^{n}\frac{w_{n,k}\,w(y)\,x_{n,k}^{2n}\,y^{2n}}{1-x_{n,k}\,y}\,dy+\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{w_{n,k}\,w_{n,j}\,x_{n,k}^{2n}\,y_{n,j}^{2n}}{1-x_{n,k}\,y_{n,j}}\,.\end{aligned} \tag{33}$$

[4] $W_n^{\frac12}=\sigma_{n,2}$ exists if and only if

$$\int_{-1}^{1}\int_{-1}^{1}\frac{w(x)\,w(y)}{1-xy}\,dx\,dy=\sum_{k=0}^{\infty}\mu_{2k}^2$$

converges.

Some numerical values of $\nu(n)$ and $W_n^{\frac{1}{2}}$ for three well-known types of numerical integration formulae are given in the table below.

We record some of the above remarks in a theorem.

Theorem 3. *Let $f(z)$ be an analytic function of z having the expansion* (2) *in* $|z| < 1 + 2\varepsilon$.

Table. *Error Constants for Theorem 3*

	$w(x)=1$; $x_{n,k}$, $w_{n,k}$ given in [8], p. 916		$w(x)=(1-x^2)^{\frac{1}{2}}$; $x_{n,k}=\cos\left(\frac{\pi k}{n+1}\right)$; $w_{n,k}=\frac{\pi}{n+1}\sin^2\left(\frac{\pi k}{n+1}\right)$		$w(x)=(1-x^2)^{-\frac{1}{2}}$; $x_{n,k}=\cos\left[\frac{(2k-1)\pi}{2n}\right]$; $w_{n,k}=\frac{\pi}{n}$	
n	$\nu(n)$	$W_n^{\frac{1}{2}}$	$\nu(n)$	$W_n^{\frac{1}{2}}$	$\nu(n)$	$W_n^{\frac{1}{2}}$
2	.21164	.55736	.098174	.19888	.67495	These do not exist; see the footnote 4
3	.10222	.39235	.039883	.10790	.44001	
4	.061014	.30368	.019654	.068058	.32799	
5	.040511	.24793	.011182	.046934	.26158	
6	.028867	.20954	.0069788	.034347	.21761	
7	.021618	.18148	.0046443	.026232	.18634	
8	.016797	.16005	.0032477	.020692	.16294	
9	.013430	.14316	.0023600	.016741	.14477	
10	.010983	.12949	.0017688	.013824	.13026	
12	.0077398	.10874	.0010682	.0098863	.10850	
16	.0044363	.082356	.00047615	.0057743	.081347	

(a) *With $\nu(n)$ defined by equation* (32), *we have*

$$|E_n(f)| \leqq \nu(n) \sum_{k=0}^{\infty} |a_{2n+2k}|. \tag{34}$$

(b) *If W_n (equation* (33)) *exists, then*

$$|E_n(f)| \leqq W_n^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} |a_{2n+2k}|^2 \right)^{\frac{1}{2}} < W_n^{\frac{1}{2}} \left[\frac{1}{2\pi} \int_0^{2\pi} F(e^{i\vartheta}) \overline{F(e^{i\vartheta})}\, d\vartheta \right]^{\frac{1}{2}} \tag{35}$$

The sharpness of the bounds (34) and (35) will depend on how well the magnitudes of the Taylor series coefficients can be estimated. The estimate $|a_{2k}| \leqq M(r)/r^{2k}$, where $M(r) = \max_{|z|=r} |f(z) + f(-z)|$, is applicable[5] for the general class $\{f\}$ we have considered. We shall illustrate with some examples.

Example 1. Let $M(r) \leqq A(1-br)^{-a}$, where $a>0$, $0<b<1$. Then $|a_{2k}| \leqq A r^{-2k}(1-br)^{-a}$, $0<r<b^{-1}$. We minimize this last inequality with respect to r

[5] This estimate is obtained from Cauchy's integral formula. A better estimate would be $|a_{2k}| \leqq \inf_{p_{2k-1}} \left\{ \sup_{|z|=r} |f(z) - p_{2k-1}(z)|\, r^{-2k} \right\}$ where p_{2k-1} is any polynomial of degree $2k-1$. In particular with p_{2k-1} the Taylor polynomial, the right side of this inequality tends to $|a_{2k}|$ as $r \to 0$.

to obtain $|a_{2k}| \leqq A\, b^{2k} e^{a^2} \left(\frac{2k}{a}\right)^a$. Next, using this estimate on $|a_{2k}|$ to make crude estimates on

$$\sum_{k=0}^{\infty} |a_{2n+2k}|, \quad \text{and} \quad \sum_{k=0}^{\infty} |a_{2n+2k}|^2,$$

we obtain

$$|E_n(f)| < \nu(n)\, A\, e^{a^2} \left[\frac{1}{a+1} + (a+1)\,|2\log b|^{-a-1}\right] \left(\frac{2n}{a}\right)^a b^{2n-2}$$

or

$$|E_n(f)| < W_n^{\frac{1}{2}}\, A\, e^{a^2} \left[\frac{1}{a+1} + (a+1)\,|4\log b|^{-a-1}\right]^{\frac{1}{2}} \left(\frac{2n}{a}\right)^a b^{2n-2}.$$

Example 2. Let $M(r) \leqq A\, e^{a r^b}$, where A, a and b are positive numbers. Using $|a_{2k}| \leqq M(r)/r^{2k}$ and minimizing with respect to r we obtain $|a_{2k}| \leqq A\left(\frac{e\,a\,b}{2k}\right)^{2k/b}$. Thus with n sufficiently large so that $\left(\frac{e\,a\,b}{2n}\right)^{2/b} < 1$, we have in this case

$$|E_n(f)| < \nu(n)\, A \left(\frac{e\,a\,b}{2n}\right)^{2n/b} \left[1 - \left(\frac{e\,a\,b}{2n}\right)^{2/b}\right]^{-1}$$

or

$$|E_n(f)| < W_n^{\frac{1}{2}}\, A \left(\frac{e\,a\,b}{2n}\right)^{2n/b} \left[1 - \left(\frac{e\,a\,b}{2n}\right)^{4/b}\right]^{-\frac{1}{2}}.$$

Example 3. Here we shall bound $E_n(f)$, given $f(z) = z^s e^z$, where s is a positive integer. It follows from this, that, since $a_k = \frac{1}{(k-s)!}$, $k \geqq s$,

$$0 < E_n(f) < \nu(n)\, \frac{e}{(2n-s)!} \sim \nu(n)\,(2\pi e)^{\frac{1}{2}} \left(\frac{e}{2n-s}\right)^{2n-s+\frac{1}{2}}.$$

The above examples illustrate that the convergence of Gaussian quadrature is much more rapid in the case when f is entire than in the case when f has a singularity in the finite plane.

Is it possible to improve the inequality (31)? The restriction that $f(z)$ be regular in $|z| < 1 + 2\varepsilon$ with $\varepsilon > 0$ can be dropped; it suffices to assume that $f(z)$ is regular in $|z| < 1$ and continuous on $|z| = 1$ for equation (16) to hold. We also observe that $E_n(f)$ is a linear functional in f, and hence with $f^*(z) = \sum_{k=0}^{\infty} a_k^* z^k$, $|z| < 1$ in the null-space of E_n (i.e. $E_n(f^*) = 0$), we have $E_n(f + f^*) = E_n(f)$. Thus for all such f^* we have

$$(36) \qquad |E_n(f)| \leqq \sigma_{n,p} \left(\sum_{k=0}^{\infty} |a_{2n+2k} - a_{2n+2k}^*|^q\right)^{1/q}.$$

For each q in $1 \leqq q < \infty$ there exists an f^* minimizing the right hand side of (36); this minimum is in fact equal to $|E_n(f)|$. However in practice finding the f^* which achieves this minimum is as difficult as finding the exact value of the original integral.[6] We can nevertheless increase the rate of convergence by "sub-

[6] The even coefficients of the minimizing f^* are given by

$$a_{2n+2k}^* = a_{2n+2k} - \frac{e_{n,k}^{p-1}\, E_n(f)}{\sigma_{n,p}^p}.$$

tracting out" singularities of f close to the region of integration. In addition it is noteworthy that any odd function, and any polynomial of degree $2n-1$ is in the null-space of $E_n(f)$.

For sake of completeness we also obtain an estimate on the bound of $E_n(f)$ when f has complex singularities close to the region of integration. The assumptions on f in the following theorem are similar to those of DAVIS [2].

Theorem 4. *Let $f(z)$ be real when z is real, and let $f(z)$ be regular in the ellipse ξ_ϱ with foci at $(\pm 1, 0)$ and semi-axis a, b where $\varrho = a+b > 1$. Let*

$$M(\varrho) = \sup_{z\in\xi_\varrho} |\operatorname{Re} f(z)|. \tag{37}$$

Then

$$|E_n(f)| < \frac{16\mu_0}{\pi} M(\varrho)\,\varrho^{-2n}. \tag{38}$$

Proof. Using equation (5) we have by the last remark of the previous paragraph, that

$$|E_n(f)| = \inf_{P_{2n-1}} \left| \int_{-1}^{1} w(x)\,[f(x) - P_{2n-1}(x)]\,dx - \sum_{k=1}^{n} w_{n,k}[f(x_{n,k}) - P_{2n-1}(x_{n,k})] \right|. \tag{39}$$

By a result of ACHIESER (see e.g. [9], p. 87)

$$\inf_{P_{2n-1}(x)} \left\{ \sup_{-1\leq x\leq 1} |f(x) - P_{2n-1}(x)| \right\} < \frac{8}{\pi} M(\varrho)\,\varrho^{-2n} \tag{40}$$

where $M(\varrho)$ is given by (37) above. Thus combining (39) and (40) we obtain (38).

Although the estimate (38) is not the best possible it is simple and easily obtained from other known results. For example if $M(\varrho) \leq A(1-c\,\varrho)^{-d}$ where $d>0$, $0<c<1$, then minimizing the right of (38) with respect to ϱ we obtain

$$|E_n(f)| < \frac{16\mu_0}{\pi}\left(\frac{2n}{d}\right)^d e^{d^2} c^{2n}.$$

It is noteworthy that in each of the above examples we can find the n required to give us a desired accuracy.

4. Conclusion

It was hoped at the undertaking of the writing of this paper that it would be possible in certain cases of Gaussian integration to obtain the results stated in Theorems 1 and 2, and also a theorem analogous to Theorem 2 but applicable to the case of $f(z)$ as in (2) with the magnitude of the a_k's decreasing monotonically and the sign of a_{2k+2} being different from that of a_{2k} for all k sufficiently large. It was hoped that in this last case it would be possible to establish that $|E_n(f)|$ tends monotonically to zero and the sign of $E_{n+1}(f)$ is different from that of $E_n(f)$ for all n sufficiently large. This last objective has not been achieved. The main difficulty is that the $e_{n,k}$ do not in general decrease monotonically as n remains fixed and k increases.

A number of the results we have obtained are nevertheless useful and new. This applies particularly to the error bounds in Section 3. We have attempted to obtain bounds suited to the Gaussian quadrature formulae, the only formulae so constructed that they exactly integrate the first $2n$ terms of the Taylor series expansion of the function $f(z)$ about $z=0$. With the exception of (38) the bounds obtained are of the form $|E_n(f)| \leqq \sigma_n \|F_n\|$, where σ_n ($\sigma_n \to 0$ as $n \to \infty$) is independent of f and $\|F_n\|$ ($\|F_n\| \to 0$ as $n \to \infty$) depends only on a_{2k} (see equation (2)), $k=n, n+1, n+2, \ldots$.

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